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RANDOM WALKS IN RANDOM DIRICHLET ENVIRONMENT ARE TRANSIENT IN DIMENSION $d \geq 3$

CHRISTOPHE SABOT

Abstract: We consider random walks in random Dirichlet environment (RWDE) which is a special type of random walks in random environment where the exit probabilities at each site are i.i.d. Dirichlet random variables. On \mathbb{Z}^d , RWDE are parameterized by a $2d$ -uplet of positive reals. We prove that for all values of the parameters, RWDE are transient in dimension $d \geq 3$. We also prove that the Green function has some finite moments and we characterize the finite moments. Our result is more general and applies for example to finitely generated symmetric transient Cayley graphs. In terms of reinforced random walks it implies that directed edge reinforced random walks are transient for $d \geq 3$.

1. INTRODUCTION

Random Walks in Random Environment (RWRE) have received a considerable attention in the last years. A lot is known for one-dimensional RWRE, but the situation is far from being so clear in dimension 2 and larger. Progress has been made for multidimensional RWRE (in particular, since the work of Sznitman and Zerner, cf [38]) essentially in two directions: for ballistic RWRE (cf [38], [36], [29] and reference therein) and more recently for small perturbations of the simple random walk in dimension $d \geq 3$. Nevertheless, many important questions as the characterization of recurrence, of ballistic behavior, invariance principle remain open. Recently, for RWRE which are small isotropic perturbations of the simple random walk on \mathbb{Z}^d , $d \geq 3$, Bolthausen and Zeitouni ([3]) made some progress in the direction of the invariance principle and proved transience using renormalization techniques (invariance principle for the corresponding continuous model has previously been obtained by Sznitman and Zeitouni, [37]). In the case of ballistic RWRE, Rassoul-Agha and Seppäläinen obtained a quenched functional central limit theorem ([31]). On very special cases as symmetric environments ([21]), or environments admitting a bounded cycle representation ([10]) an invariance principle has been obtained. On the ballistic case, some important progress has been made on the description of large deviations ([41, 43, 44, 32]). We refer to [45] or [34] for surveys.

Among random walks in random environment, random walks in random Dirichlet environment (RWDE) play a special role. It corresponds to the case where the transition probabilities at each site are chosen as i.i.d. Dirichlet random variables. RWDE have the remarkable property that the annealed law is the law of a directed edge reinforced random walk. This is a simple consequence of the representation of Polya's urns as a Dirichlet mixtures of i.i.d. sampling ([26], [13]). In [12], N. Enriquez and the author have obtained a criterion for ballistic behavior on \mathbb{Z}^d , later improved

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by L. Tournier ([40]). Besides, in the one-dimensional case, it appeared in [14] that limit theorems in the sub-ballistic regime are fully explicit in the case of Dirichlet environment, highlighting the special role of Dirichlet environment among random environments. Finally, in [30] we have described a precise relation between RWDE and hypergeometric integrals associated with certain arrangement of hyperplanes.

As already mentioned, random Dirichlet environment is very natural from the point of view of reinforced random walks. Reinforced random walks have been introduced by Diaconis (cf [8] and Pemantle's thesis [26]). There are three natural models of linearly reinforced random walks, namely vertex reinforced random walks, (undirected) edge reinforced random walks, and directed edge reinforced random walks. None of these models is completely understood yet. Vertex reinforced random walks are expected to get localized on a finite set, but this is completely solved only in dimension 1 ([28, 39]). Concerning edge reinforced random walks, Diaconis and Coppersmith proved that it can be represented as a "complicated" mixture of reversible random walks. Recurrence of edge reinforced random walks has been proved by Merkl and Rolles ([24]) on a two dimensional graph (but the question is still open on \mathbb{Z}^2 itself). By comparison, directed edge reinforced random walks correspond to the annealed law of RWDE, hence they are a simple mixture of non-reversible Markov chains. The difficulty of this model does not come from the representation as a RWRE, but lies in the non-reversibility of this RWRE. We refer to [27] for a survey on the subject.

On \mathbb{Z}^d , RWDE are parameterized by $2d$ positive reals $(\alpha_1, \dots, \alpha_{2d})$, one for each direction. We call (α_i) the weights. In this paper, we prove that RWDE on \mathbb{Z}^d , for $d \geq 3$, are transient for all values of the weights (in particular for the case of unbiased weights). In fact, we prove that the Green function has some finite moments and we explicitly compute the critical integrability exponent κ , i.e. the supremum of the reals $s > 0$ such that the Green function $G(0, 0)^s$ has finite expectation. We show that this real is the same as the one for the RWDE killed when it exists a finite large ball (which has been computed by Tournier in [40]). In some way, it means that trapping is only due to finite size traps which come from the non uniform ellipticity of the environment. Our result is in fact more general and applies for example to any finitely generated Cayley graph on which the simple random walk is transient (cf theorem 2 and corollary 1). Compared to the results of [3], our results are non-perturbative, they apply to any choice of the weights, but our method is specific to the Dirichlet environment. The proof is remarkably short, and it is quite surprising that for RWDE things simplify so much. We don't have a clear explanation for this, it may be related to the correspondence with linearly reinforced random walks (even if it does not appear in this proof) or from the relation between RWDE and hypergeometric integrals (cf [30]). We think it highlights the special role of Dirichlet environment and we think that the techniques presented in this paper will help us to understand more on RWDE.

Our proof is based on an explicit formula valid for Dirichlet environments and on a method of perturbation of the weights. Finally, the critical exponent is obtained thanks to an L_2 version of the Max-Flow Min-Cut theorem (proposition 2). The explicit formula (corollary 2) was in fact hinted in our joint work with N. Enriquez and O. Zindy ([14, 15]) where it appeared that in the case of one-dimensional RWDE, the Green function on the half-line at 0 has an explicit law (which is in fact a consequence of a result of Chamayou and Letac on explicit solutions of renewal equations, [11]).

Let us present our results for the graph \mathbb{Z}^d . Let (e_1, \dots, e_d) be the canonical base of \mathbb{Z}^d , and set $e_j = -e_{j-d}$, for $j = d+1, \dots, 2d$. The set $\{e_1, \dots, e_{2d}\}$ is the set of unit vectors of \mathbb{Z}^d . We consider a probability law λ on

$$(1.1) \quad \{(x_1, \dots, x_{2d}) \in]0, 1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}$$

We construct a Markov chain on \mathbb{Z}^d , to nearest neighbors as follows: we choose independently at each point $x \in \mathbb{Z}^d$, some transition probabilities

$$p(x) = (p(x, x + e_i))_{i=1, \dots, 2d},$$

according to the law λ . It means that the vector $(p(x))_{x \in \mathbb{Z}^d}$ is chosen according to the product measure $\mu = \otimes_{x \in \mathbb{Z}^d} \lambda$. It defines the transition probability of a Markov chain on \mathbb{Z}^d , and we denote by $P_x^{(p)}$ the law of this Markov chain starting from x :

$$P_x^{(p)}[X_{n+1} = x + e_i | X_n = x] = p(x, x + e_i).$$

Random Dirichlet environments correspond to the case where the law λ is a Dirichlet law (cf section 2 for details). More precisely, we choose some positive weights $(\alpha_1, \dots, \alpha_{2d})$ and we take $\lambda = \lambda^{(\alpha)}$ with density

$$(1.2) \quad \frac{\Gamma(\sum_{i=1}^{2d} \alpha_i)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i-1} \right) dx_1 \cdots dx_{2d-1},$$

where Γ is the usual Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. (In the previous expression $dx_1 \cdots dx_{2d-1}$ represents the image of the Lebesgue measure on \mathbb{R}^{2d-1} by the application $(x_1, \dots, x_{2d-1}) \rightarrow (x_1, \dots, x_{2d-1}, 1 - (x_1 + \cdots + x_{2d-1}))$. Obviously, the law does not depends on the specific role of x_{2d} .) We denote by $\mathbb{P}^{(\alpha)}$ the law obtained on the environment in this way. This type of environment plays a specific role, since the annealed law $\mathbb{P}_x^{(\alpha)}[\cdot] = \mathbb{E}^{(\alpha)}[P_x^{(p)}(\cdot)]$ corresponds to a directed edge reinforced random walk with an affine reinforcement, i.e.

$$\mathbb{P}_x^{(\alpha)}[X_{n+1} = X_n + e_i | \sigma(X_k, k \leq n)] = \frac{\alpha_i + N_i(X_n, n)}{\sum_{k=1}^{2d} \alpha_k + N_k(X_n, n)},$$

where $N_k(x, n)$ is the number of crossings of the directed edge $(x, x + e_k)$ up to time n (cf [13]). When the weights are constant equal to α , the environment is isotropic: when α is large, the environment is close to the deterministic environment of the simple random walk, when α is small the environment is very disordered.

