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RANDOM WALKS IN RANDOM DIRICHLET ENVIRONMENT ARE TRANSIENT IN DIMENSION $d \geq 3$

CHRISTOPHE SABOT

Abstract: We consider random walks in random Dirichlet environment (RWDE) which is a special type of random walks in random environment where the exit probabilities at each site are i.i.d. Dirichlet random variables. On \mathbb{Z}^d , RWDE are parameterized by a $2d$ -uplet of positive reals. We prove that for all values of the parameters, RWDE are transient in dimension $d \geq 3$. We also prove that the Green function has some finite moments and we characterize the finite moments. Our result is more general and applies for example to finitely generated transient Cayley graphs. In terms of reinforced random walks it implies that linearly edge-oriented reinforced random walks are transient for $d \geq 3$.

1. INTRODUCTION

Random Walks in Random Environment (RWRE) have received a considerable attention in the last years. A lot is known for one-dimensional RWRE, but the situation far from being so clear in dimension 2 and larger. Progresses have been made for multidimensional RWRE (in particular, since the work of Sznitman and Zerner, cf [24]) essentially in two directions: for ballistic RWRE (cf [24], [22], [18] and reference therein) and more recently for small perturbations of the simple random walk in dimension $d \geq 3$. Nevertheless, most of the important questions as characterization of recurrence, of ballistic behavior, invariance principle remain largely open. Recently, for RWRE which are small isotropic perturbation of the simple random walk on \mathbb{Z}^d , $d \geq 3$, Bolthausen and Zeitouni ([3]) made some progress in the direction of the invariance principle and proved transience using renormalization technics (invariance principle for the corresponding continuous model has previously been obtained by Sznitman and Zeitouni, [23]). We refer to [28] or [20] for surveys.

Among random walks in random environment, random walks in random Dirichlet environment (RWDE) play a special rôle. It corresponds to the case where the transition probabilities at each site are chosen as i.i.d. Dirichlet random variables. RWDE have the remarkable property that the annealed law is the law of an edge oriented reinforced random walk. This is a simple consequence of the representation of Polya's urns as a Dirichlet mixtures of i.i.d. sampling ([15], [7]). In [6], N. Enriquez and the author have obtained a criterion for ballistic behavior on \mathbb{Z}^d , latter improved by L. Tournier ([26]). Besides, in the one-dimensional case, it appeared in [8] that limit theorems in the sub-ballistic regime are fully explicit in the case of Dirichlet environments, highlighting the special rôle of Dirichlet environment among random environments. Finally, in [19] we have described a precise relation between RWDE and hypergeometric integrals associated with certain arrangement of hyperplanes.

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As already mentioned, random Dirichlet environment is very natural from the point of view of reinforced random walks. Reinforced random walks have been introduced by Diaconis (cf [4] and Pemantle's thesis [15]). There are three natural models of linearly reinforced random walks, namely vertex reinforced random walks, (non-oriented) edge reinforced random walks, and edge oriented reinforced random walks. None of these models is completely understood yet. Vertex reinforced random walks are expected to get localized on a finite set, but this is completely solved only in dimension 1 ([17, 25]). Concerning edge reinforced random walks, Diaconis and Coppersmith proved that it can be represented as a "complicated" mixture of reversible random walks. Recurrence of edge reinforced random walks has been proved by Merkl and Rolles ([14]) on a two dimensional graph (but the question is still open on \mathbb{Z}^2 itself). By comparison, edge-oriented reinforced random walks correspond to the annealed law of RWDE, hence they are a simple mixture of non-reversible random walks. The difficulty of this model does not come from the representation as a RWRE, but lies in the non-reversibility of this RWRE. We refer to [16] for a survey on the subject.

On \mathbb{Z}^d , RWDE are parameterized by $2d$ positive reals $(\alpha_1, \dots, \alpha_{2d})$, one for each direction. We call (α_i) the weights. In this paper, we prove that RWDE on \mathbb{Z}^d , for $d \geq 3$, are transient for all values of the weights (in particular for the case of unbiased weights). In fact, we prove something stronger. We prove that the Green function has some finite moments and we explicitly compute the critical integrability exponent κ , i.e. the supremum of the reals $s > 0$ such that the Green function $G(0, 0)^s$ has finite expectation. We show that this real is the same as the one for the RWDE killed when it exists a finite large ball (which has been computed by Tournier in [26]). In some way, it means that trapping is only due to finite size traps which come from the non uniform ellipticity of the environment. Our result is in fact more general and applies for example to any finitely generated Caley graph on which the simple random walk is transient. Compared to the results of [3], our results are non-perturbative, they apply to any choice of the weights, but our method is specific to the Dirichlet environment. The proof is remarkably short, and it is quite surprising that for RWDE, which keep the essential features of general RWRE, things simplify so much. We don't have a clear explanation for this, it may be related to the correspondance with linearly reinforced random walks (even if it does not appear in this proof) or from the relation between RWDE and hypergeometric integrals (cf [19]). We think it highlights the special rôle of Dirichlet environments and we think that the technics presented in this paper will help us to understand more on RWDE.

Our proof is based on an explicit formula valid for Dirichlet environments and on a method of perturbation of the weights. Finally, the critical exponent is obtained thanks to an L_2 version of the Max-Flow Min-Cut theorem (proposition 2). The explicit formula (corollary 1) was in fact hinted in our joint work with N. Enriquez and O. Zindy ([8, 9]) where it appeared that in the case of one-dimensional RWDE, the Green function on the half-line at 0 has an explicit law (which is in fact a consequence of a result of Chamayou and Letac on explicit solutions of renewal equations, [11]).

Let us present our results for the graph \mathbb{Z}^d . Let (e_1, \dots, e_d) be the canonical base of \mathbb{Z}^d , and set $e_j = -e_{j-d}$, for $j = d + 1, \dots, 2d$. The set $\{e_1, \dots, e_{2d}\}$ is the set of

unit vectors of \mathbb{Z}^d . We consider a probability law λ on

$$(1.1) \quad \{(x_1, \dots, x_{2d}) \in]0, 1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}$$

We construct a Markov chain on \mathbb{Z}^d , to nearest neighbors as follows: we choose independently at each point $x \in \mathbb{Z}^d$, some transition probabilities

$$p(x) = (p(x, x + e_i))_{i=1, \dots, 2d},$$

according to the law λ . It means that the vector $(p(x))_{x \in \mathbb{Z}^d}$ is chosen according to the product measure $\mu = \otimes_{x \in \mathbb{Z}^d} \lambda$. It defines the transition probability of a Markov chain on \mathbb{Z}^d , and we denote $P_x^{(p)}$ the law of this Markov chain starting from x :

$$P_x^{(p)}[X_{n+1} = x + e_i | X_n = x] = p(x, x + e_i).$$

Random Dirichlet environment corresponds to the case where the law λ is a Dirichlet law. More precisely, we choose some positive weights $(\alpha_1, \dots, \alpha_{2d})$ and we take $\lambda = \lambda^{(\alpha)}$ with density

$$\frac{\Gamma(\sum_{i=1}^{2d} \alpha_i)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i - 1} \right) dx_1 \cdots dx_{2d},$$

where Γ is the usual Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. We denote $\mathbb{P}^{(\alpha)}$ the law obtained on the environment in this way. This type of environment plays a specific rôle, since the annealed law $\mathbb{P}_x^{(\alpha)}[\cdot] = \mathbb{E}^{(\alpha)}[P_x^{(p)}(\cdot)]$ corresponds to an edge oriented reinforced random walk with an affine reinforcement, i.e.

