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# A note on Robinson consistency lemma

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## 1 Introduction

In standard model theory, Robinson's consistency property (RCP) is expressed as follow:

*Two theories  $T_1$  and  $T_2$  which have (at least) a model, and do not contradict each other on the common language (i.e. there is no formula  $\varphi$  built up from atoms occurring both in  $T_1$  and  $T_2$  such that  $T_1 \models \varphi$  and  $T_2 \models \neg\varphi$ ), then  $T_1 \cup T_2$  has a model.*

Since A. Robinson's work, this result has been shown valid for a large class of logic and recently has received generalizations in institutions [8]. One of the main reason why a logic should have RCP is because of "modular theory building". Indeed, RCP has been shown to be equivalent to Craig's interpolation property (CIP), and CIP is strongly connected to modularity in specification theory [2, 5, 6, 7, 12, 13].

Classically, RCP is proved in two steps:

1. we find two models  $\mathcal{M}_1^0$  of  $T_1$  and  $\mathcal{M}_2^0$  of  $T_2$  such that there is an elementary morphism  $g^0 : \mathcal{M}_{1|\mathcal{L}}^0 \rightarrow \mathcal{M}_{2|\mathcal{L}}^0$  (i.e. they are elementary equivalent on sentences built up on  $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$  where  $\mathcal{L}_i$  is the language of  $T_i$ , noted  $\mathcal{M}_1 \equiv_{\mathcal{L}} \mathcal{M}_2$ ).
2. we inductively build two chains of elementary morphisms  $(\mathcal{M}_i^n \xrightarrow{f_i^n} \mathcal{M}_i^{n+1})$  with  $i \in \{1, 2\}$  such that for every  $n$ , there are two elementary morphisms  $g^n : \mathcal{M}_{1|\mathcal{L}}^n \rightarrow \mathcal{M}_{2|\mathcal{L}}^n$  and  $h^n : \mathcal{M}_{2|\mathcal{L}}^n \rightarrow \mathcal{M}_{1|\mathcal{L}}^{n+1}$  such that  $f_i^n|_{\mathcal{L}} = h^n \circ g^n$  and  $f_j^n|_{\mathcal{L}} = g^{n+1} \circ h^n$ .

By applying Tarski's elementary chain theorem, for every  $i = 1, 2$ , the chain  $(\mathcal{M}_i^n \xrightarrow{f_i^n} \mathcal{M}_i^{n+1})$  has a co-limit  $\mathcal{M}'_i$ . Moreover, co-limits being preserved by forgetful,  $\mathcal{M}'_{i|\mathcal{L}}$  is also the co-limit of  $\mathcal{M}_{i|\mathcal{L}}^0 \xrightarrow{g^0} \mathcal{M}_{j|\mathcal{L}}^0 \xrightarrow{h^0} \mathcal{M}_{i|\mathcal{L}}^1 \xrightarrow{g^1} \mathcal{M}_{j|\mathcal{L}}^1 \xrightarrow{h^1} \mathcal{M}_{i|\mathcal{L}}^2 \dots$ . From there, we deduce that  $\mathcal{M}'_{1|\mathcal{L}} = \mathcal{M}'_{2|\mathcal{L}}$ . By the weak amalgamation property, there is a model  $\mathcal{M}'$  in  $T_1 \cup T_2$  such that  $\mathcal{M}'_{|\mathcal{L}_i} = \mathcal{M}'_i$ .

The second step of the RCP proof makes it usually difficult to prove and to understand. In first-order logic, there is a more direct proof of RCP which uses the deep concept of recursively saturated models [3]. In this paper, we propose to make the proof of this result simpler by directly building (i.e. without generating the three chains of elementary morphisms) a model  $\mathcal{M}'_i$  with  $i = 1, 2$  such that:

1.  $\mathcal{M}_i^0 \equiv_{\mathcal{L}_i} \mathcal{M}'_i$
2.  $\mathcal{M}'_{i|\mathcal{L}} = \mathcal{M}_{j|\mathcal{L}}^0$  with  $j \neq i$  and  $j \in \{1, 2\}$

In the paper, we will say that the pushout  $\mathcal{S}$  of  $\mathcal{L} \subseteq \mathcal{L}_1$  and  $\mathcal{L} \subseteq \mathcal{L}_2$  has the *weak Robinson property* (WRP) for  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

In Section 4, we will apply this proof process in the standard model theory but also in modal logic, first with local satisfaction and with global one. But before, in Section 2, we will review the basic notions on institutions [9] which we will use in this paper. In section 3, we will define WRP in the institution framework. We will then show that for any institutions which has WRP and the weak amalgamation property, RCP obviously holds for all theories  $T_1$  and  $T_2$  which have a model  $\mathcal{M}_i$  such that their forgetful on the common language are elementary equivalent.

## 2 Institutions

**Definition 2.1 (Institution)** *An institution  $\mathcal{I} = (Sig, Sen, Mod, \models)$  consists of*

- *a category  $Sig$ , objects of which are called signatures,*
- *a functor  $Sen : Sig \rightarrow Set$  giving for each signature a set, elements of which are called sentences,*
- *a contravariant functor  $Mod^{op} : Sig \rightarrow Cat$  giving for each signature a category, objects of which are called  $\Sigma$ -models, and*
- *a  $|Sig|$ -indexed family of relations  $\models_{\Sigma} \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$  called satisfaction relation,*

*such that the following property, called the satisfaction relation, holds:*  
 $\forall \sigma : \Sigma \rightarrow \Sigma', \forall \mathcal{M} \in Mod(\Sigma'), \forall \varphi \in Sen(\Sigma),$

$$\mathcal{M} \models_{\Sigma'} Sen(\sigma)(\varphi) \Leftrightarrow Mod(\sigma)(\mathcal{M}) \models_{\Sigma} \varphi$$

In this paper, we are especially interested in the following examples of institutions (other examples can be found in [4, 9, 11]).

