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## Multiplicities of the discrete series

Paul-Emile Paradan

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# MULTIPLICITIES OF THE DISCRETE SERIES

PAUL-EMILE PARADAN

ABSTRACT. The purpose of this paper is to show that the multiplicities of a discrete series representation relatively to a compact subgroup can be “computed” geometrically, in the way predicted by the “quantization commutes with reduction” principle of Guillemin-Sternberg.

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

In the 50’s, Harish Chandra has constructed the holomorphic discrete series representations of a real semi-simple Lie group  $G$  as quantization of certain *rigged* elliptic orbits, in a way similar to the Borel-Weil Theorem. Here the quantization procedure is the one that Kirillov calls *geometric* in [17].

The first purpose of this paper is to explain how one can “compute” geometrically the multiplicities of a holomorphic representation of a real simple Lie group relatively to a compact connected subgroup : our main result is Theorem 1.8. This computation follows the line of the orbit method [17] and is a non-compact example of the “quantization commutes with reduction” phenomenon [12, 29, 33, 39].

Next we show that this result extends to the discrete series representations for which the Harish-Chandra and Blattner parameters belong to the same chamber of strongly elliptic elements. See Theorem 1.9.

In our previous article [34], we prove that a similar result occurs : the multiplicities of any discrete series representation relatively to a maximal compact subgroup can be “computed” geometrically.

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Nevertheless, our present contribution is not a consequence of the results of [34], for two reasons:

- (1) In [34], we were working with the *metaplectic version* of the quantization (we prefer the denomination “Spin” quantization). Here we work with the *geometric version* of the quantization. See the review of Vogan [41] for a brief explanation concerning this two kinds of quantization.
- (2) The other main difference with [34] is that here we look at the multiplicities relatively to *any compact connected subgroup*, subordinated to the condition that the multiplicities are finite.

Our main tool to investigate (2) is the “formal geometric quantization” procedure that we have studied in [35].

Finally, we mention that our present paper is strongly related to the works of Kobayashi [21, 22, 23] and Duflo-Vargas [9] where they study the general setting of restrictions of irreducible representations to a reductive subgroup.

**1.1. Realisation of the holomorphic discrete series.** Let  $G$  be a connected real simple Lie group with finite center and let  $K$  be a maximal compact subgroup. We make the choice of a maximal torus  $T$  in  $K$ . Let  $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$  be the Lie algebras of  $G, K, T$ . We consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ .

We assume that  $G$  admits holomorphic discrete series representations. It is the case if and only if the real vector space  $\mathfrak{p}$  admits a  $K$ -invariant complex structure, or equivalently, if the center  $Z(K)$  of  $K$  is equal to the circle group : hence the complex structure on  $\mathfrak{p}$  is defined by the adjoint action of an element  $z_o$  in the Lie algebra of  $Z(K)$ .

Let  $\Lambda^* \subset \mathfrak{t}^*$  be the weight lattice :  $\alpha \in \Lambda^*$  if  $i\alpha$  is the differential of a character of  $T$ . Let  $\mathfrak{R} \subset \Lambda^*$  be the set of roots for the action of  $T$  on  $\mathfrak{g} \otimes \mathbb{C}$ . We have  $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$  where  $\mathfrak{R}_c$  and  $\mathfrak{R}_n$  are respectively the set of roots for the action of  $T$  on  $\mathfrak{k} \otimes \mathbb{C}$  and  $\mathfrak{p} \otimes \mathbb{C}$ . We fix a system of positive roots  $\mathfrak{R}_c^+$  in  $\mathfrak{R}_c$ . We have  $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$  where the  $K$ -module  $\mathfrak{p}^\pm$  is equal to  $\ker(\text{ad}(z_o) \mp i)$ . Let  $\mathfrak{R}_n^{\pm, z_o}$  be the set of roots for the action of  $T$  on  $\mathfrak{p}^\pm$ . The union  $\mathfrak{R}_c^+ \cup \mathfrak{R}_n^{+, z_o}$  defines then a system of positive roots in  $\mathfrak{R}$  that we denote by  $\mathfrak{R}_{\text{hol}}^+$ .

Let  $\mathfrak{t}_+^* \subset \mathfrak{t}^*$  be the Weyl chamber defined by the system of positive roots  $\mathfrak{R}_c^+$ . Let  $\mathcal{C}_{\text{hol}} := \{\xi \in \mathfrak{t}_+^* \mid (\beta, \xi) > 0, \forall \beta \in \mathfrak{R}_n^{+, z_o}\}$ , where  $(\cdot, \cdot)$  denotes the scalar product on  $\mathfrak{t}^*$  induced by the Killing form of  $\mathfrak{g}$ . So the closure  $\overline{\mathcal{C}_{\text{hol}}}$  is the Weyl chamber defined by the system of positive roots  $\mathfrak{R}_{\text{hol}}^+$ .

The complex vector space  $\mathfrak{p}^+$  is an irreducible  $K$ -representation. Hence, if  $\beta_{\min}$  is the lowest  $T$ -weight on  $\mathfrak{p}^+$ , every weight  $\beta \in \mathfrak{R}_n^{+, z_o}$  is of the form  $\beta = \beta_{\min} + \sum_{\alpha \in \mathfrak{R}_c^+} n_\alpha \alpha$  with  $n_\alpha \in \mathbb{N}$ . Then we have

$$(1.1) \quad \mathcal{C}_{\text{hol}} = \mathfrak{t}_+^* \cap \{\xi \in \mathfrak{t}^* \mid (\xi, \beta_{\min}) > 0\}.$$

Note that every  $\xi \in \mathcal{C}_{\text{hol}}$  is *strongly elliptic*: the stabilizer subgroup  $G_\xi$  is compact and coincides with the stabilizer subgroup  $K_\xi$ .

For every weight  $\Lambda \in \Lambda^* \cap \mathcal{C}_{\text{hol}}$ , we consider the coadjoint orbit

$$\mathcal{O}_\Lambda := G \cdot \Lambda \subset \mathfrak{g}^*.$$

For  $X \in \mathfrak{g}$ , let  $VX$  be the vector field on  $\mathcal{O}_\Lambda$  defined by :  $VX(\xi) := \frac{d}{dt} e^{-tX} \cdot \xi|_{t=0}$ ,  $\xi \in \mathcal{O}_\Lambda$ . We have on the coadjoint orbit  $\mathcal{O}_\Lambda$  the following data :

- (1) The Kirillov-Kostant-Souriau symplectic form  $\Omega_\Lambda$  which is defined by the relation : for any  $X, Y \in \mathfrak{g}$  and  $\xi \in \mathcal{O}_\Lambda$  we have

$$\Omega_\Lambda(VX, VY)|_\xi = \langle \xi, [X, Y] \rangle.$$

- (2) The inclusion  $\Phi_G : \mathcal{O}_\Lambda \hookrightarrow \mathfrak{g}^*$  is a moment map relative to the Hamiltonian action of  $G$  on  $(\mathcal{O}_\Lambda, \Omega_\Lambda)$ .
- (3) A  $G$ -invariant complex structure  $J_\Lambda$  characterized by the following fact. The holomorphic tangent bundle  $\mathbf{T}^{1,0}\mathcal{O}_\Lambda \rightarrow \mathcal{O}_\Lambda$  is equal, above  $\Lambda \in \mathcal{O}_\Lambda$ , to the  $T$ -module

$$\sum_{\substack{\alpha \in \mathfrak{A}_n^+ \\ \langle \alpha, \Lambda \rangle \neq 0}} \mathfrak{g}_\alpha + \underbrace{\sum_{\beta \in \mathfrak{A}_n^-} \mathfrak{g}_\beta}_{\mathfrak{p}^-}.$$

- (4) The line bundle  $\mathcal{L}_\Lambda := G \times_{K_\Lambda} \mathbb{C}_\Lambda$  over  $G/K_\Lambda \simeq \mathcal{O}_\Lambda$  with its canonical holomorphic structure. Here  $\mathbb{C}_\Lambda$  is the one dimensional representation of the stabilizer subgroup  $K_\Lambda$  attached to the weight  $\Lambda \in \Lambda^*$ .

One can check that the complex structure  $J_\Lambda$  is positive relatively to the symplectic form, e.g.  $\Omega_\Lambda(-, J_\Lambda -)$  defines a Riemannian metric on  $\mathcal{O}_\Lambda$ . Hence  $(\mathcal{O}_\Lambda, \Omega_\Lambda, J_\Lambda)$  is a Kähler manifold. Moreover the first Chern class of  $\mathcal{L}_\Lambda$  is equal to  $[\frac{\Omega_\Lambda}{2\pi}]$  : the line bundle  $\mathcal{L}_\Lambda$  is an equivariant *pre-quantum* line bundle over  $(\mathcal{O}_\Lambda, \Omega_\Lambda)$  [25, 38].

We are interested in the geometric quantization of the coadjoint orbits  $\mathcal{O}_\Lambda$ ,  $\Lambda \in \Lambda^* \cap \mathcal{C}_{\text{hol}}$ . We take on  $\mathcal{O}_\Lambda$  the invariant volume form defined by its symplectic structure. The line bundle  $\mathcal{L}_\Lambda$  is equipped with a  $G$ -invariant Hermitian metric (which is unique up to a multiplicative constant).

**Definition 1.1.** We denote  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  the Hilbert space of square integrable holomorphic sections of the line bundle  $\mathcal{L}_\Lambda \rightarrow \mathcal{O}_\Lambda$ .

The irreducible representations of  $K$  are parametrized by the set of dominant weights, that we denote

$$\widehat{K} := \Lambda^* \cap \mathfrak{t}_+^*.$$

For any  $\mu \in \widehat{K}$ , we denote  $V_\mu^K$  the irreducible representation of  $K$  with highest weight  $\mu$ .

Let  $\rho_n$  be half the sum of the elements of  $\mathfrak{A}_n^{+, z_0}$ . Let  $S(\mathfrak{p}^+)$  be the symmetric algebra of the vector space  $\mathfrak{p}^+$  : it is an admissible representation of  $K$  since the center  $Z(K)$  acts on  $\mathfrak{p}^+$  as the rotation group.

The following theorem is due to Harish Chandra [13]. See also the nice exposition [19].

**Theorem 1.2.** Let  $\Lambda \in \Lambda^* \cap \mathcal{C}_{\text{hol}}$ . Then

- If  $(\Lambda, \beta_{\min}) < 2(\rho_n, \beta_{\min})$ , the Hilbert space  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  is reduced to  $\{0\}$ .
- If  $(\Lambda, \beta_{\min}) \geq 2(\rho_n, \beta_{\min})$ , the Hilbert space  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  is an irreducible representation of  $G$  such that the subspace of  $K$ -finite vectors is isomorphic to  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$ .

The holomorphic discrete series representations of  $G$  are those of the form  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$ , for  $\Lambda \in \widehat{K} \cap \mathcal{C}_{\text{hol}}^\geq$  where

$$(1.2) \quad \mathcal{C}_{\text{hol}}^\geq = \{ \xi \in \mathfrak{t}_+^* \mid (\xi - 2\rho_n, \beta_{\min}) \geq 0 \}.$$

Here, we have parametrized the holomorphic discrete series representations  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  by their *Blattner parameter*  $\Lambda \in \widehat{K} \cap \mathcal{C}_{\text{hol}}^\geq$ . The corresponding *Harish-Chandra parameter* is  $\lambda := \Lambda + \rho_c - \rho_n$ , where  $\rho_c$  is half the sum of the elements of  $\mathfrak{R}_c^+$ . One checks that the map  $\Lambda \mapsto \Lambda + \rho_c - \rho_n$  is a one to one map between  $\widehat{K} \cap \mathcal{C}_{\text{hol}}^\geq$  and

$$(1.3) \quad \widehat{\mathcal{G}}_{\text{hol}} := \{\lambda \in \mathfrak{t}^* \mid (\lambda, \alpha) > 0 \ \forall \alpha \in \mathfrak{R}_{\text{hol}}^+ \text{ and } \lambda + \rho_n + \rho_c \in \Lambda^*\}.$$

**Example 1.3.** *Let us consider the case of the symplectic group  $G = \text{Sp}(2, \mathbb{R})$ . Here  $K$  is the unitary group  $\text{U}(2)$ , and the maximal torus is of dimension 2. In the figure 1.1, we draw the chambers  $\mathcal{C}_{\text{hol}}^\geq \subset \mathcal{C}_{\text{hol}} \subset \mathfrak{t}_+^*$ ,  $\alpha$  is the unique positive compact root, and  $\beta_1, \beta_2, \beta_3$  are the positive non-compact roots. The root  $\beta_3$  corresponds to the root  $\beta_{\text{min}}$  used in (1.1).*

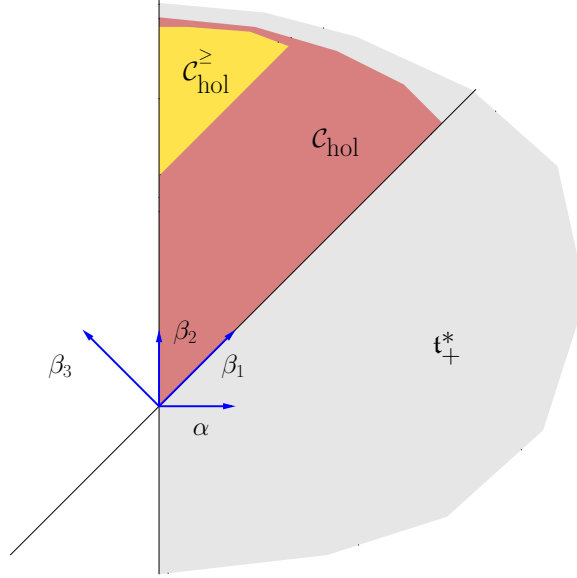


FIGURE 1. The case of  $\text{Sp}(2, \mathbb{R})$

**1.2. Main results concerning the holomorphic discrete series.** Let  $H \subset K$  be a compact connected Lie group with Lie algebra  $\mathfrak{h}$ . The  $H$ -action on  $(\mathcal{O}_\Lambda, \Omega_\Lambda)$  is Hamiltonian with moment map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  equal to the composition of  $\Phi_G : \mathcal{O}_\Lambda \rightarrow \mathfrak{g}^*$  with the projection  $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ .

**Notation 1.4.** *We denote  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  the (dense) vector subspace of  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  formed by the  $H$ -finite vectors.*

When  $\Lambda \in \mathcal{C}_{\text{hol}}^\geq$ , we know thanks to Theorem 1.2, that  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  is the “restriction” of the  $K$ -representation  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$ : we will also denote it as  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$ . We are interested in the case where the  $H$ -multiplicities in  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$  are finite, e.g.  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$  is  $H$ -admissible.

The *asymptotic  $K$ -support* of a  $K$ -representation  $E$  is the closed cone of  $\mathfrak{t}_+^*$ , denoted by  $\text{AS}_K(E)$ , formed by the limits  $\lim_{n \rightarrow \infty} \epsilon_n \mu_n$ , where  $(\epsilon_n)_{n \in \mathbb{N}}$  is a sequence of non-negative real numbers converging to 0 and  $(\mu_n)_{n \in \mathbb{N}}$  is a sequence of  $\widehat{K}$  such that  $\text{hom}_K(V_{\mu_n}^K, E) \neq 0$  for all  $n \in \mathbb{N}$ .

For any closed subgroup  $H$  of  $K$ , we denote  $\mathfrak{h}^\perp \subset \mathfrak{t}^*$  the orthogonal for the duality of the Lie algebra of  $H$ . We have the following result of T. Kobayashi.

**Proposition 1.5** ([21, 24]). *Let  $E$  be an admissible  $K$ -representation. Let  $H$  be a compact subgroup of  $K$ . Then the following two conditions are equivalent:*

- (1)  $E|_H$  is  $H$ -admissible.
- (2)  $\text{AS}_K(E) \cap K \cdot \mathfrak{h}^\perp = \{0\}$ .

Let  $\{\gamma_1, \dots, \gamma_r\}$  be a maximal family of strongly orthogonal roots (see Section 5). Schmid [36] has shown that  $S(\mathfrak{p}^+)$  is a  $K$ -representation without multiplicity, and that the representation  $V_\mu^K$  occurs in  $S(\mathfrak{p}^+)$  if and only if

$$\mu = \sum_{k=1}^r n_k (\gamma_1 + \dots + \gamma_k), \quad \text{with } n_k \in \mathbb{N}.$$

Thus, we check easily that the asymptotic  $K$ -support of  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$  is equal to

$$\sum_{k=1}^r \mathbb{R}^{\geq 0} (\gamma_1 + \dots + \gamma_k).$$

The following proposition is proved in Sections 4 and 5 (see Theorem 5.6).

**Proposition 1.6.** *Let  $\Lambda \in \mathcal{C}_{\text{hol}}$ . The following statements are equivalent.*

- (1) *The representation  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$  is admissible.*
- (2) *We have  $\sum_{k=1}^r \mathbb{R}^{\geq 0} (\gamma_1 + \dots + \gamma_k) \cap K \cdot \mathfrak{h}^\perp = \{0\}$ .*
- (3) *The map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  is proper.*

We know from Proposition 1.5 that (1) and (2) are equivalent, thus our main contribution is the equivalence with (3). Nevertheless our proof of Proposition 1.6 does not use directly the result of Proposition 1.5. We prove in Section 4 that (1) and (3) are both equivalent to the condition

$$\Delta_K(\mathfrak{p}) \cap K \cdot \mathfrak{h}^\perp = \{0\},$$

where  $\Delta_K(\mathfrak{p}) \subset \mathfrak{t}_+^*$  is the Kirwan convex set associated to the Hamiltonian action of  $K$  on  $\mathfrak{p}$ . In Section 5, a direct computation gives that

$$(1.4) \quad \Delta_K(\mathfrak{p}) = \sum_{k=1}^r \mathbb{R}^{\geq 0} (\gamma_1 + \dots + \gamma_k).$$

Another way to obtain (1.4) is by using the theorem of Schmid (which computes the  $K$ -multiplicities in  $S(\mathfrak{p}^+)$ ) together with the following fact: for any affine variety  $\mathcal{X} \subset \mathbb{C}^n$  which is invariant relative to the linear action of  $K$  on  $\mathbb{C}^n$ , the Kirwan set  $\Delta_K(\mathcal{X})$  is equal to the asymptotic  $K$ -support of the algebra  $\mathbb{C}[\mathcal{X}]$  of polynomial functions on  $\mathcal{X}$  (see the Appendix by Mumford in [31]).

**Example 1.7.** *Since the representation  $\mathcal{Q}_{Z(K)}(\mathcal{O}_\Lambda)$  is admissible, the representation  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  will be admissible for any subgroup  $H$  containing  $Z(K)$ .*

The irreducible representations of the compact Lie group  $H$  are parametrized by a set of dominant weights  $\widehat{H} \subset \mathfrak{h}^*$ . For any  $\mu \in \widehat{H}$ , we denote  $V_\mu^H$  the irreducible representation of  $H$  with highest weight  $\mu$ .

We suppose now that the moment map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  is proper, and one wants to compute the multiplicities of  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$ .

If  $\xi \in \mathfrak{h}^*$  is a regular value of  $\Phi_H$ , the Marsden-Weinstein reduction

$$(\mathcal{O}_\Lambda)_\xi := \Phi_H^{-1}(H \cdot \xi)/H$$

is a compact Kähler orbifold. If moreover  $\xi$  is integral, e.g.  $\xi = \mu \in \widehat{H}$ , there exists a holomorphic line orbibundle  $\mathcal{L}(\mu)$  that prequantizes the symplectic orbifold  $(\mathcal{O}_\Lambda)_\mu$ . In this situation, one defines the integer

$$\mathcal{Q}((\mathcal{O}_\Lambda)_\mu) \in \mathbb{Z},$$

as the holomorphic Euler characteristic of  $((\mathcal{O}_\Lambda)_\mu, \mathcal{L}(\mu))$ .

In the general case where  $\mu$  is not necessarily a regular value of  $\Phi_H$ ,  $\mathcal{Q}((\mathcal{O}_\Lambda)_\mu) \in \mathbb{Z}$  can still be defined (see [29, 33]). The integer  $\mathcal{Q}((\mathcal{O}_\Lambda)_\mu)$  only depends on the data  $(\mathcal{O}_\Lambda, \mathcal{L}_\Lambda, J_\Lambda)$  in a small neighborhood of  $\Phi_H^{-1}(\mu)$ : in particular  $\mathcal{Q}((\mathcal{O}_\Lambda)_\mu)$  vanishes when  $\mu$  does not belong to the image of  $\Phi_H$ .

Now we can state one of our main result.

**Theorem 1.8.** *Consider a holomorphic discrete series representation  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  with Blattner parameter  $\Lambda \in \mathcal{C}_{\text{hol}}^\geq$ . Let  $H \subset K$  be a compact connected Lie group such that the representation  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  is admissible. Then we have*

$$\mathcal{Q}_H(\mathcal{O}_\Lambda) = \sum_{\mu \in \widehat{H}} \mathcal{Q}((\mathcal{O}_\Lambda)_\mu) V_\mu^H.$$

*In other words, the multiplicity of  $V_\mu^H$  in the holomorphic discrete series representation  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  is equal to  $\mathcal{Q}((\mathcal{O}_\Lambda)_\mu)$ .*

A question still remains. When  $\mu \in \widehat{H}$  is a *regular* value of the moment map  $\Phi_H$ , Theorem 1.8 says that the multiplicity  $m_\Lambda(\mu)$  of the irreducible representation  $V_\mu^H$  in  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  is equal to the holomorphic Euler characteristic of line orbibundle  $\mathcal{L}(\mu) \rightarrow (\mathcal{O}_\Lambda)_\mu$ . Does the multiplicity  $m_\Lambda(\mu)$  coincides with the dimension of the vector space

$$H^0((\mathcal{O}_\Lambda)_\mu, \mathcal{L}(\mu))$$

of holomorphic sections of  $\mathcal{L}(\mu) \rightarrow (\mathcal{O}_\Lambda)_\mu$ ?

**1.3. Main result concerning the discrete series.** We work now with a real semi-simple Lie group  $G$  such that a maximal torus  $T$  in  $K$  is a Cartan subgroup of  $G$ . We know then that  $G$  has discrete series representations [14]. Nevertheless, we do not assume that  $G$  has *holomorphic* discrete series representations.