$$\mathbb{P}_x^{(\alpha)}[X_{n+1} = X_n + e_i | \sigma(X_k, k \leq n)] = \frac{\alpha_i + N_i(X_n, n)}{\sum_{k=1}^{2d} \alpha_k + N_k(X_n, n)},$$

where $N_k(x, n)$ is the number of crossings of the oriented edge $(x, x + e_k)$ up to time n (cf [7]). When the weights are constant equal to α , the environment is isotropic: when α is large, the environment is close to the deterministic environment of the simple random walk, when α is small the environment is very disordered.

Let us now describe precisely our results for \mathbb{Z}^d , $d \geq 3$. We denote $G(x, y)$ the green function in the environment $(p(x))_{x \in \mathbb{Z}^d}$.

$$G(x, y) = E_x^{(p)} \left[\sum_{k=0}^{\infty} \mathbb{1}_{X_k=y} \right].$$

Theorem 1. *For $d \geq 3$ and for any choice of weights $(\alpha_1, \dots, \alpha_{2d})$ we have*

$$\mathbb{E}^{(\alpha)} (G(0, 0)^s) < \infty$$

if and only if $s < \kappa$ where

$$\kappa = 2 \sum_{j=1}^{2d} \alpha_{e_j} - \max_{i=1, \dots, d} (\alpha_{e_i} + \alpha_{-e_i}).$$

In particular, the RWDE is transient for almost all environment.

Remark 1. *In [26], Tournier computed the critical integrability exponent for RWDE on finite graphs. For $N > 1$, let G_N be the Green function of the RWDE killed when it exists the ball $B(0, N)$: it appears that $G_N(0, 0)^s$ is integrable if and only if $s < \kappa$. Hence, it seems to mean that there is no infinite size trap and that the trapping effect comes only from finite size traps due to the non-uniform ellipticity of the environment.*

Remark 2. *The result for \mathbb{Z}^d is in fact a consequence of a more general result valid for directed graph constructed from undirected graph, under a condition on the weights (null-divergence), cf theorem 3.*

Let us describe the organization of the paper. In section 2, we describe the model of RWDE on directed graphs. In section 3, we prove the key explicit formulas. Section 4 is devoted to the proof of the main result on integrability. In section 5, we prove a generalization of the Max-Flow Min-Cut theorem for flows of finite energy. In section 6 we apply the results to directed graphs constructed from undirected graphs, and we deduce theorem 1.

2. MARKOV CHAIN IN DIRICHLET ENVIRONMENT ON DIRECTED GRAPHS

Let us first describe Random walk in Dirichlet Environment (RWDE for short) on a general graph. A directed graph is a couple $G = (V, E)$ where V is the countable set of vertices and E is the countable set of edges (i.e. E is a collection of elements of $V \times V$; for simplicity we do not allow loops). The tail of the edge e is denoted $\underline{e} = x$ and the head is denoted $\bar{e} = y$ if e corresponds to (x, y) . We suppose that the graph has bounded degree i.e. that the number of edges exiting a vertex x or pointing to a vertex x is bounded. For an integer n and a vertex x we denote $B(x, n)$ the ball with center x and radius n for the graph distance (defined independently of the orientation of the edges). We say that a subset $K \subset V$ is strongly connected if for any two vertices x and y in K there is a directed path in K from x to y .

Let us define the divergence operator on the graph G : it is the operator $\text{div} : \mathbb{R}^E \mapsto \mathbb{R}^V$ defined for a function $\theta : E \mapsto \mathbb{R}$ by

$$\text{div}(\theta)(x) = \sum_{e, \underline{e}=x} \theta(e) - \sum_{e, \bar{e}=x} \theta(e), \quad \forall x \in V.$$

The set of environments on G is defined as the set

$$\Delta = \{(p_e)_{e \in E} \in]0, 1]^E, \text{ such that } \sum_{e, \underline{e}=x} p_e = 1, \forall x \in V \text{ such that } \{e, \underline{e} = x\} \neq \emptyset\}.$$

With any environnement (p_e) we associate the Markov chain on V with law $(P_x^{(p)})$, where $P_{x_0}^{(p)}$ is the law of the Markov chain starting from x_0 with transition probabilities given by

$$P_{x_0}^{(p)}(X_{n+1} = y | X_n = x) = p_{(x,y)} = \sum_{e, \underline{e}=x, \bar{e}=y} p_e, \quad \forall x \neq y.$$

If y is a vertex such that no edge is exiting y then we put $P_{x_0}^{(p)}(X_{n+1} = y | X_n = y) = 1$, so that y is an absorbing point.

Let $(\alpha_e)_{e \in E}$ be a set of positive weights on the edges. For any vertex x we set

$$\alpha_x = \sum_{e, \underline{e}=x} \alpha_e$$

the sum of the weights of the edges with origin x . The Dirichlet environnement with parameters (α_e) is the law on Δ obtained by taking at each site x the transition probabilities $(p_e)_{\underline{e}=x}$ independently accordingly to the Dirichlet law with parameters

$(\alpha_e)_{e=x}$ i.e. with distribution

$$\left(\frac{\Gamma(\alpha_x)}{\prod_{e, \underline{e}=x} \Gamma(\alpha_e)} \right) \left(\prod_{e, \underline{e}=x} p_e^{\alpha_e-1} \right) \left(\prod_{e \in \tilde{E}, \underline{e}=x} dp_e \right).$$

where \tilde{E} is obtained from E by removing arbitrary, for each vertex x , one edge with origin x (easily, it is independent of this choice). We denote $\mathbb{P}^{(\alpha)}$ the corresponding probability measure on Δ and $\mathbb{E}^{(\alpha)}$ the associated expectation.

When the graph is finite the distribution of the Dirichlet environment is given by

$$(2.1) \quad \frac{\prod_{x \in V} \Gamma(\alpha_x)}{\prod_{e \in E} \Gamma(\alpha_e)} \left(\prod_{e \in E} p_e^{\alpha_e-1} \right) d\lambda_\Delta.$$

where $d\lambda_\Delta$ is the measure on Δ given by

$$(2.2) \quad d\lambda_\Delta = \prod_{e \in \tilde{E}} dp_e,$$

3. STABILITY BY TIME REVERSAL

3.1. The key lemma. Suppose now that the graph is finite and strongly connected i.e. that there is a directed path between any two vertices x and y . Let $\check{G} = (V, \check{E})$ be the graph obtained from G by reversing all the edges, i.e. \check{E} is obtained from E by reversing the head and the tail of the edges. For an environment (p_e) we denoted $(\check{p}_{\check{e}})$ the environment obtained by time reversal of the Markov chain, i.e. for all e in E

$$\check{p}_{\check{e}} = \frac{\pi_e}{\pi_{\check{e}}} p_e,$$

where $(\pi_x)_{x \in V}$ is the invariant probability on V for the Markov chain $P^{(p)}$. (Since we assume that the graph is strongly recurrent and that the weights p_e are positive this invariant probability exists and is unique).

