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► **To cite this version:**

Martin Deraux. Forgetful maps between Deligne-Mostow ball quotients. *Geometriae Dedicata*, 2011, 150, pp.377-389. 10.1007/s10711-010-9511-x . hal-00341618v2

HAL Id: hal-00341618

<https://hal.science/hal-00341618v2>

Submitted on 21 May 2010

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FORGETFUL MAPS BETWEEN DELIGNE-MOSTOW BALL QUOTIENTS

MARTIN DERAUX

ABSTRACT. We study forgetful maps between Deligne-Mostow moduli spaces of weighted points on \mathbb{P}^1 , and classify the forgetful maps that extend to a map of orbifolds between the stable completions. The cases where this happens include the Livné fibrations and the Mostow/Toledo maps between complex hyperbolic surfaces. They also include a retraction of a 3-dimensional ball quotient onto one of its 1-dimensional totally geodesic complex sub-manifolds.

1. INTRODUCTION

The goal of this paper is to collect some information about known maps between Deligne-Mostow ball quotients of various dimensions. Only one map given here is new (it gives a non-trivial map from a 3-dimensional ball quotient to a compact Riemann surface), but we also find it worthwhile to commit to print the fact that forgetful maps (in the sense of the present paper, see section 3) cannot yield any other examples.

The simplest examples of holomorphic maps between ball quotients are given by unbranched coverings, obtained simply by taking subgroups of finite index of the fundamental group of the relevant ball quotient. Another class of examples is given by totally geodesic maps, which are also easily constructed between ball quotients of any dimension. In fact, it is well known that there are many holomorphic totally geodesic inclusions between Deligne-Mostow quotients, and we only briefly review how to describe those maps in Proposition 2.8.

Another way to obtain non-trivial holomorphic maps is to construct branched coverings. This can of course easily be done for Riemann surfaces, which appear in the present paper in the form of surjective homomorphisms between various triangle groups. In higher dimensions however, simple branched coverings of ball quotients cannot be ball quotients themselves (see the computation of characteristic classes that comes up in the Mostow-Siu construction, see [15], [6] and [7]), and it is not clear how to determine whether a given ball quotient admits branched coverings (simple or not). Note that the maps between complex hyperbolic surfaces constructed by Mostow and Toledo (see [18]) are certainly not simple branched coverings, in fact they branch around certain complex totally geodesic curves, but they also contract some such curves.

Finally, one might hope to get certain ball quotients to fiber over ball quotients, and this was first achieved by Livné in his thesis, see [10]. His ball quotient are actually closely related to some of the Deligne-Mostow lattices, see §16 of [3], and the corresponding fibrations can then be interpreted in terms of the forgetful map construction presented in this paper; this remark was the basis for the construction of maps to Riemann surfaces used in the author's thesis (see [6]). As mentioned above, in this paper, we show that the forgetful map construction can be generalized to give a similar fibration of a 3-dimensional compact ball

quotient to a Riemann surface, but we do not know of any method to obtain fibrations for higher-dimensional examples.

The main result of this paper can be thought as shedding some light on the general question of existence of surjective holomorphic map $X^m \rightarrow Y^n$ between compact ball quotients when $m > n \geq 2$ (see the question raised by Siu in [16], p. 182). One might think of the statement of Theorem 3.1 as giving some evidence for the non-existence of such maps, but note that the ball quotients considered here are not particularly representative (there are only finitely many Deligne-Mostow ball quotients in dimension ≥ 2), and we are only considering a very specific construction of maps between them.

Our results should also be put in perspective with a recent result of Koziarz and Mok, see [9], that precludes the existence of *submersive* holomorphic maps between ball quotients. The maps obtained in this paper are indeed not submersive, and some explicit fibers are in fact singular divisors.

It should be pointed out that the ball with its Bergman metric is a *rank one* Hermitian symmetric space, which makes it very different from irreducible higher rank Hermitian symmetric spaces. Indeed, if $X = \Gamma \backslash \Omega$ is a compact manifold modelled on a higher rank bounded symmetric domain Ω , then there is no surjective holomorphic map from X onto any nonpositively curved Hermitian manifold apart from geodesic coverings (see [12]).

Another interesting feature of some of the maps that appear in this paper is that they exhibit a retraction of the relevant ball quotient onto one of its totally geodesic submanifolds. In the context of *real* hyperbolic geometry, such retractions have been obtained for certain arithmetic real hyperbolic manifolds (see [1]), without any restriction on the dimension.

This paper was written as an answer to various questions asked over the years by Domingo Toledo, Sai-Kee Yeung and Ngaiming Mok, whom the author wishes to thank for their interest in this work. The existence of a map from a 3-ball quotient to a compact hyperbolic Riemann surface as in Theorem 3.1(v) was also known to Sai-Kee Yeung.

2. REVIEW OF DELIGNE-MOSTOW THEORY

2.1. The Picard integrality condition. We start by collecting some facts from Deligne-Mostow theory (see [2], [13]), following the exposition in [8]. We state only what is needed for the purpose of this paper (for a more thorough survey see [11] for instance).

Given an integer $m \geq 1$, we would like to consider various structures on the moduli space of $m + 3$ points on \mathbb{P}^1 . In order to form a geometric invariant theory quotient of $(\mathbb{P}^1)^{m+3}$, we need to pick a line bundle \mathcal{L} on $(\mathbb{P}^1)^{m+3}$ and a lift of the PGL_2 -action to \mathcal{L} (we refer to this data as a polarization). We shall choose various polarizations, each encoded by a choice of weights $\mu = (\mu_1, \dots, \mu_{m+3}) \in]0, 1[^{m+3}$ for the $(m + 3)$ points; throughout the paper, the weights shall be taken to be rational numbers, and we assume moreover that $\sum_{j=1}^{m+3} \mu_j = 2$.