Harish-Chandra parametrizes the discrete series representations of  $G$  by a discrete subset  $\widehat{G}_d$  of regular elements of the Weyl chamber  $\mathfrak{t}_+^*$  [14]. He associates to any  $\lambda \in \widehat{G}_d$  an irreducible, square integrable, unitary representation  $\mathcal{H}_\lambda$  of  $G$ :  $\lambda$  is the *Harish-Chandra parameter* of  $\mathcal{H}_\lambda$ . The corresponding *Blattner parameter* of  $\mathcal{H}_\lambda$  is

$$\Lambda(\lambda) := \lambda - \rho_c + \rho_n(\lambda) \in \Lambda^*,$$

where  $\rho_n(\lambda)$  is half the sum of the non-compact roots  $\beta$  satisfying  $(\beta, \lambda) > 0$ .

We work under the following condition

$$(1.5) \quad (\beta, \lambda)(\beta, \Lambda(\lambda)) > 0 \quad \text{for any } \beta \in \mathfrak{R}_n.$$

The set of strongly elliptic elements of the Weyl chamber  $\mathfrak{t}_+^*$  decomposes as an union  $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_r$  of connected component : each chamber  $\mathcal{C}_i$  corresponds to a choice of positive roots  $\mathfrak{R}^{+,i} \subset \mathfrak{R}$  containing  $\mathfrak{R}_c^+$ . Condition (1.5) asks that  $\lambda$  and  $\Lambda(\lambda)$  belong to the same chamber  $\mathcal{C}_i$ .

When  $G$  admits holomorphic discrete series, there is a particular chamber  $\mathcal{C}_{\text{hol}}$  of strongly elliptic elements such that the intersection  $\widehat{G}_d \cap \mathcal{C}_{\text{hol}}$  is equal to the subset  $\widehat{\mathcal{C}}_{\text{hol}}$  defined in (1.3). We noticed already that the map  $\lambda \mapsto \Lambda(\lambda)$  defines a one to one map between  $\widehat{G}_d \cap \mathcal{C}_{\text{hol}}$  and  $\widehat{K} \cap \mathcal{C}_{\text{hol}}^{\geq}$ . In particular, any  $\lambda \in \widehat{G}_d \cap \mathcal{C}_{\text{hol}}$  satisfies Condition 1.5. We give in Section 6.2 some examples where Condition (1.5) does not hold.

Let  $\lambda \in \widehat{G}_d$  satisfying (1.5). The coadjoint orbit  $\mathcal{O}_{\Lambda(\lambda)}$  is pre-quantized by the line bundle  $G \times_{K_{\Lambda(\lambda)}} \mathbb{C}_{\Lambda(\lambda)}$ . One of the main difference with the holomorphic case is that the orbit  $\mathcal{O}_{\Lambda(\lambda)}$  is equipped with an invariant *almost complex structure*  $J_{\Lambda(\lambda)}$ , which is compatible with the symplectic form, but which is not *integrable* in general.

Let  $H$  be a compact connected Lie subgroup of  $K$ . Suppose that the moment map,  $\Phi_H : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{h}^*$  corresponding to the Hamiltonian action of  $H$  on  $\mathcal{O}_{\Lambda(\lambda)}$ , is *proper*. The reduced spaces  $(\mathcal{O}_{\Lambda(\lambda)})_{\mu} := \Phi_H^{-1}(H \cdot \mu)/H$  are in general *not Kähler*. Nevertheless, their geometric quantization  $\mathcal{Q}((\mathcal{O}_{\Lambda(\lambda)})_{\mu}) \in \mathbb{Z}$  are well defined as the index of a Dolbeault-Dirac operator (see [29, 33]).

The following theorem is proved in Section 6.

**Theorem 1.9.** *Consider a discrete series representation  $\mathcal{H}_{\lambda}$  with a Harish-Chandra parameter  $\lambda \in \widehat{G}_d$  satisfying condition (1.5). Let  $H \subset K$  be a compact connected Lie subgroup such that the moment map  $\Phi_H : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{h}^*$  is proper. Then*

- *the representation  $\mathcal{H}_{\lambda}|_H$  is admissible,*
- *we have*

$$\mathcal{H}_{\lambda}|_H = \sum_{\mu \in \widehat{H}} \mathcal{Q}((\mathcal{O}_{\Lambda(\lambda)})_{\mu}) V_{\mu}^H.$$

*In other words, the multiplicity of  $V_{\mu}^H$  in the discrete series representation  $\mathcal{H}_{\lambda}$  is equal to the quantization of the symplectic reduction  $(\mathcal{O}_{\Lambda(\lambda)})_{\mu}$ .*

Theorem 1.9 applies for (most of) the discrete series, but is less precise than the results described in Section 1.2. We *expect* that:

- (1) The properness of  $\Phi_H : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{h}^*$  should only depend of the chamber  $\mathcal{C}_i$  containing  $\Lambda(\lambda)$ .
- (2) The properness of the the moment map  $\Phi_H : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{h}^*$  should be equivalent to the admissibility of the restriction  $\mathcal{H}_{\lambda}|_H$ .

Duflo-Vargas [9] have shown that the admissibility of the restriction  $\mathcal{H}_{\lambda}|_H$  is equivalent to the properness of the moment map  $\mathcal{O}_{\lambda} \rightarrow \mathfrak{h}^*$ . Since we assume that  $\Lambda(\lambda)$  and  $\lambda$  belong to the same chamber, point (1) induces point (2).

Something which is also lacking is an effective criterium which tells us when the map  $\Phi_H : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{h}^*$  is proper. See [9] for some results in this direction.

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## 2. QUANTIZATION COMMUTES WITH REDUCTION

In this section, first we recall the “quantization commutes with reduction” phenomenon of Guillemin-Sternberg which was first proved by Meinrenken and Meinrenken-Sjamaar [28, 29]. Next we explain the functorial properties of the “formal geometric quantization” of non-compact Hamiltonian manifolds [35].

**2.1. Quantization commutes with reduction: the compact case.** Let  $M$  be a *compact* Hamiltonian  $K$ -manifold with symplectic form  $\Omega$  and moment map  $\Phi_K : M \rightarrow \mathfrak{k}^*$  characterized by the relation

$$(2.6) \quad \iota(VX)\Omega = -d\langle\Phi_K, X\rangle, \quad X \in \mathfrak{k},$$

where  $VX$  is the vector field on  $M$  generated by  $X \in \mathfrak{k}$ .

Let  $J$  be a  $K$ -invariant *almost* complex structure on  $M$  which is assumed to be compatible with the symplectic form :  $\Omega(-, J-)$  defines a Riemannian metric on  $M$ . We denote  $RR^K(M, -)$  the Riemann-Roch character defined by  $J$ . Let us recall the definition of this map.

Let  $E \rightarrow M$  be a complex  $K$ -vector bundle. The almost complex structure on  $M$  gives the decomposition  $\wedge \mathbf{T}^*M \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} \mathbf{T}^*M$  of the bundle of differential forms. Using Hermitian structure in the tangent bundle  $\mathbf{T}M$  of  $M$ , and in the fibers of  $E$ , we define a Dolbeault-Dirac operator  $\bar{\partial}_E + \bar{\partial}_E^* : \mathcal{A}^{0,even}(M, E) \rightarrow \mathcal{A}^{0,odd}(M, E)$ , where  $\mathcal{A}^{i,j}(M, E) := \Gamma(M, \wedge^{i,j} \mathbf{T}^*M \otimes_{\mathbb{C}} E)$  is the space of  $E$ -valued forms of type  $(i, j)$ . The Riemann-Roch character  $RR^K(M, E)$  is defined as the index of the elliptic operator  $\bar{\partial}_E + \bar{\partial}_E^*$ :

$$RR^K(M, E) = \text{Index}_M^K(\bar{\partial}_E + \bar{\partial}_E^*)$$

viewed as an element of  $R(K)$ , the character ring of  $K$ .

In the Kostant-Souriau framework, a Hamiltonian  $K$ -manifold  $(M, \Omega, \Phi_K)$  is pre-quantized if there is an equivariant Hermitian line bundle  $\mathcal{L}$  with an invariant Hermitian connection  $\nabla$  such that

$$(2.7) \quad L(X) - \nabla_{VX} = i\langle\Phi_K, X\rangle \quad \text{and} \quad \nabla^2 = -i\Omega,$$

for every  $X \in \mathfrak{k}$ . Here  $L(X)$  is the infinitesimal action of  $X \in \mathfrak{k}$  on the sections of  $\mathcal{L} \rightarrow M$ .  $(\mathcal{L}, \nabla)$  is also called a Kostant-Souriau line bundle. Remark that conditions (2.7) imply, via the equivariant Bianchi formula, the relation (2.6).

We will now recall the notion of geometric quantization.

**Definition 2.1.** *When  $(M, \Omega, \Phi_K)$  is prequantized by a line bundle  $\mathcal{L}$ , the geometric quantization of  $M$  is defined as the index  $RR^K(M, \mathcal{L})$  : we denote it*

$$\mathcal{Q}_K(M, \Omega) \in R(K),$$

In order to simplify the notation, we will use also the notation  $\mathcal{Q}_K(M)$  for the geometric quantization of  $(M, \Omega, \Phi_K)$ .

**Remark 2.2.** *Suppose that  $(M, \Omega, J)$  is a compact Kähler manifold pre-quantized by a holomorphic line bundle  $\mathcal{L}$ . Then*

- $\mathcal{Q}_K(M, \Omega)$  coincides with the holomorphic Euler characteristic of  $(M, \mathcal{L})$ ,
- for  $k \in \mathbb{N}$  large enough,  $\mathcal{Q}_K(M, k\Omega) \in R(K)$  is equal to the  $K$ -module formed by the holomorphic sections of  $\mathcal{L}^{\otimes k} \rightarrow M$ .

One wants to compute the  $K$ -multiplicities of  $\mathcal{Q}_K(M)$  in geometrical terms. A fundamental result of Marsden and Weinstein asserts that if  $\xi \in \mathfrak{k}^*$  is a regular value of the moment map  $\Phi$ , the reduced space (or symplectic quotient)

$$M_\xi := \Phi_K^{-1}(\xi)/K_\xi$$

is an orbifold equipped with a symplectic structure  $\Omega_\xi$ . For any dominant weight  $\mu \in \widehat{K}$  which is a regular value of  $\Phi$ ,

$$\mathcal{L}(\mu) := (\mathcal{L}|_{\Phi_K^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$$

is a Kostant-Souriau line orbundle over  $(M_\mu, \Omega_\mu)$ . The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  makes sense as in Definition 2.1. In [29], this is extended further to the case of singular symplectic quotients, using partial (or shift) de-singularization. So the integer  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  is well defined for every  $\mu \in \widehat{K}$ : in particular  $\mathcal{Q}(M_\mu) = 0$  if  $\mu \notin \Phi_K(M)$ .

The following theorem was conjectured by Guillemin-Sternberg [12] and is known as “quantization commutes with reduction” [28, 29, 39, 33]. For complete references on the subject the reader should consult [37, 40].

**Theorem 2.3** (Meinrenken, Meinrenken-Sjamaar). *We have the following equality in  $R(K)$ :*

$$\mathcal{Q}_K(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

**2.2. Formal quantization of non-compact Hamiltonian manifolds.** Suppose now that  $M$  is *non-compact* but that the moment map  $\Phi_K : M \rightarrow \mathfrak{k}^*$  is assumed to be *proper* (we will simply say “ $M$  is proper”). In this situation the geometric quantization of  $M$  as an index of an elliptic operator is not well defined. Nevertheless the integers  $\mathcal{Q}(M_\mu), \mu \in \widehat{K}$  are well defined since the symplectic quotients  $M_\mu$  are *compact*.

A representation  $E$  of  $K$  is admissible if it has finite  $K$ -multiplicities :  $\dim(\text{hom}_K(V_\mu^K, E)) < \infty$  for every  $\mu \in \widehat{K}$ . Let  $R^{-\infty}(K)$  be the Grothendieck group associated to the  $K$ -admissible representations. We have an inclusion map  $R(K) \hookrightarrow R^{-\infty}(K)$  and  $R^{-\infty}(K)$  is canonically identified with  $\text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ . Moreover the tensor product induces an  $R(K)$ -module structure on  $R^{-\infty}(K)$  since  $E \otimes V$  is an admissible representation when  $V$  and  $E$  are, respectively, a finite dimensional and an admissible representation of  $K$ .

Following [42, 35], we introduce the following

**Definition 2.4.** *Suppose that  $(M, \Omega, \Phi_K)$  is proper Hamiltonian  $K$ -manifold pre-quantized by a line bundle  $\mathcal{L}$ . The formal geometric quantization of  $(M, \Omega)$  is the element of  $R^{-\infty}(K)$  defined by*

$$\mathcal{Q}_K^{-\infty}(M, \Omega) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

When the symplectic structure  $\Omega$  is understood, we will write  $\mathcal{Q}_K^{-\infty}(M)$  for the formal geometric quantization of  $(M, \Omega, \Phi_K)$ .

For a Hamiltonian  $K$ -manifold  $M$  with proper moment map  $\Phi_K$ , the convexity Theorem [18, 26] asserts that

$$(2.8) \quad \Delta_K(M) := \Phi_K(M) \cap \mathfrak{t}_+^*$$

is a convex rational polyhedron, that one calls the *Kirwan polyhedron*.

We will need the following lemma in the next sections.

**Lemma 2.5.** *Let  $(M, \Omega_M)$  and  $(N, \Omega_N)$  be two prequantized proper Hamiltonian  $K$ -manifold. Suppose that  $\mathcal{Q}_K^{-\infty}(M, k\Omega_M) = \mathcal{Q}_K^{-\infty}(N, k\Omega_N)$  for any integer  $k \geq 1$ . Then  $\Delta_K(M) = \Delta_K(N)$ .*

*Proof.* We check that for any  $\mu \in \widehat{K}$  the multiplicity of  $V_{k\mu}^K$  in  $\mathcal{Q}_K^{-\infty}(M, k\Omega_M)$  is equal to  $\mathcal{Q}(M_\mu, k\Omega_\mu)$ . The Atiyah-Singer Riemann-Roch formula gives us the following estimate

$$\mathcal{Q}(M_\mu, k\Omega_\mu) \sim \text{cst } k^r \text{ vol}(M_\mu)$$

when  $k$  goes to infinity. Here  $\text{cst} > 0$ ,  $r = \dim M_\mu/2$  and  $\text{vol}(M_\mu)$  is the symplectic volume of  $M_\mu$ . Hence, the hypothesis “ $\mathcal{Q}_K^{-\infty}(M, k\Omega_M) = \mathcal{Q}_K^{-\infty}(N, k\Omega_N)$  for any integer  $k \geq 1$ ” implies that  $\Phi_K(M) \cap \widehat{K} = \Phi_K(N) \cap \widehat{K}$ .

Take an integer  $R \geq 1$ . By considering the multiplicities of  $V_{k\mu}^K$  in  $\mathcal{Q}_K^{-\infty}(M, kR\Omega_M)$ , we prove in the same way that  $\Phi_K(M) \cap \frac{1}{R}\widehat{K} = \Phi_K(N) \cap \frac{1}{R}\widehat{K}$ . Finally we get that

$$\Phi_K(M) \cap \left\{ \frac{\mu}{R} \mid \mu \in \widehat{K}, R \geq 1 \right\} = \Phi_K(N) \cap \left\{ \frac{\mu}{R} \mid \mu \in \widehat{K}, R \geq 1 \right\}.$$

The proof follows since  $\left\{ \frac{\mu}{R} \mid \mu \in \widehat{K}, R \geq 1 \right\}$  is a dense subset of the Weyl chamber  $\mathfrak{t}_+^*$ .  $\square$

Let  $\varphi : H \rightarrow K$  be a morphism between compact connected Lie groups. It induces a pull-back morphism  $\varphi^* : R(K) \rightarrow R(H)$ . We want to extend  $\varphi^*$  to some elements of  $R^{-\infty}(K)$ . For  $\mu \in \widehat{H}$  and  $\lambda \in \widehat{K}$ , let  $N_\mu^\lambda$  be the multiplicity of  $V_\mu^H$  in  $\varphi^*V_\lambda^K$ . Formally, the pull-back of  $E = \sum_{\lambda \in \widehat{K}} a_\lambda V_\lambda^K$  by  $\varphi$  is

$$(2.9) \quad \varphi^*E = \sum_{\mu \in \widehat{H}} b_\mu V_\mu^H \quad \text{with} \quad b_\mu = \sum_{\lambda \in \widehat{K}} a_\lambda N_\mu^\lambda.$$

**Definition 2.6.** *Let  $\varphi : H \rightarrow K$  be a morphism between compact connected Lie groups. The element  $E = \sum_{\lambda \in \widehat{K}} a_\lambda V_\lambda^K$  is  $H$ -admissible if for every  $\mu \in \widehat{H}$ , the set  $\{\lambda \in \widehat{K} \mid a_\lambda N_\mu^\lambda \neq 0\}$  is finite. Then the pull-back  $\varphi^*E \in R^{-\infty}(H)$  is defined by (2.9).*

The element  $\varphi^*E \in R^{-\infty}(H)$  is called the “restriction” of  $E$  to  $H$ , and will be sometimes simply denoted by  $E|_H$ .

We prove in [35] the following functorial properties of the formal quantization process.

**Theorem 2.7.** **[P1]** *Let  $M_1$  and  $M_2$  be respectively pre-quantized proper Hamiltonian  $K_1$  and  $K_2$ -manifolds : the product  $M_1 \times M_2$  is then a pre-quantized proper Hamiltonian  $K_1 \times K_2$ -manifold. We have*

$$(2.10) \quad \mathcal{Q}_{K_1 \times K_2}^{-\infty}(M_1 \times M_2) = \mathcal{Q}_{K_1}^{-\infty}(M_1) \widehat{\otimes} \mathcal{Q}_{K_2}^{-\infty}(M_2)$$

*in  $R^{-\infty}(K_1 \times K_2) \simeq R^{-\infty}(K_1) \widehat{\otimes} R^{-\infty}(K_2)$ .*

**[P2]** Let  $M$  be a pre-quantized proper Hamiltonian  $K$ -manifold. Let  $\varphi : H \rightarrow K$  be a morphism between compact connected Lie groups. Suppose that  $M$  is still proper as a Hamiltonian  $H$ -manifold. Then  $\mathcal{Q}_K^{-\infty}(M)$  is  $H$ -admissible and we have the following equality in  $R^{-\infty}(H)$  :

$$\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M).$$

**[P3]** Let  $N$  and  $M$  be two pre-quantized Hamiltonian  $K$ -manifolds where  $N$  is compact and  $M$  is proper. The product  $M \times N$  is then proper and we have the following equality in  $R^{-\infty}(K)$  :

$$(2.11) \quad \mathcal{Q}_K^{-\infty}(M \times N) = \mathcal{Q}_K^{-\infty}(M) \cdot \mathcal{Q}_K(N)$$

Property **[P2]** is the hard point in this theorem. In [35], we have only consider the case where  $\varphi$  is the inclusion of a subgroup. In Appendix 7.3, we check the general case of a morphism  $\varphi : H \rightarrow K$ .

**2.3. Outline of the proof of Theorem 1.8.** We come back to the setting of the introduction. We consider the holomorphic discrete series representation  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  attached to the Blattner parameter  $\Lambda \in \mathcal{C}_{\text{hol}}^{\geq}$ . Recall that the coadjoint orbit  $\mathcal{O}_\Lambda \simeq G/K_\Lambda$ , which is equipped with the Kirillov-Kostant-Souriau symplectic form  $\Omega_\Lambda$ , is pre-quantized by the line bundle  $\mathcal{L}_\Lambda := G \times_{K_\Lambda} \mathbb{C}_\Lambda$ .

Consider first the Hamiltonian action of  $K$  on  $\mathcal{O}_\Lambda$  (here  $K$  is a maximal compact subgroup of  $G$ ). One knows that the corresponding moment map  $\Phi_K : \mathcal{O}_\Lambda \rightarrow \mathfrak{k}^*$  is proper [10, 32]. Hence the formal quantization  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda)$  of the  $K$ -action on  $\mathcal{O}_\Lambda$  is well-defined.

Theorem 1.2 tells us that the restriction of the representation  $\mathcal{Q}_G(\mathcal{O}_\Lambda)$  to  $K$  is

$$\mathcal{Q}_K(\mathcal{O}_\Lambda) = V_\Lambda^K \otimes S(\mathfrak{p}^+).$$

Theorem 1.8, restricted to the case where  $H = K$ , is then equivalent to the identity

$$(2.12) \quad \mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda) = V_\Lambda^K \otimes S(\mathfrak{p}^+) \quad \text{in } R^{-\infty}(K),$$

that we prove in Section 3.

Consider now the situation of a closed connected subgroup  $H$  of  $K$ , such that the restriction  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$  is admissible, e.g. the moment map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  is proper (see Proposition 1.6). We can apply property **[P2]** of Theorem 2.7. The formal quantization  $\mathcal{Q}_H^{-\infty}(\mathcal{O}_\Lambda)$  of the  $H$ -action on  $\mathcal{O}_\Lambda$  is equal to the restriction of the formal quantization  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda)$  of the  $K$ -action on  $\mathcal{O}_\Lambda$ . Hence (2.12) implies that

$$\mathcal{Q}_H^{-\infty}(\mathcal{O}_\Lambda) = \mathcal{Q}_H(\mathcal{O}_\Lambda).$$

So Theorem 1.8 is proved for all the admissible restrictions  $\mathcal{Q}_H(\mathcal{O}_\Lambda)$ , when one proves it for the case  $H = K$ .

### 3. COMPUTATION OF $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda)$

In this section we prove the following

**Theorem 3.1.** Let  $\mathcal{O}_\Lambda$  be the coadjoint orbit passing through  $\Lambda \in \mathcal{C}_{\text{hol}}$ . We have

$$\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda) = V_\Lambda^K \otimes S(\mathfrak{p}^+).$$

Similar computation was done in [34] in the setting of a geometric quantization of the “Spin” type.

Note that the formal quantization of  $\mathcal{O}_\Lambda$  behave differently from the “true” one, defined in Definition 1.1, when  $\Lambda \in \mathcal{C}_{\text{hol}} \setminus \mathcal{C}_{\text{hol}}^\geq$  : in this case  $\mathcal{Q}_K(\mathcal{O}_\Lambda) = \{0\}$  whereas  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda) \neq \{0\}$ .