Let us now describe precisely our results for \mathbb{Z}^d , $d \geq 3$. We denote by $G(x, y)$ the green function in the environment $(p(x))_{x \in \mathbb{Z}^d}$.

$$G(x, y) = E_x^{(p)}\left[\sum_{k=0}^{\infty} \mathbb{1}_{X_k=y}\right].$$

Theorem 1. *For $d \geq 3$ and for any choice of weights $(\alpha_1, \dots, \alpha_{2d})$ we have*

$$\mathbb{E}^{(\alpha)} (G(0, 0)^s) < \infty$$

if and only if $s < \kappa$ where

$$\kappa = 2 \sum_{j=1}^{2d} \alpha_{e_j} - \max_{i=1, \dots, d} (\alpha_{e_i} + \alpha_{-e_i}).$$

In particular, the RWDE is transient for almost all environments.

Remark 1. In [40], Tournier computed the critical integrability exponent for RWDE on finite graphs. For $N > 1$, let G_N be the Green function of the RWDE killed when it exists the ball $B(0, N)$: theorem 2 of [40] implies that $G_N(0, 0)^s$ is integrable if and only if $s < \kappa$. Hence, $G_N(0, 0)$ has no higher integrable moments than the Green function $G(0, 0)$ itself. It seems to mean that there is no infinite size trap and that the trapping effect comes only from finite size traps due to the non-uniform ellipticity of the environment (cf also remark 8 after theorem 2).

Remark 2. The result for \mathbb{Z}^d is in fact a consequence of a more general result valid for directed symmetric graph, under a condition on the weights (positive divergence), cf theorem 2.

Remark 3. This theorem solves, in the special case of Dirichlet environments, problem 3 stated in Kalikow's paper [18] : Kalikow's problem is to prove that all RWRE with i.i.d. elliptic environments in dimension $d \geq 3$ are transient.

Remark 4. Let us make some comments on previous results on recurrence or transience of RWRE. In [21], Lawler proved a CLT for RWRE with symmetric environments (i.e. almost surely, the environment is symmetric at each site in each direction) and recurrence on \mathbb{Z}^2 and transience on \mathbb{Z}^d , $d \geq 3$, are proved in [45], theorem 3.3.22. Transience has been proved by Bolthausen and Zeitouni ([3]) in dimension $d \geq 3$ for small isotropic random perturbations of the simple random walk (in fact, there is no intersection between the cases treated in [3] and in this paper). In the same paper they also obtain some estimates on the exit distributions of large balls. In [37], Sznitman and Zeitouni proved an invariance principal for diffusions in isotropic environment at low disorder in dimension $d \geq 3$ (which is the continuous analogue of isotropic RWRE at low disorder investigated in [5]) and obtained transience as a by-product.

In the closely related model of (undirected) edge reinforced random walks they are very few available results. Recurrence has been proved on a graph of the type of \mathbb{Z}^2 by Merkl and Rolles ([24]). On regular trees transience or recurrence depends on the reinforcement parameter (cf [25]). On \mathbb{Z}^d , $d \geq 3$, there is, up to my knowledge, no clear conjecture.

Remark 5. The exponent κ should play an important role in the asymptotic behaviour of the RWDE. Indeed, κ is related to the tail of the expected number of visits to the point 0. In dimension 1 in the transient case, the exponent κ defined as the supremum of the $s > 0$ such that $G(0, 0)^s$ is integrable is also the exponent which governs the asymptotic behaviour of the walk (cf [20]). For RWDE, in dimension $d \geq 3$, with non-balanced weights (i.e. $\alpha_{i+d} \neq \alpha_i$ for a direction $i = 1, \dots, d$) this result leads to conjecture that the RWDE is ballistic if and only if $\kappa > 1$. It has been proved in [40], proposition 11, that the RWDE has null velocity when $\kappa \leq 1$.

Let us describe the organization of the paper. In section 2, we describe the model of RWDE on directed graphs and in section 3 we state the general results on symmetric graphs. In section 4, we prove the key explicit formulas. Section 5 is devoted to the proof of the main result on integrability and section 6 to the transience part of the result. In section 7, we prove a generalization of the Max-Flow Min-Cut theorem for flows of finite energy. In section 8 we apply the results to directed symmetric graphs and we prove theorem 2 (ii). Remark that the proof of transience on \mathbb{Z}^d , $d \geq 3$, can be understood by reading sections 2,4,5 only.

2. MARKOV CHAIN IN DIRICHLET ENVIRONMENT ON DIRECTED GRAPHS

The Dirichlet law is the multivariate generalization of the beta law. The Dirichlet law with parameters $(\alpha_1, \dots, \alpha_N)$, $\alpha_i > 0$ is the law on the simplex

$$\{(p_1, \dots, p_N) \in [0, 1]^N, \text{ such that } \sum p_i = 1\},$$

with distribution

$$\left(\frac{\Gamma(\sum_{i=1}^N \alpha_i)}{\prod_{i=1}^N \Gamma(\alpha_i)} \prod_{i=1}^N p_i^{\alpha_i - 1} \right) dx_1 \cdots dx_{N-1},$$

where as previously $dx_1 \cdots dx_{N-1}$ represents the image of the Lebesgue measure on \mathbb{R}^{N-1} by the application $(x_1, \dots, x_{N-1}) \rightarrow (x_1, \dots, x_{N-1}, 1 - (x_1 + \cdots + x_{N-1}))$ (which does not depend on the specific role of x_N). The first coordinate of a Dirichlet random variable with parameters (α, β) is by definition a beta random variable with parameter (α, β) . The following representation of Dirichlet distribution is classical (cf e.g. [42], page 180): if $(\gamma_1, \dots, \gamma_N)$ are independent gamma random variables with parameters $\alpha_1, \dots, \alpha_N$ then $(\frac{\gamma_1}{\sum \gamma_i}, \dots, \frac{\gamma_N}{\sum \gamma_i})$ is a Dirichlet random variable with parameters $(\alpha_1, \dots, \alpha_N)$. The following properties are easy consequences of this representation (cf [42] page 179-182).

(Associativity) Let I_1, \dots, I_k be a partition of $\{1, \dots, N\}$. Then the random variable $(\sum_{j \in I_i} p_j)_{i=1, \dots, k}$ is a Dirichlet random variable with parameters $(\sum_{j \in I_i} \alpha_j)_{i=1, \dots, k}$.

(Restriction) Let J be a non-empty subset of $\{1, \dots, N\}$. The random variable $(\frac{p_j}{\sum_{i \in J} p_i})_{j \in J}$ follows a Dirichlet law with parameters $(\alpha_j)_{j \in J}$ and is independent of $\sum_{i \in J} p_i$ (which follows a beta random variable with parameters $(\sum_J \alpha_i, \sum_{J^c} \alpha_i)$ by the associativity property.)

Let us first describe Random Walks in Dirichlet Environment (RWDE for short) on a general graph. A directed graph is a couple $G = (V, E)$ where V is the countable set of vertices and E is the countable set of edges. By definition, to an edge e corresponds a couple of vertices (\underline{e}, \bar{e}) : \underline{e} and \bar{e} represent respectively the tail and the head of the edge e . For convenience we allow multiple edges and loops (i.e. edges with $\underline{e} = \bar{e}$). We suppose that the graph has bounded degree i.e. that the number of edges exiting a vertex x or pointing to a vertex x is bounded. For an integer n and a vertex x we denote by $B(x, n)$ the ball with center x and radius n for the graph distance (defined independently of the orientation of the edges). We say that a subset $K \subset V$ is strongly connected if for any two vertices x and y in K there is a directed path in K from x to y .