The line bundle \mathcal{L}_μ on $(\mathbb{P}^1)^{m+3}$ associated to μ is given by $\boxtimes_j \mathcal{O}(2d\mu_j)$ where d is the common denominator of the μ_j (see section 4.6 of [2]). The corresponding geometric invariant theory quotient has a simple description, as we now recall.

We define M to be the set of $(m + 3)$ -tuples of pairwise distinct points on \mathbb{P}^1 , and the following chain of subsets

$$M \subset M_{st}^\mu \subset M_{sst}^\mu$$

of $(\mathbb{P}^1)^{m+3}$ by allowing only certain coincidences of points. M_{st}^μ denotes the subset of $(m + 3)$ -tuples where we allow coincidence of points only when the sum of the corresponding weights

is < 1 . The set M_{sst}^μ is defined similarly, allowing coincidence of points whose weights add up to ≤ 1 .

For each strictly semistable point x , there is a unique partition $\{1, \dots, m+3\} = S_1 \sqcup S_2$ such that for some $j = 1$ or 2 , the points x_i with indices $i \in S_j$ coincide. We then define the corresponding quotient spaces

$$Q \subset Q_{st}^\mu \subset Q_{sst}^\mu$$

where Q_{st}^μ is the set of PGL_2 -orbits of points of M_{st} , and two strictly semistable points x and y are identified if and only if the associated partitions coincide.

Note that the space Q_{sst}^μ is compact, but in general it is singular (whereas Q_{st}^μ is always smooth).

For convenience, we will sometimes write Q^μ for Q , and M^μ for M , even though these spaces depend on μ only through the number $m+3$ of components of μ .

Definition 2.1. We denote by D_{ij}^μ the image in Q_{st}^μ of the set of points $(x_1, \dots, x_{m+3}) \in M_{st}^\mu$ with $x_i = x_j$.

When the dependence on μ is clear, we sometimes write D_{ij} for D_{ij}^μ . This set is a divisor in Q_{st}^μ only if $\mu_i + \mu_j < 1$.

Definition 2.2. The set of weights μ is said to satisfy the Picard integrality condition if

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \quad (\text{INT})$$

whenever $i \neq j$ and $\mu_i + \mu_j < 1$.

For any $i \neq j \in \{1, \dots, m+3\}$, we shall write

$$(2.1) \quad d_{ij}^{(\mu)} = (1 - \mu_i - \mu_j)^{-1}$$

When the Picard integrality condition holds, $d_{ij}^{(\mu)}$ is always an integer (or infinity), regardless of whether or not $\mu_i + \mu_j < 1$, see [2], page 26. When no confusion arises, we shall simply write d_{ij} for $d_{ij}^{(\mu)}$.

When the Picard integrality condition is satisfied, the main result of [2] gives Q_{st}^μ the structure of a complex hyperbolic orbifold in the following sense.

Theorem 2.3. *If μ satisfies condition INT, then there is a lattice Γ_μ in $PU(m, 1)$ such that the orbifold*

$$X_\mu = \Gamma_\mu \backslash \mathbb{B}^m$$

has an underlying smooth complex manifold structure isomorphic to Q_{st}^μ . Under this identification, the singular locus of X_μ consists of the points of $Q_{st}^\mu - Q$, and the divisor D_{ij}^μ has weight $d_{ij}^{(\mu)}$.

In other words, for every torsion-free subgroup $G_\mu \subset \Gamma_\mu$ of finite index, the map $G_\mu \backslash \mathbb{B}^m \rightarrow \Gamma_\mu \backslash \mathbb{B}^m$ can be thought of as giving a description of $G_\mu \backslash \mathbb{B}^m$ as a branched covering of Q_{st}^μ , with ramification of index $d_{ij}^{(\mu)}$ above D_{ij}^μ .

The orbifold fundamental group of $X_\mu = \Gamma_\mu \backslash \mathbb{B}^m$ is of course just Γ_μ , and an explicit presentation for that group can be deduced from Lemma 2.4 below. We follow the notation used in [2] and denote by \widetilde{Q}^μ the preimage in the ball of Q^μ . Since the map $\widetilde{Q}^\mu \rightarrow Q^\mu$ is an unbranched covering, the fundamental group $\pi_1(\widetilde{Q}^\mu)$ is identified with a subgroup K_μ of $\pi_1(Q^\mu)$.

For reasons that are explained in [2], we shall call \widetilde{Q}^μ the *monodromy cover*, since its fundamental group is in fact the kernel of the monodromy representation (see [2], section 8).

The following result is a special case of Lemma 8.6.1 in [3].

Lemma 2.4. *Let γ_{ij} be a small loop that goes once around D_{ij}^μ , i.e. a loop that corresponds to x_i turning once around x_j , see Figure 1 (left). Then*

$$\Gamma_\mu \simeq \pi_1(Q^\mu)/K_\mu$$

where K_μ is the normal subgroup of $\pi_1(Q^\mu)$ generated by the $\gamma_{ij}^{d_{ij}}$, $\mu_i + \mu_j < 1$.

Remark 2.5. The lattice Γ_μ is cocompact if and only if $Q_{st}^\mu = Q_{sst}^\mu$, i.e. no subset of the weights adds up to exactly 1. When Q_{st}^μ is not compact and the corresponding lattice Γ_μ is arithmetic, the compactification Q_{sst}^μ is homeomorphic to the Baily-Borel compactification of $\Gamma_\mu \backslash \mathbb{B}^m$ (in the non-arithmetic cases, the compactification is obtained by adding a finite number of cusps).