The proof of Theorem 3.1 is conducted as follows. We introduce in Section 3.2 a  $K$ -transversally elliptic symbol  $\sigma_\Lambda$  on  $\mathcal{O}_\Lambda$ . A direct computation, done in Section 3.3, shows that the  $K$ -equivariant index of  $\sigma_\Lambda$  is equal to  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$ . In Section 3.4, we use a deformation argument based on the shifting trick to show that the index of  $\sigma_\Lambda$  coincides with  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda)$ . Putting these results together completes the proof of Theorem 3.1.

**3.1. Transversally elliptic symbols.** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah and Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [6, 7] and for a short introduction see [33].

Let  $M$  be a *compact*  $K$ -manifold. Let  $p : \mathbf{T}M \rightarrow M$  be the projection, and let  $(-, -)_M$  be a  $K$ -invariant Riemannian metric. If  $E^0, E^1$  are  $K$ -equivariant complex vector bundles over  $M$ , a  $K$ -equivariant morphism  $\sigma \in \Gamma(\mathbf{T}M, \text{hom}(p^*E^0, p^*E^1))$  is called a *symbol* on  $M$ . The subset of all  $(m, v) \in \mathbf{T}M$  where  $\sigma(m, v) : E_m^0 \rightarrow E_m^1$  is not invertible is called the *characteristic set* of  $\sigma$ , and is denoted by  $\text{Char}(\sigma)$ .

In the following, the product of a symbol  $\sigma$  by a complex vector bundle  $F \rightarrow M$ , is the symbol

$$\sigma \otimes F$$

defined by  $\sigma \otimes F(m, v) = \sigma(m, v) \otimes \text{Id}_{F_m}$  from  $E_m^0 \otimes F_m$  to  $E_m^1 \otimes F_m$ . Note that  $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$ .

Let  $\mathbf{T}_K M$  be the following subset of  $\mathbf{T}M$  :

$$\mathbf{T}_K M = \{(m, v) \in \mathbf{T}M, (v, VX(m))_M = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol  $\sigma$  is *elliptic* if  $\sigma$  is invertible outside a compact subset of  $\mathbf{T}M$  ( $\text{Char}(\sigma)$  is compact), and is *transversally elliptic* if the restriction of  $\sigma$  to  $\mathbf{T}_K M$  is invertible outside a compact subset of  $\mathbf{T}_K M$  ( $\text{Char}(\sigma) \cap \mathbf{T}_K M$  is compact). An elliptic symbol  $\sigma$  defines an element in the equivariant  $K$ -theory of  $\mathbf{T}M$  with compact support, which is denoted by  $\mathbf{K}_K(\mathbf{T}M)$ , and the index of  $\sigma$  is a virtual finite dimensional representation of  $K$ , that we denote  $\text{Index}_M^K(\sigma) \in R(K)$  [2, 3, 4, 5].

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the  $R(K)$ -submodule formed by all the infinite sum  $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$  where the map  $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$  has at most a *polynomial* growth. The  $R(K)$ -module  $R_{tc}^{-\infty}(K)$  is the Grothendieck group associated to the *trace class* virtual  $K$ -representations: we can associate to any  $V \in R_{tc}^{-\infty}(K)$ , its trace  $k \rightarrow \text{Tr}(k, V)$  which is a generalized function on  $K$  invariant by conjugation. In Section 3.3, we use the fact that the trace defines a morphism of  $R(K)$ -module

$$(3.13) \quad R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K.$$

A *transversally elliptic* symbol  $\sigma$  defines an element of  $\mathbf{K}_K(\mathbf{T}_K M)$ , and the index of  $\sigma$  is defined as a trace class virtual representation of  $K$ , that we still denote

$\text{Index}_M^K(\sigma) \in R_{tc}^{-\infty}(K)$ . See [1] for the analytic index and [6, 7] for the cohomological one. Remark that any elliptic symbol of  $\mathbf{T}M$  is transversally elliptic, hence we have a restriction map  $\mathbf{K}_K(\mathbf{T}M) \rightarrow \mathbf{K}_K(\mathbf{T}_K M)$ , and a commutative diagram

$$(3.14) \quad \begin{array}{ccc} \mathbf{K}_K(\mathbf{T}M) & \longrightarrow & \mathbf{K}_K(\mathbf{T}_K M) \\ \text{Index}_M^K \downarrow & & \downarrow \text{Index}_M^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map  $\text{Index}_U^K : \mathbf{K}_K(\mathbf{T}_K U) \rightarrow R_{tc}^{-\infty}(K)$  is still defined when  $U$  is a  $K$ -invariant relatively compact open subset of a  $K$ -manifold (see [33][section 3.1]).

**3.2. The transversally elliptic symbol  $\sigma_\Lambda$ .** Let  $\Lambda \in \mathcal{C}_{\text{hol}}$ . Let us first describe the principal symbol of the Dolbeault-Dirac operator  $\bar{\partial}_{\mathcal{L}_\Lambda} + \bar{\partial}_{\mathcal{L}_\Lambda}^*$  defined on the coadjoint orbit  $\mathcal{O}_\Lambda$ . The complex vector bundle  $(\mathbf{T}^*\mathcal{O}_\Lambda)^{0,1}$  is  $G$ -equivariantly identified with the tangent bundle  $\mathbf{T}\mathcal{O}_\Lambda$  equipped with the complex structure  $J_\Lambda$ .

Let  $h$  be the Hermitian structure on  $\mathbf{T}\mathcal{O}_\Lambda$  defined by :  $h(v, w) = \Omega_\Lambda(v, J_\Lambda w) - i\Omega_\Lambda(v, w)$  for  $v, w \in \mathbf{T}M$ . The symbol

$$\text{Thom}(\mathcal{O}_\Lambda, J_\Lambda) \in \Gamma(\mathcal{O}_\Lambda, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} \mathbf{T}\mathcal{O}_\Lambda), p^*(\wedge_{\mathbb{C}}^{\text{odd}} \mathbf{T}\mathcal{O}_\Lambda)))$$

at  $(m, v) \in \mathbf{T}\mathcal{O}_\Lambda$  is equal to the Clifford map

$$(3.15) \quad \text{Cl}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} \mathbf{T}_m \mathcal{O}_\Lambda \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} \mathbf{T}_m \mathcal{O}_\Lambda,$$

where  $\text{Cl}_m(v).w = v \wedge w - c_h(v)w$  for  $w \in \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m \mathcal{O}_\Lambda$ . Here  $c_h(v) : \wedge_{\mathbb{C}}^{\bullet} \mathbf{T}_m \mathcal{O}_\Lambda \rightarrow \wedge_{\mathbb{C}}^{\bullet-1} \mathbf{T}_m \mathcal{O}_\Lambda$  denotes the contraction map relative to  $h$ . Since  $\text{Cl}_m(v)^2 = -|v|^2 \text{Id}$ , the map  $\text{Cl}_m(v)$  is invertible for all  $v \neq 0$ . Hence the characteristic set of  $\text{Thom}(\mathcal{O}_\Lambda, J_\Lambda)$  corresponds to the 0-section of  $\mathbf{T}\mathcal{O}_\Lambda$ .

It is a classical fact that the principal symbol of the Dolbeault-Dirac operator  $\bar{\partial}_{\mathcal{L}_\Lambda} + \bar{\partial}_{\mathcal{L}_\Lambda}^*$  is equal to<sup>1</sup>

$$(3.16) \quad \tau_\Lambda := \text{Thom}(\mathcal{O}_\Lambda, J_\Lambda) \otimes \mathcal{L}_\Lambda,$$

see [11]. Here also we have  $\text{Char}(\tau_\Lambda) = 0$  – section of  $\mathbf{T}\mathcal{O}_\Lambda$ . So  $\tau_\Lambda$  is not an elliptic symbol since the coadjoint orbit  $\mathcal{O}_\Lambda$  is non-compact.

Following [33, 34], we deform  $\tau_\Lambda$  in order to define a  $K$ -transversally elliptic symbol on  $\mathcal{O}_\Lambda$ . Consider the moment map  $\Phi_K : \mathcal{O}_\Lambda \rightarrow \mathfrak{k}^*$ . With the help of the  $K$ -invariant scalar product on  $\mathfrak{k}^*$  induced by the Killing form on  $\mathfrak{g}$ , we define the  $K$ -invariant function

$$\|\Phi_K\|^2 : \mathcal{O}_\Lambda \rightarrow \mathbb{R}.$$

Let  $\mathcal{H}$  be the Hamiltonian vector field for  $\frac{-1}{2} \|\Phi_K\|^2$ , i.e. the contraction of the symplectic form by  $\mathcal{H}$  is equal to the 1-form  $\frac{-1}{2} d \|\Phi_K\|^2$ . The vector field  $\mathcal{H}$  has the following nice description. The scalar product on  $\mathfrak{k}^*$  gives an identification  $\mathfrak{k}^* \simeq \mathfrak{k}$ , hence  $\Phi_K$  can be consider as a map from  $\mathcal{O}_\Lambda$  to  $\mathfrak{k}$ . We have then

$$(3.17) \quad \mathcal{H}_m = (V\Phi_K(m))|_m, \quad m \in \mathcal{O}_\Lambda ,$$

where  $V\Phi_K(m)$  is the vector field on  $\mathcal{O}_\Lambda$  generated by  $\Phi_K(m) \in \mathfrak{k}$ .

<sup>1</sup>Here we use an identification  $\mathbf{T}^*\mathcal{O}_\Lambda \simeq \mathbf{T}\mathcal{O}_\Lambda$  given by an invariant Riemannian metric.

**Definition 3.2.** Let  $\tau_\Lambda$  be the symbol on  $\mathcal{O}_\Lambda$  defined in (3.16). The symbol  $\tau_\Lambda$  pushed by the vector field  $\mathcal{H}$  is the symbol  $\sigma_\Lambda$  defined by the relation

$$\sigma_\Lambda(m, v) := \tau_\Lambda(m, v - \mathcal{H}_m)$$

for any  $(m, v) \in \mathbf{T}\mathcal{O}_\Lambda$ .

The characteristic set of  $\sigma_\Lambda$  corresponds to  $\{(m, v) \in \mathbf{T}\mathcal{O}_\Lambda, v = \mathcal{H}_m\}$ , the graph of the vector field  $\mathcal{H}$ . Since  $\mathcal{H}$  belongs to the set of tangent vectors to the  $K$ -orbits, we have

$$\begin{aligned} \text{Char}(\sigma_\Lambda) \cap \mathbf{T}_K \mathcal{O}_\Lambda &= \{(m, 0) \in \mathbf{T}\mathcal{O}_\Lambda, \mathcal{H}_m = 0\} \\ &\cong \{m \in \mathcal{O}_\Lambda, d \|\Phi_K\|_m^2 = 0\}. \end{aligned}$$

Therefore the symbol  $\sigma_\Lambda$  is  $K$ -transversally elliptic if and only if the set  $\text{Cr}(\|\Phi_K\|^2)$  of critical points of the function  $\|\Phi_K\|^2$  is compact.

We have the following result.

**Lemma 3.3** ([10, 32]). *The set  $\text{Cr}(\|\Phi_K\|^2) \subset \mathcal{O}_\Lambda$  is equal to the orbit  $K \cdot \Lambda$ .*

**Corollary 3.4.** *The symbol  $\sigma_\Lambda$  is  $K$ -transversally elliptic.*

**3.3. Computation of  $\text{Index}^K(\sigma_\Lambda)$ : the direct approach.** The equivariant index of the symbol  $\sigma_\Lambda$  can be defined by different manners.

On one hand, since  $\mathcal{O}_\Lambda$  can be imbedded  $K$ -equivariantly in a compact manifold, one can consider  $\text{Index}_{\mathcal{O}_\Lambda}^K(\sigma_\Lambda) \in R_{tc}^{-\infty}(K)$ .

On the other hand, for any  $K$ -invariant relatively compact open neighborhood  $\mathcal{U} \subset \mathcal{O}_\Lambda$  of  $\text{Cr}(\|\Phi_K\|^2)$ , the restriction of  $\sigma_\Lambda$  to  $\mathcal{U}$  defines a class  $\sigma_\Lambda|_{\mathcal{U}} \in \mathbf{K}_K(\mathbf{T}_K \mathcal{U})$ . Since the index map is well defined on  $\mathcal{U}$ , we can take its index  $\text{Index}_{\mathcal{U}}^K(\sigma_\Lambda|_{\mathcal{U}})$ . A direct application of the excision property shows that  $\text{Index}_{\mathcal{O}_\Lambda}^K(\sigma_\Lambda) = \text{Index}_{\mathcal{U}}^K(\sigma_\Lambda|_{\mathcal{U}})$ . In order to simplify our notation, the index of  $\sigma_\Lambda$  is denoted

$$\text{Index}^K(\sigma_\Lambda) \in R_{tc}^{-\infty}(K).$$

The aim of this section is the following

**Proposition 3.5.** *Let  $\Lambda \in \mathcal{C}_{\text{hol}}$ . We have*

$$\text{Index}^K(\sigma_\Lambda) = S(\mathfrak{p}^+) \otimes V_\Lambda^K \quad \text{in} \quad R_{tc}^{-\infty}(K).$$

The rest of this section is devoted to the computation of  $\text{Index}^K(\sigma_\Lambda)$ . A similar computation is done in Section 5.2 of [34] in the context of a ‘‘Spin’’ quantization.

Let

$$(3.18) \quad \Upsilon : \mathcal{O}_\Lambda \longrightarrow \mathcal{O}'_\Lambda := K \cdot \Lambda \times \mathfrak{p}$$

be the  $K$ -equivariant diffeomorphism defined by  $\Upsilon(g \cdot \Lambda) = (k \cdot \Lambda, X)$  where  $g = e^X k$ , with  $k \in K$  and  $X \in \mathfrak{p}$ , is the Cartan decomposition of  $g \in G$ .

The data  $(\Omega_\Lambda, J_\Lambda, \mathcal{L}_\Lambda, \mathcal{H}, \sigma_\Lambda)$ , transported to the manifold  $\mathcal{O}'_\Lambda$  through  $\Upsilon$ , is denoted  $(\Omega'_\Lambda, J'_\Lambda, \mathcal{L}'_\Lambda, \mathcal{H}', \sigma'_\Lambda)$ . It is easy to check that the line bundle  $\mathcal{L}'_\Lambda$  is the pull-back of the line bundle  $K \times_{K_\Lambda} \mathbb{C}_\Lambda \rightarrow K \cdot \Lambda$  to  $\mathcal{O}'_\Lambda$ .

We consider on  $\mathcal{O}'_\Lambda$  the following  $K$ -equivariant data:

- (1) The complex structure  $J''_\Lambda$  which is the product  $J_{K \cdot \Lambda} \times -\text{ad}(z_o)$ . Here  $J_{K \cdot \Lambda}$  is the restriction of  $J_\Lambda$  to the Kähler submanifold  $K \cdot \Lambda \subset G \cdot \Lambda$ , and  $\text{ad}(z_o)$  is the complex structure on  $\mathfrak{p}$  defined in the introduction.
- (2) The vector field  $\mathcal{H}''$  defined by:  $\mathcal{H}''_{\xi, X} = -(0, [\xi, X])$  for  $\xi \in K \cdot \Lambda$  and  $X \in \mathfrak{p}$ .

**Definition 3.6.** We consider on  $\mathcal{O}'_\Lambda$  the symbols:

- $\tau''_\Lambda := \text{Thom}(\mathcal{O}'_\Lambda, J''_\Lambda) \otimes \mathcal{L}'_\Lambda$ ,
- $\sigma''_\Lambda$  which is the symbol  $\tau''_\Lambda$  pushed by the vector field  $\mathcal{H}''$  (see Def. 3.2).

**Proposition 3.7.** • The symbol  $\sigma''_\Lambda$  is a  $K$ -transversally elliptic symbol on  $\mathcal{O}'_\Lambda$ .

• If  $\mathcal{U}$  is a sufficiently small  $K$ -invariant neighborhood of  $K \cdot \Lambda \times \{0\}$  in  $\mathcal{O}'_\Lambda$ , the restrictions  $\sigma'_\Lambda|_{\mathcal{U}}$  and  $\sigma''_\Lambda|_{\mathcal{U}}$  define the same class in  $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$ .

*Proof.* The first point is due to the fact that the vector field  $\mathcal{H}''$  is tangent to the  $K$ -orbits in  $\mathcal{O}'_\Lambda$ . Hence

$$\begin{aligned} \text{Char}(\sigma''_\Lambda) \cap \mathbf{T}_K\mathcal{O}'_\Lambda &\simeq \{(\xi, X) \in \mathcal{O}'_\Lambda \mid \mathcal{H}''_{\xi, X} = 0\} \\ &= K \cdot \Lambda \times \{0\}. \end{aligned}$$

Here we have used that  $[\xi, X] = 0$  for  $\xi \in K \cdot \Lambda$  and  $X \in \mathfrak{p}$  if and only if  $X = 0$ .

We will prove the second point by using some homotopy arguments. First we consider the family of vector fields  $\mathcal{H}_t := (1-t)\mathcal{H}' + t\mathcal{H}''$ ,  $t \in [0, 1]$ . Let  $\sigma_t$  be the symbol  $\tau'_\Lambda$  pushed by  $\mathcal{H}_t$ . One checks easily that there exists  $c > 0$  such that

$$(3.19) \quad \mathcal{H}'_{\xi, X} = \mathcal{H}''_{\xi, X} + o(\|X\|^2) \quad \text{and} \quad \|\mathcal{H}''_{\xi, X}\|^2 \geq c \|X\|^2$$

holds on  $\mathcal{O}'_\Lambda$ . With the help of (3.19) it is now easy to prove that there exists a  $K$ -invariant neighborhood  $\mathcal{V}$  of  $K \cdot \Lambda \times \{0\}$  in  $\mathcal{O}'_\Lambda$  such that

$$\text{Char}(\sigma_t|_{\mathcal{V}}) \cap \mathbf{T}_K\mathcal{V} = K \cdot \Lambda \times \{0\}.$$

for any  $t \in [0, 1]$ . Hence  $\sigma'_\Lambda|_{\mathcal{U}} = \sigma_0|_{\mathcal{U}}$  defines the same class than  $\sigma_1|_{\mathcal{U}}$  in  $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$  for any  $K$ -invariant neighborhood  $\mathcal{U}$  of  $K \cdot \Lambda \times \{0\}$  that is contained in  $\mathcal{V}$ .

In order to compare the symbols  $\sigma''_\Lambda|_{\mathcal{U}}$  and  $\sigma_1|_{\mathcal{U}}$ , we use a deformation argument similar to the one that we use in the proof of Lemma 2.2 in [33].

Note first that the complex structures  $J'_\Lambda$  and  $J''_\Lambda$  are equal on  $K \cdot \Lambda \times \{0\} \subset \mathcal{O}'_\Lambda$ . We consider the family of equivariant bundle maps  $A_u \in \Gamma(\mathcal{O}'_\Lambda, \text{End}(\mathbf{T}\mathcal{O}'_\Lambda))$ ,  $u \in [0, 1]$ , defined by

$$A_u := \text{Id} - uJ'_\Lambda J''_\Lambda.$$

Since  $A_u = (1+u)\text{Id}$  on  $K \cdot \Lambda \times \{0\}$ , there exists a  $K$ -invariant neighborhood  $\mathcal{U}$  of  $K \cdot \Lambda \times \{0\}$  (contained in  $\mathcal{V}$ ), such that  $A_u$  is invertible over  $\mathcal{U}$  for any  $u \in [0, 1]$ .

Thus  $A_u$ ,  $u \in [0, 1]$  defines over  $\mathcal{U}$  a family of bundle isomorphisms :  $A_0 = \text{Id}$  and the map  $A_1$  is a bundle complex isomorphism

$$\underline{A}_1 : (\mathbf{T}\mathcal{U}, J''_\Lambda) \longrightarrow (\mathbf{T}\mathcal{U}, J'_\Lambda).$$

We extend  $\underline{A}_1$  to a complex isomorphism  $\underline{A}_1^\wedge : \wedge_{J''_\Lambda} \mathbf{T}\mathcal{U} \longrightarrow \wedge_{J'_\Lambda} \mathbf{T}\mathcal{U}$ . Then  $\underline{A}_1^\wedge$  induces an isomorphism between the symbols  $\text{Thom}(\mathcal{U}, J''_\Lambda)$  and  $\underline{A}_1^*(\text{Thom}(\mathcal{U}, J'_\Lambda))$  :  $(x, v) \mapsto \text{Thom}(\mathcal{U}, J'_\Lambda)(x, A_1(x)v)$ . In the same way  $\underline{A}_1^\wedge$  induces an isomorphism between the symbols  $\sigma''_\Lambda|_{\mathcal{U}}$  and  $\underline{A}_1^*(\sigma_1|_{\mathcal{U}})$  :  $(x, v) \mapsto \tau'_\Lambda(x, A_1(x)(v - \mathcal{H}''_x))$ . One checks easily that  $\underline{A}_u^*(\sigma_1|_{\mathcal{U}})$ ,  $u \in [0, 1]$  is an homotopy of transversally elliptic symbols.

Finally we have proved that  $\sigma''_\Lambda|_{\mathcal{U}}$ ,  $\sigma_1|_{\mathcal{U}}$  and  $\sigma'_\Lambda|_{\mathcal{U}}$  define the same class in  $\mathbf{K}_K(\mathbf{T}_K\mathcal{U})$ .  $\square$

Here also, the equivariant index of the transversally elliptic symbol  $\sigma''_\Lambda$  can be defined either as the  $\text{Index}_{\mathcal{O}'_\Lambda}^K(\sigma''_\Lambda)$  taken on  $\mathcal{O}'_\Lambda$ , or as the index  $\text{Index}_{\mathcal{U}}^K(\sigma''_\Lambda|_{\mathcal{U}})$  taken



on any  $K$ -invariant relatively compact open neighborhood  $\mathcal{U} \subset \mathcal{O}'_\Lambda$  of  $K \cdot \Lambda \times \{0\}$ . We denote simply

$$\text{Index}^K(\sigma''_\Lambda) \in R_{tc}^{-\infty}(K).$$

the equivariant index of  $\sigma''_\Lambda$ . The second point of Proposition 3.7 shows that  $\text{Index}_{\mathcal{U}}^K(\sigma''_\Lambda|_{\mathcal{U}}) = \text{Index}_{\mathcal{U}}^K(\sigma'_\Lambda|_{\mathcal{U}})$ . Hence we know that

$$\text{Index}^K(\sigma_\Lambda) = \text{Index}^K(\sigma''_\Lambda).$$

In order to compute  $\text{Index}^K(\sigma''_\Lambda)$ , we use the induction morphism

$$j_* : \mathbf{K}_{K_\Lambda}(\mathbf{T}_{K_\Lambda}\mathfrak{p}) \longrightarrow \mathbf{K}_K(\mathbf{T}_K(\mathcal{O}'_\Lambda))$$

defined by Atiyah in [1] (see also [33][Section 3]). The map  $j_*$  enjoys two properties: first,  $j_*$  is an isomorphism and the  $K$ -index of  $\sigma \in \mathbf{K}_K(\mathbf{T}_K(\mathcal{O}'_\Lambda))$  can be computed via the  $K_\Lambda$ -index of  $j_*^{-1}(\sigma)$ .