- **PL.** The institution of propositional logic. Signatures and signature morphisms are sets of propositional variables and functions between sets of propositional variables.

Given a signature  $\Sigma$ , a  $\Sigma$ -model is a mapping  $\nu$  from  $\Sigma$  to the truth-values  $\{0, 1\}$ . Morphisms between  $\Sigma$ -models are identities. Given a signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ , the forgetful functor  $Mod(\sigma)$  maps a  $\Sigma'$ -model  $\nu'$  to the  $\Sigma$ -model  $\nu : p \in \Sigma \mapsto \nu'(\sigma(p))$ .

The set of  $\Sigma$ -sentences is the least set of sentences obtained from propositional variables in  $\Sigma$  by applying a finite number of times Boolean connectives in  $\{\vee, \neg\}$ .  $Sen(\sigma)$  translates  $\Sigma$ -formulae to  $\Sigma'$ -formulae by renaming propositional variables according to the signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$ . Finally, satisfaction is the usual propositional satisfaction.

- **FOL.** The institution of (many-sorted) first-order predicate logic (with equality). The signature are pairs  $(S, F, P)$  where  $S$  is a set of sorts, and  $F$  and  $P$  are, respectively, function and predicate names with arity in  $S$ . Signature morphisms  $\sigma = (\sigma^{sort}, \sigma^{fun}, \sigma^{pred}) : (S, F, P) \rightarrow (S', F', P')$  consists of three functions between sets of sorts, sets of functions and sets of predicates which preserve arities.

Given a signature  $\Sigma = (S, F, P)$ , a  $\Sigma$ -model  $\mathcal{M}$  is a family  $M = (M_s)_{s \in S}$  where  $M_s$  is a set for each  $s \in S$ , equipped with a function  $f^{\mathcal{M}} : M_{s_1} \times \dots \times M_{s_n} \rightarrow M_s$  for each  $f : s_1 \times \dots \times s_n \rightarrow s \in F$  and with a n-ary relation  $p^{\mathcal{M}} \subseteq M_{s_1} \times \dots \times M_{s_n}$  for each  $p : s_1 \times \dots \times s_n \in P$ . Given a signature morphism  $\sigma : \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$  and a  $\Sigma'$ -model  $\mathcal{M}'$ ,  $Mod(\sigma)(\mathcal{M}')$  is the  $\Sigma$ -model  $\mathcal{M}$  defined for each  $s \in S$  by  $M_s = M'_s$ , and for each function name  $f \in F$  and each predicate name  $p \in P$ , by  $f^{\mathcal{M}} = \sigma(f)^{\mathcal{M}'}$  and by  $p^{\mathcal{M}} = p^{\mathcal{M}'}$ .

The set of  $\Sigma$ -formulae is the least set of formulae obtained from atoms of the form  $t_1 = t_2$  where  $t_1, t_2 \in T_F(X)_s$  for  $s \in S$  and of the form  $p(t_1, \dots, t_n)$  where  $t_i \in T_F(X)_{s_i}$  for each  $i$ , <sup>1</sup>  $1 \leq i \leq n$ , and  $p : s_1 \times \dots \times s_n \in P$ , by applying a finite number of times Boolean connectives in  $\{\vee, \neg\}$  and the quantifier  $\forall$ .  $Sen(\Sigma)$  is the set of all closed  $\Sigma$ -formulae. <sup>2</sup>  $Sen(\sigma)$  translates  $\Sigma$ -sentences by renaming function and predicate names according to the signature morphism  $\sigma$ . Finally, satisfaction is the usual first-order satisfaction.

- **MFOL.** The institution of the modal first order logic (with global satisfaction). The signatures are just the **FOL** signatures. The set of  $\Sigma$ -formulae is the least set of formulae obtained from atoms  $p(t_1, \dots, t_n)$  where  $t_i \in T_F(X)_{s_i}$  for each  $i$ ,  $1 \leq i \leq n$ , and  $p : s_1 \times \dots \times s_n \in P$ , by applying a finite number of times Boolean connectives in  $\{\vee, \neg\}$ , the modality  $\Box$  and the quantifier  $\forall$ .  $Sen(\Sigma)$  is the set of all closed  $\Sigma$ -formulae. Given a signature  $\Sigma = (S, F, P)$ , a  $\Sigma$ -model  $(W, R)$ , called Kripke frame, consists of a family  $W = (W^i)_{i \in I}$  of possible worlds, which are  $\Sigma$ -models in

<sup>1</sup> $T_F(X)$  is the term algebra over  $F$  with variables from  $X$ .

<sup>2</sup>A  $\Sigma$ -formula  $\varphi$  is closed when all the occurrences of variables occurring in  $\varphi$  are in the scope of the quantifier  $\forall$ .

