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# FORGETFUL MAPS BETWEEN DELIGNE-MOSTOW BALL QUOTIENTS

MARTIN DERAUX

ABSTRACT. We study forgetful maps between Deligne-Mostow moduli spaces of weighted points on  $\mathbb{P}^1$ , and classify the forgetful maps that extend to a map of orbifolds between the stable completions. The cases where this happens include the Livné fibrations and the Mostow/Toledo maps between complex hyperbolic surfaces. They also include a retraction of a 3-dimensional ball quotient onto one of its 1-dimensional totally geodesic complex submanifolds.

## 1. INTRODUCTION

The goal of this paper is to collect some information about known maps between Deligne-Mostow ball quotients of various dimensions. Only one map given here is new (it gives a non-trivial map from a 3-dimensional ball quotient to a compact Riemann surface), but we also find it worthwhile to commit to print the fact that forgetful maps (in the sense of the present paper, see section 3) cannot yield any other examples.

The simplest examples of holomorphic maps between ball quotients are given by unbranched coverings, obtained simply by taking subgroups of finite index of the fundamental group of the relevant ball quotient. Another class of examples is given by totally geodesic maps, which are also easily constructed between ball quotients of any dimension. In fact, it is well known that there are many holomorphic totally geodesic inclusions between Deligne-Mostow quotients, and we only briefly review how to describe those maps in Proposition 2.8.

Another way to obtain non-trivial holomorphic maps is to construct (simple) branched coverings. This can of course easily be done for Riemann surfaces, and in fact the maps coming from covering relations between triangle groups can be described by the constructions in this paper. In higher dimensions however, simple branched coverings of ball quotients cannot be ball quotients themselves (see the computation of characteristic classes that comes up in the Mostow-Siu construction, see [14], [5] and [6]), and it is not clear how to construct branched coverings, simple or not, of a given ball quotient. Note that the maps between complex hyperbolic surfaces constructed

by Mostow and Toledo (see [17]) are certainly not simple branched coverings, in fact they branch around certain complex totally geodesic curves, but they also contract some such curves.

Finally, one might hope to get certain ball quotients to fiber over ball quotients, and this was first achieved by Livné in his thesis, see [9]. His ball quotients are actually closely related to some of the Deligne-Mostow lattices, see §16 of [3], and the corresponding fibrations can then be interpreted in terms of the forgetful map construction presented in this paper; this remark was the basis for the construction of maps to Riemann surfaces used in the author's thesis (see [5]). As mentioned above, we show in this paper that the forgetful map construction can be generalized to give a similar fibration of a 3-dimensional compact ball quotient to a Riemann surface, but we do not know of any method to obtain fibrations for higher-dimensional examples.

The main result of this paper can be thought as shedding some light on the general question of existence of surjective holomorphic map  $X^m \rightarrow Y^n$  between compact ball quotients when  $m > n \geq 2$  (see the question raised by Siu in [15], p. 182). One might think of the statement of Theorem 3.1 as giving some evidence for the non-existence of such maps, but note that the ball quotients considered here are not particularly representative (there are only finitely many Deligne-Mostow ball quotients), and we are only considering a very specific construction of maps between them.

Our results should also be put in perspective with a recent result of Koziarz and Mok, see [8], that precludes the existence of *submersive* holomorphic maps between ball quotients. The maps obtained in this paper are indeed not submersive, and some explicit fibers are in fact singular divisors.

It should be pointed out that the ball with its Bergman metric is a *rank one* Hermitian symmetric space, which makes it very different from irreducible higher rank Hermitian symmetric spaces. Indeed, if  $X = \Gamma \backslash \Omega$  is a compact manifold modelled on a higher rank bounded symmetric domain  $\Omega$ , then there is no non constant holomorphic map from  $X$  to any nonpositively curved Kähler manifold (see [11]).

Another interesting feature of some of the maps that appear in this paper is that they exhibit a retraction of the relevant ball quotient onto one of its totally geodesic submanifolds. In the context of *real* hyperbolic geometry, such retractions have been obtained for certain arithmetic real hyperbolic manifolds (see [1]), without any restriction on the dimension.

This paper was written as an answer to various questions asked over the years by Domingo Toledo, Sai-Kee Yeung and Ngaiming Mok, whom the author wishes to thank for their interest in this work. The existence of a map from a 3-ball quotient to a compact hyperbolic Riemann surface as in Theorem 3.1(2) was also known to Sai-Kee Yeung (see Remark 3.6).

## 2. REVIEW OF DELIGNE-MOSTOW THEORY

**2.1. The Picard integrality condition.** We start by collecting some facts from Deligne-Mostow theory (see [2], [12]), following the exposition in [7]. We state only what is needed for the purpose of this paper (for a more thorough survey see [10] for instance).

Given an integer  $m \geq 1$ , we would like to consider various structures on the moduli space of  $m+3$  points on  $\mathbb{P}^1$ . In order to form a geometric invariant theory quotient of  $(\mathbb{P}^1)^{m+3}$ , we need to pick a line bundle  $\mathcal{L}$  on  $(\mathbb{P}^1)^{m+3}$  and a lift of the  $PGL_2$ -action to  $\mathcal{L}$  (we refer to this data as a polarization). We shall choose various polarizations, each encoded by a choice of weights  $\mu = (\mu_1, \dots, \mu_{m+3}) \in ]0, 1[^{m+3}$  for the  $(m+3)$  points; throughout the paper, the weights shall be taken to be rational numbers, and we assume moreover that  $\sum_{j=1}^{m+3} \mu_j = 2$ .