Lemma 1. *Suppose that the weights (α_e) have divergence null, i.e.*

$$\operatorname{div}(\alpha)(x) = 0, \quad \forall x \in V,$$

then $(\check{p}_{\check{e}})_{\check{e} \in \check{E}}$ is a Dirichlet environment on \check{G} with parameters (α_e) .

Remark 3. *By this we mean that $(\check{p}_{\check{e}})$ is distributed according to a Dirichlet environment with parameters $(\check{\alpha}_{\check{e} \in \check{E}})$ where $\check{\alpha}_{\check{e}} = \alpha_e$ if \check{e} is the reversed edge of e . We will often identify the edges in E with their reversed edges in \check{E} .*

Proof. The proof is based on a change of variable. Let e_0 be a specified edge of the graph. Let \mathcal{H}_{e_0} be the affine space defined by

$$\mathcal{H}_{e_0} = \{(z_e)_{e \in E} \in \mathbb{R}^E, \quad z_{e_0} = 1, \quad \operatorname{div}(z) \equiv 0\}.$$

and \mathcal{H} the vector space

$$\mathcal{H} = \{(z_e)_{e \in E} \in \mathbb{R}^E, \quad \operatorname{div}(z) \equiv 0\}.$$

Let $\tilde{\Delta}_{e_0} = \mathcal{H}_{e_0} \cap (\mathbb{R}_+^*)^E$. The strategy is to make the change of variable

$$\begin{aligned} \Delta &\mapsto \tilde{\Delta}_{e_0} \\ (p_e)_{e \in E} &\rightarrow (z_e = \frac{\pi_e p_e}{\pi_{e_0} p_{e_0}})_{e \in E}. \end{aligned}$$

Hence, (z_e) is the occupation time of the graph normalized so that $z_{e_0} = 1$. It is easy to see that the previous change of variable is a C^∞ -diffeomorphism. Let T be a spanning tree of the graph G such that $e_0 \notin T$. (This is possible since the graph is strongly connected and thus e_0 belongs to at least one directed cycle of the graph.) We denote $B = T \cup \{e_0\}$. Then $(z_e)_{e \in T^c}$ is a base of \mathcal{H} and $(z_e)_{e \in B^c}$ is a base of \mathcal{H}_{e_0} . Let $x_0 \in V$ be any vertex, and set $U = V \setminus \{x_0\}$. We first prove the following lemma.

Lemma 2. *Let ψ be a positive test function on Δ . Then*

$$\int_{\Delta} \Psi((p_e)) \left(\prod_{e \in E} p_e^{\alpha_e - 1} \right) d\lambda_{\Delta} = \int_{\tilde{\Delta}_{e_0}} \Psi\left(\left(\frac{z_e}{z_{\underline{e}}}\right)\right) \left(\frac{\prod_{e \in E} z_e^{\alpha_e - 1}}{\prod_{x \in V} z_x^{\alpha_x}} \right) \det(Z|_{U \times U}) \prod_{e \in B^c} dz_e.$$

where as usual $z_x = \sum_{e, \underline{e}=x} z_e$ and Z is the $V \times V$ matrix defined by

$$Z_{x,x} = z_x, \quad \forall x \in V, \quad Z_{x,y} = -z_{x,y} = - \sum_{e, \underline{e}=x, \bar{e}=y} z_e, \quad \forall x \neq y.$$

Remark 4. *This formula is essentially the same as the one which gives the correspondence between RWDE and hypergeometric integrals associated with an arrangement of hyperplanes in [19].*

Remark 5. *This lemma expresses the law of the random environment in the variables (z_e) which correspond to the occupation densities of the edges (properly renormalized). We can remark that this formula is reminiscent of the distribution discovered by Diaconis and Coppersmith ([4, 5]) which expresses edge-reinforced random walk as a mixture of reversible Markov chains.*

We see that lemma 1 is a direct consequence of the previous result. Indeed we see that lemma 2 applied to the reversed graph (\check{G}, \check{E}) , starting with the weights $\check{\alpha}_{\check{e}} = \alpha_e$ gives the same integrand with α_x replaced by $\bar{\alpha}_x = \sum_{e, \bar{e}=x} \alpha_e$. The two coincide after the change of variables exactly when $\text{div}(\alpha) \equiv 0$. \square

Proof. of lemma 2. Let us first remark that $\det(Z|_{U \times U})$ does not depend on the choice of x_0 : indeed, the lines and columns of Z has sum 0. By addition of lines and columns we can switch from x_0 to y_0 . Hence, we can freely choose $x_0 = \underline{e}_0$. Remark now that \mathcal{H} and \mathcal{H}_{e_0} are the affine subspaces of \mathbb{R}^E defined by

$$\mathcal{H} = \cap_{x \in U} \{\text{div}(z)(x) = 0\}, \quad \mathcal{H}_{e_0} = \{z_{e_0} = 1\} \cap_{x \in U} \{\text{div}(z)(x) = 0\}.$$

For simplification, we write

$$h_x(z) = \text{div}(z)(x),$$

for all x in U . It is not easy to compute directly the Jacobian of the change of variables, the strategy is to use Fourier transform to make the change of variables on free variables. We first prove the following lemma.

Lemma 3. *For any function $\phi : \mathbb{R}^E \rightarrow \mathbb{C}$, C^∞ , with compact support in $(\mathbb{R}_+^*)^E$ we have that*

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e = \int_{\mathbb{R}^U \times \mathbb{R}} \int_{\mathbb{R}^E} \phi(z) \exp\left(2i\pi \left(u_0(z_{e_0} - 1) + \sum_{x \in U} u_x h_x(z)\right)\right) \left(\prod_{e \in E} dz_e\right) \left(du_0 \prod_{x \in U} du_x\right).$$

Proof. of lemma 3. We will use several times the following simple fact (which is a consequence of the inverse Fourier transform). Let $g : \mathbb{R}^{N+k} \rightarrow \mathbb{R}$ be a C^∞ function with compact support, then

$$\begin{aligned} & \int_{\mathbb{R}^N} g(x_1, \dots, x_N, 0, \dots, 0) dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{N+k}} \exp\left(2i\pi \sum_{j=1}^k u_j x_{N+j}\right) g(x_1, \dots, x_{N+k}) (dx_1 \cdots dx_{N+k}) (du_1 \cdots du_k) \end{aligned}$$

N.B.: These integrals are well-defined as integrals in the Schwartz space.