Let  $\sigma : p^*(E^+) \rightarrow p^*(E^-)$  be a  $K$ -transversally elliptic symbol on  $\mathcal{O}'_\Lambda$ , where  $p : \mathbf{T}\mathcal{O}'_\Lambda \rightarrow \mathcal{O}'_\Lambda$  is the projection, and  $E^+, E^-$  are equivariant vector bundles over  $\mathcal{O}'_\Lambda$ . So for any  $(\xi, X) \in K \cdot \Lambda \times \mathfrak{p}$ , we have a collection of linear maps  $\sigma(\xi, X; v, Y) : E_{(\xi, X)}^+ \rightarrow E_{(\xi, X)}^-$  depending on the tangent vectors  $(v, Y) \in \mathbf{T}_\xi(K \cdot \Lambda) \times \mathfrak{p}$ . The  $K_\Lambda$ -equivariant symbol  $j_*^{-1}(\sigma)$  is defined by

$$(3.20) \quad j_*^{-1}(\sigma)(X, Y) = \sigma(\Lambda, X; 0, Y) : E_{(\Lambda, X)}^+ \longrightarrow E_{(\Lambda, X)}^- \quad \text{for any } (X, Y) \in \mathbf{T}\mathfrak{p}.$$

In the case of the symbol  $\sigma''_\Lambda$ , the super vector bundle  $E^+ \oplus E^-$  over  $\mathcal{O}'_\Lambda$  is  $\wedge_{\mathbb{C}}^{\bullet} \mathbf{T}\mathcal{O}'_\Lambda \otimes \mathcal{L}'_\Lambda$ . For any  $X \in \mathfrak{p}$ , the super vector space  $E_{(\Lambda, X)}^+ \oplus E_{(\Lambda, X)}^-$  is equal to

$$\wedge_{\mathbb{C}}^{\bullet} \mathfrak{p}^- \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda.$$

Let  $\text{Thom}(\mathfrak{p}^-)$  be the Thom symbol of the complex vector space  $\mathfrak{p}^- \simeq (\mathfrak{p}, -\text{ad}(z_o))$ . Let  $\tilde{\Lambda}$  be the vector field on  $\mathfrak{p}^-$  which is generated by  $\Lambda \in \mathfrak{k}^* \simeq \mathfrak{k}$ . Let

$$\text{Thom}^\Lambda(\mathfrak{p}^-)$$

be the symbol  $\text{Thom}(\mathfrak{p}^-)$  pushed by the vector field  $\tilde{\Lambda}$  (see Definition 3.6). Since the vector field  $\tilde{\Lambda}$  vanishes only at  $0 \in \mathfrak{p}^-$ , the symbol  $\text{Thom}^\Lambda(\mathfrak{p}^-)$  is  $K_\Lambda$ -transversally elliptic. We have

$$(3.21) \quad (j_*)^{-1}(\sigma''_\Lambda) = \text{Thom}^\Lambda(\mathfrak{p}^-) \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda.$$

In (3.21), our notation uses the structure of  $R(K_\Lambda)$ -module for  $\mathbf{K}_{K_\Lambda}(\mathbf{T}_{K_\Lambda}\mathfrak{p})$ , hence we can multiply  $\text{Thom}^\Lambda(\mathfrak{p}^-)$  by  $\wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda$ .

Let  $\mathcal{C}^{-\infty}(K_\Lambda)^{K_\Lambda}$ ,  $\mathcal{C}^{-\infty}(K)^K$  be respectively the vector spaces of generalized functions on  $K_\Lambda$  and  $K$  which are invariant relative to the conjugation action. Let

$$(3.22) \quad \text{Ind}_{K_\Lambda}^K : \mathcal{C}^{-\infty}(K_\Lambda)^{K_\Lambda} \longrightarrow \mathcal{C}^{-\infty}(K)^K .$$

be the induction map that is defined as follows : for  $\phi \in \mathcal{C}^{-\infty}(K_\Lambda)^{K_\Lambda}$ , we have

$$\int_K \text{Ind}_{K_\Lambda}^K(\phi)(k)f(k)dk = \frac{\text{vol}(K, dk)}{\text{vol}(K_\Lambda, dk')} \int_{K_\Lambda} \phi(k')f|_{K_\Lambda}(k')dk',$$

for every  $f \in \mathcal{C}^\infty(K)^K$ .

Theorem 4.1 of Atiyah in [1] tells us that

$$(3.23) \quad \begin{array}{ccc} K_{K_\Lambda}(\mathbf{T}_{K_\Lambda} \mathfrak{p}) & \xrightarrow{j_*} & K_K(\mathbf{T}_K \mathcal{O}'_\Lambda) \\ \text{Index}^{K_\Lambda} \downarrow & & \downarrow \text{Index}^K \\ \mathcal{C}^{-\infty}(K_\Lambda)^{K_\Lambda} & \xrightarrow{\text{Ind}_{K_\Lambda}^K} & \mathcal{C}^{-\infty}(K)^K . \end{array}$$

is a commutative diagram<sup>2</sup>. In other words,  $\text{Index}^K(\sigma) = \text{Ind}_{K_\Lambda}^K(\text{Index}^{K_\Lambda}(j_*^{-1}(\sigma)))$ . With (3.21) we get

$$\text{Index}^K(\sigma''_\Lambda) = \text{Ind}_{K_\Lambda}^K \left( \text{Index}^{K_\Lambda}(\text{Thom}^\Lambda(\mathfrak{p}^-)) \otimes \wedge_{\mathbb{C}}^\bullet \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda \right).$$

We know from [33][Section 5.1] that the  $K_\Lambda$ -index of  $\text{Thom}^\Lambda(\mathfrak{p}^-)$  is equal to the symmetric algebra  $S(\mathfrak{p}^+)$  viewed as a  $K_\Lambda$ -module. Since  $S(\mathfrak{p}^+)$  is a  $K$ -module, we have

$$\begin{aligned} \text{Index}^K(\sigma''_\Lambda) &= \text{Ind}_{K_\Lambda}^K (S(\mathfrak{p}^+)|_{K_\Lambda} \otimes \wedge_{\mathbb{C}}^\bullet \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda) \\ &= S(\mathfrak{p}^+) \otimes \text{Ind}_{K_\Lambda}^K (\wedge_{\mathbb{C}}^\bullet \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda) \\ &= S(\mathfrak{p}^+) \otimes V_\Lambda^K . \end{aligned}$$

The proof of Proposition 3.5 is then completed. See the Appendix in [33] for the relation  $\text{Ind}_{K_\Lambda}^K (\wedge_{\mathbb{C}}^\bullet \mathfrak{k}/\mathfrak{k}_\Lambda \otimes \mathbb{C}_\Lambda) = V_\Lambda^K$ .

**3.4. Computation of  $\text{Index}^K(\sigma_\Lambda)$ : the shifting trick.** This section is devoted to the proof of the following

**Proposition 3.8.** *Let  $\mathcal{O}_\Lambda$  be the coadjoint orbit passing through  $\Lambda \in \mathcal{C}_{\text{hol}}$ .*

*For any  $\mu \in \widehat{K}$ , the multiplicity of  $V_\mu^K$  in  $\text{Index}^K(\sigma_\Lambda)$  is equal to  $\mathcal{Q}((\mathcal{O}_\Lambda)_\mu)$ . In other words we have*

$$\text{Index}^K(\sigma_\Lambda) = \mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda).$$

The proof, which follows the same line of Section 4.1 in [34], starts with the classical “shifting trick”. For any  $V \in R^{-\infty}(K)$ , we denote  $[V]^K \in \mathbb{Z}$  the multiplicity of the trivial representation in  $V$ .

By definition the multiplicity  $m_\Lambda(\mu)$  of  $V_\mu^K$  in  $\text{Index}^K(\sigma_\Lambda)$  is equal to  $[\text{Index}^K(\sigma_\Lambda) \otimes (V_\mu^K)^*]^K$ , where  $(V_\mu^K)^*$  is the (complex) dual of  $V_\mu^K$ . The Borel-Weil Theorem tells us that the representation  $V_\mu^K$  is equal to the  $K$ -equivariant Riemann-Roch character

$$RR^K(K \cdot \mu, \mathbb{C}_{[\mu]}),$$

where  $\mathbb{C}_{[\mu]} \simeq K \times_{K_\mu} \mathbb{C}_\mu$  is the prequantum line bundle over the coadjoint orbit  $K \cdot \mu$ . Note that  $K \cdot \mu$  is equipped with the Kähler structure  $(\Omega_{K \cdot \mu}, J_{K \cdot \mu})$  where  $\Omega_{K \cdot \mu}$  is the Kirillov-Kostant-Souriau symplectic form, and  $J_{K \cdot \mu}$  is the  $K$ -invariant compatible (integrable) complex structure.

Hence the dual  $(V_\mu^K)^*$  is equal to  $RR^K(\overline{K \cdot \mu}, \mathbb{C}_{[-\mu]})$ , where  $\overline{K \cdot \mu}$  is the coadjoint orbit  $\overline{K \cdot \mu}$  equipped with the opposite Kähler structure  $(-\Omega_{K \cdot \mu}, -J_{K \cdot \mu})$ . Let  $\text{Thom}(\overline{K \cdot \mu})$  be the equivariant Thom symbol on  $(\overline{K \cdot \mu}, -J_{K \cdot \mu})$ . Then  $(V_\mu^K)^*$  is equal to  $\text{Index}_{K \cdot \mu}^K(\text{Thom}(\overline{K \cdot \mu}) \otimes \mathbb{C}_{[-\mu]})$ .

<sup>2</sup>Here we look at  $R_{t_c}^{-\infty}(K_\Lambda)$  and  $R_{t_c}^{-\infty}(K)$  as subspaces of  $\mathcal{C}^{-\infty}(K_\Lambda)^{K_\Lambda}$  and  $\mathcal{C}^{-\infty}(K)^K$  by using the trace map (see 3.13).

Let  $\text{Thom}(\mathcal{O}_\Lambda)$  be the Thom symbol on  $(\mathcal{O}_\Lambda, J_\Lambda)$ . Like in section 3.2, let  $\mathcal{H}$  be the Hamiltonian vector field of  $\frac{1}{2} \|\Phi_K\|^2: \mathcal{O}_\Lambda \rightarrow \mathbb{R}$ . We denote by  $\text{Thom}^{\mathcal{H}}(\mathcal{O}_\Lambda)$  the symbol  $\text{Thom}(\mathcal{O}_\Lambda)$  pushed by the vector field  $\mathcal{H}$ :

$$\text{Thom}^{\mathcal{H}}(\mathcal{O}_\Lambda)(m, v) := \text{Thom}(\mathcal{O}_\Lambda)(m, v - \mathcal{H}_m), \quad (m, v) \in \mathbf{T}\mathcal{O}_\Lambda.$$

Since  $\text{Index}^K(\sigma_\Lambda)$  is equal to  $\text{Index}_{\mathcal{O}_\Lambda}^K \left( \text{Thom}^{\mathcal{H}}(\mathcal{O}_\Lambda) \otimes \mathcal{L}_\Lambda \right)$ , the multiplicative property of the index [1][Theorem 3.5] gives

$$(3.24) \quad m_\Lambda(\mu) = \left[ \text{Index}_{\mathcal{O}_\Lambda \times K \cdot \mu}^K \left( (\text{Thom}^{\mathcal{H}}(\mathcal{O}_\Lambda) \otimes \mathcal{L}_\Lambda) \odot (\text{Thom}(\overline{K \cdot \mu}) \otimes \mathbb{C}_{[-\mu]}) \right) \right]^K.$$

See [1, 33], for the definition of the exterior product

$$\odot : \mathbf{K}_K(\mathbf{T}_K \mathcal{O}_\Lambda) \times \mathbf{K}_K(\mathbf{T}(K \cdot \mu)) \rightarrow \mathbf{K}_K(\mathbf{T}_K(\mathcal{O}_\Lambda \times K \cdot \mu)).$$

It is easy to check that the product  $\text{Thom}(\mathcal{O}_\Lambda) \odot \text{Thom}(\overline{K \cdot \mu})$  is equal to the Thom symbol  $\text{Thom}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  on the manifold  $(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}, J_\Lambda \times -J_{K \cdot \mu})$ . Hence the product  $\text{Thom}^{\mathcal{H}}(\mathcal{O}_\Lambda) \odot \text{Thom}(\overline{K \cdot \mu})$  is equal to the Thom symbol  $\text{Thom}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  pushed by the vector field  $(\mathcal{H}, 0)$ : let us denote it  $\text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$ .

The tensor product

$$\mathcal{L} := \mathcal{L}_\Lambda \otimes \mathbb{C}_{[-\mu]}$$

is a prequantum line bundle over the symplectic manifold  $\mathcal{O}_\Lambda \times \overline{K \cdot \mu}$ .

Finally (3.24) can be rewritten as

$$(3.25) \quad m_\Lambda(\mu) = \left[ \text{Index}_{\mathcal{O}_\Lambda \times K \cdot \mu}^K \left( \text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \otimes \mathcal{L} \right) \right]^K.$$

The moment map relative to the Hamiltonian  $K$ -action on  $\mathcal{O}_\Lambda \times \overline{K \cdot \mu}$  is

$$(3.26) \quad \begin{aligned} \Phi_1 : \mathcal{O}_\Lambda \times \overline{K \cdot \mu} &\longrightarrow \mathfrak{k}^* \\ (m, \xi) &\longmapsto \Phi_K(m) - \xi \end{aligned}$$

For any  $t \in \mathbb{R}$ , we consider the map  $\Phi_t : \mathcal{O}_\Lambda \times \overline{K \cdot \mu} \rightarrow \mathfrak{k}^*$ ,  $\Phi_t(m, \xi) := \Phi(m) - t\xi$ . Let  $\mathcal{H}_t$  be the Hamiltonian vector field of  $\frac{1}{2} \|\Phi_t\|^2: \mathcal{O}_\Lambda \times \overline{K \cdot \mu} \rightarrow \mathbb{R}$ . We denoted  $\text{Thom}^{\mathcal{H}_t}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  the symbol  $\text{Thom}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  pushed by the vector field  $\mathcal{H}_t$ .

We have the fundamental

**Proposition 3.9.** • *There exists a compact subset  $\mathcal{K}$  of  $\mathcal{O}_\Lambda$  such that*

$$\text{Cr}(\|\Phi_t\|^2) \subset \mathcal{K} \times K \cdot \mu$$

for any  $t \in [0, 1]$ .

• *The symbols  $\text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  and  $\text{Thom}^{\mathcal{H}_t}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$ ,  $t \in [0, 1]$  are  $K$ -transversally elliptic.*

• *The symbols  $\text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  and  $\text{Thom}^{\mathcal{H}_t}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$ ,  $t \in [0, 1]$  define the same class in  $\mathbf{K}_K(\mathbf{T}_K(\mathcal{O}_\Lambda \times K \cdot \mu))$ .*

*Proof.* The proof of the first point is given in [34][Section 5.3.] when  $\Lambda$  is regular. In the Appendix, we propose another (simpler) proof that we learn from Michèle Vergne. For the second point we check that

$$\text{Char} \left( \text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \right) \cap \mathbf{T}_K(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \simeq \text{Cr}(\|\Phi_K\|^2) \times K \cdot \mu$$

and

$$\text{Char} \left( \text{Thom}^{\mathcal{H}_t}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \right) \cap \mathbf{T}_K(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \simeq \text{Cr}(\|\Phi_t\|^2)$$

are compact subsets of the 0-section of  $\mathbf{T}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$ .

Since  $\text{Cr}(\|\Phi_t\|^2)$  stay in the compact set  $\mathcal{K} \times K \cdot \mu$  for any  $t \in [0, 1]$ , the family  $\text{Thom}^{\mathcal{H}_t}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  is an homotopy of transversally elliptic symbol : hence they define the same class in  $\mathbf{K}_K(\mathbf{T}_K(\mathcal{O}_\Lambda \times K \cdot \mu))$ .

The vector field  $\mathcal{H}_0$  on  $\mathcal{O}_\Lambda \times K \cdot \mu$  is equal to  $(\mathcal{H}, V)$  where

$$V(m, \xi) \in \mathbf{T}_\xi(K \cdot \mu)$$

for any  $(m, \xi) \in \mathcal{O}_\Lambda \times K \cdot \mu$ . We use the deformation  $(\mathcal{H}, sV), s \in [0, 1]$ : let  $\text{Thom}^{(\mathcal{H}, sV)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  be the Thom symbol pushed by the vector field  $(\mathcal{H}, sV)$ . It is easy to check that there exists a compact subset of  $\mathcal{K}' \subset \mathbf{T}(\mathcal{O}_\Lambda \times K \cdot \mu)$  such that

$$\text{Char} \left( \text{Thom}^{(\mathcal{H}, sV)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \right) \cap \mathbf{T}_K(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \subset \mathcal{K}'$$

for any  $s \in [0, 1]$ . The family  $\text{Thom}^{(\mathcal{H}, sV)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}), s \in [0, 1]$  is then an homotopy of transversally elliptic symbols : hence  $\text{Thom}^{(\mathcal{H}, 0)}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  and  $\text{Thom}^{\mathcal{H}_0}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu})$  define the same class in  $\mathbf{K}_K(\mathbf{T}_K(\mathcal{O}_\Lambda \times K \cdot \mu))$ .  $\square$

Following the former proposition and (3.25), we have

$$(3.27) \quad m_\Lambda(\mu) = \left[ \text{Index}_{\mathcal{O}_\Lambda \times K \cdot \mu}^K \left( \text{Thom}^{\mathcal{H}_1}(\mathcal{O}_\Lambda \times \overline{K \cdot \mu}) \otimes \mathcal{L} \right) \right]^K.$$

We are now in the following setting :

- $\mathcal{X} := \mathcal{O}_\Lambda \times \overline{K \cdot \mu}$  is a Hamiltonian  $K$ -manifold with a *proper* moment map  $\Phi_1 : \mathcal{X} \rightarrow \mathfrak{k}^*$ ,
- $\mathcal{L}$  is a prequantum line bundle over  $\mathcal{X}$ ,
- the Hamiltonian vector field  $\mathcal{H}_1$  of the function  $\frac{-1}{2} \|\Phi_1\|^2$  vanishes on a compact subset.

Hence the “pushed” Thom symbol  $\text{Thom}^{\mathcal{H}_1}(\mathcal{X})$  is  $K$ -transversally elliptic on  $\mathcal{X}$ . In this context we can consider the equivariant index  $\text{Index}^K(\text{Thom}^{\mathcal{H}_1}(\mathcal{X}) \otimes \mathcal{L})$ , and we have the following theorem

**Theorem 3.10** ([33, 35]). *The multiplicity of the trivial representation in  $\text{Index}^K(\text{Thom}^{\mathcal{H}_1}(\mathcal{X}) \otimes \mathcal{L})$  is equal to  $\mathcal{Q}(\mathcal{X}_0)$ , where  $\mathcal{X}_0$  is the (compact) symplectic reduction of  $\mathcal{X}$  at 0.*

If we apply Theorem 3.10 to  $\mathcal{X} = \mathcal{O}_\Lambda \times \overline{K \cdot \mu}$  we have  $\mathcal{X}_0 \simeq (\mathcal{O}_\Lambda)_\mu$ , and then we can conclude that

$$m_\Lambda(\mu) = \mathcal{Q}((\mathcal{O}_\Lambda)_\mu).$$

The proof of Proposition 3.8 is then completed.

#### 4. PROPERNESS AND ADMISSIBILITY

In this section, we consider an element  $\Lambda \in \mathcal{C}_{\text{hol}}$ , and a closed connected subgroup  $H$  of  $K$ . We consider the representation  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$  of  $K$ : note that it is an admissible  $K$ -representation since the circle group  $Z(K)$  acts on  $\mathfrak{p}^+$  by multiplication. We are interested in the condition

- C1** The representation  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$  is an admissible  $H$  – representation.

Let  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  be the moment map relative to the Hamiltonian action of  $H$  on the coadjoint orbit  $\mathcal{O}_\Lambda := G \cdot \Lambda$ : the map  $\Phi_H$  is simply the composition of the moment map  $\Phi_K : \mathcal{O}_\Lambda \rightarrow \mathfrak{k}^*$  with the canonical projection  $\pi : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$ .

Let us consider the condition

**C2** The map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$  is proper.

The aim of this section is to prove that **C1**  $\iff$  **C2**. During the proof, we will obtain other equivalent conditions.

We start with the

**Lemma 4.1.** *We have **C2**  $\implies$  **C1**.*

*Proof.* We have proved in Section 3 that  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$  is equal to  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda)$ . Then, Property **P2** of Theorem 2.7 tells us that the properness of  $\Phi_H$  implies the  $H$ -admissibility of  $V_\Lambda^K \otimes S(\mathfrak{p}^+)|_H$ . Since this fact is easy to prove, let's recall it.