The line bundle  $\mathcal{L}_\mu$  on  $(\mathbb{P}^1)^{m+3}$  associated to  $\mu$  is given by  $\boxtimes_j \mathcal{O}(2d\mu_j)$  where  $d$  is the common denominator of the  $\mu_j$  (see section 4.6 of [2]). The corresponding geometric invariant theory quotient has a simple description, as we now recall.

We define  $M$  to be the set of  $(m+3)$ -tuples of pairwise distinct points on  $\mathbb{P}^1$ , and the following chain of subsets

$$M \subset M_{st}^\mu \subset M_{sst}^\mu$$

of  $(\mathbb{P}^1)^{m+3}$  by allowing only certain coincidences of points.  $M_{st}^\mu$  denotes the subset of  $(m+3)$ -tuples where we allow coincidence of points only when the sum of the corresponding weights is  $< 1$ . The set  $M_{sst}^\mu$  is defined similarly, allowing coincidence of points whose weights add up to  $\leq 1$ .

For each strictly semistable point  $x$ , there is a unique partition  $\{1, \dots, m+3\} = S_1 \sqcup S_2$  such that for some  $j = 1$  or  $2$ , the points  $x_i$  with indices  $i \in S_j$  coincide. We then define the corresponding quotient spaces

$$Q \subset Q_{st}^\mu \subset Q_{sst}^\mu$$

where  $Q_{st}^\mu$  are  $PGL_2$ -orbits of points of  $M_{st}^\mu$ , and two strictly semistable points  $x$  and  $y$  are identified if and only if the associated partitions coincide.

Note that the space  $Q_{sst}^\mu$  is compact, but in general it is singular (even though  $Q_{st}^\mu$  is always smooth).

For convenience, we will sometimes write  $Q^\mu$  for  $Q$ , and  $M^\mu$  for  $M$ , even though these spaces depend on  $\mu$  only through the number  $m+3$  of components of  $\mu$ .

**Definition 2.1.** We denote by  $D_{ij}^\mu$  the image in  $Q_{st}^\mu$  of the set of points  $(x_1, \dots, x_{m+3}) \in M_{st}^\mu$  with  $x_i = x_j$ .

When the dependence on  $\mu$  is clear, we sometimes write  $D_{ij}$  for  $D_{ij}^\mu$ . This set is a divisor in  $Q_{st}^\mu$  only if  $\mu_i + \mu_j < 1$ .

**Definition 2.2.** The set of weights  $\mu$  is said to satisfy the Picard integrality condition if

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \quad (\text{INT})$$

whenever  $i \neq j$  and  $\mu_i + \mu_j < 1$ .

For any  $i \neq j \in \{1, \dots, m+3\}$ , we shall write

$$(2.1) \quad d_{ij}^{(\mu)} = (1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \cup \{\infty\}$$

When the Picard integrality condition holds, this is always an integer (or infinity), regardless of whether or not  $\mu_i + \mu_j < 1$ , see [2], page 26. When no confusion arises, we shall simply write  $d_{ij}$  for  $d_{ij}^{(\mu)}$ .

When the Picard integrality condition is satisfied, the main result of [2] gives  $Q_{st}^\mu$  the structure of a complex hyperbolic orbifold in the following sense.

**Theorem 2.3.** *If  $\mu$  satisfies condition INT, then there is a lattice  $\Gamma_\mu$  in  $PU(m, 1)$  such that the orbifold*

$$X_\mu = \Gamma_\mu \backslash \mathbb{B}^m$$

*has an underlying smooth complex manifold structure isomorphic to  $Q_{st}^\mu$ . Under this identification, the singular locus of  $X_\mu$  consists of the points of  $Q_{st}^\mu - Q$ , and the divisor  $D_{ij}^\mu$  has weight  $d_{ij}^{(\mu)}$ .*

In other words, for every torsion-free subgroup  $G_\mu \subset \Gamma_\mu$  of finite index, the map  $G_\mu \backslash \mathbb{B}^m \rightarrow \Gamma_\mu \backslash \mathbb{B}^m$  can be thought of as giving a description of  $G_\mu \backslash \mathbb{B}^m$  as a branched covering of  $Q_{st}^\mu$ , with ramification of index  $d_{ij}^{(\mu)}$  above  $D_{ij}^\mu$ .

**Remark 2.4.** The lattice  $\Gamma_\mu$  is cocompact if and only if  $Q_{st}^\mu = Q_{sst}^\mu$ , i.e. no subset of the weights adds up to exactly 1. When  $Q_{st}^\mu$  is not compact, the compactification  $Q_{sst}^\mu$  is homeomorphic to the Baily-Borel compactification of  $\Gamma_\mu \backslash \mathbb{B}^m$ .

The orbifold fundamental group of  $X_\mu = \Gamma_\mu \backslash \mathbb{B}^m$  is of course just  $\Gamma_\mu$ , and an explicit presentation for that group can be deduced from the following result, which is a special case of Lemma (8.6.1) in [3].

**Lemma 2.5.** *Let  $\gamma_{ij}$  be a small loop that goes once around  $D_{ij}^\mu$ , i.e. a loop that corresponds to  $x_i$  turning once around  $x_j$ , see Figure 1 (left). Then*

$$\Gamma_\mu \simeq \pi_1(Q^\mu) / K_\mu$$

*where  $K_\mu$  is the normal subgroup of  $\pi_1(Q^\mu)$  generated by the  $\gamma_{ij}^{d_{ij}}$ ,  $\mu_i + \mu_j < 1$ .*

A nice feature of the above picture is that the divisors  $D_{ij}^\mu$ , themselves have a modular interpretation, as moduli spaces of  $m+2$  points on  $\mathbb{P}^1$ , with two of the weights  $\mu_i$  and  $\mu_j$  replaced by their sum.