**FOL** such that  $(W^i)_s = (W^j)_s$  for each  $i, j \in I$  and each  $s \in S$ , and an “accessibility” relation  $R \subseteq I \times I$ . The satisfaction of formulæ by the Kripke frames, noted  $(W, R) \models \varphi$ , is defined by  $(W, R) \models^i \varphi$  for each  $i \in I$ , where  $\models^i$  is defined by induction on the structure of the formula  $\varphi$  as follows:

- atoms, Boolean connectives and quantifier are handled as in **FOL** for  $W^i$ ,
- $(W, R) \models^i \Box \varphi$  when for each  $j \in I$  such that  $i R j$ ,  $(W, R) \models^j \varphi$ .

Modal propositional logic (**MPL**) is the sub-institution of **MFOL** determined by the signatures with empty set of sort symbols and empty set of operation symbols.

- **LMFOL**. The institution of the modal first order logic (with local satisfaction). Signatures and sentences are **MFOL** signatures and **MFOL** sentences. Given a signature  $\Sigma = (S, F, P)$ , a  $\Sigma$ -model is a pointed Kripke frame  $(W, R, W^i)$  for  $i \in I$ . The satisfaction of a  $\Sigma$ -sentence  $\varphi$  by a  $\Sigma$ -model  $(W, R, W^i)$ , noted  $(W, R, W^i) \models \varphi$ , is defined by:  $(W, R, W^i) \models \varphi \Leftrightarrow (W, R) \models^i \varphi$ .

- **LIMFOL**. The institution of the modal first order logic (with local satisfaction and infinite disjunction and conjunction). This institution extends **LMFOL** to sentences of the form  $\bigwedge \Phi$  and  $\bigvee \Phi$  where  $\Phi$  is a set (possibly infinite) of  $\Sigma$ -sentences. Given a pointed Kripke frame  $(W, R, W^i)$ ,

- $(W, R, W^i) \models \bigwedge \Phi \Leftrightarrow \forall \varphi \in \Phi, (W, R, W^i) \models \varphi$
- $(W, R, W^i) \models \bigvee \Phi \Leftrightarrow \exists \varphi \in \Phi, (W, R, W^i) \models \varphi$

**Definition 2.2 (Weak amalgamation square)** *Let  $\mathcal{I}$  be an institution. The commuting square of signature morphisms in  $\mathcal{I}$*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow{\sigma'_2} & \Sigma' \end{array}$$

*is a weak amalgamation square if and only if for each  $\Sigma_1$ -model  $\mathcal{M}_1$  and each  $\Sigma_2$ -model  $\mathcal{M}_2$  such that  $\text{Mod}(\sigma_1)(\mathcal{M}_1) = \text{Mod}(\sigma_2)(\mathcal{M}_2)$ , there exists a  $\Sigma'$ -model  $\mathcal{M}'$  such that  $\text{Mod}(\sigma'_1)(\mathcal{M}') = \mathcal{M}_1$  and  $\text{Mod}(\sigma'_2)(\mathcal{M}') = \mathcal{M}_2$ .  $\mathcal{I}$  has the weak amalgamation property if and only if every commuting square is a weak amalgamation square.*

**Definition 2.3 (Elementary equivalence)** *Let  $\mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$  be an institution. Let  $\Sigma$  be a signature. Two  $\Sigma$ -models  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are elementary equivalent, noted  $\mathcal{M}_1 \equiv_{\Sigma} \mathcal{M}_2$ , if and only if the following condition holds:*

$$\forall \varphi \in \text{Sen}(\Sigma), \mathcal{M}_1 \models_{\Sigma} \varphi \Leftrightarrow \mathcal{M}_2 \models_{\Sigma} \varphi$$

**Definition 2.4 (Theory)** Let  $\mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$  be an institution. Let  $\Sigma$  be a signature of  $|\text{Sig}|$ . Let  $T$  be a set of  $\Sigma$ -sentences. Let us note  $\text{Mod}(T)$  the full sub-category of  $\text{Mod}(\Sigma)$  whose objects are all  $\Sigma$ -models  $\mathcal{M}$  such that for any  $\varphi \in T$ ,  $\mathcal{M} \models_{\Sigma} \varphi$ , and  $T^{\bullet}$  the subset of  $\text{Sen}(\Sigma)$ , so-called semantic consequences of  $T$ , defined as follows:  $T^{\bullet} = \{\varphi \mid \forall \mathcal{M} \in |\text{Mod}(T)|, \mathcal{M} \models_{\Sigma} \varphi\}$ .  $T$  is a theory if and only if  $T = T^{\bullet}$ .

**Definition 2.5 (Consistency)** A  $\Sigma$ -theory is consistent if and only if  $\text{Mod}(T) \neq \emptyset$ .

**Definition 2.6 (Robinson consistency property)** Let  $\mathcal{I}$  be an institution. A commuting square  $\mathcal{S}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow{\sigma'_2} & \Sigma' \end{array}$$

has the Robinson consistency property (**RCP**) if and only if for every pair of consistent theories  $T_1$  and  $T_2$  over  $\Sigma_1$  and  $\Sigma_2$ , respectively, with “inter-consistent reducts”, i.e.  $T_1|_{\sigma_1} \cup T_2|_{\sigma_2}$  is consistent where  $T_i|_{\sigma_i} = \{\varphi \in \text{Sen}(\Sigma) \mid \text{Sen}(\sigma_i)(\varphi) \in T_i\}$ , have inter-consistent  $\Sigma'$ -translations, i.e.  $\text{Sen}(\sigma'_1)(T_1) \cup \text{Sen}(\sigma'_2)(T_2)$  is consistent.