Let us define the divergence operator on the graph G : it is the operator $\text{div} : \mathbb{R}^E \mapsto \mathbb{R}^V$ defined for a function $\theta : E \mapsto \mathbb{R}$ by

$$\text{div}(\theta)(x) = \sum_{e, \underline{e}=x} \theta(e) - \sum_{e, \bar{e}=x} \theta(e), \quad \forall x \in V.$$

The set of environments on G is defined as the set

$$\Delta = \{(p_e)_{e \in E} \in [0, 1]^E, \text{ such that } \sum_{e, \underline{e}=x} p_e = 1, \forall x \in V \text{ such that } \{e, \underline{e}=x\} \neq \emptyset\}.$$

With any environment (p_e) we associate the Markov chain on V with law $(P_x^{(p)})$, where $P_{x_0}^{(p)}$ is the law of the Markov chain starting from x_0 with transition probabilities given

by

$$P_{x_0}^{(p)}(X_{n+1} = y | X_n = x) = p_{(x,y)} = \sum_{e, \underline{e}=x, \bar{e}=y} p_e, \quad \forall x \neq y.$$

If y is a vertex such that no edge is exiting y then we put $P_{x_0}^{(p)}(X_{n+1} = y | X_n = y) = 1$, so that y is an absorbing point. We denote by $G^{(p)}(x, y)$ the Green function in the environment (p) defined for x and y in V by

$$G^{(p)}(x, y) = E_x^{(p)} \left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=y} \right).$$

(We often simply write $G(x, y)$ for $G^{(p)}(x, y)$).

Let $(\alpha_e)_{e \in E}$ be a set of positive weights on the edges. For any vertex x we set

$$\alpha_x = \sum_{e, \underline{e}=x} \alpha_e$$

the sum of the weights of the edges with origin x . The Dirichlet environment with parameters (α_e) , denoted by $\mathbb{P}^{(\alpha)}$, is the law on Δ obtained by taking at each site x the transition probabilities $(p_e)_{\underline{e}=x}$ independently accordingly to the Dirichlet law with parameters $(\alpha_e)_{\underline{e}=x}$.

When the graph is finite the distribution of the Dirichlet environment is given by

$$(2.1) \quad \frac{\prod_{x \in V} \Gamma(\alpha_x)}{\prod_{e \in E} \Gamma(\alpha_e)} \left(\prod_{e \in E} p_e^{\alpha_e - 1} \right) d\lambda_{\Delta}.$$

where $d\lambda_{\Delta}$ is the measure on Δ given by

$$d\lambda_{\Delta} = \prod_{e \in \tilde{E}} dp_e,$$

where \tilde{E} is obtained from E by removing arbitrarily, for each vertex x , one edge with origin x (easily, it is independent of this choice)

Remark 6. *We allow multiple edges for convenience (for the proof of corollary 2) but it does not play any role in terms of the process: indeed, if $G = (V, E)$ with weights $(\alpha_e)_{e \in E}$ has multiple edges we can consider the quotiented graph $\tilde{G} = (V, \tilde{E})$ with weights $(\tilde{\alpha}_e)$ obtained by replacing multiple edges by a unique edge with weight equal to the sum of the weights of the corresponding edges in G . If $(p_e)_{e \in E}$ is a Dirichlet environment on the graph G , the corresponding environment on the quotiented graph \tilde{G} obtained by summing the (p_e) of multiple edges is again a Dirichlet environment on \tilde{G} with weights $(\tilde{\alpha}_e)$. This is due to the associativity property of Dirichlet laws.*

3. STATEMENT OF THE RESULTS ON TRANSIENT SYMMETRIC GRAPHS

For us, a directed symmetric graph will be a directed graph G without multiple edges and such that if $(x, y) \in E$ then $(y, x) \in E$. To G corresponds a non-directed graph $\bar{G} = (V, \bar{E})$ where $\{x, y\} \in \bar{E}$ if and only if $(x, y) \in E$.

Theorem 2. *Let G be a connected directed symmetric graph with bounded degree. Suppose that the weights $(\alpha_e)_{e \in E}$ satisfy*

(H1) *there exists $c > 0$ and $C > c$ such that $c \leq \alpha_e \leq C$ for all e in E .*

(H2) *For all vertices x , $\text{div}(\alpha)(x) \geq 0$.*

Suppose that the simple random walk on the non-directed graph \overline{G} is transient.

(i) For any x_0 in V , there exists $\kappa_0 > 0$ such that

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty$$

for all $s < \kappa_0$. In particular, the RWDE on G with parameter (α_e) is transient for almost all environments.

(ii) Assume moreover that the following condition on G holds

(H'3) There exists a strictly increasing sequence of integers η_n such that $B(x_0, \eta_{n+1}) \setminus B(x_0, \eta_n)$ is connected in G .

Then

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty, \text{ if and only if } s < \kappa$$

where if $(x_0, x_0) \notin E$

$$\kappa = \min\{\alpha(\partial_E K), K \subset V \text{ is finite connected in } \overline{G}, x_0 \in K \text{ and } K \neq \{x_0\},\}$$

and if $(x_0, x_0) \in E$

$$\kappa = \min\{\alpha(\partial_E K), K \subset V \text{ is finite connected in } \overline{G}, x_0 \in K.\}$$

with $\partial_E K = \{e \in E, \underline{e} \in K, \bar{e} \notin K\}$ and $\alpha(\partial_E K) = \sum_{e \in \partial_E K} \alpha_e$.

Remark 7. An expression for κ_0 in terms of a L_2 -Max-Flow problem is given in the proof, cf formula 5.3.

Remark 8. The interpretation of κ is the following : the finite subsets K which appear in the infimum should be understood as finite traps and the value $\alpha(\partial_E K)$ represents the strength of the trap K : indeed, $\alpha(\partial_E K)$ governs the tail of the probability that the values of $(p_e)_{e \in \partial_E(K)}$ are all smaller than ϵ ; when the exit probabilities $(p_e)_{e \in \partial_E K}$ are small the process spends a long time in the strongly connected region K . In [40], corollary 4, Tournier computed the critical integrability exponent for RWDE on finite graphs: (ii) shows that κ is the same as the critical integrability exponent of the Green function of the RWDE killed when it exits the ball $B(x_0, N)$, for N such that $B(x_0, N)$ contains the subset K which realizes the minimum in the expression of κ . This suggests that trapping only comes from finite size traps on transient symmetric graphs that satisfy (H'3).

Remark 9. The difference between the expression of κ when there is a loop at x_0 or not comes from the fact that when $(x_0, x_0) \in E$ the RWDE can be trapped on $\{x_0\}$ but not when $(x_0, x_0) \notin E$.

Corollary 1. Let $\{e_1, \dots, e_d\}$ be a finite, symmetric (i.e. the set is stable by inversion), set of generators of a group and G be the associated Cayley graph. Let $(\alpha_1, \dots, \alpha_d)$ be positive reals. If the simple random walk on the Cayley graph is transient, then the RWDE on the Cayley graph G , with weights $(\alpha_{g,ge_i} = \alpha_i)$, is transient almost surely.

Proof. (H1) is clearly true. For all element of the group g , $\sum_{i=1}^N \alpha_{g,ge_i} = \sum_{i=1}^N \alpha_i = \sum_{i=1}^N \alpha_{ge_i^{-1},g}$. Thus (H2) is satisfied. \square

4. STABILITY BY TIME REVERSAL

4.1. **The key lemma.** Suppose now that the graph is finite and strongly connected i.e. that there is a directed path between any two vertices x and y . Let $\check{G} = (V, \check{E})$ be the graph obtained from G by reversing all the edges, i.e. \check{E} is obtained from E by reversing the head and the tail of the edges. If $e \in E$ we denote by $\check{e} \in \check{E}$ the reversed edge with tail \bar{e} and head \underline{e} . For an environment (p_e) , $(\check{p}_{\check{e}})$ denotes the environment obtained by time reversal of the Markov chain, i.e. for all e in E

$$\check{p}_{\check{e}} = \frac{\pi_{\underline{e}}}{\pi_{\bar{e}}} p_e,$$

where $(\pi_x)_{x \in V}$ is the invariant probability on V for the Markov chain $P^{(p)}$. (Since we assume that the graph is strongly connected and that the weights p_e are positive this invariant probability exists and is unique).

Lemma 1. *Let $G = (V, E)$ be a strongly connected finite directed graph. Suppose that the weights (α_e) have divergence null, i.e.*

$$\operatorname{div}(\alpha)(x) = 0, \quad \forall x \in V,$$

If $(p_e)_{e \in E}$ is a Dirichlet environment with parameters $(\alpha_e)_{e \in E}$ then the time reversed environment $(\check{p}_{\check{e}})_{\check{e} \in \check{E}}$ is a Dirichlet environment on \check{G} with parameters (α_e) .