A nice feature of the above picture is that the divisors D_{ij}^μ themselves have a modular interpretation, as moduli spaces of $m + 2$ points on \mathbb{P}^1 , with two of the weights μ_i and μ_j replaced by their sum.

More generally, the configurations obtained by letting certain subsets of the $m + 3$ points coalesce give suborbifolds of larger codimension, and they also have a modular interpretation (we only allow points to coalesce if the sum of the corresponding weights is strictly less than 1).

Definition 2.6. For any subset $I \subset \{1, \dots, m+3\}$ consisting of $r+1$ elements, the *contraction* of $\mu = (\mu_1, \dots, \mu_{m+3})$ along I is the $(n-r)$ -tuple obtained by replacing the weights μ_j , $j \in I$ by their sum, i.e. $\mu^{(I)} = (\mu_{i_1}, \dots, \mu_{i_{n-r-1}}, \sum_{i \in I} \mu_i)$, where $\{1, \dots, n\} \setminus I = \{i_1, \dots, i_{n-r-1}\}$.

We shall consider the contraction $\mu^{(I)}$ only when $\sum_{i \in I} \mu_i < 1$, in which case $\mu^{(I)}$ satisfies the running hypotheses of this section, hence defines a lattice $\Gamma_{\mu^{(I)}}$ acting on a ball of dimension $m - r$ as in Theorem 2.3.

Definition 2.7. $\mu^{(I)}$ is called a *hyperbolic contraction* of μ if $\sum_{i \in I} \mu_i < 1$.

Proposition 2.8. *Let μ satisfy condition INT, and let $\mu^{(I)}$ be a hyperbolic contraction of μ . Then $\mu^{(I)}$ also satisfies INT, and moreover there exists a totally geodesic subball $B \in \mathbb{B}^m$ of codimension $r = |I| - 1$ such that the image of B in $\Gamma_\mu \backslash \mathbb{B}^m$ is isomorphic as orbifolds to $\Gamma_{\mu^{(I)}} \backslash \mathbb{B}^{m-r}$. $\Gamma_{\mu^{(I)}}$ is isomorphic to the stabilizer of B modulo its fixed point stabilizer.*

This proposition follows from (8.8.1) in [2], see also Lemma 2.4 in [14]. It gives totally geodesic inclusions between various Deligne-Mostow orbifolds.

2.2. Condition $\frac{1}{2}$ INT. We now discuss how to generalize the results of the previous section as in [13]. The generalized version is also the one that appears in Thurston's account of this theory, see [17]. The idea is to consider moduli of $(m + 3)$ *unordered* points rather than ordered. Since we consider weighted points, we allow identification of $(m + 3)$ -tuples of points that differ by ordering only when the corresponding permutation of the indices preserves the weights.

More specifically, we fix a partition $m + 3 = i_1 + \dots + i_k$, and consider moduli of sets S_1, \dots, S_k of points on \mathbb{P}^1 , with S_j having cardinality i_j for each $j = 1, \dots, k$. In terms of

the notation used in the previous section, this moduli space is a quotient of Q^μ/Σ , where $\Sigma = \Sigma_{i_1} \times \cdots \times \Sigma_{i_k}$ is a product of symmetric groups.

In order to describe the choice of weights in this setting in terms of the notation used in the previous section, we consider $(m+3)$ -tuples μ with $0 < \mu_j < 1$ for all j and $\sum \mu_j = 2$, and break up the index set $I = \{1, \dots, m+3\}$ as a disjoint union $I_1 \sqcup \cdots \sqcup I_k$ in such a way that, for each j , the $\mu_i, i \in I_j$ are equal. In the sequel we shall always assume that the index sets I_j are arranged in increasing order, in the sense that if $j < j'$, all the elements of I_j are smaller than those of $I_{j'}$. Note that, by construction, μ is then invariant under Σ .

Definition 2.9. The pair μ, Σ as above satisfies the half-integrality condition $\frac{1}{2}\text{INT}$ if for all $i \neq j$ such that $\mu_i + \mu_j < 1$, we have

$$(1 - \mu_i - \mu_j)^{-1} \in \begin{cases} \mathbb{Z} & \text{if } i \text{ and } j \text{ are not in the same } \Sigma\text{-orbit} \\ \frac{1}{2}\mathbb{Z} & \text{if } i \text{ and } j \text{ are in the same } \Sigma\text{-orbit} \end{cases}$$

We adapt the definition of the integers d_{ij} accordingly, and set

$$(2.2) \quad d_{ij}^{(\mu, \Sigma)} = \begin{cases} (1 - \mu_i - \mu_j)^{-1} & \text{if } i \text{ and } j \text{ are not in the same } \Sigma\text{-orbit} \\ 2(1 - \mu_i - \mu_j)^{-1} & \text{if } i \text{ and } j \text{ are in the same } \Sigma\text{-orbit} \end{cases}$$

and write $D_{ij}^{\mu, \Sigma}$ for the image of $D_{ij}^\mu \subset Q_{st}^\mu$ in the quotient Q_{st}^μ/Σ . As above, when no confusion arises, we shall simply write d_{ij} instead of $d_{ij}^{(\mu, \Sigma)}$.

Remark 2.10. (1) We do not necessarily assume that $\mu_i \neq \mu_{i'}$ when $i \in I_j$ and $i' \in I_{j'}$ with $j \neq j'$. In other words, we do not assume that the sets of indices I_j are as large as possible to get μ to be Σ -invariant.