More generally, the configurations obtained by letting certain subsets of the  $m+3$  points coalesce give suborbifolds of larger codimension, and they also have a modular interpretation (we only allow points to coalesce if the sum of the corresponding weights is strictly less than 1).

**Definition 2.6.** For any subset  $I \subset \{1, \dots, m+3\}$  consisting of  $r+1$  elements, the *contraction* of  $\mu = (\mu_1, \dots, \mu_{m+3})$  along  $I$  is the  $(n-r)$ -tuple obtained by replacing the weights  $\mu_j$ ,  $j \in I$  by their sum, i.e.  $\mu^{(I)} = (\mu_{i_1}, \dots, \mu_{i_{n-r-1}}, \sum_{i \in I} \mu_i)$ , where  $\{1, \dots, n\} \setminus I = \{i_1, \dots, i_{n-r-1}\}$ .

We shall consider the contraction  $\mu^{(I)}$  only when  $\sum_{i \in I} \mu_i < 1$ , in which case  $\mu^{(I)}$  satisfies the running hypotheses of this section, hence defines a lattice  $\Gamma_{\mu^{(I)}}$  acting on a ball of dimension  $m-r$  as in Theorem 2.3.

**Definition 2.7.**  $\mu^{(I)}$  is called a *hyperbolic contraction* of  $\mu$  if  $\sum_{i \in I} \mu_i < 1$ .

**Proposition 2.8.** *Let  $\mu$  satisfy condition INT, and let  $\mu^{(I)}$  be a hyperbolic contraction of  $\mu$ . Then  $\mu^{(I)}$  also satisfies INT, and moreover there exists a totally geodesic subball  $B \in \mathbb{B}^m$  of codimension  $r = |I| - 1$  such that the image of  $B$  in  $\Gamma_\mu \backslash \mathbb{B}^m$  is isomorphic as orbifolds to  $\Gamma_{\mu^{(I)}} \backslash \mathbb{B}^{m-r}$ .  $\Gamma_{\mu^{(I)}}$  is isomorphic to the stabilizer of  $B$  modulo its fixed point stabilizer.*

This proposition follows from (8.8.1) in [2], see also Lemma 2.4 in [13]. It gives totally geodesic inclusions between various Deligne-Mostow orbifolds.

**2.2. Condition  $\frac{1}{2}$ -INT.** We now discuss how to generalize the results of the previous section as in [12]. The generalized version is also the one that appears in Thurston's account of this theory, see [16]. The idea is to consider moduli of  $(m+3)$  *unordered* points rather than ordered. Since we consider weighted points, we allow identification of  $(m+3)$ -tuples of points that differ by ordering only when the corresponding permutation of the indices preserves the weights.

More specifically, we fix a partition  $m+3 = i_1 + \dots + i_k$ , and consider moduli of sets  $S_1, \dots, S_k$  of points on  $\mathbb{P}^1$ , with  $S_j$  having cardinality  $i_j$  for each  $j = 1, \dots, k$ . In terms of the notation used in the previous section, this moduli space is a quotient of  $Q^\mu / \Sigma$ , where  $\Sigma = \Sigma_{i_1} \times \dots \times \Sigma_{i_k}$  is a product of symmetric groups.

In order to describe the choice of weights in this setting in terms of the notation used in the previous section, we consider  $(m+3)$ -tuples  $\mu$  with  $0 < \mu_j < 1$  for all  $j$  and  $\sum \mu_j = 2$ , and break up the index set  $I = \{1, \dots, m+3\}$  as a disjoint union  $I_1 \sqcup \dots \sqcup I_k$  in such a way that the  $\mu_i$ ,  $i \in I_j$  are equal. In the sequel we shall always assume that the index sets  $I_j$  are arranged in increasing order, in the sense that if  $j < j'$ , all the elements of  $I_j$  are smaller than those of  $I_{j'}$ . Note that, by construction,  $\mu$  is then invariant under  $\Sigma$ .

**Definition 2.9.** The pair  $\mu, \Sigma$  as above satisfies the half-integrality condition  $\frac{1}{2}$ INT if for all  $i \neq j$  such that  $\mu_i + \mu_j < 1$ , we have

$$(1 - \mu_i - \mu_j)^{-1} \in \begin{cases} \mathbb{Z} & \text{if } i \text{ and } j \text{ are not in the same } \Sigma\text{-orbit} \\ \frac{1}{2}\mathbb{Z} & \text{if } i \text{ and } j \text{ are in the same } \Sigma\text{-orbit} \end{cases}$$

We adapt the definition of the integers  $d_{ij}$  accordingly, and set

$$d_{ij}^{(\mu, \Sigma)} = \begin{cases} (1 - \mu_i - \mu_j)^{-1} & \text{if } i \text{ and } j \text{ are not in the same } \Sigma\text{-orbit} \\ 2(1 - \mu_i - \mu_j)^{-1} & \text{if } i \text{ and } j \text{ are in the same } \Sigma\text{-orbit} \end{cases}$$

and write  $D_{ij}^{\mu, \Sigma}$  for the image of  $D_{ij}^\mu \subset Q_{st}^\mu$  in the quotient  $Q_{st}^\mu/\Sigma$ . As above, when no confusion arises, we shall simply write  $d_{ij}$  instead of  $d_{ij}^{(\mu, \Sigma)}$

**Remark 2.10.** (1) We do not necessarily assume that  $\mu_i \neq \mu_{i'}$  when  $i \in I_j$  and  $i' \in I_{j'}$  with  $j \neq j'$ . In other words, we do not assume that the sets of indices  $I_j$  are as large as possible to get  $\mu$  to be  $\Sigma$ -invariant.