For  $\mu \in \widehat{K}$  and  $\nu \in \widehat{H}$  we denote  $N_\nu^\mu = \dim(\text{hom}_H(V_\nu^H, V_\mu^K|_H))$  the multiplicity of  $V_\nu^H$  in the restriction  $V_\mu^K|_H$ . Since  $V_\Lambda^K \otimes S(\mathfrak{p}^+) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K$  we know that the multiplicity (possibly infinite) of  $V_\nu^H$  in  $V_\Lambda^K \otimes S(\mathfrak{p}^+)$  is

$$(4.28) \quad \sum_{\mu \in \widehat{K}} N_\nu^\mu \mathcal{Q}(M_\mu).$$

Let us see that the former sum is always *finite* when **C2** holds. Since  $V_\mu^K$  is equal to the  $K$ -quantization of the coadjoint orbit  $K \cdot \mu$ , the restriction  $V_\mu^K|_H$  is equal to the quantization of  $K \cdot \mu$ , viewed as a Hamiltonian  $H$ -manifold: the corresponding moment map  $K \cdot \mu \rightarrow \mathfrak{h}^*$  is the restriction of the projection  $\pi$  to  $K \cdot \mu$ . The ‘‘quantization commutes with reduction’’ theorem tells us that the multiplicity  $N_\nu^\mu$  is equal to the quantization of the symplectic reduction of the Hamiltonian  $H$ -manifold  $K \cdot \mu$  at  $\nu$ . In particular  $N_\nu^\mu \neq 0$  implies that  $\nu \in \pi(K \cdot \mu)$ . Finally

$$N_\nu^\mu \mathcal{Q}(M_\mu) \neq 0 \implies \mu \in K \cdot \pi^{-1}(\nu) \quad \text{and} \quad \Phi_K^{-1}(\mu) \neq \emptyset.$$

These two conditions imply that we can restrict the sum of (4.28) to

$$\mu \in \widehat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$$

which is *finite* since  $\Phi_H$  is proper.  $\square$

The rest of this section is dedicated to the proof of **C1**  $\implies$  **C2**. Since  $V_\mu^K$  is finite dimensional, one notices that **C1** is equivalent to :

**C1'** The representation  $S(\mathfrak{p}^+)|_H$  is an admissible  $H$ -representation.

**4.1. Formal quantization of  $\mathfrak{p}$ .** Let us denote  $\mathfrak{p}^-$ , the real vector space  $\mathfrak{p}$  equipped with the complex structure  $-\text{ad}(z_o)$  (see the introduction). Let  $\Omega_{\mathfrak{p}}$  be the (constant) symplectic structure on  $\mathfrak{p}$  defined by

$$(4.29) \quad \Omega_{\mathfrak{p}}(X, Y) = B_{\mathfrak{g}}(X, \text{ad}(z_o)Y)$$

where  $B_{\mathfrak{g}}$  is the Killing form on  $\mathfrak{g}$ .

Let  $h$  be the Hermitian structure on  $\mathfrak{p}^-$  defined by  $h(X, Y) = B(X, Y) - i\Omega_{\mathfrak{p}}(X, Y)$ . Let  $U := U(\mathfrak{p}^-)$  be the unitary group with Lie algebra  $\mathfrak{u}$ . We use the isomorphism  $\epsilon : \mathfrak{u} \rightarrow \mathfrak{u}^*$  defined by  $\langle \epsilon(A), B \rangle = -\text{Tr}_{\mathbb{C}}(AB) \in \mathbb{R}$ . For  $X, Y \in \mathfrak{p}$ , let  $X \otimes Y^* : \mathfrak{p}^- \rightarrow \mathfrak{p}^-$  be the  $\mathbb{C}$ -linear map  $Z \mapsto h(Z, X)Y$ .

The action of  $U$  on  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$  is Hamiltonian with moment map  $\Phi_U : \mathfrak{p} \rightarrow \mathfrak{u}^*$  defined by

$$\langle \Phi_U(X), A \rangle = \Omega_{\mathfrak{p}}(A(X), X), \quad X \in \mathfrak{p}, \quad A \in \mathfrak{u}.$$

Via  $\epsilon$ , the moment map  $\Phi_U$  is defined by

$$(4.30) \quad \Phi_U(X) = \frac{1}{i} X \otimes X^*, \quad X \in \mathfrak{p}.$$

The Hamiltonian space  $(\mathfrak{p}, \Omega_{\mathfrak{p}}, \Phi_U)$  is prequantized by the trivial line bundle, equipped with the Hermitian structure  $\langle s, s' \rangle_X = e^{-\frac{\|X\|^2}{2}} \overline{ss'}$  and the Hermitian connexion  $\nabla = d - i\theta$  where  $\theta$  is the 1-form on  $\mathfrak{p}$  defined by  $\theta = \Omega_{\mathfrak{p}}(X, dX)$ .

One sees that  $\Phi_U$  is a proper map. Hence we can consider the formal quantization  $\mathcal{Q}_U^{-\infty}(\mathfrak{p}, \Omega_{\mathfrak{p}}) \in R^{-\infty}(U)$  of the  $U$ -action on the symplectic manifold  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ . We are also interested in  $\mathcal{Q}_U^{-\infty}(\mathfrak{p}, k\Omega_{\mathfrak{p}}) \in R^{-\infty}(U)$ , for any integer  $k \geq 1$ .

**Lemma 4.2** ([35]). *The symmetric space  $S(\mathfrak{p}^+)$  is an admissible  $U$ -representation. The following equality*

$$(4.31) \quad \mathcal{Q}_U^{-\infty}(\mathfrak{p}, k\Omega_{\mathfrak{p}}) = S(\mathfrak{p}^+)$$

*holds in  $R^{-\infty}(U)$ , for any  $k \geq 1$ .*

*Proof.* In [35], we consider the case  $k = 1$ . The other cases follow since the symplectic vector space  $(\mathfrak{p}, k\Omega_{\mathfrak{p}})$  is equivariantly symplectomorphic to  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ .  $\square$

**4.2. Formal quantization of  $\mathfrak{p}$  relative to the  $K$ -action.** The adjoint action of  $K$  on  $\mathfrak{p}$  defines a morphism  $\varphi : K \rightarrow U$ . Let us denote by  $\varphi : \mathfrak{k} \rightarrow \mathfrak{u}$  the corresponding morphism of Lie algebra, and by  $\varphi^* : \mathfrak{u}^* \rightarrow \mathfrak{k}^*$  the dual linear map. The moment map  $\Phi_K : \mathfrak{p} \rightarrow \mathfrak{k}^*$  is equal to the composition of  $\Phi_U : \mathfrak{p} \rightarrow \mathfrak{u}^*$  with  $\varphi^*$ . Via the identification  $\mathfrak{k}^* \simeq \mathfrak{k}$  given<sup>3</sup> by the Killing form  $B_{\mathfrak{g}}$ , the moment map  $\Phi_K$  is defined by

$$(4.32) \quad \Phi_K(X) = -[X, [z_o, X]] \in \mathfrak{k}, \quad X \in \mathfrak{p}.$$

We note that  $\langle \Phi_K(X), z_o \rangle = \| [z_o, X] \|^2 > 0$  if  $X \neq 0$ . Hence the moment map  $\Phi_K : \mathfrak{p} \rightarrow \mathfrak{k}^*$  is *proper*. We use property **[P2]** of Theorem 2.7 (see also Appendix C) to get from Lemma 4.2 the

**Corollary 4.3.** *The symmetric space  $S(\mathfrak{p}^+)$  is an admissible  $K$ -representation. The following equality*

$$(4.33) \quad \mathcal{Q}_K^{-\infty}(\mathfrak{p}, k\Omega_{\mathfrak{p}}) = S(\mathfrak{p}^+)$$

*holds in  $R^{-\infty}(K)$ , for any  $k \geq 1$ .*

We look now at the Hamiltonian action of a closed connected subgroup  $H \subset K$  on  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ . The moment map  $\Phi_H : \mathfrak{p} \rightarrow \mathfrak{h}^*$  is the composition of the map  $\Phi_K : \mathfrak{p} \rightarrow \mathfrak{k}^*$  with the canonical projection  $\pi : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$ . In this setting, we know from property **[P2]** that the properness of  $\Phi_H$  implies that  $S(\mathfrak{p}^+)|_H$  is an admissible representation of  $H$ . In [35][Section 5], we have proved the converse. Let  $\Delta_K(\mathfrak{p})$  be the Kirwan polyhedral convex set associated to the Hamiltonian action of  $K$  on  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ . Let  $\mathfrak{h}^{\perp} := \ker(\pi) \subset \mathfrak{k}^*$ . We have the

**Lemma 4.4** ([35]). *The following conditions are equivalent :*

<sup>3</sup>The map  $\xi \in \mathfrak{k}^* \mapsto \tilde{\xi} \in \mathfrak{k}$  is defined by the relation  $\langle \xi, X \rangle := -B_{\mathfrak{g}}(\tilde{\xi}, X)$ ,  $\forall X \in \mathfrak{k}$ .

- (1)  $\Delta_K(\mathfrak{p}) \cap K \cdot \mathfrak{h}^\perp = \{0\}$ ,
- (2) the map  $\Phi_H : \mathfrak{p} \rightarrow \mathfrak{h}^*$  is proper,
- (3) The subalgebra  $S(\mathfrak{p}^+)^H$  formed by the  $H$ -invariant elements is reduced to the constants.
- (4)  $\mathbf{C1}' : S(\mathfrak{p}^+)|_H$  is an admissible representation of  $H$ .

*Proof.* Since the map  $\Phi_H : \mathfrak{p} \rightarrow \mathfrak{h}^*$  is quadratic, the map  $\Phi_H$  is proper if and only if  $\Phi_H^{-1}(0) = \{0\}$ . Now it is easy to check that  $\Phi_K(\Phi_H^{-1}(0)) = K \cdot \Delta_K(\mathfrak{p}) \cap \mathfrak{h}^\perp$ . Hence (1)  $\iff \Phi_H^{-1}(0) = \{0\} \iff$  (2).

The equivalence of the last three points uses property **[P2]** and some basic results of Geometric Invariant Theory (see Lemma 5.2 in [35]).  $\square$

**4.3. Proof of C1  $\implies$  C2.** Let  $\Delta_K(\mathcal{O}_\Lambda)$  be the Kirwan polyhedral convex set associated to the Hamiltonian action of  $K$  on  $(\mathcal{O}_\Lambda, \Omega_\Lambda)$ .

To any non-empty subset  $C$  of a real vector space  $E$ , we associate its asymptotic cone  $\text{As}(C) \subset E$  formed by all the limits  $y = \lim_{k \rightarrow \infty} t_k y_k$  where  $(t_k)$  is a sequence of non-negative reals converging to 0 and  $y_k \in C$ . Recall the following basic facts:

- (1)  $\text{As}(C)$  is a closed cone,
- (2)  $\text{As}(C) = \{0\}$  if  $C$  is bounded,
- (3)  $\text{As}(C) = C$  if  $C$  is a closed cone,
- (4) If a compact Lie group  $K$  acts linearly on  $E$ , then  $\text{As}(K \cdot C) = K \cdot \text{As}(C)$ .

**Proposition 4.5.** *Let  $\Lambda \in \mathcal{C}_{\text{hol}}$ . We have*

$$\Delta_K(\mathcal{O}_\Lambda) = \Delta_K(K \cdot \Lambda \times \mathfrak{p}) \quad \text{and} \quad \text{As}(\Delta_K(\mathcal{O}_\Lambda)) = \Delta_K(\mathfrak{p}).$$

*Proof.* For any integer  $k \geq 1$ , the coadjoint orbit  $\mathcal{O}_\Lambda$ , equipped with the symplectic form  $k\Omega_\Lambda$ , is symplectomorphic to  $(\mathcal{O}_{k\Lambda}, \Omega_{k\Lambda})$ . Theorem 3.1 shows then that

$$\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda, k\Omega_\Lambda) = V_{k\Lambda}^K \otimes S(\mathfrak{p}^+).$$

Consider now the product  $\mathcal{O}'_\Lambda := K \cdot \Lambda \times \mathfrak{p}$  equipped with the symplectic structure  $\Omega'_\Lambda := \Omega_{K \cdot \Lambda} \times \Omega_\mathfrak{p}$ : here  $\Omega_{K \cdot \Lambda}$  is the Kirillov-Kostant-Souriau symplectic form and  $\Omega_\mathfrak{p}$  is defined in (4.29). For any integer  $k \geq 1$ , the symplectic manifold is  $(\mathcal{O}'_\Lambda, k\Omega'_\Lambda)$  is pre-quantized by  $(\mathcal{L}'_\Lambda)^{\otimes k}$ , where  $\mathcal{L}'_\Lambda$  is the pull-back of the line bundle  $K \times_{K_\Lambda} \mathbb{C}_\Lambda \rightarrow K \cdot \Lambda$  to  $\mathcal{O}'_\Lambda$ .

Since  $(\mathfrak{p}, \Omega_\mathfrak{p})$  has a *proper*  $K$ -moment map, we can use property **[P3]** of Theorem 2.7. We have

$$\begin{aligned} \mathcal{Q}_K^{-\infty}(\mathcal{O}'_\Lambda, k\Omega'_\Lambda) &= \mathcal{Q}_K(K \cdot \Lambda, k\Omega_{K \cdot \mu}) \otimes \mathcal{Q}_K^{-\infty}(\mathfrak{p}, k\Omega_\mathfrak{p}) \\ &= V_{k\Lambda}^K \otimes S(\mathfrak{p}^+). \end{aligned}$$

We are now in the setting of Lemma 2.5 :  $(\mathcal{O}_\Lambda, \Omega_\Lambda)$  and  $(\mathcal{O}'_\Lambda, \Omega'_\Lambda)$  are two pre-quantized *proper* Hamiltonian  $K$ -manifold such that  $\mathcal{Q}_K^{-\infty}(\mathcal{O}_\Lambda, k\Omega_\Lambda) = \mathcal{Q}_K^{-\infty}(\mathcal{O}'_\Lambda, k\Omega'_\Lambda)$  for any integer  $k \geq 1$ . This implies that  $\Delta_K(\mathcal{O}_\Lambda) = \Delta_K(\mathcal{O}'_\Lambda)$ . Hence the first point is proved.

For the other point, we first observe that  $\Lambda + \Delta_K(\mathfrak{p}) \subset \Delta_K(\mathcal{O}'_\Lambda)$ , so

$$\Delta_K(\mathfrak{p}) = \text{As}(\Lambda + \Delta_K(\mathfrak{p})) \subset \text{As}(\Delta_K(\mathcal{O}'_\Lambda)).$$

Let  $y \in \text{As}(\Delta_K(\mathcal{O}'_\Lambda))$ . We have  $y = \lim_{k \rightarrow \infty} t_k y_k$  with  $y_k = y'_k + y''_k$ , where  $y'_k \in K \cdot \Lambda$ ,  $y''_k \in \Phi_K(\mathfrak{p})$ ,  $y'_k + y''_k \in \mathfrak{t}_+^*$  and  $t_k$  is a sequence of positive number converging to 0. Since  $y'_k$  is bounded, we have

$$y = \lim_{k \rightarrow \infty} t_k y''_k \in \Phi_K(\mathfrak{p}) \cap \mathfrak{t}_+^*.$$

So we have proved that  $y \in \Delta_K(\mathfrak{p})$ . With the first point we can conclude that

$$\Delta_K(\mathfrak{p}) = \text{As}(\Delta_K(\mathcal{O}'_\Lambda)) = \text{As}(\Delta_K(\mathcal{O}_\Lambda)).$$

□

**Remark 4.6.** When  $\Lambda \in \mathcal{C}_{\text{hol}}$  is  $K$ -invariant, the Kähler manifold  $\mathcal{O}_\Lambda$  is exactly the Hermitian symmetric space  $G/K$ . In this situation, McDuff [27] has shown that  $G/K$  is symplectomorphic to the symplectic vector space  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ .

In the light of Proposition 4.5, we conjecture that for any  $\Lambda \in \mathcal{C}_{\text{hol}}$ , the coadjoint orbit  $\mathcal{O}_\Lambda$  is  $K$ -equivariantly symplectomorphic to the product  $K \cdot \Lambda \times \mathfrak{p}$  equipped with the symplectic structure  $\Omega_{K \cdot \Lambda} \times \Omega_{\mathfrak{p}}$ .

We need the following basic

**Lemma 4.7.** Let  $(M, \Omega)$  be a Hamiltonian  $K$ -manifold with a proper moment map  $\Phi_K : M \rightarrow \mathfrak{k}^*$ . Let  $H \subset K$  be a closed connected subgroup. Let  $\Phi_H : M \rightarrow \mathfrak{k}^*$  be the moment map relative to the action of  $H$  on  $M$ . Suppose that we have

$$\text{As}(\Delta_K(M)) \cap K \cdot \mathfrak{h}^\perp = \{0\}.$$

Then there exists  $c > 0$  such that  $\|\Phi_H(m)\| \geq c\|\Phi_K(m)\|$  holds outside a compact subset of  $M$ . In particular  $\Phi_H$  is a proper map.

*Proof.* Suppose that there exists a sequence  $m_i \in M$  such that

$$\lim_{i \rightarrow \infty} \|\Phi_K(m_i)\| = \infty \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{\|\Phi_H(m_i)\|}{\|\Phi_K(m_i)\|} = 0.$$

We put  $\Phi_K(m_i) = k_i \cdot y_i$  with  $k_i \in K$  and  $y_i \in \Delta_K(M)$ . We have then

$$\lim_{i \rightarrow \infty} \pi \left( k_i \cdot \frac{y_i}{\|y_i\|} \right) = 0$$

where  $\pi : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$  is the projection. Here we can assume that the sequence  $k_i$  converge to  $k \in K$ , and that the sequence  $\frac{y_i}{\|y_i\|}$  converge to  $y \in \text{As}(\Delta_K(M))$ ,  $\|y\| = 1$ . We get then that  $\pi(k \cdot y) = 0$ . In other words,  $y$  is a non-zero element in  $\text{As}(\Delta_K(M)) \cap K \cdot \ker(\pi)$ . □

We can now finish the proof of **C1**  $\implies$  **C2**. We have already check in Lemma 4.4 that

$$\mathbf{C1} \iff \mathbf{C1}' \iff \Delta_K(\mathfrak{p}) \cap K \cdot \mathfrak{h}^\perp = \{0\}.$$

We have proved in Proposition 4.5 that  $\Delta_K(\mathfrak{p}) = \text{As}(\Delta_K(\mathcal{O}_\Lambda))$ , so condition **C1** is equivalent to

$$(4.34) \quad \text{As}(\Delta_K(\mathcal{O}_\Lambda)) \cap K \cdot \mathfrak{h}^\perp = \{0\}.$$

Finally, we know after Lemma 4.7 that (4.34) implies the properness of the moment map  $\Phi_H : \mathcal{O}_\Lambda \rightarrow \mathfrak{h}^*$ .



5. DESCRIPTION OF  $\Delta_K(\mathfrak{p})$ 

The purpose of this section is the description of the Kirwan polyhedral cone  $\Delta_K(\mathfrak{p})$  which is attached to the Hamiltonian action of  $K$  on  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$ .

For any root  $\alpha \in \mathfrak{R} = \mathfrak{R}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  the corresponding root space  $\mathfrak{g}_{\alpha} \subset \mathfrak{g}_{\mathbb{C}}$  is defined as  $\{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = i\langle \alpha, H \rangle X, \forall H \in \mathfrak{t}\}$ .

For the rest of this section, we work with the system of positive roots  $\mathfrak{R}_{\text{hol}}^+ = \mathfrak{R}_{\mathbb{C}}^+ \cup \mathfrak{R}_n^{+, z_o}$  defined in the introduction. For any positive non-compact root  $\beta \in \mathfrak{R}_n^{+, z_o}$ , there are  $H_{\beta} \in \mathfrak{t}, E_{\beta} \in \mathfrak{g}_{\beta}, E_{-\beta} \in \mathfrak{g}_{-\beta}$  such that

$$(5.35) \quad \begin{aligned} [E_{\beta}, E_{-\beta}] &= iH_{\beta} \\ \overline{E_{\beta}} &= E_{-\beta} \\ B_{\mathfrak{g}}(E_{\beta}, E_{-\beta}) &= \frac{2}{\|\beta\|^2}. \end{aligned}$$

Here  $X \mapsto \overline{X}$  is the conjugation on  $\mathfrak{g}_{\mathbb{C}}$  relative to the real form  $\mathfrak{g}$ , and the norm  $\| - \|^2$  on  $\mathfrak{t}^*$  is induced by the Killing form  $B_{\mathfrak{g}}$ .

Note that conditions (5.35) implies that  $[iH_{\beta}, E_{\beta}] = 2E_{\beta}, [iH_{\beta}, E_{-\beta}] = -2E_{-\beta}$  and

$$(5.36) \quad H_{\beta} \simeq -2 \frac{\beta}{\|\beta\|^2}$$

through the isomorphism  $\mathfrak{t} \simeq \mathfrak{t}^*$ . In particular  $iH_{\beta}, E_{\beta}$  and  $E_{-\beta}$  span a subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ .

For  $\beta \in \mathfrak{R}_n^{+, z_o}$ , let  $X_{\beta} = \frac{1}{2}(E_{\beta} + E_{-\beta})$  and  $Y_{\beta} = \frac{1}{2i}(E_{\beta} - E_{-\beta})$ . Thus the set  $\{X_{\beta}, Y_{\beta}\}_{\beta \in \mathfrak{R}_n^{+, z_o}}$  is a real basis of  $\mathfrak{p}$ . Since  $\langle \beta, z_o \rangle = 1$  for any  $\beta \in \mathfrak{R}_n^{+, z_o}$ , we have  $\text{ad}(z_o)X_{\beta} = -Y_{\beta}$  and  $\text{ad}(z_o)Y_{\beta} = X_{\beta}$ .

We will now describe the restricted root system of  $G/K$ . Two roots  $\alpha, \beta \in \mathfrak{R}$  are *strongly orthogonal*, written  $\alpha \perp\!\!\!\perp \beta$ , if neither of  $\alpha \pm \beta$  is a root. One can easily check that strong orthogonality implies orthogonality with respect to the scalar product on  $\mathfrak{t}^*$ .

Consider the ‘‘cascade construction’’

$$\begin{aligned} \Psi &= \{\gamma_1, \dots, \gamma_r\}, \text{ maximal set constructed by :} \\ \gamma_1 &\text{ is the maximal root in } \mathfrak{R}_n^{+, z_o} \\ \gamma_{i+1} &\text{ is the maximal root in } \{\beta \in \mathfrak{R}_n^{+, z_o} \mid \beta \perp\!\!\!\perp \gamma_k \text{ for } k = 1, \dots, i\}. \end{aligned}$$

For the roots  $\gamma_k$ , we denote simply  $X_k, Y_k, H_k$  the elements  $X_{\gamma_k}, Y_{\gamma_k}, H_{\gamma_k}$ . We have the classical result (see [16][Prop. 7.4])

**Lemma 5.1.** *The subspace*

$$\mathfrak{a} := \sum_{k=1}^r \mathbb{R}X_k$$

*is maximal abelian in  $\mathfrak{p}$ .*

Since  $\mathfrak{p} = K \cdot \mathfrak{a}$ , it is sufficient to understand the image of  $\mathfrak{a}$  by  $\Phi_K$  to compute  $\Delta_K(\mathfrak{p})$ : in fact this Kirwan cone will be computed by describing the image by  $\Phi_K$  of a closed cone  $\mathfrak{a}_+ \subset \mathfrak{a}$ , which is a fundamental domain for the  $K$ -action on  $\mathfrak{p}$ .