(2) When i and j are in the same Σ -orbit, we do not assume that $(1 - \mu_i - \mu_j)^{-1}$ are in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

The action of Σ on M clearly descends to an action on Q , and this action extends to an action on Q_{st}^μ . In general, the quotient space Q_{st}^μ/Σ has singularities, but the content of the main result of [13] is that it carries a complex hyperbolic orbifold structure, similar to the one mentioned in the previous section:

Theorem 2.11. *If μ, Σ satisfies condition $\frac{1}{2}\text{INT}$, then there is a lattice $\Gamma_{\mu, \Sigma}$ in $PU(m, 1)$ such that the orbifold*

$$X_{\mu, \Sigma} = \Gamma_{\mu, \Sigma} \backslash \mathbb{B}^m$$

has the same underlying (singular) algebraic variety as Q_{st}^μ/Σ . Under this identification, the divisors $D_{ij}^{\mu, \Sigma}$ have weight $d_{ij}^{(\mu, \Sigma)}$, and the other divisors with weight > 1 have weight two, and are the images of codimension one fixed point sets of elements of Σ that are contained in Q^μ .

Remark 2.12. (1) There are indeed sometimes elements of Σ that fix a codimension one subset contained in Q^μ . The list of cases where that happens can be deduced from in Lemma 8.3.2 of [3] (the elements that give codimension one fixed point set contained in Q^μ are bitranspositions when $m+3=5$, and tritranspositions when $m+3=6$).

(2) The same criterion as in the previous section determines whether the relevant ball quotient is compact or not, namely $\Gamma_{\mu, \Sigma}$ is cocompact if and only if no subset of the weights adds up to exactly 1.

The analogue of Lemma 2.4 in the context of $\frac{1}{2}\text{INT}$ examples is slightly more complicated to state.

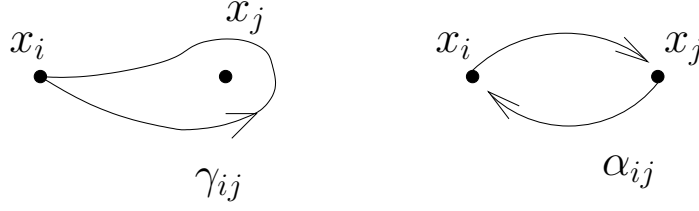


FIGURE 1. A full twist (left) and a half twist (right) between x_i and x_j . γ_{ij} induces a loop in Q_{st}^μ that goes once around D_{ij}^μ . When i and j are not in the same Σ -orbit, γ_{ij} induces a loop in Q_{st}^μ/Σ that goes once around $D_{ij}^{\mu,\Sigma}$. When i and j are in the same Σ -orbit, α_{ij} induces a loop in Q_{st}^μ/Σ that goes once around $D_{ij}^{\mu,\Sigma}$, and α_{ij}^2 induces the same loop as γ_{ij} .

Definition 2.13. Let $Q'^{\mu,\Sigma}$ denote the largest open set of Q^μ on which the action of Σ is free.

When i and j are not in the same Σ -orbit, we use the same notation as above and write γ_{ij} for a full twist between x_i and x_j . If i, j are in the same Σ -orbit, we denote by α_{ij} the corresponding half twist (see Figure 1). Note that the γ_{ij} (resp. α_{ij}) with $\mu_i + \mu_j < 1$ give “small loops” in $Q'^{\mu,\Sigma}/\Sigma$ around $D_{ij}^{\mu,\Sigma}$.

Now consider the elements of Σ that have a codimension one fixed point set contained in Q^μ (see Remark 2.12(1)), and denote by B_1, \dots, B_k the components of their image in $Q'^{\mu,\Sigma}/\Sigma$. Write β_j for a small loop in $Q'^{\mu,\Sigma}/\Sigma$ that goes once around B_j .

Lemma 2.14. *We have*

$$\Gamma_{\mu,\Sigma} \simeq \pi_1(Q'^{\mu,\Sigma}/\Sigma)/K_{\mu,\Sigma}$$

where

$$K_{\mu,\Sigma} = \langle \langle \alpha_{ij}^{d_{ij}}, \beta_i^2 \rangle \rangle$$

is the normal subgroup of $\pi_1(Q'^{\mu,\Sigma}/\Sigma)$ generated by the $\alpha_{ij}^{d_{ij}}$ such that $\mu_i + \mu_j < 1$, and by the β_i^2 , $i = 1, \dots, k$.

The group $K_{\mu,\Sigma}$ can again be interpreted as the fundamental group of a certain unbranched covering $\widetilde{Q'^{\mu,\Sigma}/\Sigma}$ of $Q'^{\mu,\Sigma}/\Sigma$, which we call the monodromy cover (see [13]). Under the identification given in Theorem 2.11, $\widetilde{Q'^{\mu,\Sigma}/\Sigma}$ identifies with an open set in the ball \mathbb{B}^m .

2.3. Obvious commensurabilities. If condition $\frac{1}{2}$ INT holds for a given μ but for two different symmetry groups, then the corresponding two lattices are commensurable, as we now explain.

If the partition I_1, \dots, I_k is a refinement of a partition J_1, \dots, J_l (i.e. every I_i is contained in some J_j), and we denote by $\Sigma^{(I)}$ (resp. $\Sigma^{(J)}$) the corresponding symmetry group preserving the partition $\{I_i\}$ (resp. preserving $\{J_j\}$), then clearly $\Sigma^{(I)}$ is a subgroup of $\Sigma^{(J)}$, hence there is a natural map

$$Q_{st}^\mu/\Sigma^{(I)} \rightarrow Q_{st}^\mu/\Sigma^{(J)}$$

Provided that $\mu, \Sigma^{(I)}$ and $\mu, \Sigma^{(J)}$ both satisfy condition $\frac{1}{2}$ INT, one gets accordingly that

$$\Gamma_{\mu,\Sigma^{(I)}} \subset \Gamma_{\mu,\Sigma^{(J)}}$$

is a subset of index $[\Sigma^{(J)} : \Sigma^{(I)}]$.