(2) When  $i$  and  $j$  are in the same  $\Sigma$ -orbit, we do not assume that  $(1 - \mu_i - \mu_j)^{-1}$  are in  $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ .

The action of  $\Sigma$  on  $M$  clearly descends to an action on  $Q$ , and this action extends to an action on  $Q_{st}^\mu$ . In general, the quotient space  $Q_{st}^\mu/\Sigma$  has singularities, but the content of the main result of [12] is that it carries a complex hyperbolic orbifold structure, similar to the one mentioned in the previous section:

**Theorem 2.11.** *If  $\mu, \Sigma$  satisfies condition  $\frac{1}{2}INT$ , then there is a lattice  $\Gamma_{\mu, \Sigma}$  in  $PU(m, 1)$  such that the orbifold*

$$X_\mu = \Gamma_{\mu, \Sigma} \setminus \mathbb{B}^m$$

*has the same underlying (singular) algebraic variety as  $Q_{st}^\mu/\Sigma$ . Under this identification, the divisors  $D_{ij}^{\mu, \Sigma}$  have weight  $d_{ij}^{(\mu, \Sigma)}$ , and the other divisors with weight  $> 1$  have weight two, and are the images of codimension one fixed point sets of elements of  $\Sigma$  that are contained in  $Q^\mu$ .*

**Remark 2.12.** (1) There are indeed sometimes elements of  $\Sigma$  that fix a codimension one subset contained in  $Q^\mu$ . The list of cases where that happens can be deduced from in Lemma 8.3.2 of [3] (the elements that give codimension one fixed point set contained in  $Q^\mu$  are bitranspositions when  $m + 3 = 5$ , and tritranspositions when  $m + 3 = 6$ ).

(2) The same criterion as in the previous section determines whether the relevant ball quotient is compact or not, namely  $\Gamma_{\mu, \Sigma}$  is cocompact if and only if no subset of the weights adds up to exactly 1.

The analogue of Lemma 2.5 in the context of  $\frac{1}{2}INT$  example is slightly more complicated to state.

**Definition 2.13.** Let  $Q^{\mu, \Sigma}$  denote the largest open set of  $Q^\mu$  on which the action of  $\Sigma$  is free.

When  $i$  and  $j$  are not in the same  $\Sigma$ -orbit, we use the same notation as above and write  $\gamma_{ij}$  for a full twist between  $x_i$  and  $x_j$ . If  $i, j$  are in the same  $\Sigma$ -orbit, we denote by  $\alpha_{ij}$  the corresponding half twist (see Figure 1). Note

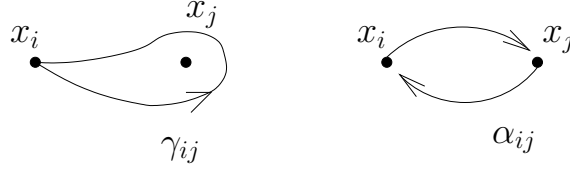


FIGURE 1. A full twist (left) and a half twist (right) between  $x_i$  and  $x_j$ .  $\gamma_{ij}$  induces a loop in  $Q_{st}^\mu$  that goes once around  $D_{ij}^\mu$ . When  $i$  and  $j$  are not in the same  $\Sigma$ -orbit,  $\gamma_{ij}$  induces a loop in  $Q_{st}^\mu/\Sigma$  that goes once around  $D_{ij}^{\mu,\Sigma}$ . When  $i$  and  $j$  are in the same  $\Sigma$ -orbit,  $\alpha_{ij}$  induces a loop in  $Q_{st}^\mu/\Sigma$  that goes once around  $D_{ij}^{\mu,\Sigma}$ , and  $\alpha_{ij}^2$  induces the same loop as  $\gamma_{ij}$ .

that the  $\gamma_{ij}$  (resp.  $\alpha_{ij}$ ) with  $\mu_i + \mu_j < 1$  give “small” loops in  $Q^{\mu,\Sigma}$  around  $D_{ij}^{\mu,\Sigma}$ .

Now consider the elements of  $\Sigma$  that have a codimension one fixed point set contained in  $Q^\mu$  (see Remark 2.12(1)), and denote by  $B_1, \dots, B_k$  the components of their image in  $Q^{\mu,\Sigma}/\Sigma$ . Write  $\beta_j$  for a small loop in  $Q^{\mu,\Sigma}/\Sigma$  that goes once around  $B_j$ .

**Lemma 2.14.** *We have*

$$\Gamma_{\mu,\Sigma} \simeq \pi_1(Q^{\mu,\Sigma})/K_{\mu,\Sigma}$$

where

$$K_{\mu,\Sigma} = \langle \langle \alpha_{ij}^{d_{ij}}, \beta_i^2 \rangle \rangle$$

is the normal subgroup of  $\pi_1(Q^{\mu,\Sigma})$  generated by the  $\alpha_{ij}^{d_{ij}}$  such that  $\mu_i + \mu_j < 1$ , and by the  $\beta_i^2$ ,  $i = 1, \dots, k$ .

If condition  $\frac{1}{2}$ INT holds for a given  $\mu$  but for two different symmetry groups, then the corresponding two lattices are commensurable, as we now explain.