For  $\lambda \in \mathfrak{a}^*$ , we write

$$\mathfrak{g}^{\lambda} := \{X \in \mathfrak{g} \mid [H, X] = \langle \lambda, H \rangle X \text{ for all } H \in \mathfrak{a}\}.$$

If  $\mathfrak{g}^\lambda \neq 0$  and  $\lambda \neq 0$ , we call  $\lambda$  a *restricted root* of  $\mathfrak{g}$ . The set of restricted roots is denoted  $\Sigma$ . Let  $W_\Sigma$  be the group generated by the orthogonal symmetries along the hyperplane  $\ker(\lambda)$ ,  $\lambda \in \Sigma$ . A proof of the following classic result can be found in [20][Sec. VI.5].

**Proposition 5.2.** •  $\Sigma$  is an abstract root system on  $\mathfrak{a}^*$ .

• The group  $W_\Sigma$  is finite and is canonically identify with the quotient  $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ , where  $N_K(\mathfrak{a})$  is the normalizer subgroup of  $\mathfrak{a}$  in  $K$  and  $Z_K(\mathfrak{a})$  is the centralizer subgroup of  $\mathfrak{a}$  in  $K$ .

With the help of a system of positive roots  $\Sigma^+$ , we define the closed chamber

$$\mathfrak{a}_+ := \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle \geq 0 \text{ for all } \lambda \in \Sigma^+\}.$$

Proposition 5.2 tell us then that any  $K$ -orbit in  $\mathfrak{p}$  intersects  $\mathfrak{a}_+$  in a unique point.

We have the fundamental

**Proposition 5.3.** For a particular system of positive roots  $\Sigma^+$ , we have

$$\mathfrak{a}_+ = \sum_{k=1}^r \mathbb{R}^{\geq 0}(X_1 + \cdots + X_k).$$

*Proof.* The proof is done in Appendix B. □

An element  $X$  of the chamber  $\sum_{k=1}^r \mathbb{R}^{\geq 0}(X_1 + \cdots + X_k)$  is of the form  $X = \sum_{k=1}^r t_k X_k$  with  $t_1 \geq \cdots \geq t_r \geq 0$ . Then  $\Phi_K(X)$ , view as an element of  $\mathfrak{k}$ , is equal to

$$\begin{aligned} \Phi_K(X) &= -[X, [z_o, X]] \\ &= \sum_{k,l} t_k t_l [X_k, Y_l] \\ &= -\frac{1}{2} \sum_{k=1}^r (t_k)^2 H_k. \end{aligned}$$

Here we have used the fact  $[X_k, Y_l] = 0$  for  $k \neq l$  since  $[\mathfrak{g}_{\gamma_k}, \mathfrak{g}_{\pm\gamma_l}] = 0$ . When  $k = l$ , one sees that  $[X_k, Y_k] = \frac{i}{2}[E_{\gamma_k}, E_{-\gamma_k}] = -\frac{1}{2}H_k$ .

Since the vector  $-\frac{1}{2}H_k \in \mathfrak{t}$  corresponds to  $\frac{\gamma_k}{\|\gamma_k\|^2}$  through the identification  $\mathfrak{t} \simeq \mathfrak{t}^*$ . We conclude that

$$\Phi_K(X) = \sum_{k=1}^r (t_k)^2 \frac{\gamma_k}{\|\gamma_k\|^2} \in \mathfrak{t}^*$$

for  $X = \sum_{k=1}^r t_k X_k$ .

Let  $\mathfrak{t}_+^* \subset \mathfrak{t}^*$  be the Weyl chamber defined the by the system of positive compact roots  $\mathfrak{R}_c^+$ . Let  $\overline{\mathcal{C}_{\text{hol}}} \subset \mathfrak{t}_+^*$  be the Weyl chamber defined by the system of positive roots  $\mathfrak{R}_{\text{hol}}^+$ . The following proposition will be proved in Appendix B.

**Proposition 5.4.** All the roots  $\gamma_k$  have the same lengths, and we have

$$\overline{\mathcal{C}_{\text{hol}}} \cap \text{Vect}(\gamma_1, \dots, \gamma_r) = \sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \cdots + \gamma_k).$$

In particular, the weight  $\gamma_1 + \cdots + \gamma_k$  is dominant for any  $k = 1, \dots, r$ .

We know then that  $\Phi_K(X) = \frac{1}{\|\gamma_1\|^2} \sum_{k=1}^r (t_k)^2 \gamma_k$  belongs to the Weyl chamber  $\mathfrak{t}_+^*$  if  $X = \sum_{k=1}^r t_k X_k$  belongs to the chamber  $\mathfrak{a}_+$ . Hence, the moment map  $\Phi_K : \mathfrak{p} \rightarrow \mathfrak{t}_+^*$  defines a one to one map between  $\mathfrak{a}_+$  and the cone  $\sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k) \subset \mathfrak{t}_+^*$ . Using now the fact that  $\mathfrak{a}_+$  and  $\mathfrak{t}_+^*$  are respectively fundamental domains for the  $K$ -action on  $\mathfrak{p}$  and  $\mathfrak{t}_+^*$ , we get the following

**Proposition 5.5.** • *The Kirwan polyhedral cone  $\Delta_K(\mathfrak{p})$  is equal to*

$$(5.37) \quad \sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k).$$

• *The  $K$ -Hamiltonian space  $(\mathfrak{p}, \Omega_{\mathfrak{p}})$  is without multiplicities : for any  $\xi \in \mathfrak{t}_+^*$ , the fiber  $\Phi_K^{-1}(K \cdot \xi) \subset \mathfrak{p}$  is a  $K$ -orbit.*

We can summarize the results of Sections 4 and 5 in the following

**Theorem 5.6.** *Let  $V_{\Lambda}^K \otimes S(\mathfrak{p}^+)$  be the admissible  $K$ -representation attached to  $\Lambda \in \mathcal{C}_{\text{hol}}$ . Let  $H$  be a closed connected Lie subgroup of  $K$ . Let  $\Phi_H : \mathcal{O}_{\Lambda} \rightarrow \mathfrak{h}^*$  be the moment map relative to the action of  $H$  on the coadjoint orbit  $\mathcal{O}_{\Lambda}$ . The following statement are equivalent:*

- (1) *The map  $\Phi_H : \mathcal{O}_{\Lambda} \rightarrow \mathfrak{h}^*$  is proper.*
- (2) *The  $H$ -multiplicities in  $V_{\Lambda}^K \otimes S(\mathfrak{p}^+)$  are finite.*
- (3) *The subalgebra  $S(\mathfrak{p}^+)^H$  formed by the  $H$ -invariant elements is reduced to the constants.*
- (4) *We have*

$$(5.38) \quad \left( \sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k) \right) \cap K \cdot \mathfrak{h}^{\perp} = \{0\}.$$

**Remark 5.7.** *Note that the condition (5.38) holds trivially when  $H = K$  since then  $K \cdot \mathfrak{h}^{\perp} = \{0\}$ . When  $H$  is equal to the center  $Z(K) \subset K$ , the set  $K \cdot \mathfrak{h}^{\perp} = \text{Lie}(Z(K))^{\perp}$  intersects  $\sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k)$  only at 0 since  $\langle \gamma_k, z_o \rangle = 1$  for all  $k = 1, \dots, r$ .*

We finish this section by considering the example of  $\text{SU}(p, q)$ , with  $1 \leq p \leq q$ . A maximal compact subgroup of  $\text{SU}(p, q)$  is  $K = \text{S}(\text{U}(p) \times \text{U}(q))$ . The maximal torus  $T \subset K$  is composed by the diagonal matrices. The dual of its Lie algebra is

$$\mathfrak{t}^* := \{(x_1, \dots, x_{p+q}) \in \mathbb{R}^{p+q} \mid \sum_j x_j = 0\}.$$

The vector space  $\mathfrak{p}^+$  is the complex vector space  $\text{M}_{p,q}(\mathbb{C})$  of complex  $p \times q$  matrices. The action of  $K = \text{S}(\text{U}(p) \times \text{U}(q))$  on  $\mathfrak{p}^+ = \text{M}_{p,q}(\mathbb{C})$  is defined by  $(g, h) \cdot M = gMh^{-1}$ .

The Weyl chamber relative to a system of positive compact roots  $\mathfrak{A}_{\mathfrak{c}}^+$  is

$$\mathfrak{t}_+^* := \{(x_1, \dots, x_{p+q}) \in \mathfrak{t}^* \mid x_1 \geq \dots \geq x_p \text{ and } x_{p+1} \geq \dots \geq x_{p+q}\}.$$

The Weyl chamber relative to a system of positive roots  $\mathfrak{A}_{\text{hol}}^+$  is

$$\overline{\mathcal{C}_{\text{hol}}} := \{(x_1, \dots, x_{p+q}) \in \mathfrak{t}^* \mid x_1 \geq \dots \geq x_p \geq x_{p+1} \geq \dots \geq x_{p+q}\}.$$

A family of strongly orthogonal roots is  $\Psi = \{\gamma_1, \dots, \gamma_p\}$  where<sup>4</sup>

$$\gamma_j = e_j - e_{p+q-j+1}.$$

Hence the cone  $\sum_{k=1}^p \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k)$  is equal to

$$\mathcal{D} := \left\{ (x_1, \dots, x_p, \underbrace{0, \dots, 0}_{q-p \text{ times}}, -x_p, \dots, -x_1) \mid x_1 \geq \dots \geq x_p \geq 0 \right\}.$$

Let us consider the normal subgroups  $SU(p)$  and  $SU(q)$  of  $K$ . If  $H = SU(p)$ , it is not hard to see that

$$\mathfrak{h}^\perp \cap \mathfrak{t}^* = \left\{ (\underbrace{x, \dots, x}_p \text{ times}, y_1, \dots, y_q) \mid px + \sum_j y_j = 0 \right\}.$$

so  $\mathfrak{h}^\perp \cap \mathcal{D}$  contains the non-zero element  $(\underbrace{1, \dots, 1}_p \text{ times}, \underbrace{0, \dots, 0}_{q-p \text{ times}}, -1, \dots, -1)$ . Thus we

know from Theorem 5.6 that

- (1) the holomorphic discrete series representations of  $SU(p, q)$  does not have a admissible restriction to  $SU(p)$ ,
- (2) the algebra  $S(M_{p \times q}(\mathbb{C}))$  has an homogeneous  $SU(p)$ -invariant element with strictly positive degree.

Consider now the case where  $H = SU(q)$  with  $p < q$ . We see that

$$\mathfrak{h}^\perp \cap \mathfrak{t}^* = \left\{ (x_1, \dots, x_p, \underbrace{y, \dots, y}_q \text{ times}) \mid \sum_k x_k + qy = 0 \right\}.$$

and that  $\mathfrak{h}^\perp \cap \mathcal{D} = \{0\}$ . From Theorem 5.6 we have then that, if  $p < q$ ,

- (1) the holomorphic discrete series representations of  $SU(p, q)$  have an admissible restriction to  $SU(q)$ ,
- (2) the algebra  $S(M_{p \times q}(\mathbb{C}))$  does not have an homogeneous  $SU(q)$ -invariant element with strictly positive degree.

## 6. MULTIPLICITIES OF THE DISCRETE SERIES

Let  $G$  be a real, connected, semi-simple Lie group with finite center. Let  $K$  be a maximal compact subgroup of  $G$ , and  $T$  be a maximal torus in  $K$ . For the remainder of this section, we assume that  $T$  is a Cartan subgroup of  $G$ . The discrete series of  $G$  is then non-empty and is parametrized by a subset  $\widehat{G}_d$  in the dual  $\mathfrak{t}^*$  of the Lie algebra of  $T$  [14].

Let us fix some notation. Let  $\mathfrak{R}_c \subset \mathfrak{R} \subset \Lambda^*$  be respectively the set of (real) roots for the action of  $T$  on  $\mathfrak{k} \otimes \mathbb{C}$  and  $\mathfrak{g} \otimes \mathbb{C}$ . We choose a system of positive roots  $\mathfrak{R}_c^+$  for  $\mathfrak{R}_c$ , we denote by  $\mathfrak{t}_+^*$  the corresponding Weyl chamber, and we let  $\rho_c$  be half the sum of the elements of  $\mathfrak{R}_c^+$ .

An element  $\lambda \in \mathfrak{t}^*$  is called *regular* if  $(\lambda, \alpha) \neq 0$  for every  $\alpha \in \mathfrak{R}$ , or equivalently, if the stabilizer subgroup of  $\lambda$  in  $G$  is  $T$ . Given a system of positive roots  $\mathfrak{R}^+$  for  $\mathfrak{R}$ , consider the subset  $\Lambda^* + \frac{1}{2} \sum_{\alpha \in \mathfrak{R}^+} \alpha$  of  $\mathfrak{t}^*$ . It does not depend on the choice of  $\mathfrak{R}^+$ , and we denote it by  $\Lambda_\rho^*$  [8].

The discrete series of  $G$  are parametrized by

$$(6.39) \quad \widehat{G}_d := \{ \lambda \in \mathfrak{t}^*, \lambda \text{ regular} \} \cap \Lambda_\rho^* \cap \mathfrak{t}_+^* .$$

<sup>4</sup>Here  $\{e_1, \dots, e_{p+q}\}$  is the canonical basis of  $\mathbb{R}^{p+q}$ .

An element  $\lambda \in \widehat{G}_d$  determines a choice  $\mathfrak{R}^{+, \lambda}$  of positive roots for the  $T$ -action on  $\mathfrak{g} \otimes \mathbb{C} : \alpha \in \mathfrak{R}^{+, \lambda} \iff (\alpha, \lambda) > 0$ . We have  $\mathfrak{R}^{+, \lambda} = \mathfrak{R}_c^+ \cup \mathfrak{R}_n^{+, \lambda}$  and we define

$$\rho_n(\lambda) := \frac{1}{2} \sum_{\beta \in \mathfrak{R}_n^{+, \lambda}} \beta,$$

Note that the *Blattner parameter*

$$\Lambda(\lambda) := \lambda - \rho_c + \rho_n(\lambda)$$

is a dominant weight for any  $\lambda \in \widehat{G}_d$ . We work in this section under Condition (1.5), which states that  $\beta \in \mathfrak{R}_n^{+, \lambda} \iff (\beta, \Lambda(\lambda)) > 0$ . This implies in particular, that the dominant weight  $\Lambda(\lambda)$  does not belong to the non-compact walls.

Let us consider the coadjoint orbit  $\mathcal{O}_{\Lambda(\lambda)} := G \cdot \Lambda(\lambda)$ . It is a  $G$ -Hamiltonian manifold which is prequantized by the line bundle  $\mathcal{L}_{\Lambda(\lambda)} := G \times_{K_{\Lambda(\lambda)}} \mathbb{C}_{\Lambda(\lambda)}$ . We equip  $\mathcal{O}_{\Lambda(\lambda)}$  with the  $G$ -invariant *almost complex structure*  $J_{\Lambda(\lambda)}$  which is characterized by the following fact. The bundle  $\mathbf{T}^{1,0} \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathcal{O}_{\Lambda(\lambda)}$  is equal, above  $\Lambda(\lambda) \in \mathcal{O}_{\Lambda(\lambda)}$ , to the  $T$ -module

$$\sum_{\substack{\alpha \in \mathfrak{R}_c \\ (\alpha, \Lambda(\lambda)) > 0}} \mathfrak{g}_\alpha \oplus \underbrace{\sum_{\substack{\beta \in \mathfrak{R}_n \\ (\beta, \Lambda(\lambda)) < 0}} \mathfrak{g}_\beta}_{\mathfrak{p}(\lambda)^-}$$

Similarly we note  $\mathfrak{p}(\lambda)^+ := \sum_{\beta \in \mathfrak{R}_n^{+, \lambda}} \mathfrak{g}_\beta \subset \mathfrak{p} \otimes \mathbb{C}$ . Note that the *almost complex structure*  $J_{\Lambda(\lambda)}$  is compatible with the symplectic structure on  $\mathcal{O}_{\Lambda(\lambda)}$ , but in general  $J_{\Lambda(\lambda)}$  is not integrable.

Let  $\mathcal{H}_\lambda$  be a discrete series representation attached to  $\lambda \in \widehat{G}_d$ . Recall that the restriction  $\mathcal{H}_\lambda|_K$  is an admissible representation.

The main result of this section is

**Theorem 6.1.** *If  $\lambda \in \widehat{G}_d$  satisfy condition (1.5) we have*

$$(6.40) \quad \mathcal{H}_\lambda|_K = \mathcal{Q}_K^{-\infty}(\mathcal{O}_{\Lambda(\lambda)}).$$

Like we did before, if we use (6.40) together with the property [P2], we get Theorem 1.9.

The proof of Theorem 6.1 is similar to the proof of Theorem 3.1. We introduce, like in Section 3.2, a  $K$ -transversal elliptic symbol  $\sigma_{\Lambda(\lambda)}$  on  $\mathcal{O}_{\Lambda(\lambda)}$  built from the data  $(\mathcal{L}_{\Lambda(\lambda)}, J_{\Lambda(\lambda)})$  and the moment map  $\Phi_K : \mathcal{O}_{\Lambda(\lambda)} \rightarrow \mathfrak{k}^*$ . The same deformation argument as the one used in Section 3.4 shows that

$$\text{Index}^K(\sigma_{\Lambda(\lambda)}) = \mathcal{Q}_K^{-\infty}(\mathcal{O}_{\Lambda(\lambda)}).$$

Thus Theorem 6.1 follows from the following

**Proposition 6.2.** *If  $\lambda \in \widehat{G}_d$  satisfy condition (1.5), we have*

$$\text{Index}^K(\sigma_{\Lambda(\lambda)}) = \mathcal{H}_\lambda|_K \quad \text{in} \quad R_{tc}^{-\infty}(K).$$

**6.1. Proof of Proposition 6.2.** The proof is an adaptation to the proof of Proposition 3.5. Here we consider the the  $K$ -invariant diffeomorphism

$$(6.41) \quad \widehat{\Upsilon} : \mathcal{O}_{\Lambda(\lambda)} \longrightarrow \widehat{\mathcal{O}}_{\Lambda(\lambda)} := K \times_{K_{\Lambda(\lambda)}} \mathfrak{p},$$

defined by  $\widehat{\Upsilon}(ke^X \cdot \Lambda(\lambda)) := [k, X]$ .

The data  $(J_{\Lambda(\lambda)}, \mathcal{L}_{\Lambda(\lambda)}, \mathcal{H}, \sigma_{\Lambda(\lambda)})$ , transported to the manifold  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$  through  $\widehat{\Upsilon}$ , is denoted  $(\widehat{J}_{\Lambda(\lambda)}, \widehat{\mathcal{L}}_{\Lambda(\lambda)}, \widehat{\mathcal{H}}, \widehat{\sigma}_{\Lambda(\lambda)})$ . The line bundle  $\widehat{\mathcal{L}}_{\Lambda(\lambda)}$  is the pull-back of the line bundle  $K \times_{K_{\Lambda(\lambda)}} \mathbb{C}_{\Lambda(\lambda)} \rightarrow K \cdot \Lambda(\lambda)$  to  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$ .

The tangent bundle  $\mathbf{T}\widehat{\mathcal{O}}_{\Lambda(\lambda)}$  is  $K$ -equivariantly isomorphic to  $K \times_{K_{\Lambda(\lambda)}} (\mathfrak{r}_{\lambda} \oplus \mathbf{T}\mathfrak{p})$  where  $\mathfrak{r}_{\lambda} := [\mathfrak{k}, \Lambda(\lambda)]$  is the  $K_{\Lambda(\lambda)}$ -invariant complement of  $\mathfrak{k}_{\Lambda(\lambda)}$ .

Let  $J_{\lambda}$  be the (linear) complex structure on the vector space  $\mathfrak{p}$  such that  $(\mathfrak{p}, J_{\lambda}) \simeq \mathfrak{p}(\lambda)^+$ . Note that  $J_{\lambda}$  is  $K_{\Lambda(\lambda)}$ -invariant since  $\lambda$  satisfies condition (1.5).

Let  $J_{K \cdot \Lambda(\lambda)}|_e$  be the (linear)  $K_{\Lambda(\lambda)}$ -invariant complex structure on the vector space  $\mathfrak{r}_{\lambda}$  defined by the Kähler structure  $J_{K \cdot \Lambda(\lambda)}$  on the coadjoint orbit  $K \cdot \Lambda(\lambda)$ .

We consider on  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$  the following  $K$ -equivariant data:

- (1) The almost complex structure  $\widehat{J}'_{\Lambda}$  such that

$$\widehat{J}'_{\Lambda}|_{(e,v)} = J_{K \cdot \Lambda(\lambda)}|_e \times -J_{\lambda} \quad \text{for every } v \in \mathfrak{p}.$$

- (2) The vector field  $\widehat{\mathcal{H}}'$  defined by:  $\widehat{\mathcal{H}}'_{[k,v]} = -(0, k \cdot [\Lambda(\lambda), v])$  for  $[k, v] \in \widehat{\mathcal{O}}_{\Lambda(\lambda)}$ .

**Definition 6.3.** We consider on  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$  the symbols:

- $\widehat{\tau}'_{\Lambda(\lambda)} := \text{Thom}(\widehat{\mathcal{O}}_{\Lambda(\lambda)}, \widehat{J}'_{\Lambda}) \otimes \widehat{\mathcal{L}}_{\Lambda(\lambda)}$ ,
- $\widehat{\sigma}'_{\Lambda(\lambda)}$  which is the symbol  $\widehat{\tau}'_{\Lambda(\lambda)}$  pushed by the vector field  $\widehat{\mathcal{H}}'$  (see Def. 3.2).