### 3 Weak Robinson property

**Definition 3.1 (Weak Robinson property)** Let  $\mathcal{I}$  be an institution. Let  $\mathcal{S}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow{\sigma'_2} & \Sigma' \end{array}$$

be a commuting square. Let  $\mathcal{M}_1 \in |\text{Mod}(T_1)|$  and  $\mathcal{M}_2 \in |\text{Mod}(T_2)|$  be two models such that  $\text{Mod}(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} \text{Mod}(\sigma_2)(\mathcal{M}_2)$ .  $\mathcal{S}$  has the Weak Robinson Property (**WRP**) for  $\mathcal{M}_1$  and  $\mathcal{M}_2$  if and only if there exists  $i \in \{1, 2\}$ , there exists a  $\Sigma_i$ -model  $\mathcal{M}'_i \in |\text{Mod}(\Sigma_i)|$  such that:

1.  $\mathcal{M}_i \equiv_{\Sigma_i} \mathcal{M}'_i$ ,
2.  $\text{Mod}(\sigma_i)(\mathcal{M}'_i) = \text{Mod}(\sigma_j)(\mathcal{M}_j)$  where  $j \neq i$  and  $j \in \{1, 2\}$ .

An institution  $\mathcal{I}$  has WRP if every commuting square  $\mathcal{S}$  has WRP for every pair of models  $\mathcal{M}_1$  and  $\mathcal{M}_2$  such that  $\text{Mod}(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} \text{Mod}(\sigma_2)(\mathcal{M}_2)$ .

Obviously, we have the following result:

**Theorem 3.2** *Let  $\mathcal{I}$  be an institution which has WRP and the weak amalgamation property. Let  $\mathcal{S}$*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow[\sigma'_2]{} & \Sigma' \end{array}$$

*be a commuting square. If for every consistent  $\Sigma_1$ -theory  $T_1$  and every consistent  $\Sigma_2$ -theory with inter-consistent reducts, there is  $\mathcal{M}_i \in |\text{Mod}(T_i)|$  such that  $\text{Mod}(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} \text{Mod}(\sigma_2)(\mathcal{M}_2)$ , then  $\mathcal{S}$  has RCP.*

**Proof** By WRP, there then exists a  $\Sigma_i$ -model  $\mathcal{M}'_i \in |\text{Mod}(\Sigma_i)|$  for  $i \in \{1, 2\}$  such that:

1.  $\mathcal{M}_i \equiv_{\Sigma_i} \mathcal{M}'_i$ ,
2.  $\text{Mod}(\sigma_i)(\mathcal{M}'_i) = \text{Mod}(\sigma_j)(\mathcal{M}_j)$  where  $j \neq i$  and  $j \in \{1, 2\}$ .

By the weak amalgamation property, there then exists  $\mathcal{M}' \in |\text{Mod}(\Sigma')|$  such that  $\text{Mod}(\sigma'_i)(\mathcal{M}') = \mathcal{M}'_i$  and  $\text{Mod}(\sigma'_j)(\mathcal{M}') = \mathcal{M}_j$ . By the satisfaction condition, we then have that  $\mathcal{M}' \in |\text{Mod}(\text{Sen}(\sigma'_1)(T_1) \cup \text{Sen}(\sigma_2)(T_2))|$ .  $\square$

**Proposition 3.3** *In PL, for every commuting square  $\mathbf{S}$ , every consistent  $\Sigma_1$ -theory and every consistent  $\Sigma_2$ -theory with inter-consistent reducts, there are a  $\Sigma_1$ -valuation  $\nu_1$  and a  $\Sigma_2$ -valuation  $\nu_2$  such that  $\nu_1|_{\sigma_1} \equiv_{\Sigma} \nu_2|_{\sigma_2}$  (and then  $\nu_1|_{\sigma_1} = \nu_2|_{\sigma_2}$ ).*

*In FOL, LMFOL, LIMFOL, and MFOL for every commuting square satisfying the four conditions  $C_1, C_2, C'_1$  and  $C'_2$  of Proposition 3 in [8], every consistent  $\Sigma_1$ -theory  $T_1$  and every consistent  $\Sigma_2$ -theory  $T_2$  with inter-consistent reducts, there are  $\mathcal{M}_1 \in |\text{Mod}(T_1)|$  and  $\mathcal{M}_2 \in |\text{Mod}(T_2)|$  such that  $\text{Mod}(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} \text{Mod}(\sigma_2)(\mathcal{M}_2)$ .*