If the partition  $I_1, \dots, I_k$  is a refinement of a partition  $J_1, \dots, J_l$  (i.e. every  $I_i$  is contained in some  $J_j$ ), and we denote by  $\Sigma^{(I)}$  (resp.  $\Sigma^{(J)}$ ) the corresponding symmetry group preserving the partition  $\{I_i\}$  (resp. preserving  $\{J_j\}$ ), then clearly  $\Sigma^{(I)}$  is a subgroup of  $\Sigma^{(J)}$ , hence there is a natural map

$$Q_{st}^\mu/\Sigma^{(I)} \rightarrow Q_{st}^\mu/\Sigma^{(J)}$$

Provided that  $\mu, \Sigma^{(I)}$  and  $\mu, \Sigma^{(J)}$  both satisfy condition  $\frac{1}{2}$ INT, one gets accordingly that

$$\Gamma_{\mu,\Sigma^{(I)}} \subset \Gamma_{\mu,\Sigma^{(J)}}$$

is a subset of index  $[\Sigma^{(J)} : \Sigma^{(I)}]$ .

For each  $\mu$  that satisfies  $\frac{1}{2}$ INT for some symmetry group, there is a *finest* partition of the indices for which condition  $\frac{1}{2}$ INT holds, and it is obtained



by requiring that  $(1 - \mu_i + \mu_j)^{-1}$  be in  $\frac{1}{2}\mathbb{Z}$  but *not* in  $\mathbb{Z}$  for each  $i \neq j$  in a common subset of the partition.

### 3. FORGETFUL MAPS

The rough idea of our construction is to consider the obvious maps from the moduli space of  $m' = m + 3$  points to the moduli space of  $n' = n + 3$  points on  $\mathbb{P}^1$ , obtained by forgetting  $m - n$  points (here we assume that  $m \geq n$ ). This is only a rough idea, because we are actually interested in moduli spaces of *weighted* points on  $\mathbb{P}^1$ .

Without loss of generality, we may assume that the points that get forgotten are the last  $m - n$ , i.e. we consider the map  $(\mathbb{P}^1)^{m'} \rightarrow (\mathbb{P}^1)^{n'}$  given by

$$(3.1) \quad (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_{m'}) \mapsto (x_1, \dots, x_{n'}).$$

This map is clearly  $PGL_2$ -equivariant, hence it always induces a holomorphic map  $Q^\mu \rightarrow Q^\nu$  for any set of weights  $\mu = (\mu_1, \dots, \mu_{m+3})$  and  $\nu = (\nu_1, \dots, \nu_{n+3})$ .

We consider the following questions, for various choices of  $\mu, \Sigma$  and  $\nu, \mathcal{T}$  that satisfy condition  $\frac{1}{2}\text{INT}$ .

- (1) Does the forgetful map descend to a map  $f : Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T}$ ?
- (2) When does  $f$  extend to a holomorphic map  $Q_{st}^\mu/\Sigma \rightarrow Q_{st}^\nu/\mathcal{T}$ ?
- (3) If such an extension exists, does it induce a map of orbifolds  $X_\mu \rightarrow X_\nu$  when we use the identification of Theorem 2.11?

When these three conditions hold, we shall simply say that the forgetful map induces a map of orbifolds (this really means that the forgetful map has a well-defined extension to stable points, and that this extension gives a map of orbifolds between the corresponding ball quotients).

The fact that these questions can be given a positive answer for certain choices of  $\mu, \Sigma$  and  $\nu, \mathcal{T}$  was already noticed in a couple of places in the literature. For  $m = 2$  and  $n = 1$  this was used in [4] (see also [5]). For  $m = n = 2$ , it was used by Toledo in [17]. Note that in the equidimensional case, the number of weighted points is the same for both moduli spaces in the source and target, so the forgetful map is simply the identity on the level of  $Q^\mu = Q^\nu$  (but it turns out that its extension to stable moduli spaces contracts some divisors).

The goal of the present paper is to give the list of all other cases where the forgetful maps give maps of orbifolds between the orbifold ball quotients. For simplicity we focus on the case of compact orbifolds, i.e. we consider sets of weights  $\mu$  such that  $Q_{st}^\mu = Q_{sst}^\mu$  (equivalently that no sum  $\sum_{j \in J} \mu_j$  adds up to exactly one for any  $J \subset \{1, \dots, m + 3\}$ ).

One way to summarize the results of the classification is the following.

**Theorem 3.1.** *Suppose  $m \geq n$ ,  $\mu, \Sigma$  and  $\nu, \mathcal{T}$  satisfy condition  $\frac{1}{2}\text{INT}$ , and assume that the orbifolds  $X_{\mu, \Sigma}$  and  $X_{\nu, \mathcal{T}}$  are compact, of dimension  $m$  and  $n$  respectively.*

- (i) If  $(m, n)$  is not  $(1, 1)$ ,  $(2, 2)$ ,  $(2, 1)$  or  $(3, 1)$ , then the forgetful map never induces a map of orbifolds.
- (ii) When  $(m, n) = (1, 1)$ , the forgetful maps that induce maps of orbifolds correspond to covering relations between triangle groups.
- (iii) When  $(m, n) = (2, 2)$ , the forgetful maps that induce maps of orbifolds correspond to the Mostow/Toledo maps.
- (iv) Many forgetful maps induce maps of orbifolds in the case  $(m, n) = (2, 1)$ , including some that correspond to the Livné fibrations.
- (v) When  $(m, n) = (3, 1)$ , up to symmetry and commensurability (see section 2.2), there is precisely one forgetful map that yields a map of orbifolds, corresponding to the weights  $\mu = (3, 3, 3, 3, 3, 1)/8$  and  $\nu = (3, 3, 3, 7)/8$ .

As noted in the introduction, Koziarz and Mok have recently shown that there is no submersive map  $X^m \rightarrow X^n$  between ball quotients, compact or not (see [8]). The map  $X^3 \rightarrow X^1$  that appears in part (v) of the theorem is of course not a submersion (for a description of the non-submersive locus, see Remark 3.8). The fact that the above construction should produce maps from a compact 3-ball quotient to compact a hyperbolic Riemann surface seems not to have appeared anywhere in the literature.