Let us compute the Jacobian of the linear change of variables

$$(3.1) \quad \begin{aligned} \mathbb{R}^E &\mapsto \mathbb{R}^U \times \mathbb{R}^{T^c} \\ ((z_e)_{e \in E}) &\rightarrow ((h_x(z))_{x \in U}, (z_e)_{e \in T^c}). \end{aligned}$$

Denoting $T = \{e_{j_1}, \dots, e_{j_{|U|}}\}$ and $T^c = \{e_{i_1}, \dots, e_{i_{|E|-|U|}}\}$, $U = \{x_1, \dots, x_{|U|}\}$:

$$|J| = \left| \det \begin{pmatrix} \frac{\partial h_{x_1}}{\partial z_{e_{j_1}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_1}}} & & \\ \vdots & & \vdots & & 0 \\ \frac{\partial h_{x_1}}{\partial z_{e_{j_{|U|}}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_{|U|}}}} & & \\ \frac{\partial h_{x_1}}{\partial z_{e_{i_1}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{i_1}}} & & \\ \vdots & & \vdots & & \text{Id}_{T^c \times T^c} \\ \frac{\partial h_{x_1}}{\partial z_{e_{i_{|E|-|U|}}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{i_{|E|-|U|}}}} & & \end{pmatrix} \right|.$$

So, we get

$$|J| = \left| \det \begin{pmatrix} \frac{\partial h_{x_1}}{\partial z_{e_{j_1}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_1}}} \\ \vdots & & \vdots \\ \frac{\partial h_{x_1}}{\partial z_{e_{j_{|U|}}}} & \cdots & \frac{\partial h_{x_{|U|}}}{\partial z_{e_{j_{|U|}}}} \end{pmatrix} \right|.$$

The previous matrix is the incidence matrix on U of the spanning tree T , indeed we have if $x \in U$

$$\frac{\partial h_x}{\partial z_e} = \begin{cases} +1 & \text{if } \underline{e} = x, \\ -1 & \text{if } \bar{e} = x, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that this determinant is equal to ± 1 . Let us now denote $\tilde{\phi} : \mathbb{R}^U \times \mathbb{R}^{T^c} \mapsto \mathbb{C}$ the function defined by

$$\phi((z_e)_{e \in E}) = \tilde{\phi}((h_x(z))_{x \in U}, (z_e)_{e \in T^c}).$$

By formula (3.1) and since $\tilde{\phi}(0, (z_e)_{e \in T^c}) = \phi((z_e)_{e \in E})$ on \mathcal{H} , we get

$$\begin{aligned} \int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}}((z_e)) \prod_{e \in B^c} dz_e &= \int_{\mathbb{R}} \int_{\mathcal{H}} \phi|_{\mathcal{H}}((z_e)) e^{2i\pi u_0(z_{e_0}-1)} \left(\prod_{e \in T^c} dz_e \right) du_0 \\ &= \int_{\mathbb{R}} \int_{\mathcal{H}} \tilde{\phi}(0, (z_e)_{e \in T^c}) e^{2i\pi u_0(z_{e_0}-1)} \left(\prod_{e \in T^c} dz_e \right) du_0 \end{aligned}$$

Using again formula 3.1 we see that the previous integral is equal to

$$= \int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R}^U \times \mathbb{R}^{T^c}} \exp \left(2i\pi \left(u_0(z_{e_0} - 1) + \sum_{x \in U} u_x h_x \right) \right) \tilde{\phi}((h_x)_{x \in U}, (z_e)_{e \in T^c}) \\ \left(\prod_{x \in U} dh_x \prod_{e \in T^c} dz_e \right) (du_0 \prod_{x \in U} du_x).$$

Then, the change of variables (3.1) gives lemma 3. \square

We make the following change of variables:

$$\begin{aligned} (\mathbb{R}_+^*)^E &\rightarrow (\mathbb{R}_+^*)^V \times \Delta \\ (z_e) &\mapsto ((v_x)_{x \in V}, (p_e)_{e \in E}), \end{aligned}$$

given by

$$v_x = \sum_{\underline{e}=x} z_e, \quad p_e = \frac{z_e}{v_{\underline{e}}}.$$

(More precisely, it is the change of variables onto the set $(\mathbb{R}_+^*)^V \times]0, 1]^{\tilde{E}}$ where \tilde{E} is defined in (2.2). It means that we choose $(p_e)_{e \in \tilde{E}}$ as the coordinate system on Δ .) We have $z_e = v_{\underline{e}} p_e$ and the Jacobian is given by

$$\prod_{x \in V} v_x^{n_x - 1},$$

where $n_x = |\{e, \underline{e} = x\}|$. Implementing this change of variables in lemma 3 gives that

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e$$

is equal to

$$\int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R}^V \times \Delta} \left(\prod_{x \in V} v_x^{n_x - 1} \right) \phi((v_{\underline{e}} p_e)) \exp \left(iu_0(v_{\underline{e}_0} p_{e_0} - 1) + i \sum_{x \in U} u_x h_x \right) \\ \left(\left(\prod_{x \in V} dv_x \right) d\lambda_{\Delta} \right) (du_0 \prod_{x \in U} du_x),$$

where $d\lambda_{\Delta}$ is the measure on Δ defined in (2.2) and with

$$h_x((v_x), (p_e)) = v_x - \sum_{e, \bar{e}=x} v_{\underline{e}} p_e.$$

The strategy is now to apply formula (3) to get rid of variables $(u_0, (u_x)_{x \in U})$. For this we need to change to variables $(v_{e_0} p_{e_0}, (h_x)_{x \in U})$. We make the following change of variables.

$$\begin{aligned} \mathbb{R}^V \times \Delta &\mapsto \mathbb{R} \times \mathbb{R}^U \times \Delta \\ ((v_x)_{x \in V}, (p_e)) &\rightarrow (k_0(v, p), (h_x(v, p))_{x \in U}, (p_e)), \end{aligned}$$

with $k_0 = v_{\underline{e}_0} p_{e_0}$. This change of variables can be inverted by

$$(v_x)_{x \in V} = (M^{(p)})^{-1}(k_0, (h_x)),$$

where

$$M_{x,y}^{(p)} = (I - P)_{x,y}, \quad \forall x \in U, \forall y \in V,$$

and $M_{0,y} = p_{e_0} \mathbb{1}_{y=\underline{e}_0}$ (where P is the transition matrix in the environment (p)). Since we have chosen $x_0 = \underline{e}_0$, the Jacobian of the change of variables is

$$|J| = p_{e_0} \det(I - P)_{U \times U}.$$

By this change of variables, the integral becomes

$$\int_{\mathbb{R} \times \mathbb{R}^U} \int_{\mathbb{R} \times \mathbb{R}^{U \times \Delta}} \phi((v_e p_e)) \frac{(\prod_{x \in V} v_x^{n_x - 1})}{p_{e_0} \det(I - P)_{U \times U}} \exp(2i\pi(u_0(k_0 - 1) + \sum_{x \in U} u_x h_x)) \\ ((dk_0 \prod_{x \in U} dh_x) d\lambda_\Delta) (du_0 \prod_{x \in U} du_x),$$