For each μ that satisfies $\frac{1}{2}$ INT for some symmetry group, there is a *finest* partition of the indices for which condition $\frac{1}{2}$ INT holds, and it is obtained by requiring that $(1 - \mu_i + \mu_j)^{-1}$ be in $\frac{1}{2}\mathbb{Z}$ but *not* in \mathbb{Z} for each $i \neq j$ in a common subset of the partition.

Remark 2.15. There is an extra relation between 1-dimensional Deligne-Mostow moduli spaces, alluded to in [2], namely the moduli space for $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ is isomorphic to the one for $(1 - \mu_1, 1 - \mu_2, 1 - \mu_3, 1 - \mu_4)$. We shall refer to these two sets of weights as being *dual* to each other, see [2], p. 84.

3. FORGETFUL MAPS

The rough idea of our construction is to consider the obvious maps from the moduli space of $m' = m + 3$ points to the moduli space of $n' = n + 3$ points on \mathbb{P}^1 , obtained by forgetting $m - n$ points (here we assume that $m \geq n$). This is only a rough idea, because we are actually interested in moduli spaces of *weighted* points on \mathbb{P}^1 .

Without loss of generality, we may assume that the points that get forgotten are the last $m - n$, i.e. we consider the map $(\mathbb{P}^1)^{m'} \rightarrow (\mathbb{P}^1)^{n'}$ given by

$$(3.1) \quad (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_{m'}) \mapsto (x_1, \dots, x_{n'}).$$

This map is clearly PGL_2 -equivariant, hence it always induces a holomorphic map $Q^\mu \rightarrow Q^\nu$ for any set of weights $\mu = (\mu_1, \dots, \mu_{m+3})$ and $\nu = (\nu_1, \dots, \nu_{n+3})$.

We now fix two choices μ, Σ and ν, \mathcal{T} of weights with symmetry that satisfy condition $\frac{1}{2}$ INT, and we wish to investigate the following question. *When does the above forgetful map induce a map of orbifolds between the corresponding moduli spaces $X_{\mu, \Sigma}$ and $X_{\nu, \mathcal{T}}$?*

The fact that the forgetful maps do indeed induce maps of orbifolds for certain choices of μ, Σ and ν, \mathcal{T} was already noticed in a couple of places in the literature. For $m = 2$ and $n = 1$ this was used in [5] (see also [6]). For $m = n = 2$, it was used by Toledo in [18]. Note that in the equidimensional case, the number of weighted points is the same for both moduli spaces in the source and target, so the forgetful map is simply the identity on the level of $Q^\mu = Q^\nu$ (but it turns out that its extension to stable moduli spaces contracts some divisors).

The goal of the present paper is to give the list of all other cases where the forgetful maps give maps of orbifolds between the orbifold ball quotients. For simplicity we focus on the case of compact orbifolds, i.e. we consider sets of weights μ such that $Q_{st}^\mu = Q_{sst}^\mu$ (equivalently that no sum $\sum_{j \in J} \mu_j$ is equal to one, for any $J \subset \{1, \dots, m + 3\}$).

One way to summarize the results of the classification is the following.

Theorem 3.1. *Suppose $m \geq n$, μ, Σ and ν, \mathcal{T} satisfy condition $\frac{1}{2}$ INT, and assume that the orbifolds $X_{\mu, \Sigma}$ and $X_{\nu, \mathcal{T}}$ are compact, of dimension m and n respectively.*

- (i) *If (m, n) is not $(1, 1)$, $(2, 2)$, $(2, 1)$ or $(3, 1)$, then the forgetful map never induces a map of orbifolds.*
- (ii) *When $(m, n) = (1, 1)$, there are many forgetful maps that induce maps of orbifolds (giving among others surjective homomorphisms between triangle groups).*
- (iii) *When $(m, n) = (2, 2)$, the forgetful maps that induce maps of orbifolds correspond to the Mostow/Toledo maps.*
- (iv) *Many forgetful maps induce maps of orbifolds in the case $(m, n) = (2, 1)$, some corresponding to the Livné fibrations.*

(v) When $(m, n) = (3, 1)$, up to symmetry and obvious commensurability (see section 2.2), there is precisely one forgetful map that yields a map of orbifolds, corresponding to the weights $\mu = (3, 3, 3, 3, 3, 1)/8$ and $\nu = (3, 3, 3, 7)/8$.

As noted in the introduction, Koziarz and Mok have recently shown that there are no submersive maps $X^m \rightarrow X^n$ between compact ball quotients apart from unbranched coverings (this holds for finite volume ball quotients as well provided $n \geq 2$, see [9]). The map $X^3 \rightarrow X^1$ that appears in part (v) of the theorem is of course not a submersion (for a description of the non-submersive locus, see Remark 3.8). The fact that the above construction should produce maps from a compact 3-ball quotient to a compact hyperbolic Riemann surface seems not to have appeared anywhere in the literature.

We shall not attempt to use general results from geometric invariant theory, since the above question can be answered in a fairly elementary way by using Theorem 2.11 and Lemma 2.14. In order for the forgetful map to induce a map between the orbifold quotients, we shall require that the forgetful map be compatible with the symmetry groups, in the sense that

$$Q'^{\mu, \Sigma} \text{ maps into } Q'^{\nu, \mathcal{T}}, \text{ and this map descends to a map } Q'^{\mu, \Sigma}/\Sigma \rightarrow Q'^{\nu, \mathcal{T}}/\mathcal{T}.$$

In order for the forgetful map to induce a map of orbifolds, we need to require moreover that the map $Q'^{\mu, \Sigma}/\Sigma \rightarrow Q'^{\nu, \mathcal{T}}/\mathcal{T}$ lifts to monodromy covers, which in view of Lemma 2.14 can be equivalently expressed by the fact that

$$K_{\mu, \Sigma} \text{ maps into } K_{\nu, \mathcal{T}}.$$

If the latter condition holds, then the lift defines a holomorphic map from the complement of a discrete union of subballs in \mathbb{B}^m , which extends to the whole \mathbb{B}^m by Hartogs' theorem.