**Proposition 6.4.** • The symbol  $\widehat{\sigma}'_{\Lambda(\lambda)}$  is a  $K$ -transversaly elliptic symbol on  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$ .

• If  $\mathcal{U}$  is a sufficiently small  $K$ -invariant neighborhood of  $K \times_{K_{\Lambda(\lambda)}} \{0\}$  in  $\widehat{\mathcal{O}}_{\Lambda(\lambda)}$ , the restrictions  $\widehat{\sigma}_{\Lambda(\lambda)}|_{\mathcal{U}}$  and  $\widehat{\sigma}'_{\Lambda(\lambda)}|_{\mathcal{U}}$  define the same class in  $\mathbf{K}_K(\mathbf{T}_K \mathcal{U})$ .

*Proof.* The proof works as the proof of Proposition 3.7.  $\square$

Proposition 6.4 shows that  $\text{Index}^K(\sigma_{\Lambda(\lambda)}) = \text{Index}^K(\widehat{\sigma}_{\Lambda(\lambda)}) = \text{Index}^K(\widehat{\sigma}'_{\Lambda(\lambda)})$ . In order to compute  $\text{Index}^K(\widehat{\sigma}'_{\Lambda(\lambda)})$ , we use the induction morphism

$$i_* : \mathbf{K}_{K_{\Lambda}}(\mathbf{T}_{K_{\Lambda}} \mathfrak{p}) \longrightarrow \mathbf{K}_K(\mathbf{T}_K(\widehat{\mathcal{O}}_{\Lambda(\lambda)}))$$

defined by Atiyah in [1] (see also [33][Section 3]). Here  $i_*$  differs from the induction morphism  $j_*$  used in Section 3.3, by the isomorphism

$$\mathbf{K}_K(\mathbf{T}_K(\widehat{\mathcal{O}}_{\Lambda(\lambda)})) \simeq \mathbf{K}_K(\mathbf{T}_K(K \cdot \Lambda(\lambda) \times \mathfrak{p}))$$

induced by the  $K$ -diffeomorphism  $\widehat{\mathcal{O}}_{\Lambda(\lambda)} \simeq K \cdot \Lambda(\lambda) \times \mathfrak{p}$ ,  $[k, X] \mapsto (k \cdot \Lambda(\lambda), k \cdot X)$ .

Let  $\text{Thom}(\mathfrak{p}(\lambda)^-)$  be the  $K_{\Lambda}(\lambda)$ -equivariant Thom symbol of the complex vector space  $\mathfrak{p}(\lambda)^- \simeq (\mathfrak{p}, -J_{\lambda})$ . Let  $\widetilde{\Lambda}(\lambda)$  be the vector field on  $\mathfrak{p}$  which is generated by  $\Lambda(\lambda) \in \mathfrak{k}^* \simeq \mathfrak{k}$ . Let

$$\text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-)$$

be the symbol  $\text{Thom}(\mathfrak{p}(\lambda)^-)$  pushed by the vector field  $\widetilde{\Lambda}(\lambda)$  (see Definition 3.6).

Since  $\Lambda(\lambda)$  does not belongs to the non-compact walls (see condition (1.5)), the vector field  $\widetilde{\Lambda}(\lambda)$  vanishes only at  $0 \in \mathfrak{p}$ : hence the symbol  $\text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-)$  is  $K_{\Lambda(\lambda)}$ -transversaly elliptic.

One checks easily that

$$(6.42) \quad (i_*)^{-1}(\widehat{\sigma}'_{\Lambda(\lambda)}) = \text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-) \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_{\Lambda(\lambda)} \otimes \mathbb{C}_{\Lambda(\lambda)}.$$

Let  $\text{Ind}_{K_{\Lambda(\lambda)}}^K : \mathcal{C}^{-\infty}(K_{\Lambda(\lambda)})^{K_{\Lambda(\lambda)}} \longrightarrow \mathcal{C}^{-\infty}(K)^K$  be the induction map introduced in (3.22). Equality (6.42) and the commutative diagram (3.23) give

$$\begin{aligned} \text{Index}^K(\widehat{\sigma}'_{\Lambda(\lambda)}) &= \text{Ind}_{K_{\Lambda(\lambda)}}^K \left( \text{Index}^{K_{\Lambda(\lambda)}} \left( \text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-) \right) \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_{\Lambda(\lambda)} \otimes \mathbb{C}_{\Lambda(\lambda)} \right) \\ &= \text{Ind}_T^K \left( \text{Index}^T \left( \text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-) \right) \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t} \otimes \mathbb{C}_{\Lambda(\lambda)} \right) \end{aligned}$$

In the last equality, we use two facts (see [33]) :

- Since the symbol  $\text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-)$  is  $T$ -transversally elliptic, the index  $\text{Index}^{K_{\Lambda(\lambda)}} \left( \text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-) \right)$  is  $T$ -admissible, and its restriction to  $T$  is equal to  $\text{Index}^T \left( \text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-) \right)$ .

- For any  $K_{\Lambda(\lambda)}$ -module  $E$  which is  $T$ -admissible we have

$$\text{Ind}_{K_{\Lambda(\lambda)}}^K (E \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{k}_{\Lambda(\lambda)}) = \text{Ind}_T^K (E|_T \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t}).$$

We know from [33][Section 5.1] that the  $T$ -index of  $\text{Thom}^{\Lambda(\lambda)}(\mathfrak{p}(\lambda)^-)$  is equal to the symmetric algebra  $S(\mathfrak{p}(\lambda)^+)$  viewed as a  $T$ -module. Here we use in a crucial way Condition (1.5): for every weight  $\beta$  relative to the  $T$ -action on the complex vector spaces  $\mathfrak{p}(\lambda)^-$ , we have  $(\beta, \Lambda(\lambda)) < 0$ . The  $T$ -module  $S(\mathfrak{p}(\lambda)^+)$  is denoted

$$\left[ \prod_{\beta \in \mathfrak{R}_n^+, \lambda} (1 - t^\beta) \right]_\lambda^{-1} \in R_{tc}^{-\infty}(T).$$

in [34]. So we have proved that

$$(6.43) \quad \text{Index}^K(\sigma_{\Lambda(\lambda)}) = \text{Ind}_T^K \left( \left[ \prod_{\beta \in \mathfrak{R}_n^+, \lambda} (1 - t^\beta) \right]_\lambda^{-1} \otimes \mathbb{C}_{\Lambda(\lambda)} \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t} \right).$$

We have proved in [34][Section 5.1] that the Blattner formulas [15] which computes the  $K$ -multiplicities of the discrete series representation  $\mathcal{H}_\lambda$  are equivalent to the following relation

$$(6.44) \quad \mathcal{H}_\lambda|_K = \text{Hol}_T^K \left( \left[ \prod_{\beta \in \mathfrak{R}_n^+, \lambda} (1 - t^\beta) \right]_\lambda^{-1} \otimes \mathbb{C}_{\Lambda(\lambda)} \right) \quad \text{in } R_{tc}^{-\infty}(K),$$

where the ‘‘holomorphic’’ induction map  $\text{Hol}_T^K$  is equal to  $\text{Ind}_T^K(- \otimes \wedge_{\mathbb{C}}^{\bullet} \mathfrak{k}/\mathfrak{t})$ .

We see that (6.43) and (6.44) complete the proof of Proposition 6.2.

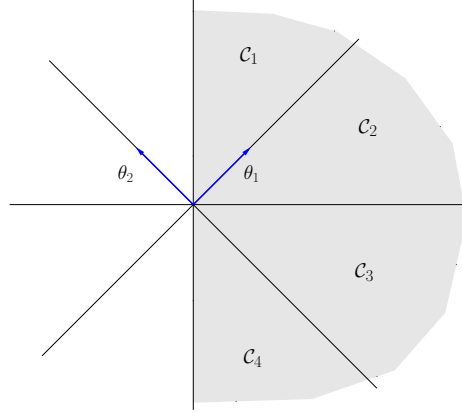
## 6.2. Examples.

6.2.1. *The case of  $\text{Sp}(2, \mathbb{R})$ .* We examined this case in Example (1.3). Let  $\theta_1, \theta_2$  be the  $\mathbb{Z}$ -basis of of the lattice  $\Lambda^*$ . The set of compact roots is  $\mathfrak{R}_c = \{\pm(\theta_1 - \theta_2)\}$ , and the set of non-compact roots is  $\mathfrak{R}_n = \{\pm(\theta_1 + \theta_2), \pm 2\theta_1, \pm 2\theta_2\}$ . We choose  $\theta_1 - \theta_2$  as the positive compact root, hence  $\mathfrak{t}_+^* = \{\theta_1 \geq \theta_2\}$ .

The set of strongly elliptic elements in the Weyl chamber  $\mathfrak{t}_+^*$  has four chambers (see Figure (6.2.1)):  $\mathcal{C}_1 = \{\theta_1 \geq \theta_2 > 0\}$ ,  $\mathcal{C}_2 = \{\theta_1 > -\theta_2 > 0\}$ ,  $\mathcal{C}_3 = \{-\theta_2 > \theta_1 > 0\}$ , and  $\mathcal{C}_4 = \{-\theta_2 \geq -\theta_1 > 0\}$ .

For  $\lambda \in \mathfrak{t}_+^*$  which is regular, the term  $\rho_n(\lambda)$  only depends of the chamber  $\mathcal{C}_i$  where  $\lambda$  stands: let us denoted it  $\rho_n(\mathcal{C}_i)$ .

We check that  $-\rho_c + \rho_n(\mathcal{C}_i) \in \overline{\mathcal{C}_i}$  for  $i = 2, 3$ . Hence, for  $i = 2, 3$  and any Harish-Chandra parameter  $\lambda \in \mathcal{C}_i$ , we have  $\Lambda(\lambda) = \lambda - \rho_c + \rho_n(\mathcal{C}_i) \in \mathcal{C}_i$ .


 FIGURE 2. Chambers for  $\mathrm{Sp}(2, \mathbb{R})$ 

We know already that any regular weight of the holomorphic chamber  $\mathcal{C}_1$  satisfies condition (1.5). It is also the case for the *anti-holomorphic* chamber  $\mathcal{C}_4$ .

Finally we see that condition (1.5) holds for any Harish-Chandra parameter of a discrete series of  $\mathrm{Sp}(2, \mathbb{R})$ .

6.2.2. *The case of  $\mathrm{Sp}(4, \mathbb{R})$ .* Let  $\theta_1, \dots, \theta_4$  be the canonical basis of  $\mathbb{R}^4 \simeq \mathfrak{t}^*$ . The compact positive roots are  $\theta_i - \theta_j, 1 \leq i < j \leq 4$ , so that the corresponding Weyl chamber is  $\mathfrak{t}_+^* := \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4\}$ , and  $\rho_c = \frac{1}{2}(3, 1, -1, -3)$ . The set of non-compact roots is  $\{2\theta_i\} \cup \{\theta_i + \theta_j, i < j\}$ .

We consider the chamber  $\mathcal{C} := \{\lambda_1 \geq \lambda_2 > -\lambda_4 > \lambda_3 > 0\}$  of the Weyl chamber  $\mathfrak{t}_+^*$ . We have  $\rho_n(\mathcal{C}) = \frac{1}{2}(5, 5, 3, -1)$  and then

$$-\rho_c + \rho_n(\mathcal{C}) = (1, 2, 2, 1).$$

We check that  $\lambda = (5, 3, 1, -2)$  is a Harish-Chandra parameter belonging to  $\mathcal{C}$ , but  $\Lambda(\lambda) = (6, 5, 3, -1)$  does not belong to  $\bar{\mathcal{C}}$ .

6.2.3. *The case of  $\mathrm{SU}(3, 2)$ .* Let  $T$  be the torus of  $\mathrm{SU}(3, 2)$  formed by all the diagonal matrices. The dual of Lie algebra of  $T$  is  $\mathfrak{t}^* = \{(\lambda_1, \dots, \lambda_5) \in \mathbb{R}^5 \mid \sum_i \lambda_i = 0\}$ . Let  $e_1, \dots, e_5$  be the canonical basis of  $\mathbb{R}^5$ . The choice of positive compact roots  $\mathfrak{R}_c^+$  is  $\{e_1 - e_2, e_1 - e_3, e_2 - e_3, e_4 - e_5\}$  so that the Weyl chamber is

$$\mathfrak{t}_+^* := \left\{ \lambda_1 \geq \lambda_2 \geq \lambda_3 \text{ and } \lambda_4 \geq \lambda_5 \right\}.$$

We have  $\rho_c = (1, 0, -1, \frac{1}{2}, -\frac{1}{2})$ . The non-compact roots are  $\pm(e_i - e_j), i = 1, 2, 3, j = 4, 5$ .

• Let  $\lambda = (3, 1, -1, 0, -3)$  be in the chamber  $\mathcal{C}_1 := \{\lambda_1 \geq \lambda_2 > \lambda_4 > \lambda_3 > \lambda_5\}$ . We have  $\rho_n(\mathcal{C}_1) = (1, 1, 0, -\frac{1}{2}, -\frac{3}{2})$ , and then

$$\Lambda(\lambda) = \lambda - \rho_c + \rho_n(\mathcal{C}_1) = (3, 2, 0, -1, -4)$$

is a regular element which does not belong to  $\mathcal{C}_1$ .

• Let us consider the chamber  $\mathcal{C}_2 := \{\lambda_1 > \lambda_4 > \lambda_2 > \lambda_5 > \lambda_3\}$ . We see that  $\rho_n(\mathcal{C}_2) = \rho_c$ , hence any Harish-Chandra parameter of the chamber  $\mathcal{C}_2$  satisfies condition (1.5).



## 7. APPENDICES

Let  $G$  be a connected real semi-simple Lie group with finite center. Let  $K$  be a maximal compact Lie subgroup of  $G$ . Let  $T$  be a maximal torus in  $K$ . Let  $\mathfrak{t}, \mathfrak{k}, \mathfrak{g}$  be the respective Lie algebras of  $T, K, G$ . We assume that  $\mathfrak{t}$  is a Cartan subalgebra of  $\mathfrak{g}$ .

In Appendix A, we use the identification  $X \mapsto B_{\mathfrak{g}}(X, -), \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$  given by the Killing form. Hence the *coadjoint* orbits of  $G$  considered in the previous sections will be replaced by *adjoint* orbits.

**7.1. Appendix A.** Let  $\mathcal{O} = G \cdot \lambda$  be an adjoint orbit of  $G$  passing through  $\lambda \in \mathfrak{t}$ . Let  $K \cdot \mu$  be an adjoint orbit of  $K$  passing through  $\mu \in \mathfrak{t}$ . We consider the maps

$$\Phi_t : \mathcal{O} \times K \cdot \mu \longrightarrow \mathfrak{k}, \quad t \in [0, 1]$$

defined by  $\Phi_t(m, \xi) = \pi_{\mathfrak{k}}(m) - t\xi$ . Here  $\pi_{\mathfrak{k}} : \mathfrak{g} \rightarrow \mathfrak{k}$  is the orthogonal projection. The maps  $\Phi_t, t \in [0, 1]$  generates the vector fields  $\mathcal{H}_t, t \in [0, 1]$  on  $\mathcal{O} \times K \cdot \mu$  by  $\mathcal{H}_t(n) = (V\Phi_t(n))|_n$  for  $n \in \mathcal{O} \times K \cdot \mu$ .

The aim of this section is the following

**Proposition 7.1.** *There exists a compact subset  $\mathcal{K}$  of  $\mathcal{O}$  such that*

$$\{\mathcal{H}_t = 0\} \subset \mathcal{K} \times K \cdot \mu$$

for any  $t \in [0, 1]$ .

*Proof.* The proof is given in [34][Section 5.3] in the case where  $\lambda$  is a regular element of  $\mathfrak{g}$ . Here we propose another proof, which is technically simpler, that was communicated to us by Michèle Vergne.

By definition, we have

$$\mathcal{H}_t(m, \xi) = -\left([\pi_{\mathfrak{k}}(m) - t\xi, m], [\pi_{\mathfrak{k}}(m), \xi]\right) \in \mathbf{T}_n\mathcal{O} \times \mathbf{T}_{\xi}(K \cdot \mu)$$

Let us denote  $\mathcal{C}_t$  the subset  $\{\mathcal{H}_t = 0\}$ . We have

$$\begin{aligned} \mathcal{C}_t &= \{(m, \xi) \in \mathcal{O} \times K \cdot \mu \mid [\pi_{\mathfrak{k}}(m) - t\xi, m] = 0 \text{ and } [\pi_{\mathfrak{k}}(m), \xi] = 0\} \\ &= K \cdot \{(m, \mu) \in \mathcal{O} \times K \cdot \mu \mid [\pi_{\mathfrak{k}}(m) - t\mu, m] = 0 \text{ and } [\pi_{\mathfrak{k}}(m), \mu] = 0\}. \end{aligned}$$

The condition  $[\pi_{\mathfrak{k}}(m), \mu] = 0$  means that  $\pi_{\mathfrak{k}}(m)$  belongs to the subalgebra  $\mathfrak{k}_{\mu}$  that stabilizes  $\mu \in \mathfrak{t}$ . We have  $\mathfrak{k}_{\mu} = K_{\mu} \cdot \mathfrak{t}$ , hence  $\mathcal{C}_t \subset K \cdot \mathcal{D}_t \times K \cdot \mu$  where

$$\mathcal{D}_t = \{m \in \mathcal{O} \mid \pi_{\mathfrak{k}}(m) \in \mathfrak{t} \text{ and } m \in \mathfrak{g}_{\pi_{\mathfrak{k}}(m) - t\mu}\}.$$

Here  $\mathfrak{g}_{\pi_{\mathfrak{k}}(m) - t\mu}$  is the subalgebra that stabilizes  $\pi_{\mathfrak{k}}(m) - t\mu$ . The proof will be settled if one proves that  $\cup_{t \in [0, 1]} \mathcal{D}_t$  is contained in a compact subset of  $\mathcal{O}$ .

The subalgebras  $\mathfrak{g}_X, X \in \mathfrak{t}$  describe a finite subset that we enumerate  $\mathfrak{g}_i, i = 1, \dots, r$ . For each subalgebra  $\mathfrak{g}_i$ , let  $G_i$  be the corresponding closed connected subgroup of  $G$ . Note that  $\mathfrak{t}$  is contained in each  $\mathfrak{g}_i$ , and that the center  $z(\mathfrak{g}_i)$  of  $\mathfrak{g}_i$  is contained in (the Cartan subalgebra)  $\mathfrak{t}$ . Note that the condition  $\mathfrak{g}_{\pi_{\mathfrak{k}}(m) - t\mu} = \mathfrak{g}_i$  implies that  $\pi_{\mathfrak{k}}(m) - t\mu \in z(\mathfrak{g}_i)$ . It gives that  $\mathcal{D}_t \subset \cup_{i=1}^r \mathcal{D}_t^i$  with

$$\mathcal{D}_t^i = \{m \in \mathcal{O} \cap \mathfrak{g}_i \mid \pi_{\mathfrak{k}}(m) - t\mu \in z(\mathfrak{g}_i)\}.$$

It is a classical result that the intersection  $\mathcal{O} \cap \mathfrak{g}_i$  is equal to a *finite* collection of adjoint  $G_i$  orbit:

$$\mathcal{O} \cap \mathfrak{g}_i = \bigcup_{\alpha \in A_i} G_i \cdot \alpha.$$

Let  $\pi_i : \mathfrak{g} \rightarrow z(\mathfrak{g}_i)$  be the orthogonal projection. If  $\pi_{\mathfrak{k}}(m) - t\mu \in z(\mathfrak{g}_i)$ , we have

$$\begin{aligned}\pi_{\mathfrak{k}}(m) - t\mu &= \pi_i(\pi_{\mathfrak{k}}(m) - t\mu) \\ &= \pi_i(m) - t\pi_i(\mu).\end{aligned}$$

But the map  $\pi_i$  is constant on each connected component  $G_i \cdot \alpha$ . So finally,

$$\mathcal{D}_t^i = \bigcup_{\alpha \in A_i} \{m \in G_i \cdot \alpha \mid \pi_{\mathfrak{k}}(m) - t\mu = \pi_i(\alpha) - t\pi_i(\mu)\}.$$

and then

$$\begin{aligned}\mathcal{D}_t^i &= \bigcup_{\alpha \in A_i} G_i \cdot \alpha \cap \pi_{\mathfrak{k}}^{-1}(\theta_{i,\alpha,t}) \\ &\subset \bigcup_{\alpha \in A_i} \mathcal{O} \cap \pi_{\mathfrak{k}}^{-1}(\theta_{i,\alpha,t})\end{aligned}$$

with  $\theta_{i,\alpha,t} = \pi_i(\alpha) + t(\mu - \pi_i(\mu))$ . We get finally that

$$\bigcup_{t \in [0,1]} \mathcal{D}_t \subset \mathcal{O} \cap \pi_{\mathfrak{k}}^{-1}(C)$$

where

$$C = \{\theta_{i,\alpha,t}, t \in [0,1], i = 1, \dots, r, \alpha \in A_i\}$$

is a compact subset of  $\mathfrak{t}$ . Since the map  $\pi_{\mathfrak{k}}$  is *proper* when restricted to  $\mathcal{O}$ , the set  $\mathcal{O} \cap \pi_{\mathfrak{k}}^{-1}(C)$  is compact.  $\square$

**7.2. Appendix B.** Here we suppose that  $G/K$  is an irreducible Hermitian symmetric spaces, and we use the notations of Section 5. Our aim is the proof of Propositions 5.3 and 5.4. Our (classical) arguments uses the knowledge of the restricted root system  $\Sigma$  and the Cayley transform.

We denote  $(-, -)_{\mathfrak{t}}$  the scalar product on  $\mathfrak{t}$  defined by  $(X, Y)_{\mathfrak{t}} := -B_{\mathfrak{g}}(X, Y)$  for  $X, Y \in \mathfrak{t}$ . Let  $(-, -)_{\mathfrak{t}^*}$  be the scalar product on  $\mathfrak{t}^*$  which make the map  $X \mapsto (X, -)_{\mathfrak{t}}$ , from  $\mathfrak{t}$  to  $\mathfrak{t}^*$ , unitary.

Let  $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_j$  be the maximal abelian algebra of  $\mathfrak{p}$  attached to the maximal family  $\Psi = \{\gamma_1, \dots, \gamma_r\}$  of strongly orthogonal roots (see Section 5).