Remark that if the following equalities are satisfied

$$h_x((v_x), (p_e)) = 0, \quad \forall x \in U, \quad v_{e_0} p_{e_0} = 1$$

it implies that $v_x = \frac{\pi_x}{\pi_{e_0} p_{e_0}}$ for all x . Integrating over the variables $(u_0, (u_x))$ by formula (3.1) we get

$$\int_{\mathcal{H}_{e_0}} \phi|_{\mathcal{H}_{e_0}} \prod_{e \in B^c} dz_e = \int_{\Delta} \prod_{x \in V} \left(\frac{\pi_x}{\pi_{e_0} p_{e_0}} \right)^{n_x - 1} \frac{\phi((z_e))}{p_{e_0} \det(I - P)_{U \times U}} d\lambda_\Omega,$$

where $z_e = \frac{\pi_e p_e}{\pi_{e_0} p_{e_0}}$. Then we just have to replace ϕ by

$$\phi((z_e)) = \psi \left(\left(\frac{z_e}{z_{\underline{e}}} \right) \right) \frac{\prod_{e \in E} \mathbb{1}_{z_e > 0} z_e^{\alpha_e - 1}}{\prod_{x \in V} z_x^{\alpha_x}} \det(Z|_{U \times U})$$

This can be done by monotone convergence. \square

3.2. Applications. Consider now a finite graph with a cemetery point, i.e. we suppose that $G = (V, E)$, that V and E are finite and that V can be written $V = U \cup \{\delta\}$ where

- no edge is exiting δ (δ is called the cemetery point)
- for any point x in U there is a directed path from x to δ .

It means that δ is absorbing for the Markov chain on G with law $P^{(p)}$. For x and y in U we denote $G^{(p)}(x, y)$ the Green function in the environment (p_e)

$$G^{(p)}(x, y) = E_x^{(p)} \left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=y} \right).$$

Corollary 1. (i) Suppose that $\text{div}(\alpha)(x) = 0$ for all x in U such that $x \neq x_0$ then $\text{div}(\alpha)(x_0) > 0$ and $G^{(p)}(x_0, x_0)$ is distributed as $\frac{1}{W}$ where W is a beta random variable with parameter $(\text{div}(\alpha)(x_0), \alpha_{x_0} - \text{div}(\alpha)(x_0))$.

(ii) Suppose that $\text{div}(\alpha)(x)$ is non-negative for all x in U . For all x_0 in U such that $\text{div}(\alpha)(x_0) > 0$, $G^{(p)}(x_0, x_0)$ is stochastically dominated by $\frac{1}{W}$ where W is a beta random variable with parameter $(\text{div}(\alpha)(x_0), \alpha_{x_0} - \text{div}(\alpha)(x_0))$.

Remark 6. The explicit formula in (i) was in fact suggested by our joint work with N. Enriquez and O. Zindy ([8]) and also by the correspondance established in [19]. It appeared in [8] that in the case of sub-ballistic one-dimensional RWRE limit theorems are fully explicit in the case of Dirichlet environments. This is a consequence of the fact that the Green function at 0 of the RWDE on the half line \mathbb{Z}_+ is equal in law to $1/W$ where W is a beta random variable with appropriate weights. In the case of one-dimensional RWDE, this explicit formula is a consequence of a result of Chamayou and Letac ([11]) on explicit solutions for renewal equations. From the point of view of the correspondance with hypergeometric integrals described in [19], the condition of null divergence corresponds to a condition of resonance of the weights of the arrangement. Resonant arrangements are not well understood yet.

Remark 7. *By the principle of conservation of mass we know that $\sum_{x \in U} \operatorname{div}(\alpha)(x) = \sum_{\bar{e}=\delta} \alpha_e$ so that there is always a vertex x such that $\operatorname{div}(\alpha)(x) > 0$.*

(i) We can freely suppose that any y in U can be reached following a directed path from x_0 (indeed, the part of the graph which cannot be reached from x_0 does not play any role in $G(x_0, x_0)$). Suppose that $\operatorname{div}(\alpha)(x_0) = \gamma$, $\gamma > 0$. It means that $\operatorname{div}(\alpha)(\delta) = -\gamma$. Consider now the graph $\tilde{G} = (U, E)$ obtained by identification of the vertices δ and x_0 . The edges of \tilde{G} are just obtained by identification of δ and x_0 in the edges of G (with eventually multiples edges in \tilde{G}). The graph \tilde{G} is clearly strongly connected and if we keep the same weights on the edges we have $\operatorname{div}^{\tilde{G}}(\alpha) \equiv 0$. Consider the invariant probability $(\pi_x)_{x \in U}$ for the RWDE on \tilde{G} . It gives the occupation time on the edges $z_e = \pi_e p_e$, and the time reversal transition probabilities $\check{p}_e = \frac{z_e}{\pi_e}$. If $G^{(p)}(x_0, x_0)$ is the Green function on the graph G (which is the graph with the cemetery point)

$$G^{(p)}(x_0, x_0) = \frac{\pi_{x_0}}{\sum_{\bar{e}=\delta} z_e} = \frac{1}{\sum_{\bar{e}=\delta} \check{p}_e}.$$

(where $\bar{e} = \delta$ is relative to the edges in the initial graph G). By the previous lemma we know that $(\check{p}_e)_{\bar{e}=\delta}$ or $\bar{e}=x_0$ has the law of a Dirichlet random variable with parameters $(\alpha_e)_{\bar{e}=\delta}$ or $\bar{e}=x_0$. (Indeed, the vertices δ and x_0 are identified in the quotiented graph \tilde{G} .) The conclusion comes from the associativity property of Dirichlet random variables (cf for instance [26] p. 2, or [27]).

(ii) Consider the graph $\tilde{G} = (\tilde{U} \cup \{\tilde{\delta}\}, \tilde{E})$ obtained as follows. The set of vertices is defined by $\tilde{U} = U \cup \{\delta\}$ and $\tilde{\delta}$ is the new cemetery point. The set of edges \tilde{E} is obtained by adding to the edges E of G some new edges with origin δ : for any $x \neq x_0$ such that $\operatorname{div}(\alpha)(x) > 0$ we add the edge (δ, x) with weight $\operatorname{div}(\alpha)(x)$; we also add the edge $(\delta, \tilde{\delta})$ with weight $\operatorname{div}(\alpha)(x_0)$. Clearly, the new graph \tilde{G} with the previous choice of weights satisfies the condition of (i). Moreover if $(p_e)_{e \in E}$ is a Dirichlet environment on G with weights (α_e) it can be extended to a Dirichlet environment $(\check{p}_e)_{e \in \tilde{E}}$ on \tilde{G} just by choosing independently the transition probabilities on the edges exiting δ according to a Dirichlet random variable with the appropriate weights. Remark that

$$G^{(p)}(x_0, x_0) < G^{(\check{p})}(x_0, x_0).$$

Indeed, the Markov chains on G and \tilde{G} behave the same as long as they are on U but the Markov chain on G is stuck on δ although the Markov chain on \tilde{G} can come back to U from δ . The conclusion is a consequence of (i) since (i) implies that $G^{(\check{p})}(x_0, x_0)$ has the law of $1/W$ with W a beta random variable with appropriate weights. \square

4. INTEGRABILITY CONDITION

Suppose now that $G = (V, E)$ is a countable connected directed graph with bounded degree. Suppose for simplicity that there is at least one edge exiting each vertex.