3.1. Compatibility of the symmetry groups. As in the previous paragraphs, we fix two sets of weights with symmetry μ, Σ and ν, \mathcal{T} that both satisfy condition $\frac{1}{2}$ INT. In order for the map (3.1) to induce a map

$$Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T},$$

we need to require a compatibility condition between the partition $I = I_1 \sqcup \cdots \sqcup I_k$ (resp. $J = J_1 \sqcup \cdots \sqcup J_l$) corresponding to μ, Σ (resp. to ν, \mathcal{T}).

Lemma 3.2. *The forgetful map $Q^\mu \rightarrow Q^\nu$ descends to a map $Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T}$ if and only if for each i , I_i is either entirely forgotten (i.e. $j > n'$ for all $j \in I_i$), or contained in J_j for some j .*

Finally, the condition that $Q'^{\mu, \Sigma}$ be mapped to $Q'^{\nu, \mathcal{T}}$, which corresponds to saying that the forgetful map needs to map smooth points to smooth points (this is to be the case if we want the map to be a map of orbifolds), can be checked by finding an explicit description of these two open sets. Indeed, one can easily find a list of the fixed points of the action of Σ (resp. \mathcal{T}) on Q^μ (resp. Q^ν), using the technique of Lemma 8.3.2 of [3].

We do not go through the trouble of writing down a general combinatorial version of this condition, because the divisibility conditions stated in the next section turn out to be enough to prove the result of Theorem 3.1. Indeed, at least for higher-dimensional targets, there are very few cases where the divisibility conditions hold (see Proposition 3.5). For each case where the divisibility conditions do hold, we shall check whether we have a well-defined map $Q'^{\mu, \Sigma} \rightarrow Q'^{\nu, \mathcal{T}}$.

Remark 3.3. The particular case of forgetful maps between INT examples corresponds to the case when Σ and \mathcal{T} are both trivial. In that case, the condition stated in Lemma 3.2 is of course always trivially satisfied, and $Q'^{\mu, \Sigma} = Q^\mu$ and $Q'^{\nu, \mathcal{T}} = Q^\nu$.

3.2. Lifting to monodromy covers. Recall from section 2.2 that, in certain cases, the map $\mathbb{B}^m \rightarrow Q_{st}^\mu/\Sigma$ ramifies over points of Q^μ/Σ . In order to get an unramified covering, one needs to get rid of the fixed points in Q^μ of the action of Σ and work with the open set $Q^{\mu,\Sigma}/\Sigma \subset Q^\mu/\Sigma$ instead (see Definition 2.13). In most cases, $\pi_1(Q^{\mu,\Sigma}/\Sigma) \simeq \pi_1(Q^\mu/\Sigma)$, but it can happen that some codimension one component of the fixed point of some $\sigma \in \Sigma$ is contained in Q^μ , see Remark 2.12(1).

In any case, in order to lift the map to monodromy covers, we require that $Q^{\mu,\Sigma}$ be mapped into $Q^{\nu,\mathcal{T}}$, and denote by $f_* : \pi_1(Q^{\mu,\Sigma}) \rightarrow \pi_1(Q^{\nu,\mathcal{T}})$ the induced map on the level of fundamental groups. From the discussion in section 2.2, we have:

Proposition 3.4. *Suppose that $f(Q^{\mu,\Sigma}) \subset Q^{\nu,\mathcal{T}}$, and that the symmetries are compatible in the sense of Lemma 3.2. Then the map lifts to a map $\tilde{f} : \widetilde{Q^{\mu,\Sigma}/\Sigma} \subset \widetilde{Q^\mu/\Sigma}$ if and only if $f_*(K_{\mu,\Sigma}) \subset K_{\nu,\mathcal{T}}$.*

Moreover, Lemma 2.14 gives an explicit way to check the condition

$$f_*(K_{\mu,\Sigma}) \subset K_{\nu,\mathcal{T}},$$

since it reduces to verifying a divisibility condition between the orbifold weights of the source and the target.

Specifically, in order to get a map of orbifolds $X_\mu \rightarrow X_\nu$, we need to require that the weights in the target divide the weights in the source, whenever a codimension one fixed point set of elements of Σ in Q^μ gets mapped onto a codimension one fixed point set of elements of \mathcal{T} in Q^ν .

In other words, whenever $i \neq j$, $i, j \leq n + 3$, and $\mu_i + \mu_j < 1$, we require that

$$(3.2) \quad d_{ij}^{\nu,\mathcal{T}} \text{ divides } d_{ij}^{\mu,\Sigma}.$$

Recall from equation (2.2) that condition (3.2) means that:

- $(1 - \nu_i - \nu_j)^{-1}$ divides $(1 - \mu_i - \mu_j)^{-1}$ if i and j are not in the same Σ orbit nor in the same \mathcal{T} -orbit;
- $(1 - \nu_i - \nu_j)^{-1}$ divides $2(1 - \mu_i - \mu_j)^{-1}$ if i and j are in the same \mathcal{T} -orbit but not in the same Σ -orbit;
- $2(1 - \nu_i - \nu_j)^{-1}$ divides $2(1 - \mu_i - \mu_j)^{-1}$ if i and j are in the same Σ orbit and in the same \mathcal{T} -orbit.