**Proof** In PL, we find a  $\Sigma_1$ -valuation  $\nu_1 \models T_1$  such that  $\nu_1|_{\sigma_1} \models T_2|_{\sigma_2}$ . If such a model does not exist, then by compactness there exists a finite set  $\{\varphi_1, \dots, \varphi_n\} \subseteq T_2|_{\sigma_2}$  such that  $T_1 \models \neg \bigwedge_{i \geq n} \varphi_i$ , making  $T_1|_{\sigma_1} \cup T_2|_{\sigma_2}$  inconsistent, a contradiction. After, we find a  $\Sigma_2$ -valuation  $\nu_2$  such that  $\nu_1|_{\sigma_1} \equiv_{\Sigma} \nu_2|_{\sigma_2}$ . For this, let us consider the complete  $\Sigma$ -theory  $\text{Th}(\nu_1|_{\sigma_1})$  and find a  $\Sigma \cup \Sigma_2$ -valuation  $\nu'$  of  $\text{Th}(\nu_1|_{\sigma_1}) \cup T_2$ . But if this is inconsistent, then by compactness, we would find  $\varphi \in \text{Th}(\nu_1|_{\sigma_1})$  such that  $T_2 \models \neg \varphi$ . This means that  $\neg \varphi \in T_2|_{\sigma_2}$  and then  $\neg \varphi \in \text{Th}(\nu_1|_{\sigma_1})$ , contradicting the fact that  $\nu_1|_{\sigma_1} \models \varphi$ . Consequently,  $\nu'|_{\Sigma_2 \leftarrow \Sigma \cup \Sigma_2 \circ \sigma_2} \equiv_{\Sigma} \nu_1|_{\sigma_1}$ .

For FOL, instantiating the two first steps of Proposition 1 in [8] by replacing  $A_1$  and  $A_2$  by  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

In LMFOL and LIMFOL, for a signature  $\Sigma = (S, F, R)$ , let us define the FOL signature  $\bar{\Sigma} = (\bar{S}, \bar{F}, \bar{R})$  as follow:

- $\bar{S} = S \cup \{ind\}$
- $\bar{F} = \{f : ind \times s_1 \times \dots \times s_n \rightarrow s \mid f : s_1 \times \dots \times s_n \rightarrow s \in F\} \cup \{i : \rightarrow ind\}$
- $\bar{R} = \{r : ind \times s_1 \times \dots \times s_n \mid r : s_1 \times \dots \times s_n \in R\} \cup \{R : ind \times ind\}$

Given a pointed Kripke frame  $(W, R, W^i)$  over  $\Sigma$ , let us define the **FO**L  $\bar{\Sigma}$ -model  $\mathcal{M}_W$  as follows:

- $M_{ind} = I$  and  $\forall s \in S, M_s = (W^i)_s$
- $i^{\mathcal{M}} = i$  and  $R^{\mathcal{M}} = R$
- $\forall f : ind \times s_1 \times \dots \times s_n \rightarrow s \in F_W, \forall (j, a_1, \dots, a_n) \in I \times M_{s_1} \times \dots \times M_{s_n},$   
 $f^{\mathcal{M}}(j, a_1, \dots, a_n) = f^{W^j}(a_1, \dots, a_n)$
- $\forall r : ind \times s_1 \times \dots \times s_n, (j, a_1, \dots, a_n) \in r^{\mathcal{M}} \Leftrightarrow (a_1, \dots, a_n) \in r^{W^j}$

Let  $X$  be a  $\bar{S}$ -indexed family of sets of variables. For every  $x \in X_{ind} \cup I$ , let us define  $FO_x : Sen_{\mathbf{MFOL}}(\Sigma) \times T_F(X) \rightarrow Sen_{\mathbf{FO}}(\bar{\Sigma}) \times T_{\bar{F}}(X)$  inductively on the formula structure as follows:

- $f(t_1, \dots, t_n) \mapsto f(x, t_1, \dots, t_n)$
- $r(t_1, \dots, t_n) \mapsto r(x, t_1, \dots, t_n)$
- $\varphi \vee \psi \mapsto FO_x(\varphi) \vee FO_x(\psi)$
- $\neg \varphi \mapsto \neg FO_x(\varphi)$
- $\forall y. \varphi \mapsto \forall y. FO_x(\varphi)$
- $\Box \varphi \mapsto \forall y, x R y \Rightarrow FO_y(\varphi)$

Observe that  $(W, R, W^i) \models \varphi \Leftrightarrow \mathcal{M}_W \models FO_i(\varphi)$ .

Given a  $\Sigma$ -theory  $T$ ,  $FO_x(T) = (\{FO_x(\varphi) \mid \varphi \in T\})^\bullet$ . By construction, if  $T$  is a consistent theory then so does  $FO_i(T)$  as well in **FO**L. Therefore, as  $T_1$  and  $T_2$  have inter-consistent reducts,  $FO_i(T_1)_{\sigma_1} \cup FO_i(T_2)_{\sigma_2}$  is consistent. Hence, by following the same process than above in **FO**L, we can find a  $\bar{\Sigma}_1$ -model  $\mathcal{M}_1$  and a  $\bar{\Sigma}_2$ -model  $\mathcal{M}_2$  such that  $Mod(\bar{\sigma}_1)(\mathcal{M}_1) \equiv_{\bar{\Sigma}} Mod(\bar{\sigma}_2)(\mathcal{M}_2)$ . For each  $\mathcal{M}_i$ , by applying the opposite process than above, we can build a pointed Kripke frame  $(W_i, R_i, W_i^{p_i})$  over  $\Sigma_i$  such that:  $\mathcal{M}_i \models FO_i(\varphi) \Leftrightarrow (W_i, R_i, W_i^{p_i}) \models \varphi$ . Hence,  $(W_i, R_i, W_i^{p_i})$  belongs to  $|Mod(T_i)|$ , and  $Mod(\sigma_1)((W_1, R_1, W_1^{p_1})) \equiv_{\Sigma} Mod(\sigma_2)((W_2, R_2, W_2^{p_2}))$ .