We recall that a flow from a vertex x_0 to infinity (cf [12]) is a positive function θ on the edges such that

$$\operatorname{div}(\theta)(x) = 0, \forall x \neq x_0,$$

and

$$\operatorname{div}(\theta)(x_0) \geq 0.$$

The strength of the flow is the value

$$\text{strength}(\theta) = \text{div}(\theta)(x_0).$$

A unit flow is a flow of strength 1.

We say that the flow θ has finite energy if it is square integrable i.e. if

$$\sum_{e \in E} \theta_e^2 < \infty.$$

Theorem 2. *Let $(\alpha_e)_{e \in E}$ be a family of positive weights on the edges which satisfy*

(H1) *there exists $c > 0$ and $C \geq c$ such that $c \leq \alpha_e \leq C$ for all e in E .*

(H2) *For all vertex x , $\text{div}(\alpha)(x) = 0$.*

Suppose that θ is a unit flow with finite energy from x_0 to infinity then

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty$$

as soon as

$$s < \inf_{e \in E} \frac{\alpha_e}{\theta_e}$$

Proof. Let γ be a positive real. We define the weights

$$\alpha^\gamma = \alpha + \gamma\theta.$$

We clearly have

$$\text{div}(\alpha^\gamma) = \gamma\delta_{x_0}.$$

For a positive integer N , let U_N be the ball with center x_0 and radius N in G . We define the graph $G_N = (U_N \cup \{\delta\}, V_N)$ as follows. We contract all the vertices of $(U_N)^c$ to the cemetery point δ . The edges E_N are obtained from E as follows: in E we delete all the edges exiting a point of U_N^c and we define E_N from the remaining edges by contraction of U_N^c to the single vertex δ . By the bounded degree property we see that G_N is a finite graph. We keep the same weights on the edges and we see that on the graph G_N we have

$$\text{div}^N(\alpha)(x) = \sum_{e \in E, \bar{e}=x, e \notin U_N} \alpha_e.$$

(With div^N the divergence operator on the graph G_N). Hence $\text{div}^N(\alpha) \geq 0$, and for N large enough (when all the edges pointing to x_0 are in E_N) we have

$$\text{div}^N(\alpha^\gamma)(x_0) = \gamma.$$

Denote by $G_N^{(p)}(x_0, x_0)$ the Green function of the Markov chain killed when it exits U_N . From lemma 1 we see that under $\mathbb{P}^{(\alpha^\gamma)}$ the Green function $G_N^{(p)}(x_0, x_0)$ is stochastically dominated by $\frac{1}{W}$ where W is a beta random variable with parameters $(\gamma, \alpha_{x_0} + \gamma\theta_{x_0} - \gamma)$ (for N large enough).

Consider now $s > 0$. Considering formula (2.1) we have (we write simply $G_N(x_0, x_0)$ for $G_N^{(p)}(x_0, x_0)$ the Green function in environment (p_e))

$$\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s) = \frac{\prod_{x \in U_N} \Gamma(\alpha_x)}{\prod_{e \in E_N} \Gamma(\alpha_e)} \frac{\prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \mathbb{E}^{(\alpha^\gamma)} \left(G_N(x_0, x_0)^s \prod_{e \in E_N} p_e^{-\gamma\theta_e} \right).$$

Using Hölder's inequality for $q > 1$ and $p = \frac{q}{q-1}$ we get that $\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s)$ is lower than

$$\frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \left(\mathbb{E}^{(\alpha^\gamma)}(G_N(x_0, x_0)^{ps}) \right)^{1/p} \left(\mathbb{E}^{(\alpha^\gamma)} \left(\prod_{e \in E_N} p_e^{-q\gamma\theta_e} \right) \right)^{1/q}.$$

Remark that the second expectation is finite if and only if $q\gamma\theta_e < \alpha_e^\gamma$ for all e in E_N , or equivalently

$$q - 1 < \frac{\alpha_e}{\gamma\theta_e}$$

for all e in E_N . We now choose q such that

$$(4.1) \quad q - 1 < \inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}$$

so that the previous condition is fulfilled for all e in E . In terms of p it is equivalent to

$$(4.2) \quad p > \frac{\inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e} + 1}{\inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}}.$$

Now we compute the second expectation. We have

$$\begin{aligned} & \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)} \left(\mathbb{E}^{(\alpha^\gamma)} \left(\prod_{e \in E_N} p_e^{-q\gamma\theta_e} \right) \right)^{1/q} \\ &= \frac{\prod_{x \in U_N} \Gamma(\alpha_x) \prod_{e \in E_N} \Gamma(\alpha_e^\gamma)^{1-\frac{1}{q}} \prod_{e \in E_N} \Gamma(\alpha_e - (q-1)\gamma\theta_e)^{\frac{1}{q}}}{\prod_{e \in E_N} \Gamma(\alpha_e) \prod_{x \in U_N} \Gamma(\alpha_x^\gamma)^{1-\frac{1}{q}} \prod_{x \in U_N} \Gamma(\alpha_x - (q-1)\gamma\theta_x)^{\frac{1}{q}}} \end{aligned}$$

Considering the function

$$\nu(\alpha, u) = \frac{1}{q} \ln \Gamma(\alpha - (q-1)u) + \left(1 - \frac{1}{q}\right) \ln \Gamma(\alpha + u) - \ln \Gamma(\alpha),$$

we see that the previous expression is equal to

$$\exp \left(\sum_{e \in E_N} \nu(\alpha_e, \gamma\theta_e) - \sum_{x \in U_N} \nu(\alpha_x, \gamma\theta_x) \right).$$

Now, $\nu(\alpha, 0) = 0$ and clearly $\frac{\partial}{\partial u} \nu(\alpha, 0) = 0$. The function $\nu(\alpha, u)$ is C^∞ on the domain $D = \{\alpha > 0\} \cap \{u < \alpha/(q-1)\}$. By conditions (H1) and (4.1) we know that $(\alpha_e, \gamma\theta_e)_{e \in E}$ and $(\alpha_x, \gamma\theta_x)_{x \in V}$ are in a compact subset of D . Hence, we can find a constant $C > 0$ such that for all $N > 0$

$$\sum_{e \in E_N} \nu(\alpha_e, \gamma\theta_e) - \sum_{x \in U_N} \nu(\alpha_x, \gamma\theta_x) \leq C \left(\sum_{e \in E_N} (\gamma\theta_e)^2 + \sum_{x \in U_N} (\gamma\theta_x)^2 \right).$$

Since (θ_e) is square integrable and the graph G has bounded degree, (θ_x) is square integrable. Hence we have a constant $C' > 0$, such that for all N

$$\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s) \leq C' \left(\mathbb{E}^{(\alpha^\gamma)}(G_N(x_0, x_0)^{ps}) \right)^{1/p}$$