Remark 10. *By this we mean that $(\check{p}_{\check{e}})$ is distributed according to a Dirichlet environment with parameters $(\check{\alpha}_{\check{e} \in \check{E}})$ where $\check{\alpha}_{\check{e}} = \alpha_e$ if \check{e} is the reversed edge of e . We will often identify the edges in E with their reversed edges in \check{E} .*

Proof. Two proofs are available for this lemma. The original proof given in this paper is analytic and based on a change of variable. Later, in collaboration with L. Tournier, we obtained a shorter probabilistic proof, cf [33]. The analytic proof has nevertheless an interest since it leads to an interesting distribution on the space of occupation density (cf lemma and remark below).

Let e_0 be a specified edge of the graph. Let \mathcal{H}_{e_0} be the affine space defined by

$$\mathcal{H}_{e_0} = \{(z_e)_{e \in E} \in \mathbb{R}^E, \quad z_{e_0} = 1, \quad \operatorname{div}(z) \equiv 0\}.$$

and \mathcal{H} the vector space

$$\mathcal{H} = \{(z_e)_{e \in E} \in \mathbb{R}^E, \quad \operatorname{div}(z) \equiv 0\}.$$

Let $\tilde{\Delta}_{e_0} = \mathcal{H}_{e_0} \cap (\mathbb{R}_+^*)^E$. The strategy is to make the change of variable

$$\begin{aligned} \Delta &\mapsto \tilde{\Delta}_{e_0} \\ (p_e)_{e \in E} &\rightarrow (z_e = \frac{\pi_{\underline{e}} p_e}{\pi_{\underline{e_0}} p_{e_0}})_{e \in E}. \end{aligned}$$

Hence, (z_e) is the occupation time of the graph normalized so that $z_{e_0} = 1$. It is easy to see that the previous change of variable is a C^∞ -diffeomorphism. Let T be a spanning tree of the graph G such that $e_0 \notin T$. (This is possible since the graph is strongly connected and thus e_0 belongs to at least one directed cycle of the graph.) We denote by $B = T \cup \{e_0\}$. Then $(z_e)_{e \in T^c}$ is a base of \mathcal{H} and $(z_e)_{e \in B^c}$ is a base of \mathcal{H}_{e_0} . Let $x_0 \in V$ be any vertex, and set $U = V \setminus \{x_0\}$. We need the following lemma.

Lemma 2. *Let ψ be a positive test function on Δ . Then*

$$\int_{\Delta} \Psi((p_e)) \left(\prod_{e \in E} p_e^{\alpha_e - 1} \right) d\lambda_{\Delta} = \int_{\tilde{\Delta}_{e_0}} \Psi\left(\left(\frac{z_e}{z_{\bar{e}}}\right)\right) \left(\frac{\prod_{e \in E} z_e^{\alpha_e - 1}}{\prod_{x \in V} z_x^{\alpha_x}} \right) \det(Z|_{U \times U}) \prod_{e \in B^c} dz_e.$$

where as usual $z_x = \sum_{e, \bar{e}=x} z_e$ and Z is the $V \times V$ matrix defined by

$$Z_{x,x} = z_x, \quad \forall x \in V, \quad Z_{x,y} = -z_{x,y} = - \sum_{e, \bar{e}=x, \bar{e}=y} z_e, \quad \forall x \neq y.$$

Remark 11. *This formula is essentially the same as the one which gives the correspondence between RWDE and hypergeometric integrals associated with an arrangement of hyperplanes in [30].*

Remark 12. *This lemma expresses the law of the random environment in the variables (z_e) which correspond to the occupation densities of the edges (properly renormalized). We can remark that this formula is reminiscent of the distribution discovered by Diaconis and Coppersmith ([7, 8, 19, 9]) which expresses edge-reinforced random walk as a mixture of reversible Markov chains.*

We see that lemma 1 is a direct consequence of the previous result. Indeed we see that lemma 2 applied to the reversed graph (\check{G}, \check{E}) , starting with the weights $\check{\alpha}_{\bar{e}} = \alpha_e$ gives the same integrand with α_x replaced by $\bar{\alpha}_x = \sum_{e, \bar{e}=x} \alpha_e$. The two coincide after the change of variables exactly when $\text{div}(\alpha) \equiv 0$. \square

The proof of lemma 2 needs some lengthy computation and is deferred to the appendix in section 8.

4.2. Applications. Consider now a finite graph with a cemetery point, i.e. we suppose that $G = (V, E)$, that V and E are finite and that V can be written $V = U \cup \{\delta\}$ where

- no edge is exiting δ (δ is called the cemetery point)
- for any point x in U there is a directed path from x to δ .

It means that δ is absorbing for the Markov chain on G with law $P^{(p)}$. For x and y in U we denote by $G^{(p)}(x, y)$ the Green function in the environment (p_e)

$$G^{(p)}(x, y) = E_x^{(p)}\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=y}\right).$$

Corollary 2. *(i) Suppose that $\text{div}(\alpha)(x) = 0$ for all x in U such that $x \neq x_0$ then $\text{div}(\alpha)(x_0) > 0$ and $G^{(p)}(x_0, x_0)$ is distributed as $\frac{1}{W}$ where W is a beta random variable with parameter $(\text{div}(\alpha)(x_0), \alpha_{x_0} - \text{div}(\alpha)(x_0))$.*

(ii) Suppose that $\text{div}(\alpha)(x)$ is non-negative for all x in U . Let x_0 be a vertex in U such that $\text{div}(\alpha)(x_0) \geq \gamma > 0$, then $G^{(p)}(x_0, x_0)$ is stochastically dominated by $\frac{1}{W}$ where W is a beta random variable with parameter $(\gamma, \alpha_{x_0} - \gamma)$.

Remark 13. *The explicit formula in (i) was in fact suggested by our joint work with N. Enriquez and O. Zindy ([14]) and also by the correspondence established in [30]. It appeared in [14] that in the case of sub-ballistic one-dimensional RWRE limit theorems are fully explicit in the case of Dirichlet environments. This is a consequence of the fact that the Green function at 0 of the RWDE on the half line \mathbb{Z}_+ is equal in law to*

$1/W$ where W is a beta random variable with appropriate weights. In the case of one-dimensional RWDE, this explicit formula is a consequence of a result of Chamayou and Letac ([11]) on explicit solutions for renewal equations. From the point of view of the correspondence with hypergeometric integrals described in [30], the condition of null divergence corresponds to a condition of resonance of the weights of the arrangement. Resonant arrangements are not well understood yet.

Proof. (of corollary 2) (i) We can freely suppose that any y in U can be reached following a directed path from x_0 (indeed, the part of the graph which cannot be reached from x_0 does not play any role in $G(x_0, x_0)$). Suppose that $\text{div}(\alpha)(x_0) = \gamma$, $\gamma > 0$. It means that $\text{div}(\alpha)(\delta) = -\gamma$. Consider now the graph $\tilde{G} = (U, E)$ obtained by identification of the vertices δ and x_0 . The edges of \tilde{G} are just obtained by identification of δ and x_0 in the edges of G (with possibly creation of multiples edges and loops in \tilde{G}), and we denote by \tilde{x}_0 the point corresponding to the indentification of δ and x_0 . The graph \tilde{G} is clearly strongly connected and if we keep the same weights on the edges we have $\text{div}^{\tilde{G}}(\alpha) \equiv 0$. Consider the invariant probability $(\pi_x)_{x \in U}$ for the RWDE on \tilde{G} . It gives the occupation time on the edges $z_e = \pi_{\underline{e}} p_e$, and the time reversal transition probabilities $\check{p}_e = \frac{z_e}{\pi_{\bar{e}}}$. If $G^{(p)}(x_0, x_0)$ is the Green function on the graph G (which is the graph with the cemetery point)

$$G^{(p)}(x_0, x_0) = \frac{1}{\sum_{\bar{e}=\delta} \check{p}_{\bar{e}}}.$$

(where $\bar{e} = \delta$ is relative to the edges in the initial graph G). Indeed, we have

$$\frac{1}{G^{(p)}(x_0, x_0)} = \sum_{\sigma \in \Sigma} p_{\sigma},$$

where Σ is the set of direct paths σ in G from x_0 to δ i.e. $\sigma = (e_0, \dots, e_{n-1})$ with $\bar{e}_{i-1} = \underline{e}_i$ for $i = 1, \dots, n-1$ and $\underline{e}_0 = x_0$, $\bar{e}_{n-1} = \delta$ and $\underline{e}_i \notin \{x_0, \delta\}$ for $i = 1, \dots, n-1$. Since in \tilde{G} $\sigma \in \Sigma$ is a cycle from \tilde{x}_0 to \tilde{x}_0 , we have

$$\frac{1}{G^{(p)}(x_0, x_0)} = \sum_{\sigma \in \Sigma} \check{p}_{\check{\sigma}}.$$

where $\check{\sigma}$ is the returned path $(\check{e}_{n-1}, \dots, \check{e}_0)$ if $\sigma = (e_0, \dots, e_{n-1})$. But since there is no edge exiting δ in the graph G we have

$$\sum_{\sigma \in \Sigma} \check{p}_{\check{\sigma}} = \sum_{\bar{e}=\delta} \check{p}_{\bar{e}}.$$