3.3. Combinatorial check. The proof of part (1) of Theorem 3.1 amounts to a combinatorial check on the list of tuples of weights that satisfy condition $\frac{1}{2}$ INT, using the results of the next few sections.

Specifically, the necessary conditions of Lemma 3.2 and Proposition 3.4 can be checked using a computer, so one can easily find the list of examples of forgetful maps between any finite list of Deligne-Mostow moduli spaces.

Recall that the list of n -tuples satisfying condition $\frac{1}{2}$ INT is finite for $n \geq 5$ (it can be found in [17], for instance), but there are infinitely many 4-tuples that satisfy $\frac{1}{2}$ INT (corresponding to the fact that there are infinitely many hyperbolic triangle groups).

However, from the translation in terms of divisibility conditions of Proposition 3.4 (see the end of section 3.2), one easily gets a bound on the least common denominator of the target weights, so in order to get maps to strictly smaller dimension, we need only consider finitely many 1-dimensional moduli spaces as targets. Specifically, since all $\frac{1}{2}$ INT k -tuples with $k \geq 5$

have least common denominator ≤ 42 , in order to get maps from dimension m to dimension n with $m > n = 1$, we need only consider 4-tuples with denominators ≤ 84 .

Proposition 3.5. *Suppose μ, Σ and ν, \mathcal{T} satisfy the compatibility conditions of Lemma 3.2 and the divisibility conditions (3.2). If X_μ is cocompact and $m \geq 3$, then up to obvious commensurabilities and symmetry, the pair μ, ν is one of the following:*

$$\begin{aligned} \mu &= (3, 3, 3, 3, 3, 1)/8 \text{ and } \nu = (3, 3, 3, 7)/8, & \Sigma = \mathcal{T} &= \{Id\}; \\ \mu &= (3, 3, 3, 3, 3, 1)/8 \text{ and } \nu = (5, 5, 5, 1)/8, & \Sigma = \mathcal{T} &= \{Id\}; \\ \mu &= (3, 3, 3, 3, 6, 2)/10 \text{ and } \nu = (3, 3, 3, 3, 8)/10, & \Sigma = \mathcal{T} &= S_4. \end{aligned}$$

Note that the first two cases are essentially identical, since the groups for $(3, 3, 3, 7)/8$ and $(5, 5, 5, 1)/8$ are isomorphic because the sum of their respective weights is one (see Remark 2.15).

Proposition 3.5 is proved by direct case by case verification (reduced to a finite problem because of the discussion of the beginning of this section). The author did this by writing a computer program that generates all Deligne-Mostow sets of weights, as well as their permutations (or rather all essential permutations, meaning that we take the symmetry of the weights into account). For a given pair μ, ν , it is of course straightforward to check the compatibility and divisibility conditions Proposition 3.5. Computer code that produces this list is available on the author's webpage, see [4].

We now finish the proof of Theorem 3.1. First note that the forgetful map corresponding to $\mu = (3, 3, 3, 3, 6, 2)/10$, $\nu = (3, 3, 3, 3, 8)/10$, with $\Sigma = \mathcal{T} = S_4$ does not yield a map of orbifolds, because $Q^{\mu, \Sigma}$ does not map to $Q^{\nu, \mathcal{T}}$. More specifically, the fixed point sets in Q^μ and Q^ν of bitranspositions have codimension one in Q^ν only, not in Q^μ (see Lemma 8.3.2 of [3], and also Remark 2.12(1)). If the map were a map of orbifolds, the induced homomorphism between the orbifold fundamental groups would have to map the trivial element to a nontrivial one.

Finally we need to show that the map $X_\mu \rightarrow X_\nu$ for $\mu = (3, 3, 3, 3, 3, 1)/8$, $\nu = (3, 3, 3, 7)/8$ (and $\Sigma = \mathcal{T} = \{Id\}$) does yield a map of orbifolds. In that case we clearly have a map $Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T}$, that lifts to monodromy covers because the divisibility condition is satisfied (see Proposition 3.4). As mentioned in [18], this map has a holomorphic extension to the Fox completions $Q_{st}^\mu \rightarrow Q_{st}^\nu$ because of Hartog's theorem, and this extension is still equivariant, so it induces a map of orbifolds $X_\mu \rightarrow X_\nu$.

We shall give a more concrete description of the corresponding map $X_\mu \rightarrow X_\nu$ in section 3.4.

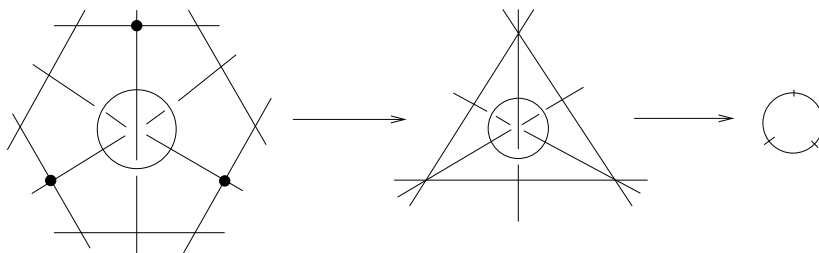
3.4. An example. We now give some detail on the example that appears in Theorem 3.1, part (v). Consider

$$\mu = (3, 3, 3, 3, 3, 1)/8$$

and

$$\nu = (3, 3, 3, 7)/8.$$

We shall choose the symmetry groups Σ, \mathcal{T} to be trivial (there are several maps obtained for various non-trivial symmetry groups, but as mentioned above, the corresponding groups are commensurable, see the discussion in the end of section 2.2). Accordingly, we use the notation of section 2.1 rather than of section 2.2.

FIGURE 2. Naïve description of the map $\widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$.