In **MFOL**, the process is almost the same than for **LMFOL** and **LIMFOL** except that  $\bar{F}$  does not contain  $i : \rightarrow ind$  anymore. Given a Kripke frame  $(W, R)$  over  $\Sigma$ , we then have:

$$(W, R) \models^i \varphi \Leftrightarrow \forall \iota : X \rightarrow M, \iota(x) = i \Rightarrow \mathcal{M}_W \models_{\iota} FO_x(\varphi)$$



Given a  $\Sigma$ -theory  $T$ ,  $FO_x(T) = \{\forall x.FO_x(\varphi) \mid \varphi \in T\}$ . Therefore, as  $T_1$  and  $T_2$  have inter-consistent reducts,  $FO_x(T_1)_{\sigma_1} \cup FO_x(T_2)_{\sigma_2}$  is consistent. Hence, by following the same process than above in **FO**L, we can find a  $\bar{\Sigma}_1$ -model  $\mathcal{M}_1$  and a  $\bar{\Sigma}_2$ -model  $\mathcal{M}_2$  such that  $Mod(\bar{\sigma}_1)(\mathcal{M}_1) \equiv_{\bar{\Sigma}} Mod(\bar{\sigma}_2)(\mathcal{M}_2)$ . For each  $\mathcal{M}_i$ , by applying the opposite process than above, we can build a Kripke frame  $(W_i, R_i)$  over  $\Sigma_i$  such that:  $\mathcal{M}_i \models \forall x.FO_x(\varphi) \iff (W_i, R_i) \models \varphi$ . Hence,  $(W_i, R_i)$  belongs to  $|Mod(T_i)|$ , and  $Mod(\sigma_1)((W_1, R_1)) \equiv_{\Sigma} Mod(\sigma_2)((W_2, R_2))$ .  $\square$

The four conditions on commuting square allow us to remove the following counter-example adapted from [1]: suppose the commuting diagram  $\mathcal{S}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow[\sigma'_2]{} & \Sigma' \end{array}$$

where  $\Sigma = (\{s_1, s_2\}, \{a := s_1, b := s_2\}, \emptyset)$ ,  $\Sigma_1 = (\{s\}, \{a := s\}, \emptyset)$ , and  $\Sigma_2 = (\{s\}, \{a, b := s\}, \emptyset)$ , and  $\sigma_i(s_j) = s$  with  $i, j = 1, 2$ ,  $\sigma_1(a) = \sigma_1(b) = a$ ,  $\sigma_2(a) = a$ , and  $\sigma_2(b) = b$ . Then, let us suppose  $T_1 = \emptyset^\bullet$  and  $T_2 = \{a \neq b\}^\bullet$ . Each  $T_i$  is consistent. However, we cannot find two models  $\mathcal{M}_1$  and  $\mathcal{M}_2$  which are elementary equivalent on  $\Sigma$  because models of  $T_2$  have  $a \neq b$  while models in  $T_1$  require that  $a = b$ .

## 4 Applications

**Theorem 4.1** **PL** has *WRP*.

**Proof** Obvious because for every set of propositional variable  $\Sigma$ ,  $\nu \equiv_{\Sigma} \nu'$  means that  $\nu = \nu'$ .  $\square$

**Theorem 4.2** **FOL** has *WRP*.

**Proof** Let  $\mathcal{S}$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow[\sigma'_2]{} & \Sigma' \end{array}$$

be a commuting square where  $\Sigma = (S, F, P)$ ,  $\Sigma' = (S', F', P')$ , and  $\Sigma_i = (S_i, F_i, P_i)$  for  $i = 1, 2$ . Let  $\mathcal{M}_1 \in |Mod(\Sigma_1)|$  and  $\mathcal{M}_2 \in |Mod(\Sigma_2)|$  be two first-order structures such that  $Mod(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} Mod(\sigma_2)(\mathcal{M}_2)$ . For every sort  $s \in S$ , note  $\alpha_s$  the cardinality of  $(M_i)_{\sigma_i(s)}$ . Here, two cases have to be considered:

- $\alpha_s < \aleph_0$  (i.e.  $(M_i)_{\sigma_i(s)}$  is a finite set). By definition,  $(Mod(\sigma_i)(M_i))_s$  is also a finite set of cardinality  $\alpha_s$ . By the hypothesis that  $Mod(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} Mod(\sigma_2)(\mathcal{M}_2)$ ,  $(M_j)_{\sigma_j(s)}$  is also a finite set of cardinality  $\alpha_s$ . Hence  $(M_i)_{\sigma_i(s)}$  and  $(M_j)_{\sigma_j(s)}$  are bijective for some bijection  $g_s$ .
- $\alpha_s \geq \aleph_0$ . By definition,  $(Mod(\sigma_i)(M_i))_s$  is also a set of cardinality  $\alpha_s$ . By the hypothesis that  $Mod(\sigma_1)(\mathcal{M}_1) \equiv_{\Sigma} Mod(\sigma_2)(\mathcal{M}_2)$ ,  $(M_j)_{\sigma_j(s)}$  is then an infinite set of cardinality  $\beta_s \geq \aleph_0$ . Moreover, by applying the Lowenheim-Skolem theorem on the theory  $Th(\mathcal{M}_j)$ , there exists  $\mathcal{M}_j'' \in |Mod(\Sigma_2)|$  such that:
  - for each  $s \in S$  such that  $\alpha_s \geq \aleph_0$ ,  $(M_j'')_{\sigma_j(s)}$  is an infinite of cardinality  $\alpha_s$ .
  - for all the other sorts  $s \in S_j$  (i.e. every  $s \in S$  such that  $\alpha_s < \aleph_0$  and every  $s \in S_j \setminus \sigma_j(S)$ ),  $(M_j'')_s = (M_j)_s$ .