Using corollary 1 we have

$$\mathbb{E}^{(\alpha)}(G_N(x_0, x_0)^s) \leq C' \left(\mathbb{E}(W^{-ps}) \right)^{1/p}$$

where W is a beta random variable with parameter $(\gamma, \alpha_{x_0} + \gamma\theta_{x_0} - \gamma)$. Since W^{-ps} is integrable for $ps < \gamma$ we see that

$$\mathbb{E}^{(\alpha)} (G(x_0, x_0)^s) = \sup_N \mathbb{E}^{(\alpha)} (G_N(x_0, x_0)^s) < \infty$$

for all s such that $sp < \gamma$. This is true for any choice of p which satisfies (4.2), so $G(x_0, x_0)^s$ is integrable as soon as

$$s < \frac{\inf_{e \in E} \frac{\alpha_e}{\theta_e}}{1 + \inf_{e \in E} \frac{\alpha_e}{\gamma\theta_e}}.$$

Letting γ tend to infinity we get the result. \square

5. MAX-FLOW OF FINITE ENERGY

Let us recall some notions about Max-Flow Min-Cut theorem (cf [12], section 2.6, [2]). Let G be a countable directed graph and x_0 a vertex such that there is an infinite directed simple path starting at x_0 . Let $(c(e))_{e \in E}$ be a family of non-negative reals, called the capacities.

Definition 1. A flow θ from x_0 to ∞ is compatible with the capacities $(c(e))_{e \in E}$ if

$$\theta(e) \leq c(e), \quad \forall e \in E.$$

A cutset is a subset $S \subset E$ such that any infinite directed simple path from x_0 contains at least one edge of S .

The well-known Maw-Flow Min-Cut theorem says that the maximum flow equals the minimal cutset sum (cf [10]). We give here a version for countable graphs ([12], theorem 2.19, cf also [2]).

Proposition 1. The maximum compatible flow equals the infimum of the cutset sum, i.e.

$$\begin{aligned} & \max\{\text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ to } \infty \text{ compatible with } (c(e))\} \\ &= \inf\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}. \end{aligned}$$

where

$$c(S) = \sum_{e \in S} c(e).$$

In theorem 2, we see that the max strength of flows of finite energy gives a lower bound on the critical integrability exponent of the Green function. More precisely, let

$$\kappa_0 = \sup\{\text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ of finite energy compatible with } (\alpha_e)\},$$

Then $G(x_0, x_0)^s$ is integrable for $s < \kappa_0$. Naturally, the question is to know whether κ_0 is also equal to the min-cut. It is not true in general (cf the following remark) but it is true under fairly general conditions.

Proposition 2. Let $(c(e))_{e \in E}$ be a family of capacities. Suppose that

$$\inf_{e \in E} c(e) > 0,$$

and that the following holds

(H3) There exists a strictly increasing sequence of integers η_n such that $B(x_0, \eta_{n+1}) \setminus B(x_0, \eta_n)$ is strongly connected in G .

If there exists a unit flow of finite energy on G from x_0 to ∞ then

$$\begin{aligned} & \max\{\text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ of finite energy compatible with } (c(e))\} \\ &= \min\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}. \end{aligned}$$

Remark 8. : If condition (H3) fails the equality maybe wrong. The following counter-example is due to R. Aharoni, [1]: let G be a binary tree glued by its root to a copy of \mathbb{Z}_+ . Take capacities constant equal to 1. Then any flow of finite energy necessarily vanishes on the copy of \mathbb{Z}_+ and the equality cannot hold.

Proof. Let θ be a unit flow from x_0 to ∞ of finite energy. Let us denote

$$c(G) = \inf\{c(S), S \text{ is a cutset separating } x_0 \text{ from } \infty\}.$$

The strategy is to modify the capacities c using θ . For a positive integer r , we denote $B_E(x_0, r)$ the set of edges

$$B_E(x_0, r) = \{e \in E, \underline{e} \in B(x_0, r), \bar{e} \in B(x_0, r)\}.$$

and

$$\underline{B}_E(x_0, r) = \{e \in E, \underline{e} \in B(x_0, r)\}.$$

Let N_0 be such that

$$(5.1) \quad \sup_{e \notin B_E(x_0, \eta_{N_0})} \theta(e) \leq \frac{\inf_E c}{2c(G)}.$$

Let N_1 be such that

$$(N_1 - N_0) \inf_E c > 2c(G).$$

Consider now the capacities c' defined by

$$c'(e) = \begin{cases} c(e), & \text{if } e \in \underline{B}_E(x_0, \eta_{N_1}) \\ 2c(G)\theta(e), & \text{if } e \notin \underline{B}_E(x_0, \eta_{N_1}). \end{cases}$$

By condition (5.1), we clearly have

$$c'(e) \leq c(e), \forall e \in E,$$

and

$$\sum_E (c'(e))^2 < \infty.$$

We now want to check that the min cutset sum is the same for c and c' . Let S be a minimal cutset for inclusion. If $S \subset \underline{B}_E(x_0, \eta_{N_1})$ then $c'(S) = c(S) \geq c(G)$. If $S \subset B_E(x_0, \eta_{N_0})^c$ then by (5.1) $c'(S) \geq 2c(G)\theta(S) \geq 2c(G)$ since θ is a unit flow. Otherwise, it means that S has one edge e_0 in $B_E(x_0, \eta_{N_0})$ and one edge e_1 in $\underline{B}_E(x_0, \eta_{N_1})^c$. Let K be the set of vertices that can be reached by a directed path in $E \setminus S$ from x_0 . Since S is minimal for inclusion, it means that there is a simple path from \bar{e}_0 to ∞ in K^c . Hence, there is a directed path in K^c from $B(x_0, N_0)$ to ∞ , and there is a sequence $y_1, \dots, y_{N_1-N_0}$ in K^c such that $y_k \in U_k$ where

$$U_k = B(x_0, \eta_{k+1}) \setminus B(x_0, \eta_k).$$

Similarly, there is a directed path in K from x_0 to \underline{e}_1 . It implies that there is a sequence $z_1, \dots, z_{N_1-N_0}$ in K such that $z_k \in U_k$. By assumption (H3) there is a directed path in U_k from z_k to y_k . This directed path necessarily contains an edge of S . Hence, $|S| \geq N_1 - N_0$ so that $c'(S) > 2c(G)$. Then, we apply the Max-Flow Min-Cut theorem to the capacities c' . It gives a flow of finite energy (since c' is squared integrable) compatible with c' and consequently with c , and with strength

$c(G)$. Moreover the proof implies that $c(G)$ is reached for a cut-set $S \subset \underline{B}_E(x_0, \eta_{N_1})$, and the infimum is a minimum. \square

6. APPLICATIONS

Theorem 2 tells us that $G(x_0, x_0)^s$ is integrable for $s < \kappa_0$ where κ_0 is the max-flow of finite energy, compatible with the capacities (α_e) . Proposition 2 tells us that this max-flow coincides with the min-cut under condition (H3). But, it is easy to see that even under assumption (H3), κ_0 may not coincide with the critical integrability exponent κ defined as the supremum of the reals $s > 0$ such that $G(x_0, x_0)^s$ is integrable. Indeed, for example if there is no directed path ending at x_0 , then $G(x_0, x_0) = 1$ and $\kappa = \infty$ whereas $\kappa_0 < \infty$. We strongly conjecture that when the conditions (H1), (H2), (H3) are valid and when there is a flow of finite energy from x_0 , then κ should be the same as the integrability exponent obtained on finite large enough boxes (as computed in [26]). In the case of directed graphs constructed from undirected graphs, it is easy to deduce this conjecture from previous results. This is in fact the most interesting case since it contains \mathbb{Z}^d and more generally finitely generated Caley graphs.