Let  $\mathfrak{t}_1 \subset \mathfrak{t}$  be the subspace orthogonal (for the duality) to the vector subspace spanned by  $\gamma_1, \dots, \gamma_r$ :  $\mathfrak{t}_1$  is also the centralizer of  $\mathfrak{a}$  in  $\mathfrak{t}$ . Let  $\mathfrak{t}_2 \subset \mathfrak{t}$  be the orthogonal of  $\mathfrak{t}_1$  (relatively to the scalar product on  $\mathfrak{t}$ ). We check easily that

$$\mathfrak{t}_2 = \text{Vect}(H_1, \dots, H_r).$$

We have then the orthogonal decomposition  $\mathfrak{t}^* = \mathfrak{t}_1^* \oplus \mathfrak{t}_2^*$  with  $\mathfrak{t}_2^* = \text{Vect}(\gamma_1, \dots, \gamma_r)$ .

Let  $\mathfrak{R} = \mathfrak{R}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  be the roots system associated to the Cartan subalgebra  $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ . Let  $\mathfrak{R}_{\text{hol}}^+ = \mathfrak{R}_{\mathbb{C}}^+ \cup \mathfrak{R}_n^{+,z_0}$  be the system of positive roots consider in the introduction. Let  $\mathcal{C}_{\text{hol}} := \{\xi \in \mathfrak{t}^* \mid (\xi, \alpha)_{\mathfrak{t}^*} \geq 0, \forall \alpha \in \mathfrak{R}_{\text{hol}}^+\}$  be the corresponding Weyl chamber.

Let  $\pi' : \mathfrak{t}^* \rightarrow \mathfrak{t}_2^*$  be the canonical projection, and let us consider

$$\Sigma' := \pi'(\mathfrak{R}) \setminus \{0\} \quad \text{and} \quad (\Sigma')^+ := \pi'(\mathfrak{R}_{\text{hol}}^+) \setminus \{0\}.$$

We see that  $\mathcal{C} \cap \text{Vect}(\gamma_1, \dots, \gamma_r) = \mathcal{D}$ , with

$$(7.45) \quad \mathcal{D} := \{\xi \in \mathfrak{t}_2^* \mid (\xi, \alpha)_{\mathfrak{t}^*} \geq 0, \forall \alpha \in (\Sigma')^+\}.$$

Now we use the description of  $\Sigma'$  given by Harish-Chandra and Moore.

**Proposition 7.2** ([13, 30]). • *All the  $\gamma_k$  have the same length.*

- *For any  $i < j$ , there is an  $\alpha \in \mathfrak{A}_c^+$ , such that  $\pi'(\alpha) = \frac{1}{2}(\gamma_i - \gamma_j)$ .*
- *They are two possibilities for  $\Sigma' := \pi'(\mathfrak{A}) \setminus \{0\}$ :*

$$\Sigma' := \left\{ \pm \frac{1}{2}(\gamma_i + \gamma_j), \pm \frac{1}{2}(\gamma_i - \gamma_j), 1 \leq i < j \leq r \right\} \cup \left\{ \pm \gamma_i, 1 \leq i \leq r \right\},$$

or

$$\Sigma' := \left\{ \pm \frac{1}{2}(\gamma_i + \gamma_j), \pm \frac{1}{2}(\gamma_i - \gamma_j), 1 \leq i < j \leq r \right\} \cup \left\{ \pm \frac{1}{2}\gamma_i, \pm \gamma_i, 1 \leq i \leq r \right\}.$$

Since the  $\gamma_k$  belongs to  $(\Sigma')^+$ , the last two point of Proposition shows that

$$(7.46) \quad (\Sigma')^+ = \left\{ \frac{1}{2}(\gamma_i + \gamma_j), \frac{1}{2}(\gamma_i - \gamma_j), 1 \leq i < j \leq r \right\} \cup \{\gamma_1, \dots, \gamma_r\} \cup \Xi$$

where  $\Xi = \emptyset$  or  $\Xi = \{\frac{1}{2}\gamma_1, \dots, \frac{1}{2}\gamma_r\}$ .

Since the  $\gamma_k$  have the same length, it is now easy to see that the set  $\mathcal{D}$  defined in (7.45) is equal to  $\sum_{i=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_r)$ . Thus the second point of Proposition 5.4 is proved: we have

$$(7.47) \quad \mathcal{C} \cap \text{Vect}(\gamma_1, \dots, \gamma_r) = \sum_{k=1}^r \mathbb{R}^{\geq 0}(\gamma_1 + \dots + \gamma_k).$$

**Remark 7.3.** *We know from (5.36) that  $\|H_k\| = 2\|\gamma_k\|^{-1}$ . Thus, all the  $H_k$  have the same length.*

Now, we go into the proof of Proposition 5.3: we will compute a fundamental domain  $\mathfrak{a}_+$  for the action of  $K$  on  $\mathfrak{p}$ .

In the complex semi-simple algebra  $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ , we consider the Cartan algebra

$$\mathfrak{h} := \mathfrak{t}_1 \oplus i\mathfrak{a}.$$

that we equip with the scalar product  $(X, Y)_{\mathfrak{h}} := -B_{\mathfrak{g}}(X, Y)$ ,  $\forall X, Y \in \mathfrak{h}$ . We take on  $\mathfrak{h}^*$  the scalar product such that the map  $\mathfrak{h} \rightarrow \mathfrak{h}^*$ ,  $X \mapsto (X, -)_{\mathfrak{h}}$  is orthogonal.

Let  $\mathfrak{A}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \mathfrak{h}^*$  be the set of roots relative to the adjoint action of  $\mathfrak{h}_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$ . The projection  $\pi : \mathfrak{h}^* \rightarrow (i\mathfrak{a})^*$  sends  $\mathfrak{A}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  onto  $\widetilde{\Sigma} \cup \{0\}$ , where  $\Sigma$  is the restricted root system, and  $\xi \mapsto \widetilde{\xi}, \mathfrak{a}^* \simeq (i\mathfrak{a})^*$  is the one to one map defined by  $\langle \widetilde{\xi}, iX \rangle := \langle \xi, X \rangle$ .

The Cayley transform

$$\mathbf{c} := \exp \left( -\frac{i\pi}{2} \text{ad} \left( \sum_{k=1}^r Y_k \right) \right).$$

is an automorphism of the complex Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ . One checks that  $\mathbf{c}(Y) = Y$  for any  $Y \in \mathfrak{t}_1$  and that

$$\mathbf{c}(iX_k) = \frac{1}{2}H_k, \quad \forall k = 1, \dots, r.$$

Hence the Cayley transform sends the subalgebra  $\mathfrak{h}$  onto the subalgebra  $\mathfrak{t}$ . Moreover one checks easily that  $\mathbf{c} : \mathfrak{h} \rightarrow \mathfrak{t}$  is an orthogonal map, thus we know that all the  $X_k$  have the same length. Let us denoted  $\mathbf{c}^* : \mathfrak{t}^* \rightarrow \mathfrak{h}^*$  the dual orthogonal map.

Since  $\mathbf{c}$  is an automorphism of  $\mathfrak{g}_{\mathbb{C}}$ , the image of the root system  $\mathfrak{A} := \mathfrak{A}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  by  $\mathbf{c}^*$  is equal to the root system  $\mathfrak{A}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Since  $\mathbf{c}$  is the identity map on  $\mathfrak{t}_1$ , we have  $\mathbf{c}^*(\Sigma') = \widetilde{\Sigma}$ .

If we choose systems of positive roots such that

$$\mathfrak{A}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ := \mathbf{c}^*(\mathfrak{A}_{\text{hol}}^+) \quad \text{and} \quad \widetilde{\Sigma}^+ := \mathbf{c}^*((\Sigma')^+),$$

we get

$$\begin{aligned} \mathfrak{a}_+ &:= \{X \in \mathfrak{a} \mid \langle \beta, X \rangle \geq 0, \forall \beta \in \Sigma^+\} \\ &= \{X \in \mathfrak{a} \mid \langle \mathbf{c}^*(\alpha), iX \rangle \geq 0, \forall \alpha \in (\Sigma')^+\} \\ &= \{X = \sum_k a_k X_k \mid \sum_k a_k \langle \alpha, H_k \rangle \geq 0, \forall \alpha \in (\Sigma')^+\}. \end{aligned}$$

From the description (7.46) of  $(\Sigma')^+$  we finally found that

$$\mathfrak{a}_+ = \sum_{k=1}^r \mathbb{R}^{\geq 0} (X_1 + \cdots + X_k).$$

**7.3. Appendix C.** Let  $\varphi : H \rightarrow K$  be a morphism of compact connected Lie group. Let  $d\varphi : \mathfrak{h} \rightarrow \mathfrak{k}$  be the corresponding morphism of Lie algebras. Any Hamiltonian  $K$ -manifold  $(M, \Phi_K)$  can be seen as a Hamiltonian  $H$ -manifold, with moment map  $\Phi_H = d\varphi^* \circ \Phi_K$ .

The morphism  $\varphi$  induces a map  $\varphi^* : R(K) \rightarrow R(H)$ . When  $E \in R^{-\infty}(K)$  is  $H$ -admissible (see Definition 2.6), one can define its “restriction” to  $H$ , that we denote by  $\varphi^*E$  (or simply  $E|_H$ ).

The aim of this appendix is to check that the following version of **[P2]** holds.

**Proposition 7.4.** *Let  $M$  be a pre-quantized proper Hamiltonian  $K$ -manifold. If  $M$  is still proper as a Hamiltonian  $H$ -manifold. Then  $\mathcal{Q}_K^{-\infty}(M)$  is  $H$ -admissible and we have the following equality in  $R^{-\infty}(H)$  :*

$$\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M).$$

*Proof.* The proof is given in [35] when  $\varphi$  is the inclusion of a subgroup. Let us generalize this result to a general morphism  $\varphi : H \rightarrow K$ . Let  $L := \varphi(H)$ . We write  $\varphi = i \circ j$  where  $i : L \hookrightarrow K$  is the one to one map given by the inclusion, and  $j : H \rightarrow L$  is the onto morphism induced by  $\varphi$ .

We consider the one to one linear map  $j^* : \mathfrak{l}^* \rightarrow \mathfrak{h}^*$ . We can choose compatible system of positive roots for  $H$  and  $L$ , so that  $j^*$  defines a one to one map from  $\widehat{L}$  to  $\widehat{H}$ . Then  $j^*V_\mu^L = V_{j^*(\mu)}^H$  for any highest weight  $\mu \in \widehat{L}$ .

Let  $M$  be a proper Hamiltonian  $K$ -manifold which is prequantized by a line bundle  $\mathcal{L}$ . Since  $j : H \rightarrow L$  is onto we have:

- Any  $E \in R^{-\infty}(K)$  is  $H$ -admissible if and only if  $E$  is  $L$ -admissible, and  $E|_H = j^*(E|_L)$ .
- $M$  is proper as a Hamiltonian  $H$ -manifold if and only if it is proper as a Hamiltonian  $L$ -manifold.

Hence

$$\begin{aligned} \mathcal{Q}_K^{-\infty}(M)|_H = j^*(\mathcal{Q}_K^{-\infty}(M)|_L) &= j^*(\mathcal{Q}_L^{-\infty}(M)) \\ &= j^*\left(\sum_{\mu \in \widehat{L}} \mathcal{Q}(M_{\mu,L}) V_\mu^L\right) \\ &= \sum_{\mu \in \widehat{L}} \mathcal{Q}(M_{\mu,L}) V_{j^*(\mu)}^H, \end{aligned}$$

where  $M_{\mu,L}$  is the symplectic reduction at  $\mu$  relative to the action of  $L$  on  $M$ . Our proof is then finished if we check that

$$(7.48) \quad \mathcal{Q}(M_{\mu,L}) = \mathcal{Q}(M_{j^*(\mu),H})$$

holds for any  $\mu \in \widehat{L}$ .

The one to one map  $j^* : \mathfrak{l}^* \rightarrow \mathfrak{h}^*$  satisfies  $h \cdot j^*(\xi) = j^*(j(h) \cdot \xi)$  for any  $h \in H$  and  $\xi \in \mathfrak{l}^*$ . Hence the map  $j^*$  defines a  $\varphi$ -equivariant symplectomorphism between the coadjoint orbits  $L \cdot \xi$  and  $H \cdot j^*(\xi)$ .

Let  $\mu \in \widehat{L}$ . We work now with the *proper* Hamiltonian  $L$ -manifold  $\mathcal{X} := M \times \overline{L \cdot \mu}$  which is prequantized by the line bundle  $\mathcal{L}_{\mathcal{X}} := \mathcal{L} \otimes \mathbb{C}_{[-\mu]}$ . Let  $\Phi_L : M \times \overline{L \cdot \mu} \rightarrow \mathfrak{l}^*$  be the moment map relative to the  $L$ -action. Let  $\mathcal{H}_L$  be the Hamiltonian vector field of the function  $\frac{-1}{2} \|\Phi_L\|^2$ .

The ‘‘pushed’’ Thom symbol  $\text{Thom}^{\mathcal{H}_L}(\mathcal{X})$  is  $L$ -transversally elliptic when we restrict it to a  $L$ -invariant relatively compact open subset  $\mathcal{U}$  such that

$$\partial\mathcal{U} \cap \text{Cr}(\|\Phi_L\|^2) = \emptyset.$$

Then we may consider the equivariant index  $\text{Index}_{\mathcal{U}}^L(\text{Thom}^{\mathcal{H}_L}(\mathcal{X})|_{\mathcal{U}} \otimes \mathcal{L}_{\mathcal{X}})$ . We know from Theorem 3.10 that

$$(7.49) \quad \left[ \text{Index}_{\mathcal{U}}^L(\text{Thom}^{\mathcal{H}_L}(\mathcal{X})|_{\mathcal{U}} \otimes \mathcal{L}_{\mathcal{X}}) \right]^L = \mathcal{Q}(M_{\mu, L})$$

when  $\Phi_L^{-1}(0) \subset \mathcal{U}$ .

Now we look at  $\mathcal{X}$  as a Hamiltonian  $H$ -manifold through the onto morphism  $j : H \rightarrow L$ : then  $\mathcal{X} \simeq M \times \overline{H \cdot j^*(\mu)}$ . Let  $\Phi_H = j^* \circ \Phi_L$  be the corresponding moment map. Since  $j^*$  is one to one, the functions  $\|\Phi_L\|^2$  and  $\|\Phi_H\|^2$  coincides if we choose appropriate invariant scalar products on  $\mathfrak{l}^*$  and  $\mathfrak{h}^*$ . Then we have  $\Phi_L^{-1}(0) = \Phi_H^{-1}(0)$  and  $\mathcal{H}_L = \mathcal{H}_H$ . As before Theorem 3.10 gives

$$(7.50) \quad \left[ \text{Index}_{\mathcal{U}}^H(\text{Thom}^{\mathcal{H}_H}(\mathcal{X})|_{\mathcal{U}} \otimes \mathcal{L}_{\mathcal{X}}) \right]^H = \mathcal{Q}(M_{j^*(\mu), H}).$$

Since  $[E]^L = [j^*E]^H$  for any  $E \in R^{-\infty}(L)$ , the relations (7.49) and (7.50) imply finally (7.48). □

## REFERENCES

- [1] M.F. ATIYAH, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, **401**.
- [2] M.F. ATIYAH, G.B. SEGAL, The index of elliptic operators II, *Ann. Math.* **87**, 1968, p. 531-545.
- [3] M.F. ATIYAH, I.M. SINGER, The index of elliptic operators I, *Ann. Math.* **87**, 1968, p. 484-530.
- [4] M.F. ATIYAH, I.M. SINGER, The index of elliptic operators III, *Ann. Math.* **87**, 1968, p. 546-604.
- [5] M.F. ATIYAH, I.M. SINGER, The index of elliptic operators IV, *Ann. Math.* **93**, 1971, p. 139-141.
- [6] N. BERLINE and M. VERGNE, The Chern character of a transversally elliptic symbol and the equivariant index, *Invent. Math.* **124**, 1996, p. 11-49.
- [7] N. BERLINE and M. VERGNE, L’indice équivariant des opérateurs transversalement elliptiques, *Invent. Math.* **124**, 1996, p. 51-101.
- [8] M. DUFLO, Représentations de carré intégrable des groupes semi-simples réels, *Séminaire Bourbaki*, Vol. 1977/78, Exposé No.508, Lect. Notes Math. 710, 22-40 (1979).
- [9] M. DUFLO and J. VARGAS, Proper maps and multiplicities, preprint
- [10] M. DUFLO, G. HECKMAN and M. VERGNE, Projection d’orbites, formule de Kirillov et formule de Blattner, *Mémoires de la S.M.F.* **15**, 1984, p. 65-128.
- [11] J. J. DUISTERMAAT, *The heat lefschetz fixed point formula for the Spin<sup>c</sup>-Dirac operator*, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.

- [12] V. GUILLEMIN and S. STERNBERG, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67**, 1982, p. 515-538.
- [13] HARISH-CHANDRA, Representations of semisimple Lie groups IV, V, VI, *Amer. J. Math.*, **77** (1955), p. 743-777; **78** (1956) p. 1-41 and p. 564-628.
- [14] HARISH-CHANDRA Discrete series for semi-simple Lie group, I and II, *Acta Mathematica*, **113** (1965) p. 242-318, and **116** (1966) p. 1-111.
- [15] H. HECHT and W. SCHMID, A proof of Blattner's conjecture, *Invent. Math.*, **31**, 1975, p. 129-154.
- [16] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces*, Pure and applied mathematics, Academic Press, INC (London), 1978.
- [17] A. A. KIRILLOV, *Lecture on the orbit method*, Graduate Studies in Mathematics, vol. 64, 2004.
- [18] F. KIRWAN, Convexity properties of the moment mapping III, *Invent. Math.* **77**, 1984, p. 547-552.
- [19] A. W. KNAPP, Bounded symmetric domains and holomorphic discrete series, *Symmetric Spaces* ( W.M. Boothby and G.L. Weiss, eds.), Marcel Dekker, New York, 1972, p. 211-246.
- [20] A. W. KNAPP, *Lie groups Beyond an Introduction, Second Edition*, Progress in Math, vol. 140, Birkhäuser, 2004.
- [21] T. KOBAYASHI, Discrete decomposability of the restriction of  $A_q(\lambda)$  with respect to reductive subgroups. II. Micro-local analysis and asymptotic  $K$ -support. *Ann. of Math. (2)* **147** (1998), no. 3, 709-729.
- [22] T. KOBAYASHI, Discrete decomposability of the restriction of  $A_q(\lambda)$  with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties. *Invent. Math.* **131** (1998), no. 2, 229-256.
- [23] T. KOBAYASHI, Discrete decomposability of the restriction of  $A_q(\lambda)$  with respect to reductive subgroups and its applications. *Invent. Math.* **117** (1994), no. 2, 181-205.
- [24] T. KOBAYASHI, Admissible restrictions of unitary representations of reductive Lie groups discrete decomposability and symplectic geometry, in preparation.
- [25] B. KOSTANT, Quantization and unitary representations, in *Modern Analysis and Applications*, Lecture Notes in Math., Vol. 170, Springer-Verlag, 1970, p. 87-207.
- [26] E. LERMAN, E. MEINRENKEN, S. TOLMAN and C. WOODWARD, Non-Abelian convexity by symplectic cuts, *Topology* **37**, 1998, p. 245-259.
- [27] D. McDUFF, The symplectic structure of Kähler manifolds on nonpositive curvature, *J. Diff. Geom.*, **28**, 1988, p. 467-475.
- [28] E. MEINRENKEN, Symplectic surgery and the Spin<sup>c</sup>-Dirac operator, *Adv. in Math.* **134**, 1998, p. 240-277.
- [29] E. MEINRENKEN, S. SJAMAAR, Singular reduction and quantization, *Topology* **38**, 1999, p. 699-762.
- [30] C. C. MOORE Compactification of symmetric spaces, II. The Cartan Domains. *Amer. J. Math.*, **86**, 1964, 358-378.
- [31] L. NESS A stratification of the null cone via the moment map. *Amer. J. Math.*, **106**, 1984, 1281-1329. With an Appendix by D. Mumford.
- [32] P-E. PARADAN, The Fourier transform of semi-simple coadjoint orbits, *J. Funct. Anal.* **163**, 1999, p. 152-179.
- [33] P-E. PARADAN, Localization of the Riemann-Roch character, *J. Funct. Anal.* **187**, 2001, p. 442-509.
- [34] P-E. PARADAN, Spin<sup>c</sup> quantization and the  $K$ -multiplicities of the discrete series, *Annales Scientifiques de l'E. N. S.*, **36**, 2003, p. 805-845.
- [35] P-E. PARADAN, Formal geometric quantization, will appear in *Annales de l'Institut Fourier*, arXiv:math/0702224v1.
- [36] W. SCHMID, Die Randwerte holomorpher Funktionen auf hermiteschen Räumen. (German), *Invent. Math.* **9**, 1969/1970, p. 61-80.
- [37] R. SJAMAAR, Symplectic reduction and Riemann-Roch formulas for multiplicities, *Bull. Amer. Math. Soc.* **33**, 1996, p. 327-338.
- [38] J.-M. SOURIAU, *Structure des systèmes dynamiques, Maîtrise de mathématiques*, Dunod, Paris, 1970; English transl. by C. H. Cushman-de Vries in *Structure of Dynamical Systems: A Symplectic View of Physics*, Progress in Mathematics, vol. 149, Birkhäuser Boston, Boston, MA, 1997.

- [39] Y. TIAN, W. ZHANG, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, *Invent. Math.* **132**, 1998, p. 229-259.
- [40] M. VERGNE, Quantification géométrique et réduction symplectique, *Séminaire Bourbaki* **888**, 2001.
- [41] D. VOGAN, Review of “Lectures on the orbit method” by A.A. Kirillov, Bulletin of the A. M. S., 1997
- [42] J. WEITSMAN, Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds. EuroConférence Moshé Flato 2000, Part I (Dijon). *Lett. Math. Phys.*, **56**, 2001, no. 1, p. 31-40.
- [43] WOODHOUSE, *Geometric quantization*, 2nd ed. Oxford Mathematical Monographs. Oxford: Clarendon Press, 1997.

INSTITUT DE MATHÉMATIQUES ET MODÉLISATION DE MONTPELLIER (I3M), UNIVERSITÉ MONTPELLIER 2

*E-mail address:* Paul-Emile.Paradan@math.univ-montp2.fr