The proof of Theorem 3.1 amounts to a combinatorial check on the list of tuples of weights that satisfy condition  $\frac{1}{2}$ INT, using the results of the next few sections (specifically Lemma 3.2, Proposition 3.3 and Proposition 3.4).

**3.1. Compatibility of the symmetry groups.** As in the previous paragraphs, we fix two sets of weights with symmetry  $\mu, \Sigma$  and  $\nu, \mathcal{T}$  that both satisfy condition  $\frac{1}{2}$ INT. In order for the map (3.1) to induce a map

$$Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T},$$

we need to require a compatibility condition between the partition  $I = I_1 \sqcup \cdots \sqcup I_k$  (resp.  $J = J_1 \sqcup \cdots \sqcup J_l$ ) corresponding to  $\mu, \Sigma$  (resp. to  $\nu, \mathcal{T}$ ).

**Lemma 3.2.** *The forgetful map  $Q^\mu \rightarrow Q^\nu$  descends to a map  $Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T}$  if and only if for each  $i$ ,  $I_i$  is either entirely forgotten (i.e.  $j > n'$  for all  $j \in I_i$ ), or there is some  $j$  such that  $I_i \subset J_j$ .*

The particular case of forgetful maps between INT examples corresponds to the case when  $\Sigma$  and  $\mathcal{T}$  are both trivial, and then the condition stated in Lemma 3.2 is of course always trivially satisfied.

**3.2. Extension to stable points.** We start by finding the cases when  $Q^\mu \rightarrow Q^\nu$  extends to the stable compactifications.

**Proposition 3.3.** *If  $|\nu| \geq 5$ , the forgetful map (3.1) extends to a holomorphic map  $Q_{st}^\mu \rightarrow Q_{st}^\nu$  if and only if  $M_{st}^\mu$  maps into  $M_{st}^\nu$ , i.e. for any  $J \subset \{1, \dots, n'\}$  such that  $\sum_{j \in J} \mu_j < 1$ , we have  $\sum_{j \in J} \nu_j < 1$ .*

*If  $|\nu| = 4$ , the map extends if and only if  $\sum_{j \in J} \mu_j > 1$  for every  $J \subset \{1, \dots, 4\}$  with  $|J| = 3$ .*

**Proof:** For the case  $|\nu| = 4$ , we define a map  $Q_{st}^\mu \rightarrow \mathbb{P}^1$  by mapping  $(x_1, \dots, x_n)$  to the cross-ratio of the four points  $x_1, x_2, x_3$ , and  $x_4$ , i.e.

$$(3.2) \quad \frac{x_3 - x_2}{x_3 - x_1} \cdot \frac{x_4 - x_1}{x_4 - x_2}.$$

This defines a holomorphic map to  $\mathbb{P}^1$  provided that no triple of points in  $\{x_1, x_2, x_3, x_4\}$  coincide.  $\square$

If the map extends, we still refer to its extension as a forgetful map, and with a little abuse of notation we denote by the same symbol the forgetful map  $f : Q^\mu \rightarrow Q^\nu$  and its extension  $f : Q_{st}^\mu \rightarrow Q_{st}^\nu$ .

### 3.3. Maps of orbifolds.

**Proposition 3.4.** *Suppose the map  $f : Q_{st}^\mu/\Sigma \rightarrow Q_{st}^\nu/\mathcal{T}$  is well defined. The following conditions are equivalent:*

- (1)  $f$  viewed as a map  $\Gamma_{\mu,\Sigma} \backslash \mathbb{B}^m = X_\mu \rightarrow X_\nu = \Gamma_{\nu,\mathcal{T}} \backslash \mathbb{B}^n$  lifts to an equivariant holomorphic map  $\tilde{f} : \mathbb{B}^m \rightarrow \mathbb{B}^n$ .
- (2)  $Q'^{\mu,\Sigma}$  maps into  $Q'^{\nu,\mathcal{T}}$ , and the induced homomorphism  $\pi_1(Q'^{\mu,\Sigma}) \rightarrow \pi_1(Q'^{\nu,\mathcal{T}})$  maps  $K_{\mu,\Sigma}$  into  $K_{\nu,\Sigma}$ .

**Proof:** The equivalence of (1) and (2) follows from the proof of the main result in [12], where it is shown that the ball is obtained as the Fox completion of the so-called monodromy cover  $\widetilde{Q'^{\mu,\Sigma}/\Sigma}$  over  $Q_{st}^{\mu,\Sigma}/\Sigma$  (the map  $\widetilde{Q'^{\mu,\Sigma}/\Sigma} \rightarrow Q'^{\mu,\Sigma}/\Sigma$  is etale, but its extension to the Fox completion branches over stable points).

By construction that monodromy cover has fundamental group given by the kernel of the monodromy representation, and Lemma (8.6.1) of [3] states that this kernel is precisely the group  $K_{\mu,\Sigma}$  that appears in Lemma 2.14.