Let $\overline{G} = (V, \overline{E})$ be a connected undirected graph with bounded degree (for simplicity we do not allow loops, cf remark latter). Classically, we can associate a directed graph $G = (V, E)$ by duplicating an edge of \overline{E} in 2 edges with opposite directions. Let (α_e) be a family of weights on the set of directed edges E .

Theorem 3. *Suppose that the weights (α_e) satisfy*

(H1) *there exists $c > 0$ and $C > c$ such that $c \leq \alpha_e \leq C$ for all e in E .*

(H2) *For all vertex x , $\text{div}(\alpha)(x) = 0$.*

Suppose that the simple random walk on the undirected graph \overline{G} is transient. For any x_0 in V , let

$$\kappa_0 = \max\{\text{strength}(\theta), \theta \text{ is a flow from } x_0 \text{ to } \infty \text{ of finite energy such that } \theta \leq \alpha\}$$

Then $\kappa_0 > 0$ and

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty$$

for all $s < \kappa_0$. In particular, the RWDE on G with parameter (α_e) is transient for almost all environment.

(ii) *Assume moreover that the following condition on \overline{G} holds*

(H'3) *There exists a strictly increasing sequence of integers η_n such that $B(x_0, \eta_{n+1}) \setminus B(x_0, \eta_n)$ is connected in \overline{G} .*

Then

$$\mathbb{E}^{(\alpha)}(G(x_0, x_0)^s) < \infty, \text{ if and only if } s < \kappa$$

where

$$\kappa = \min\{\alpha(\partial_E K), K \subset V \text{ is finite connected in } \overline{G}, x_0 \in K \text{ and } K \neq \{x_0\}.\}$$

(With as before $\partial_E K = \{e \in E, \underline{e} \in K, \overline{e} \notin K\}$.)

Remark 9. *In [26], corollary 4, Tournier computed the critical integrability exponent for RWDE on finite graphs: (ii) shows that κ is the same as the critical integrability exponent of the Green function of the RWDE killed when it exits the ball $B(x_0, N)$, for*

N large enough. This suggests that trapping only comes from finite size traps which are due to the non-uniform ellipticity of the environment.

Remark 10. The condition $K \neq \{x_0\}$ comes from the fact that we do not allow loops, so that $\{x_0\}$ itself cannot be a trap. If there is a loop at x_0 it is easy to see from the proof that the same result holds if we remove the restriction $K \neq \{x_0\}$: in this case, under condition (H'3), $\kappa = \kappa_0$.

Proof. Let us recall the definition of a flow on an undirected graph. We choose an arbitrary orientation of the edges of \overline{G} . A flow from x_0 to infinity is a (non-necessarily positive) function $\overline{\theta}$ on the edges such that for the orientation chosen

$$\operatorname{div}(\overline{\theta})(x) = 0, \quad \forall x \neq x_0.$$

The flow $\overline{\theta}$ is a unit flow if moreover $\operatorname{div}(\overline{\theta})(x_0) = 1$. To any flow $\overline{\theta}$ on the undirected graph \overline{G} we can associate a flow θ on the directed graph as follows: for two opposite edges of the directed graph, θ is null on one of them and on the other one it is equal to the absolute value of $\overline{\theta}$ on the corresponding undirected edge. The choice of the edge with positive flow is of course made according to the sign of the flow $\overline{\theta}$ and the orientation of the edges (cf [12], section 2.6). By construction the L_2 norm of θ and $\overline{\theta}$ are the same. Then the result comes from a classical result on electrical networks (cf [12] proposition 2.10, or [13]) which says that the undirected graph \overline{G} is transient if and only if there exists a unit flow with finite energy from a point x_0 to infinity. Hence it implies that $\kappa_0 > 0$.

(ii) For any environment (p) , we have

$$\sum_{\underline{e}=x_0} p_e = 1,$$

hence $G(x_0, x_0)^s$ is integrable if and only if $p_e^s G(x_0, x_0)^s$ is integrable for any e such that $\underline{e} = x_0$. Let e_0 be an edge exiting x_0 and $\alpha^{(e_0)}$ be the weight obtained from α by adding s to α_{e_0} . For any cutset S containing e_0 , $\alpha^{(e_0)}(S) > s$. Hence, by theorem 2 and proposition 2, $p_{e_0}^s G(x_0, x_0)^s$ is integrable if s is smaller than the minimal cutset sum (for (α)) among cutsets which do not contain e_0 . Since the graph has no loop it means that $G(x_0, x_0)^s$ is integrable if

$$(6.1) \quad s < \min\{\alpha(S), S \text{ cutset}, S \not\supset \{e\}_{\underline{e}=x_0}\}.$$

Let S be a cutset which does not contain $\{e\}_{\underline{e}=x_0}$. Let K be the set of vertices that can be reached from x_0 by a directed path using edges of $E \setminus S$. Then, $\partial_E K$ is a cutset contained in S and K is connected in \overline{G} . Moreover $K \neq \{x_0\}$ thanks to the condition on S . This implies that κ is equal to the right hand side of (6.1).

If $s \geq \kappa$, then by taking a large enough box $B(x_0, N)$, we know that the cutset which achieves the minimum in the definition of κ is included in $B(x_0, N)$ (the infimum is reached, cf the proof of proposition 2). From [26], corollary 4, it implies that $G(x_0, x_0)^s$ is not integrable. \square

6.1. The case of \mathbb{Z}^d . Of course theorem 3 (i) and (ii) apply to the model of RWDE on \mathbb{Z}^d described in the introduction, for $d \geq 3$. It is easy to see that on \mathbb{Z}^d , the critical exponent κ is obtained for $K = \{0, e_{i_0}\}$ for some i_0 in $\{1, \dots, d\}$. Hence we have

$$\kappa = 2 \left(\sum_{i=1, \dots, d, i \neq i_0} \alpha_{e_i} + \alpha_{-e_i} \right) + \alpha_{e_{i_0}} + \alpha_{-e_{i_0}}.$$

This proves theorem 1.

6.2. Finitely generated Caley graphs. Let $\{e_1, \dots, e_d\}$ be a finite, symmetric (i.e. the set is stable by inversion), set of generators of a group. We associate with any generator e_i a positive weight $\alpha_i > 0$. The RWDE on the associated Caley graph, with weights $(\alpha_{(g,ge_i)} = \alpha_i)$, has the property (H1) and (H2). Hence, if the simple random walk on the Caley graph is transient, so are the RWDE's.

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