By the previous lemma we know that $(\check{p}_{\bar{e}})_{\bar{e}=\delta}$ or $_{\bar{e}=x_0}$ has the law of a Dirichlet random variable with parameters $(\alpha_e)_{\bar{e}=\delta}$ or $_{\bar{e}=x_0}$. (Indeed, the vertices δ and x_0 are identified in the quotiented graph \tilde{G} .) The conclusion comes from the associativity property of Dirichlet random variables (cf section 2).

(ii) Consider the graph $\tilde{G} = (\tilde{U} \cup \{\tilde{\delta}\}, \tilde{E})$ obtained as follows. The set of vertices is defined by $\tilde{U} = U \cup \{\delta\}$ and $\tilde{\delta}$ is the new cemetery point. The set of edges \tilde{E} is obtained by adding to the edges E of G some new edges with origin δ : for $x \neq x_0$ in U such that $\text{div}(\alpha)(x) > 0$ we add the edge (δ, x) with weight $\text{div}(\alpha)(x)$; for x_0 , if $\text{div}(\alpha)(x_0) > \gamma$ we put a new edge (δ, x_0) with weight $\text{div}(\alpha)(x_0) - \gamma$; we also add the edge $(\delta, \tilde{\delta})$ with weight γ . Clearly, the new graph \tilde{G} with the previous choice of weights (denoted by $\tilde{\alpha}$) satisfies the condition of (i) since $\text{div}(\tilde{\alpha})_{|\tilde{U}} = \gamma \delta_{x_0}$. Moreover

if $(p_e)_{e \in E}$ is a Dirichlet environment on G with weights (α_e) it can be extended to a Dirichlet environment $(\tilde{p}_e)_{e \in \tilde{E}}$ on \tilde{G} just by choosing independently the transition probabilities on the edges exiting δ according to a Dirichlet random variable with the appropriate weights. Remark that

$$G^{(p)}(x_0, x_0) \leq G^{(\tilde{p})}(x_0, x_0).$$

Indeed, the Markov chains on G and \tilde{G} behave the same as long as they are on U but the Markov chain on G is stuck on δ although the Markov chain on \tilde{G} can come back to U from δ . The conclusion is a consequence of (i) since (i) implies that $G^{(\tilde{p})}(x_0, x_0)$ has the law of $1/W$ with W a beta random variable with weights $(\gamma, \alpha_{x_0} - \gamma)$. \square

5. INTEGRABILITY CONDITION

Suppose now that $G = (V, E)$ is a countable connected directed graph with bounded degree. Suppose for simplicity that there is at least one edge exiting each vertex.

We recall that a flow from a vertex x_0 to infinity (cf [22]) is a positive function θ on the edges such that

$$\operatorname{div}(\theta)(x) = 0, \quad \forall x \neq x_0,$$

and

$$\operatorname{div}(\theta)(x_0) \geq 0.$$

The strength of the flow is the value

$$\operatorname{strength}(\theta) = \operatorname{div}(\theta)(x_0).$$

A unit flow is a flow of strength 1.

We say that the flow θ has finite energy if it is square integrable i.e. if

$$\sum_{e \in E} \theta_e^2 < \infty.$$

Theorem 3. *Let $(\alpha_e)_{e \in E}$ be a family of positive weights on the edges which satisfy*

(H1) *there exists $c > 0$ and $C \geq c$ such that $c \leq \alpha_e \leq C$ for all e in E .*

(H2) *For all vertices x , $\operatorname{div}(\alpha)(x) \geq 0$.*

Suppose that θ is a unit flow with finite energy from x_0 to infinity then

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty$$

as soon as

$$s < \inf_{e \in E} \frac{\alpha_e}{\theta_e}$$

Proof. Let γ be a positive real. We define the weights

$$\alpha^\gamma = \alpha + \gamma\theta.$$

We clearly have

$$\operatorname{div}(\alpha^\gamma) \geq \gamma\delta_{x_0}.$$

For a positive integer N , let U_N be the ball with center x_0 and radius N in G . We define the graph $G_N = (U_N \cup \{\delta\}, E_N)$ as follows. We contract all the vertices of $(U_N)^c$ to the cemetery point δ . The edges E_N are obtained from E as follows: in E we delete all the edges exiting a point of U_N^c and we define E_N from the remaining edges by contraction of U_N^c to the single vertex δ . By the bounded degree property

we see that G_N is a finite graph. We keep the same weights on the edges and we see that on the graph G_N we have for x in U_N

$$\operatorname{div}^N(\alpha^\gamma)(x) = \sum_{e \in E, e=x} \alpha_e^\gamma - \sum_{e \in E, \bar{e}=x, e \in U_N} \alpha_e^\gamma \geq \operatorname{div}(\alpha^\gamma)(x)$$

(With div^N the divergence operator on the graph G_N). Hence $\operatorname{div}^N(\alpha^\gamma) \geq 0$, and

$$\operatorname{div}^N(\alpha^\gamma)(x_0) \geq \gamma.$$

Denote by $G_N^{(p)}(x_0, x_0)$ the Green function of the Markov chain killed when it exits U_N . From corollary 2 (ii) we see that under $\mathbb{P}^{(\alpha^\gamma)}$ the Green function $G_N^{(p)}(x_0, x_0)$ is stochastically dominated by $\frac{1}{W}$ where W is a beta random variable with parameters $(\gamma, \alpha_{x_0} + \gamma\theta_{x_0} - \gamma)$ (for N large enough).

Considering formula (2.1) we see that on the graph G_N , the measure $\mathbb{P}^{(\alpha)}$ is absolutely continuous with respect to the measure $\mathbb{P}^{(\alpha^\gamma)}$ and that

$$\frac{d\mathbb{P}^{(\alpha)}}{d\mathbb{P}^{(\alpha^\gamma)}} = \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \prod_{e \in E_N} p_e^{-\gamma\theta_e}.$$

Consider now $s > 0$. We have (we write simply $G_N(x_0, x_0)$ for $G_N^{(p)}(x_0, x_0)$ the Green function in environment (p_e))

$$\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s) = \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \mathbb{E}^{(\alpha^\gamma)} \left(G_N(x_0, x_0)^s \prod_{e \in E_N} p_e^{-\gamma\theta_e} \right).$$

Using Hölder's inequality for $q > 1$ and $p = \frac{q}{q-1}$ we get that $\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s)$ is lower than

$$\frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \left(\mathbb{E}^{(\alpha^\gamma)}(G_N(x_0, x_0)^{ps}) \right)^{1/p} \left(\mathbb{E}^{(\alpha^\gamma)} \left(\prod_{e \in E_N} p_e^{-q\gamma\theta_e} \right) \right)^{1/q}.$$

Remark that the second expectation is finite if and only if $q\gamma\theta_e < \alpha_e^\gamma$ for all e in E_N , or equivalently

$$q - 1 < \frac{\alpha_e}{\gamma\theta_e}$$

for all e in E_N . We now choose q such that

$$(5.1) \quad q - 1 < \inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}$$

so that the previous condition is fulfilled for all e in E . In terms of p it is equivalent to

$$(5.2) \quad p > \frac{\inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e} + 1}{\inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}}.$$

Now we compute the second expectation. We have

$$\begin{aligned} & \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \left(\mathbb{E}^{(\alpha^\gamma)} \left(\prod_{e \in E_N} p_e^{-q\gamma\theta_e} \right) \right)^{1/q} \\ &= \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)^{1-\frac{1}{q}} \prod_{e \in E_N} \Gamma(\alpha_e - (q-1)\gamma\theta_e)^{\frac{1}{q}}}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)^{1-\frac{1}{q}} \prod_{x \in U_N} \Gamma(\alpha_x - (q-1)\gamma\theta_x)^{\frac{1}{q}}} \end{aligned}$$