The condition that  $K_{\mu,\Sigma}$  maps into  $K_{\nu,\mathcal{T}}$  is simply the obvious condition for a map to lift to etale coverings. If the map  $Q'^{\mu,\Sigma}/\Sigma \rightarrow Q'^{\nu,\mathcal{T}}/\mathcal{T}$  lifts to  $\widetilde{Q'^{\mu,\Sigma}/\Sigma} \rightarrow \widetilde{Q'^{\nu,\mathcal{T}}/\mathcal{T}}$ , it extends to a map between Fox completions by Hartogs' theorem to give a holomorphic map  $\mathbb{B}^m \rightarrow \mathbb{B}^n$ .  $\square$

In view of Lemma 2.14, we can formulate the condition of Proposition 3.4 in terms of divisibility conditions on the orbifold weights. Specifically, in order to get a map of orbifolds  $X_\mu \rightarrow X_\nu$ , we need the codimension one fixed point sets of elements of  $\Sigma$  in  $Q^\mu$  to map to codimension one fixed point sets of elements of  $\mathcal{T}$  in  $Q^\nu$ , and moreover for all  $i \neq j$  with  $\mu_i + \mu_j < 1$ ,

- $d_{ij}^\nu$  divides  $d_{ij}^\mu$  if  $i$  and  $j$  are not in the same  $\Sigma$  orbit nor in the same  $\mathcal{T}$ -orbit.
- $d_{ij}^\nu$  divides  $2d_{ij}^\mu$  if  $i$  and  $j$  are in the same  $\mathcal{T}$ -orbit but not in the same  $\Sigma$ -orbit.
- $d_{ij}^\nu$  divides  $d_{ij}^\mu$  if  $i$  and  $j$  are in the same  $\Sigma$  orbit and in the same  $\mathcal{T}$ -orbit.

**Remark 3.5.** Another way to obtain an equivariant map  $\mathbb{B}^n \rightarrow \mathbb{B}^{n-r}$  is to use harmonic map theory. Indeed, Lemma (8.6.1) of [3] gives a group presentation for the Deligne-Mostow lattices; the condition that  $K_\mu$  map to  $K_\nu$  amounts to requiring that there is a (surjective) homomorphism  $\Gamma_\mu \rightarrow \Gamma_\nu$ , and there is a unique harmonic map twisted by that morphism.

**Remark 3.6.** Sai-Kee Yeung pointed out to the author that, when the dimension of the target has dimension one, the extension to stable compactifications automatically follows from the condition that  $K_{\mu,\Sigma}$  maps into  $K_{\nu,\mathcal{T}}$ . Indeed, in that case, the map  $\widetilde{Q^{\mu,\Sigma}/\Sigma} \rightarrow \widetilde{Q^{\nu,\mathcal{T}}/\mathcal{T}}$  can be shown to extend by general principles, using hyperbolicity and the Schwarz lemma.

One way to state his remark is the following; if  $M$  is a compact ball quotient and  $U \subset M$  is a Zariski open set, any holomorphic map  $U \rightarrow U_1$  to a Zariski open set  $U_1 \subset M_1$  in a compact 1-ball quotient extends to a holomorphic map  $M \rightarrow M_1$ .

**Proof** (of Theorem 3.1): In order to check the statement of the Theorem, one needs to list all possible pairs of weights with symmetry  $\mu, \Sigma$  and  $\nu, \mathcal{T}$ , and test whether the hypotheses of Lemma 3.2, Proposition 3.3 and Proposition 3.4 hold.

Note that one needs to allow permutations of the sets of weights given in the list in Deligne-Mostow, and to allow for various sets of symmetry when appropriate.

In any case, this is clearly a finite combinatorial problem when the dimension of the corresponding ball quotients is at least two, since there are only finitely many Deligne-Mostow ball quotients of dimension at least two.

In order to handle the 1-dimensional Deligne-Mostow ball quotients (there are infinitely many of these, corresponding to hyperbolic triangle groups), we recall that we are after maps to lower-dimensional ball quotients, hence we need only consider 1-dimensional ball quotients as targets. In that case, the divisibility conditions in the hypotheses of Proposition 3.4 imply that one needs only consider a finite set of 4-tuples. Indeed, the relevant weights  $\mu$  are tuples of rational numbers, and because of the fact that  $d_{ij}^{(\mu)}$  is  $(1 - \mu_i - \mu_j)^{-1}$  or twice that number, an upper bound on the  $d_{ij}$  implies an upper bound on the least common denominator of  $\mu_1, \dots, \mu_{m+3}$ .  $\square$

**Remark 3.7.** (1) There is only one map with  $m \geq n$  and either  $m > 2$  or  $n > 1$  that satisfies the compatibility condition (Lemma 3.2) and the divisibility condition (Proposition 3.4) but does not satisfy the extension property. It corresponds to  $\mu = (3, 3, 3, 3, 6, 2)/10$  and  $\nu = (3, 3, 3, 3, 8)/10$ , with symmetry group  $\Sigma = \mathcal{T} = S_4$ .

In particular, the fixed point sets in  $Q^\mu$  or  $Q^\nu$  of bitranspositions in  $\Sigma = \mathcal{T}$  have codimension one only in  $Q^\mu$ , not in  $Q^\nu$ . In other words, the map  $Q^\mu/\Sigma \rightarrow Q^\nu/\mathcal{T}$  maps divisors with weight 2 to divisors in with weight 1, so it is not a map of orbifolds.

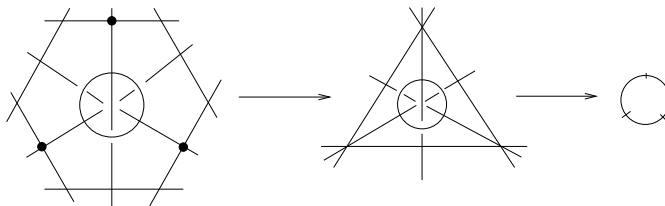


FIGURE 2. Naïve description of the map  $\widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$ .