Note that Q_{st}^ν is simply a copy of P^1 , with three orbifold points (see [2], p. 29). The isomorphism is provided simply by the cross ratio

$$(3.3) \quad \frac{x_3 - x_2}{x_3 - x_1} \cdot \frac{x_4 - x_1}{x_4 - x_2},$$

which is well-defined as long as no triple of points in $\{x_1, x_2, x_3, x_4\}$ coincide (this never happens in M_{st}^ν , since any three weights of ν add up to more than one).

Now the map that sends (x_1, \dots, x_6) to the cross-ratio of x_1, x_2, x_3, x_4 , see equation (3.3), is well-defined on M_{st}^μ (again because no triple of weights of μ add up to more than one), and clearly descends to Q_{st}^μ .

The following result is contained in Theorem 4.1 of [8].

Lemma 3.6. Q_{st}^μ is a \mathbb{P}^1 -bundle over $\widehat{\mathbb{P}^2}$, where $\widehat{\mathbb{P}^2}$ denotes \mathbb{P}^2 blown-up at a generic quadruple of points.

Proof: Consider the map forgetting x_6 from M_{st}^μ into $(\mathbb{P}^1)^5$. Since the stability condition allows any pair among the first five points to coalesce, M_{st}^μ maps into M_{st}^λ , where $\lambda = (2, 2, 2, 2, 2)/5$. The corresponding quotient Q_{st}^λ is known to be $\widehat{\mathbb{P}^2}$ (which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in three distinct points of the diagonal), see [2], Example 1, p. 33 for instance.

It is easy to check that the fibers of the corresponding map $M_{st}^\mu \rightarrow M_{st}^\lambda$ are all projective lines, and the stabilizer of a fiber (for the diagonal action of PGL_2) is trivial, so the fibers of the induced map $Q_{st}^\mu \rightarrow Q_{st}^\lambda$ are also projective lines. \square

Remark 3.7. $\widehat{\mathbb{P}^2}$ is in fact homeomorphic to Q_{st}^θ for various 5-tuples θ , but the corresponding map $Q_{st}^\mu \rightarrow Q_{st}^\theta$ is never a map of orbifolds (see Theorem 3.1).

Now there is an obvious map $\widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$, coming from the fibration of $\mathbb{P}^2 \setminus \{p\}$ over the \mathbb{P}^1 of lines through p (the former map contracts three of the exceptional divisors of $\widehat{\mathbb{P}^2}$, and maps onto the other exceptional divisor).

We claim that the composition $Q_{st}^\mu \rightarrow \widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$ can be made into a map of orbifolds, where \mathbb{P}^1 is the orbifold ball quotient Q_{st}^ν . We denote by $f : Q_{st}^\mu \rightarrow Q_{st}^\nu$ the corresponding holomorphic map. The set of points of Q_{st}^μ where df is not surjective consists of three \mathbb{P}^1 's, which are the fibers over three points of $\widehat{\mathbb{P}^2}$, represented by the solid dots on Figure 2 (left).

Note also that the preimage of each of the three singular points on Q_{st}^ν consists of two divisors, for instance D_{12} and D_{34} map to the same point in Q_{st}^ν .

We close this section by noting that the divisibility conditions (3.2) are trivially satisfied in this example (recall that we take Σ and \mathcal{T} to be trivial, so K_μ is the same as $K_{\mu,\Sigma}$, etc). Indeed, recall that K_μ is the normal subgroup generated by the $\gamma_{ij}^{d_{ij}}$, where d_{ij} is either 2 or 4. Note that $d_{ij} = 2$ only when one of i or j is equal to 6, but then the loop γ_{ij} has trivial image. Since all the $(1 - \nu_i - \nu_j)^{-1}$ are ± 4 , K_μ maps into K_ν .

Remark 3.8. Note that the divisors D_{i5} , $i = 1, \dots, 4$, D_{i6} , $i = 1, \dots, 5$, surject onto $Q_{st}^\nu \simeq P^1$, whereas the other divisors get mapped to the three orbifold points. For instance, the fiber over $D_{12}^\nu \in \mathbb{P}^1$, which corresponds to $x_1 = x_2$ in Q_{st}^ν , is given by the union of the divisors D_{12}^μ and D_{34}^μ in Q_{st}^μ (these are both projective planes and their intersection is a projective line).

4. MAPS BETWEEN NON-COMPACT EXAMPLES

The same construction works for non compact moduli spaces, and one gets maps that have the same behavior as the ones between compact ones. Here the divisibility condition is easily adapted to allow for infinite weights if some pairs of weights add up to exactly one. The list of pairs of weights with symmetry (at least one of which is non-compact, with $m > 2$ or $n > 1$) that satisfy the compatibility and divisibility conditions is the following:

- $\mu = (2, 2, 2, 3, 3)/6$, $\Sigma = S_3$
 $\nu = (2, 2, 2, 1, 5)/6$, $\mathcal{T} = S_3$;
- $\mu = (4, 4, 4, 5, 7)/12$, $\Sigma = S_3$
 $\nu = (2, 2, 2, 1, 5)/6$, $\mathcal{T} = S_3$;
- $\mu = (2, 2, 3, 3, 1, 1)/6$, $\Sigma = S_2$
 $\nu = (1, 7, 7, 9)/12$, $\mathcal{T} = \{id\}$;
- $\mu = (2, 2, 3, 3, 1, 1)/6$, $\Sigma = S_2$
 $\nu = (1, 3, 4, 4)/6$, $\mathcal{T} = \{id\}$.

Note that in particular one does not get more pairs (m, n) of dimensions that are related by a surjective map of orbifolds coming from a forgetful map than in the compact case; in other words part (i) of the statement of Theorem 3.1 remains true for non-compact examples.

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