Considering the function

$$\nu(\alpha, u) = \frac{1}{q} \ln \Gamma(\alpha - (q-1)u) + \left(1 - \frac{1}{q}\right) \ln \Gamma(\alpha + u) - \ln \Gamma(\alpha),$$

we see that the previous expression is equal to

$$\exp \left(\sum_{e \in E_N} \nu(\alpha_e, \gamma\theta_e) - \sum_{x \in U_N} \nu(\alpha_x, \gamma\theta_x) \right).$$

Now, $\nu(\alpha, 0) = 0$ and one can easily compute and see that $\frac{\partial}{\partial u} \nu(\alpha, 0) = 0$. The function $\nu(\alpha, u)$ is C^∞ on the domain $D = \{\alpha > 0\} \cap \{u < \alpha/(q-1)\}$. By conditions (H1) of theorem 3 and (5.1) we know that $(\alpha_e, \gamma\theta_e)_{e \in E}$ and $(\alpha_x, \gamma\theta_x)_{x \in V}$ are in a compact subset of D . Hence, we can find a constant $C > 0$ such that for all $N > 0$

$$\sum_{e \in E_N} \nu(\alpha_e, \gamma\theta_e) - \sum_{x \in U_N} \nu(\alpha_x, \gamma\theta_x) \leq C \left(\sum_{e \in E_N} (\gamma\theta_e)^2 + \sum_{x \in U_N} (\gamma\theta_x)^2 \right).$$

Since (θ_e) is square integrable and the graph G has bounded degree, (θ_x) is square integrable. Hence we have a constant $C' > 0$, such that for all N

$$\mathbb{E}^{(\alpha)} (G_N(x_0, x_0)^s) \leq C' (\mathbb{E}^{(\alpha\gamma)} (G_N(x_0, x_0)^{ps}))^{1/p}$$

Using corollary 2 we have

$$\mathbb{E}^{(\alpha)} (G_N(x_0, x_0)^s) \leq C' (\mathbb{E} (W^{-ps}))^{1/p}$$

where W is a beta random variable with parameter $(\gamma, \alpha_{x_0} + \gamma\theta_{x_0} - \gamma)$. Since W^{-ps} is integrable for $ps < \gamma$ we see that

$$\mathbb{E}^{(\alpha)} (G(x_0, x_0)^s) = \sup_N \mathbb{E}^{(\alpha)} (G_N(x_0, x_0)^s) < \infty$$

for all s such that $sp < \gamma$. This is true for any choice of p which satisfies (5.2), so $G(x_0, x_0)^s$ is integrable as soon as

$$s < \frac{\inf_{e \in E} \frac{\alpha_e}{\theta_e}}{1 + \inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}}.$$

Letting γ tend to infinity we get the result. \square

Maximizing on the L_2 unit-flows, we see that under the conditions of theorem 3, $\mathbb{E}^{(\alpha)} (G(x_0, x_0)^s)$ is finite for all $s < \kappa_0$ where

$$\kappa_0 = \sup_{\theta \text{ } L_2\text{-unit flow}} \inf_E \frac{\alpha_e}{\theta_e}.$$

Remark that if θ is a unit flow with finite energy then $\inf_E \frac{\alpha_e}{\theta_e} > 0$ if condition (H1) is satisfied. Hence, under conditions (H1) and (H2), the existence of a unit flow with finite energy ensures that $G(x_0, x_0)$ has some finite moments and hence the transience of the RWDE. We see that κ_0 can be rewritten as a Max-Flow problem (cf section 7) with an extra L_2 condition.

(5.3)

$$\kappa_0 = \sup \{ \text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ to } \infty \text{ of finite energy such that } \theta \leq \alpha \}$$

6. PROOF OF TRANSIENCE ON SYMMETRIC TRANSIENT GRAPHS

This section is devoted to the proof of theorem 2 (i). Let G be a symmetric graph and \overline{G} the associated undirected graph. Let us recall the definition of a flow on an undirected graph. We choose an arbitrary orientation of the edges of \overline{G} . A flow from x_0 to infinity is a (non-necessarily positive) function $\overline{\theta}$ on the edges such that for the orientation chosen

$$\operatorname{div}(\overline{\theta})(x) = 0, \quad \forall x \neq x_0.$$

The flow $\overline{\theta}$ is a unit flow if moreover $\operatorname{div}(\overline{\theta})(x_0) = 1$. To any flow $\overline{\theta}$ on the undirected graph \overline{G} we can associate a flow θ on the directed graph as follows: for two opposite edges of the directed graph, θ is null on one of them and on the other one it is equal to the absolute value of $\overline{\theta}$ on the corresponding undirected edge. The choice of the edge with positive flow is of course made according to the sign of the flow $\overline{\theta}$ and the orientation of the edges (cf [22], section 2.6). By construction the L_2 norm of θ and $\overline{\theta}$ are the same. Then the result comes from a classical result on electrical networks (cf [22] proposition 2.10, or [23]) which says that the undirected graph \overline{G} is transient if and only if there exists a unit flow with finite energy from a point x_0 to infinity. Hence, it implies that $\kappa_0 > 0$.

7. MAX-FLOW OF FINITE ENERGY

Let us recall some notions about Max-Flow Min-Cut theorem (cf [22], section 2.6, [2]). Let G be a countable directed graph and x_0 a vertex such that there is an infinite directed simple path starting at x_0 . Let $(c(e))_{e \in E}$ be a family of non-negative reals, called the capacities.

Definition 1. A flow θ from x_0 to ∞ is compatible with the capacities $(c(e))_{e \in E}$ if

$$\theta(e) \leq c(e), \quad \forall e \in E.$$

A cutset is a subset $S \subset E$ such that any infinite directed simple path from x_0 contains at least one edge of S .

The well-known Max-Flow Min-Cut theorem says that the maximum flow equals the minimal cutset sum (cf [16]). We give here a version for countable graphs ([22], theorem 2.19, cf also [2]).

Proposition 1. The maximum compatible flow equals the infimum of the cutset sum, i.e.

$$(7.1) \quad \max\{\operatorname{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ to } \infty \text{ compatible with } (c(e))\} \\ = \inf\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}.$$

where

$$c(S) = \sum_{e \in S} c(e).$$

Theorem 3 tells us that κ_0 , defined as the max strength of flows of finite energy (cf (5.3)), gives a lower bound on the critical integrability exponent of the Green function. It is natural to ask whether κ_0 is also equal to the min-cut. It is not true in general (cf the following remark) but it is true under fairly general conditions.

Proposition 2. *Let $(c(e))_{e \in E}$ be a family of capacities. Suppose that*

$$\inf_{e \in E} c(e) > 0,$$

and that the following holds

(H3) *There exists a strictly increasing sequence of integers η_n such that $B(x_0, \eta_{n+1}) \setminus B(x_0, \eta_n)$ is strongly connected in G .*

If there exists a unit flow of finite energy on G from x_0 to ∞ then the infimum in (7.1) is reached and

$$\begin{aligned} & \max\{\text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ of finite energy compatible with } (c(e))\} \\ &= \min\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}. \end{aligned}$$

Remark 14. *: If condition (H3) fails the equality may be wrong. The following counter-example is due to R. Aharoni, [1]: let G be a binary tree glued by its root to a copy of \mathbb{Z}_+ . Take capacities constant equal to 1. Then any flow of finite energy necessarily vanishes on the copy of \mathbb{Z}_+ and the equality cannot hold.*

Proof. Let θ be a unit flow from x_0 to ∞ of finite energy. Set

$$c(G) = \inf\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}.$$

The strategy is to modify the capacities c using θ . For a positive integer r , $B_E(x_0, r)$ denotes the set of edges

$$B_E(x_0, r) = \{e \in E, \underline{e} \in B(x_0, r), \bar{e} \in B(x_0, r)\}.$$

and

$$\underline{B}_E(x_0, r) = \{e \in E, \underline{e} \in B(x_0, r)\}.$$

Let N_0 be such that

$$(7.2) \quad \sup_{e \notin B_E(x_0, \eta_{N_0})} \theta(e) \leq \frac{\inf_E c}{2c(G)}.$$