Finally,  $Th(\mathcal{M}_j)$  is a complete theory. Therefore, we have  $\mathcal{M}_j'' \equiv_{\Sigma_j} \mathcal{M}_j$ .

Hence, from both above cases, we have a bijection  $g : M_i \rightarrow M_j''$  from which we can define the following  $\Sigma_j$ -model  $\mathcal{M}_j'$  as follows:

1.  $\forall s \in S, (M_j')_{\sigma_j(s)} = (M_i)_{\sigma_i(s)}$
2.  $\forall s \in S_j \setminus \sigma_j(S), (M_j')_s = (M_j'')_s$
3.  $\forall f \in F, \sigma_j(f)^{\mathcal{M}_j'} = \sigma_i(f)^{\mathcal{M}_i}$
4.  $\forall f : s_1 \times \dots \times s_n \rightarrow s \in F_j \setminus \sigma_j(F)$ ,
$$f^{\mathcal{M}_j'} : (M_j')_{s_1} \times \dots \times (M_j')_{s_n} \rightarrow (M_j')_s$$

$$(m_1, \dots, m_n) \mapsto g^{-1}(f^{\mathcal{M}_j''}(g(m_1), \dots, g(m_n)))$$
5.  $\forall r \in R, \sigma_j(r)^{\mathcal{M}_j'} = \sigma_i(r)^{\mathcal{M}_i}$
6.  $\forall r : s_1 \times \dots \times s_n \in R_j \setminus \sigma_j(R)$ ,
$$r^{\mathcal{M}_j'} = \{(m_1, \dots, m_n) | (g(m_1), \dots, g(m_n)) \in r^{\mathcal{M}_j''}\}$$

In Points 4. et 6.,  $g$  is the identity on every sort  $s \in S_j \setminus \sigma_j(S)$ .

By construction, we have both:

- $\mathcal{M}_j''$  is isomorphic to  $\mathcal{M}_j'$ , and then  $\mathcal{M}_j \equiv_{\Sigma_j} \mathcal{M}_j'$ , and
- $Mod(\sigma_i)(\mathcal{M}_i) = Mod(\sigma_j)(\mathcal{M}_j')$ .

□

**Theorem 4.3** *In LMFOL and LIMFOL, every commuting square  $\mathcal{S}$  has WRP for every pair of pointed Kripke frames  $(W_1, R_1, W_1^{p_1}) \in |Mod(\Sigma_1)|$  and  $(W_2, R_2, W_2^{p_2}) \in |Mod(\Sigma_2)|$  such that  $W_1^{p_1}$  and  $W_2^{p_2}$  are bisimilar.*

**Proof** Define  $(W'_i, R'_i, W_i^{p_i})$  for  $i \in \{1, 2\}$  such that:

- $(W'_i, R'_i, W_i^{p_i}) \equiv_{\Sigma_i} (W_i, R_i, W_i^{p_i})$
- $Mod(\sigma_i)((W'_i, R'_i, W_i^{p_i})) = Mod(\sigma_j)((W_j, R_j, W_j^{p_j}))$  where  $j \neq i$  and  $j \in \{1, 2\}$

as follow:

- $W'_i = (W_i^{l_k})_{k \in J}$  where for  $k \in J$ ,
  - $k = p'_i$ .  $W_i^{p'_i}$  is defined as in the proof of Theorem 4.2 where  $W_i^{p_i}$ ,  $W_i^{p'_i}$  and  $W_j^k$  are respectively  $\mathcal{M}_j$ ,  $\mathcal{M}'_j$  and  $\mathcal{M}_i$ .
  - $k \neq p'_i$ . Two cases can occur:
    1. if there is  $l \in I$  such that  $Mod(\sigma_j)(W_j^k)$  and  $Mod(\sigma_i)(W_i^l)$  are bisimilar then  $W_i^{l_k}$  is defined as in the proof of Theorem 4.2 where  $W_i^l$ ,  $W_i^{l_k}$  and  $W_j^k$  are respectively  $\mathcal{M}_j$ ,  $\mathcal{M}'_j$  and  $\mathcal{M}_i$ .
    2. Otherwise (i.e. for all  $l \in I$ ,  $Mod(\sigma_j)(W_j^k)$  and  $Mod(\sigma_i)(W_i^l)$  are not bisimilar),  $W_i^{l_k}$  is defined as follows:
      - \*  $\forall s \in S, (W_i^{l_k})_{\sigma_1(s)} = (W_j^k)_{\sigma_1(s)}$
      - \*  $\forall s \in S_1 \setminus Sen(\sigma_1)(S), (W_i^{l_k})_s = (W_i^{p_i})_s$
      - \*  $\forall f \in F, \sigma_1(f)^{W_i^{l_k}} = \sigma_1(f)^{W_j^k}$
      - \*  $\forall f : \rightarrow s \in F_1 \setminus \sigma_1(F), f^{W_i^{l_k}} = f^{W_i^{p_i}}$
      - \*  $\forall f : s_1 \times \dots \times s_n \rightarrow s \in F_1 \setminus \sigma_1(F),$   
 $f^{W_i^{l_k}} \in ((W_i^{l_k})_s)((W_j^k)_{s_1} \times \dots \times (W_j^k)_{s_n})$
      - \*  $\forall r : s_1 \times \dots \times s_n \in R_1 \setminus \sigma_1(R), r^{W_i^{l_k}} \subseteq (W_j^k)_{s_1} \times \dots \times (W_j^k)_{s_n}$
- $R'_i = R_j$
- $p'_i = p_j$