- (2) If we replace  $\mathcal{T}$  by  $\mathcal{T}' \supset \mathcal{T}$ , we get a surjective map of orbifolds  $Q_{st}^\nu/\mathcal{T} \rightarrow Q_{st}^\nu/\mathcal{T}'$ , so we are after maps with  $\Sigma$  minimal and  $\mathcal{T}$  maximal.

**3.4. An example.** We now give some detail on the example that appears in Theorem 3.1, part (v). Consider

$$\mu = (3, 3, 3, 3, 3, 1)/8$$

and

$$\nu = (3, 3, 3, 7)/8.$$

We shall choose the symmetry groups  $\Sigma$ ,  $\mathcal{T}$  to be trivial (there are several maps obtained for various non-trivial symmetry groups, but as mentioned above, the corresponding groups are commensurable, see the discussion in the end of section 2.2). Accordingly, we use the notation of section 2.1 rather than of section 2.2.

The map that sends  $(x_1, \dots, x_6)$  to the cross-ratio of  $x_1, x_2, x_3, x_4$  (see equation (3.2)) descends to  $Q_{st}^\mu$  (see Proposition 3.3).

To be more explicit, note that  $Q_{st}^\mu$  is a  $P^1$ -bundle over  $\widehat{\mathbb{P}^2}$ , where  $\widehat{\mathbb{P}^2}$  denotes  $\mathbb{P}^2$  blown-up at a generic quadruple of points. Now there is an obvious map  $\widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$ , coming from the fibration of  $\mathbb{P}^2 \setminus \{p\}$  over the  $\mathbb{P}^1$  of lines through  $p$  (the former map contracts three of the exceptional divisors of  $\widehat{\mathbb{P}^2}$ , and maps onto the other exceptional divisor).

Note that the blown-up  $\widehat{\mathbb{P}^2}$  is homeomorphic to  $Q_{st}^\theta$  for various 5-tuples  $\theta$ , but the corresponding map  $Q_{st}^\mu \rightarrow Q_{st}^\theta$  is never a map of orbifolds (see Theorem 3.1). However the composition  $Q_{st}^\mu \rightarrow \widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^1$  can be made into a map of orbifolds, where  $\mathbb{P}^1$  is the orbifold ball quotient  $Q_{st}^\nu$ . We denote by  $f : Q_{st}^\mu \rightarrow Q_{st}^\nu$  the corresponding holomorphic map. The set of points of  $Q_{st}^\mu$  where  $df$  is not surjective consists of three  $\mathbb{P}^1$ 's, which are the fibers over three points of  $\widehat{\mathbb{P}^2}$ , represented by the solid dots on Figure 2 (left).

Note also that the preimage of each of the three singular points on  $Q_{st}^\nu$  consists of two divisors, for instance  $D_{12}$  and  $D_{34}$  map to the same point in  $Q_{st}^\nu$ .

We close this section by noting that the condition of Proposition 3.4 is trivially satisfied in this example (recall that we take  $\Sigma$  and  $\mathcal{T}$  to be trivial, so  $K_\mu$  is the same as  $K_{\mu, \Sigma}$ , etc). Indeed, recall that  $K_\mu$  is the normal

subgroup generated by the  $\gamma_{ij}^{d_{ij}}$ , where  $d_{ij}$  is either 2 or 4. Note that  $d_{ij} = 2$  only when one of  $i$  or  $j$  is equal to 6, but then the loop  $\gamma_{ij}$  has trivial image. Since all the  $(1 - \nu_i - \nu_j)^{-1}$  are  $\pm 4$ ,  $K_\mu$  maps into  $K_\nu$ .

**Remark 3.8.** Note that the divisors  $D_{i5}$ ,  $i = 1, \dots, 4$ ,  $D_{i6}$ ,  $i = 1, \dots, 5$  surject onto  $Q_{st}^\nu \simeq P^1$ , whereas the other divisors get mapped to the three orbifold points. For instance, the fiber over  $D_{12}^\nu \in \mathbb{P}^1$ , which corresponds to  $x_1 = x_2$  in  $Q_{st}^\nu$ , is given by the union of the divisors  $D_{12}^\mu$  and  $D_{34}^\mu$  in  $Q_{st}^\mu$  (these are both projective planes and their intersection is a projective line).

#### 4. MAPS BETWEEN NON-COMPACT EXAMPLES

The same construction works for non compact moduli spaces, and one gets maps that have the same behavior as the ones between compact ones. Here the divisibility condition is easily adapted to allow for infinite weights if some pairs of weights add up to exactly one. The list of pairs of weights with symmetry (at least one of which is non-compact, with  $m > 2$  or  $n > 1$ ) that satisfy the compatibility and divisibility conditions is the following:

- $\mu = (2, 2, 2, 3, 3)/6$ ,  $\Sigma = S_3$   
 $\nu = (2, 2, 2, 1, 5)/6$ ,  $\mathcal{T} = S_3$ .
- $\mu = (4, 4, 4, 5, 7)/12$ ,  $\Sigma = S_3$   
 $\nu = (2, 2, 2, 1, 5)/6$ ,  $\mathcal{T} = S_3$ .
- $\mu = (2, 2, 3, 3, 1, 1)/6$ ,  $\Sigma = S_2$   
 $\nu = (1, 7, 7, 9)/12$ ,  $\mathcal{T} = \{id\}$ .
- $\mu = (2, 2, 3, 3, 1, 1)/6$ ,  $\Sigma = S_2$   
 $\nu = (1, 3, 4, 4)/6$ ,  $\mathcal{T} = \{id\}$ .

Note that in particular one does not get more pairs  $(m, n)$  of dimensions that are related by a surjective map of orbifolds coming from a forgetful map than in the compact case; in other words part (i) of the statement of Theorem 3.1 remains true for non-compact examples.

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