Let N_1 be such that

$$(N_1 - N_0) \inf_E c > 2c(G).$$

Consider now the capacities c' defined by

$$c'(e) = \begin{cases} c(e), & \text{if } e \in \underline{B}_E(x_0, \eta_{N_1}) \\ 2c(G)\theta(e), & \text{if } e \notin \underline{B}_E(x_0, \eta_{N_1}). \end{cases}$$

By condition (7.2), we clearly have

$$c'(e) \leq c(e), \forall e \in E,$$

and

$$\sum_E (c'(e))^2 < \infty.$$

We now want to check that the min cutset sum is the same for c and c' . Let S be a minimal cutset for inclusion. If $S \subset \underline{B}_E(x_0, \eta_{N_1})$ then $c'(S) = c(S) \geq c(G)$. If $S \subset B_E(x_0, \eta_{N_0})^c$ then by (7.2) $c'(S) \geq 2c(G)\theta(S) \geq 2c(G)$ since θ is a unit flow. Otherwise, it means that S has one edge e_0 in $B_E(x_0, \eta_{N_0})$ and one edge e_1 in $\underline{B}_E(x_0, \eta_{N_1})^c$. Let K be the set of vertices that can be reached by a directed path in $E \setminus S$ from x_0 . Since S is minimal for inclusion, it means that there is a simple path

from \bar{e}_0 to ∞ in K^c . Hence, there is a directed path in K^c from $B(x_0, N_0)$ to ∞ , and there is a sequence $y_1, \dots, y_{N_1-N_0}$ in K^c such that $y_k \in U_k$ where

$$U_k = B(x_0, \eta_{k+1}) \setminus B(x_0, \eta_k).$$

Similarly, there is a directed path in K from x_0 to \underline{e}_1 . It implies that there is a sequence $z_1, \dots, z_{N_1-N_0}$ in K such that $z_k \in U_k$. By assumption (H3) there is a directed path in U_k from z_k to y_k . This directed path necessarily contains an edge of S . Hence, $|S| \geq N_1 - N_0$ so that $c'(S) > 2c(G)$. Then, we apply the Max-Flow Min-Cut theorem to the capacities c' . It gives a flow of finite energy (since c' is squared integrable) compatible with c' and consequently with c , and with strength $c(G)$. Moreover the proof implies that $c(G)$ is reached for a cut-set $S \subset \underline{B}_E(x_0, \eta_{N_1})$, and the infimum is a minimum. \square

8. PROOF OF THEOREM 2 (II)

Let κ_0 be defined as in section 5 (5.3), and κ as in theorem 2 (ii). Clearly, if the non-directed graph \bar{G} satisfies (H'3) then the directed graph G satisfies (H3). Let us first prove that $G^s(x_0, x_0)$ is integrable if $s < \kappa$. Under condition (H3) we have $\kappa_0 \leq \kappa$. Indeed, if K is a finite connected subset containing x_0 then $\partial_E(K)$ is a cut-set separating x_0 from infinity and $\alpha(\partial_E(K)) \geq \kappa_0$ by proposition 2.

Case 1. If (x_0, x_0) is in E then we have $\kappa_0 = \kappa$. Indeed, let S_0 be a cut-set which realizes the minimum in proposition 2, so that $\kappa_0 = \alpha(S_0)$. Let $K_0 \subset V$ be the set of vertices that can be reached from x_0 by a directed path using edges of $E \setminus S_0$. By minimality we have $S_0 = \partial_E(K_0)$ and $\kappa_0 = \alpha(\partial_E K_0)$. The subset K_0 is finite and connected and it contains x_0 ; it implies that $\kappa = \kappa_0$. It implies by theorem 3 that $\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty$ for $s < \kappa$.

Case 2. Suppose that $(x_0, x_0) \notin E$. For any environment (p) , we have

$$\sum_{\underline{e}=x_0} p_e = 1,$$

hence $G(x_0, x_0)^s$ is integrable if and only if $p_e^s G(x_0, x_0)^s$ is integrable for any e such that $\underline{e} = x_0$. Let e_0 be an edge exiting x_0 and $\alpha^{(e_0)}$ be the weight obtained from α by adding s to α_{e_0} . For any cutset S containing e_0 , $\alpha^{(e_0)}(S) > s$. Hence, by theorem 3 and proposition 2, $p_{e_0}^s G(x_0, x_0)^s$ is integrable if s is smaller than the minimal cutset sum (for (α)) among cutsets which do not contain e_0 . It means that $G(x_0, x_0)^s$ is integrable if

$$(8.1) \quad s < \min\{\alpha(S), S \text{ cutset}, S \not\ni \{e\}_{\underline{e}=x_0}\}.$$

Let S be a cutset which does not contain $\{e\}_{\underline{e}=x_0}$. Let K be the set of vertices that can be reached from x_0 by a directed path using edges of $E \setminus S$. Then, $\partial_E K$ is a cutset contained in S and K is connected in \bar{G} . Moreover $K \neq \{x_0\}$ thanks to the condition that $(x_0, x_0) \notin E$ (indeed, if $K = \{x_0\}$ then $\partial_E(K) = \{e\}_{\underline{e}=x_0}$ if $(x_0, x_0) \notin E$). This implies that $G^s(x_0, x_0)$ is integrable if $s < \kappa$.

If $s \geq \kappa$, then by taking a large enough box $B(x_0, N)$, we know that the cutset which achieves the minimum in the definition of κ is included in $B(x_0, N)$ (the infimum is reached, cf proposition 2). From [40], corollary 4, it implies that $G(x_0, x_0)^s$ is not integrable (the statement is written for $(x_0, x_0) \notin E$ but it clearly extends to the case where there is a loop at x_0 with the κ defined in theorem 2).

8.1. **The case of \mathbb{Z}^d .** Of course theorem 2 (i) and (ii) apply to the model of RWDE on \mathbb{Z}^d described in the introduction, for $d \geq 3$. It is easy to see that on \mathbb{Z}^d , the critical exponent κ is obtained for $K = \{0, e_{i_0}\}$ for some i_0 in $\{1, \dots, d\}$. Hence we have

$$\kappa = \min_{i_0=1, \dots, d} \left(2 \left(\sum_{i=1, \dots, d, i \neq i_0} \alpha_{e_i} + \alpha_{-e_i} \right) + \alpha_{e_{i_0}} + \alpha_{-e_{i_0}} \right).$$

This proves theorem 1.

9. APPENDIX

Proof. (of lemma 2) Let us first remark that $\det(Z|_{U \times U})$ does not depend on the choice of x_0 : indeed, the lines and columns of Z have sum 0. By addition of lines and columns we can switch from x_0 to y_0 . Hence, we can freely choose $x_0 = \underline{e}_0$. Remark now that \mathcal{H} and \mathcal{H}_{e_0} are the affine subspaces of \mathbb{R}^E defined by

$$\mathcal{H} = \cap_{x \in U} \{\operatorname{div}(z)(x) = 0\}, \quad \mathcal{H}_{e_0} = \{z_{e_0} = 1\} \cap_{x \in U} \{\operatorname{div}(z)(x) = 0\}.$$

For simplification, we write

$$h_x(z) = \operatorname{div}(z)(x),$$

for all x in U . It is not easy to compute directly the Jacobian of the change of variables, the strategy is to use Fourier transform to make the change of variables on free variables. We first prove the following lemma.