By construction, we have both that:

1.  $Mod(\sigma_i)((W'_i, R'_i, W_i^{p_i})) = Mod(\sigma_j)((W_j, R_j, W_j^{p_j}))$ , and
2.  $W_i^{p'_i}$  and  $W_i^{p_i}$  are bisimilar.

Therefore, we conclude that  $(W'_i, R'_i, W_i^{p_i}) \equiv_{\Sigma_i} (W_i, R_i, W_i^{p_i})$ . □

**Theorem 4.4** *In MFOL, every commuting square  $\mathcal{S}$  has WRP for every pair of Kripke frames which are globally bisimilar.*

**Proof** Define  $(W'_i, R'_i)$  for  $i \in \{1, 2\}$  such that:

- $(W'_i, R'_i) \equiv_{\Sigma_i} (W_i, R_i)$

- $Mod(\sigma_i)((W'_i, R'_i)) = Mod(\sigma_j)((W_j, R_j))$  where  $j \neq i$  and  $j \in \{1, 2\}$

as follow:

- $W'_i = (W_i'^k)_{k \in J}$  where for  $k \in J$ ,  $W_i'^k$  is defined as in the proof of Theorem 4.2 where  $W_i^l$ ,  $W_i'^k$  and  $W_j^k$  are respectively  $\mathcal{M}_j$ ,  $\mathcal{M}'_j$  and  $\mathcal{M}_i$  for some  $l \in J$  such that  $W_i^l$  and  $W_j^k$  are bisimilar.
- $R'_i = R_j$ .

By construction, we have both that:

1.  $Mod(\sigma_i)((W'_i, R'_i)) = Mod(\sigma_j)((W_j, R_j))$ , and
2. for every  $W_i'^{p_i}$  and  $W_i^{p_i}$  are bisimilar.

Therefore, we conclude that  $(W'_i, R'_i, W_i'^{p_i}) \equiv_{\Sigma_i} (W_i, R_i, W_i^{p_i})$ .  $\square$

**Corollary 4.5** *In PL every commuting square has RCP.*

*In FOL, LMFOL and LIMFOL, every commuting square satisfying the four conditions  $C_1$ ,  $C_2$ ,  $C'_1$  and  $C'_2$  of Proposition 3.3 is RCP.*

**Proof** In **PL** and **FOL**, this is obvious. In **LIMFOL**, this is a direct consequence of Karp's theorem which expresses that pointed Kripke frames which are elementary equivalent are bisimilar. In **LMFOL**, from results of the standard model theory, this is a consequence of the fact that the ultrapower of any pointed Kripke frame with respect to a regular ultrafilter <sup>3</sup> is an  $\omega$ -saturated elementary extension of this pointed Kripke frame (see Corollary 4.3.14 in [3]). And, by a classic theorem in model theory of modal logic, it is well-known that if two pointed Kripke frames are elementary equivalent, then any pair of their  $\omega$ -saturated ultrapowers are bisimilar (a simple consequence of an extension of the Hennessy-Milner property extended to modally saturated Kripke frames - see [10] for details on these notions).  $\square$

In **MFOL**, we have not the equivalent of the Hennessy-Milner property. The problem is because we require that the two Kripke frames are globally bisimilar. There are many counter-examples of elementary equivalent Kripke frames which are not globally bisimilar. The only result that we can give is the following:

**Proposition 4.6** *In MFOL, given a commuting square  $\mathcal{S}$*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ \sigma_2 \downarrow & & \downarrow \sigma'_1 \\ \Sigma_2 & \xrightarrow{\sigma'_2} & \Sigma' \end{array}$$

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<sup>3</sup>It is well-known that for any set of power  $\alpha$ , there exists an  $\alpha$ -regular ultrafilter over it (Proposition 4.3.5 in [3]).

such that  $\sigma_1$  is injective, and two consistent theories  $T_1$  and  $T_2$  over  $\Sigma_1$  and  $\Sigma_2$ , respectively with inter-consistent reducts and such that  $T_1|_{\sigma_1} = T_2|_{\sigma_2}$  and  $T_1 = \text{Sen}(\sigma_1)(T_1|_{\sigma_1})$ , then  $\mathcal{S}$  has RCP.

**Proof** Let  $\mathcal{M}_2$  be a  $\Sigma_2$ -model of  $T_2$ . By the satisfaction condition,  $\text{Mod}(\sigma_2)(\mathcal{M}_2)$  is a model of  $T_2|_{\sigma_2}$  and then of  $T_1|_{\sigma_1}$ . As  $\sigma_1$  is injective, there exists a  $\sigma_1$ -extension  $\mathcal{M}_1$  of  $\text{Mod}(\sigma_2)(\mathcal{M}_2)$ , that is  $\text{Mod}(\sigma_1)(\mathcal{M}_1) = \text{Mod}(\sigma_2)(\mathcal{M}_2)$ .  $\square$

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