Lemma 3. *For any function $\phi : \mathbb{R}^E \rightarrow \mathbb{C}$, C^∞ , with compact support in $(\mathbb{R}_+^*)^E$ we have that*

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e = \int_{\mathbb{R}^U \times \mathbb{R}} \int_{\mathbb{R}^E} \phi(z) \exp \left(2i\pi \left(u_0(z_{e_0} - 1) + \sum_{x \in U} u_x h_x(z) \right) \right) \left(\prod_{e \in E} dz_e \right) \left(du_0 \prod_{x \in U} du_x \right).$$

Proof. of lemma 3. We will use several times the following simple fact (which is a consequence of the inverse Fourier transform). Let $g : \mathbb{R}^{N+k} \rightarrow \mathbb{R}$ be a C^∞ function with compact support, then

$$\begin{aligned} & \int_{\mathbb{R}^N} g(x_1, \dots, x_N, 0, \dots, 0) dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{N+k}} \exp \left(2i\pi \sum_{j=1}^k u_j x_{N+j} \right) g(x_1, \dots, x_{N+k}) (dx_1 \cdots dx_{N+k}) (du_1 \cdots du_k) \end{aligned}$$

N.B.: These integrals are well-defined as integrals in the Schwartz space. Let us compute the Jacobian of the linear change of variables

$$(9.1) \quad \begin{aligned} \mathbb{R}^E &\mapsto \mathbb{R}^U \times \mathbb{R}^{T^c} \\ ((z_e)_{e \in E}) &\rightarrow ((h_x(z))_{x \in U}, (z_e)_{e \in T^c}). \end{aligned}$$

Denoting $T = \{e_{j_1}, \dots, e_{j_{|U|}}\}$ and $T^c = \{e_{i_1}, \dots, e_{i_{|E|-|U|}}\}$, $U = \{x_1, \dots, x_{|U|}\}$, the Jacobian of the change of variable is:

$$|J| = \left| \det \begin{pmatrix} \frac{\partial h_{x_1}}{\partial z_{e_{j_1}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_1}}} & & \\ \vdots & & \vdots & & 0 \\ \frac{\partial h_{x_1}}{\partial z_{e_{j_{|U|}}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_{|U|}}}} & & \\ \frac{\partial h_{x_1}}{\partial z_{e_{i_1}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{i_1}}} & & \\ \vdots & & \vdots & & \text{Id}_{T^c \times T^c} \\ \frac{\partial h_{x_1}}{\partial z_{e_{i_{|E|-|U|}}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{i_{|E|-|U|}}}} & & \end{pmatrix} \right|.$$

So, we get

$$|J| = \left| \det \begin{pmatrix} \frac{\partial h_{x_1}}{\partial z_{e_{j_1}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_1}}} \\ \vdots & & \vdots \\ \frac{\partial h_{x_1}}{\partial z_{e_{j_{|U|}}}} & \dots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_{|U|}}}} \end{pmatrix} \right|.$$

The previous matrix is the incidence matrix on U of the spanning tree T , indeed we have if $x \in U$

$$\frac{\partial h_x}{\partial z_e} = \begin{cases} +1 & \text{if } \underline{e} = x, \\ -1 & \text{if } \bar{e} = x, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that this determinant is equal to ± 1 : indeed, it is a special case of the Kirchhoff's matrix-tree theorem (see [17]) where the graph is only composed of the edges of the spanning tree T ; in this case there is a unique spanning tree, T itself, and the incident matrix of the graph is the matrix below). Let $\tilde{\phi} : \mathbb{R}^U \times \mathbb{R}^{T^c} \mapsto \mathbb{C}$ denote the function defined by

$$\phi((z_e)_{e \in E}) = \tilde{\phi}((h_x(z))_{x \in U}, (z_e)_{e \in T^c}).$$

By formula (9.1) and since $\tilde{\phi}(0, (z_e)_{e \in T^c}) = \phi((z_e)_{e \in E})$ on \mathcal{H} , we get

$$\begin{aligned} \int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}}((z_e)) \prod_{e \in B^c} dz_e &= \int_{\mathbb{R}} \int_{\mathcal{H}} \phi|_{\mathcal{H}}((z_e)) e^{2i\pi u_0(z_{e_0}-1)} \left(\prod_{e \in T^c} dz_e \right) du_0 \\ &= \int_{\mathbb{R}} \int_{\mathcal{H}} \tilde{\phi}(0, (z_e)_{e \in T^c}) e^{2i\pi u_0(z_{e_0}-1)} \left(\prod_{e \in T^c} dz_e \right) du_0 \end{aligned}$$

Using again formula 9.1 we see that the previous integral is equal to

$$\begin{aligned} &= \int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R}^U \times \mathbb{R}^{T^c}} \exp(2i\pi(u_0(z_{e_0}-1) + \sum_{x \in U} u_x h_x)) \tilde{\phi}((h_x)_{x \in U}, (z_e)_{e \in T^c}) \\ &\quad \left(\prod_{x \in U} dh_x \prod_{e \in T^c} dz_e \right) (du_0 \prod_{x \in U} du_x). \end{aligned}$$

Then, the change of variables (9.1) gives lemma 3. \square

We make the following change of variables:

$$\begin{aligned} (\mathbb{R}_+^*)^E &\rightarrow (\mathbb{R}_+^*)^V \times \Delta \\ (z_e) &\mapsto ((v_x)_{x \in V}, (p_e)_{e \in E}), \end{aligned}$$

given by

$$v_x = \sum_{\underline{e}=x} z_e, \quad p_e = \frac{z_e}{v_{\underline{e}}}.$$

(More precisely, it is the change of variables onto the set $(\mathbb{R}_+^*)^V \times]0, 1]^{\tilde{E}}$ where \tilde{E} is defined in (2.1). It means that we choose $(p_e)_{e \in \tilde{E}}$ as the coordinate system on Δ .) We have $z_e = v_{\underline{e}} p_e$ and the Jacobian is given by

$$\prod_{x \in V} v_x^{n_x - 1},$$

where $n_x = |\{e, \underline{e} = x\}|$. Implementing this change of variables in lemma 3 gives that

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e$$

is equal to

$$\int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R}^V \times \Delta} \left(\prod_{x \in V} v_x^{n_x - 1} \right) \phi((v_{\underline{e}} p_e)) \exp(iu_0(v_{\underline{e_0}} p_{e_0} - 1) + i \sum_{x \in U} u_x h_x) \\ ((\prod_{x \in V} dv_x) d\lambda_{\Delta}) (du_0 \prod_{x \in U} du_x),$$

where $d\lambda_{\Delta}$ is the measure on Δ defined in (2.1) and with

$$h_x((v_x), (p_e)) = v_x - \sum_{e, \bar{e}=x} v_{\underline{e}} p_e.$$

The strategy is now to apply formula (3) to get ride of variables $(u_0, (u_x)_{x \in U})$. For this we need to change to variables $(v_{e_0} p_{e_0}, (h_x)_{x \in U})$. We make the following change of variables.

$$\mathbb{R}^V \times \Delta \mapsto \mathbb{R} \times \mathbb{R}^U \times \Delta \\ ((v_x)_{x \in V}, (p_e)) \rightarrow (k_0(v, p), (h_x(v, p))_{x \in U}, (p_e)),$$

with $k_0 = v_{\underline{e_0}} p_{e_0}$. This change of variables can be inverted by

$$(v_x)_{x \in V} = (M^{(p)})^{-1}(k_0, (h_x)),$$

where

$$M_{x,y}^{(p)} = (I - P)_{x,y}, \quad \forall x \in U, \forall y \in V,$$

and $M_{0,y} = p_{e_0} \mathbb{1}_{y=\underline{e_0}}$ (where P is the transition matrix in the environment (p)). Since we have chosen $x_0 = \underline{e_0}$, the Jacobian of the change of variables is

$$|J| = p_{e_0} \det(I - P)_{U \times U}.$$

By this change of variables, the integral becomes

$$\int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R} \times \mathbb{R}^U \times \Delta} \phi((v_{\underline{e}} p_e)) \frac{(\prod_{x \in V} v_x^{n_x - 1})}{p_{e_0} \det(I - P)_{U \times U}} \exp(2i\pi (u_0(k_0 - 1) + \sum_{x \in U} u_x h_x)) \\ ((dk_0 \prod_{x \in U} dh_x) d\lambda_{\Delta}) (du_0 \prod_{x \in U} du_x),$$

Remark that if the following equalities are satisfied

$$h_x((v_x), (p_e)) = 0, \quad \forall x \in U, \quad v_{\underline{e_0}} p_{e_0} = 1$$

it implies that $v_x = \frac{\pi_x}{\pi_{e_0} p_{e_0}}$ for all x . Integrating over the variables $(u_0, (u_x))$ by formula (9.1) we get

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e = \int_{\Delta} \prod_{x \in V} \left(\frac{\pi_x}{\pi_{e_0} p_{e_0}} \right)^{n_x - 1} \frac{\phi((z_e))}{p_{e_0} \det(I - P)_{U \times U}} d\lambda_{\Omega},$$

where $z_e = \frac{\pi_e p_e}{\pi_{e_0} p_{e_0}}$. Then we just have to replace ϕ by

$$\phi((z_e)) = \psi \left(\left(\frac{z_e}{z_e} \right) \right) \frac{\prod_{e \in E} \mathbb{1}_{z_e > 0} z_e^{\alpha_e - 1}}{\prod_{x \in V} z_x^{\alpha_x}} \det(Z|_{U \times U})$$

This can be done by monotone convergence. \square

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