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Hedging of Defaultable Contingent Claims using BSDE with uncertain time horizon.

Christophette BLANCHET-SCALLIET¹, Anne EYRAUD-LOISEL², Manuela ROYER-CARENZI³

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Abstract

This article focuses on the mathematical problem of existence and uniqueness of BSDE with a random terminal time which is a general random variable but not a stopping time, as it has been usually the case in the previous literature of BSDE with random terminal time. The main motivation of this work is a financial or actuarial problem of hedging of defaultable contingent claims or life insurance contracts, for which the terminal time is a default time or a death time, which are not stopping times. We have to use progressive enlargement of the Brownian filtration, and to solve the obtained BSDE under this enlarged filtration. This work gives a solution to the mathematical problem and proves the existence and uniqueness of solutions of such BSDE under certain general conditions. This approach is applied to the financial problem of hedging of defaultable contingent claims, and an expression of the hedging strategy is given for a defaultable contingent claim or a life insurance contract.

Keywords : Progressive Enlargement of filtration, BSDE, Uncertain time horizon, Defaultable contingent claims

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Introduction

In the present work, we study backward stochastic differential equations with uncertain time horizon : the terminal time of the problem is a random variable τ , which is not a stopping time, as usually stated in the previous literature, but a general random variable. Hedging problems with random variables as terminal time are among possible application fields : for example defaultable contingent claims or life insurance contracts fit into this framework, as the terminal time is a default time or a death time, which are not stopping times.

BSDEs were first introduced by E. Pardoux and S. Peng in 1990 [23]. Such equations are frequently used, and have a large panel of application areas, especially in mathematical finance. They also appear in several cases such as stochastic control (see S. Peng [25], N. El Karoui, S. Peng and M.C. Quenez [10], and X. Zhou and J. Yong [27]) or problems linked with PDEs (see E. Pardoux [21] and G. Barles, R. Buckdahn and E. Pardoux [2]). BSDEs are useful in our framework since these equations naturally appear when describing hedging problems. As we study hedging of contingent claims with random exercise time, we have to model it with BSDEs with random terminal time. Such equations were introduced by S. Peng (1991) [24], and developed by R. Darling and E. Pardoux (1997) [9], P. Briand and Y. Hu (1998) [8], E. Pardoux (1999) [22], M. Royer (2004) [26] among others, and by E. Pardoux (1995) [20] for BSDEs with jumps and random terminal time.

As the terminal horizon of our problem is not a stopping time, the filtration that appears to be convenient to work with is not the Brownian filtration \mathcal{F}_t , but the smallest filtration that contains \mathcal{F}_t and that makes τ a stopping time. This method is well-known as progressive enlargement of filtration. It has been introduced in T. Jeulin (1980) [15], T. Jeulin and M. Yor (1978,1985) [16, 17], and further developed in J. Azema, T. Jeulin, F. Knight and M. Yor (1992) [1]. This framework has been used in default risk models, as the default time is not a stopping time. Works on default risk models has been developed for example in C. Blanchet-Scalliet and M. Jeanblanc (2004) [7], T. Bielecki, M. Jeanblanc and M. Rutkowski (2004) [3], M. Jeanblanc and Y. Le Cam (2007) [14]. A useful representation theorem in this framework is given in S. Kusuoka (1999) [18]. Existence of solutions of BSDE under enlarged filtration has already been studied by A. Eyraud-Loisel (2005) [11] for deterministic horizon, and by A. Eyraud-Loisel and M. Royer-Carenzi (2006) [12] for random terminal time (stopping time), but only under an initially enlarged filtration, as used for asymmetrical information and insider trading modeling.

In a first part, we present the financial and actuarial motivation of this work and introduce several notations. In the second part, the problem of existence and uniqueness of the BSDE under the enlarged filtration \mathcal{G} is solved. The last section is devoted to an application of the previous results to hedging against a defaultable contingent claim. We give an explicit hedging strategy in the defaultable world, under traditional hypothesis **(H)**.

1 Financial motivation

Let $(S_t)_{0 \leq t \leq T}$ be the price process and $(\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by the price processes.

Suppose that an agent has to hedge against some contract with random terminal condition, where the random terminal condition is not a stopping time. It is the case for defaultable contingent claims, where the terminal time is a default time, or for a life insurance contract, where the terminal time is a death time. Let us give an example : an agent sells an option with maturity T based on a defaultable asset (defaultable contingent claim). This type of contract generally involves two different kind of possible payoffs : the seller commits itself to give the payoff of a regular option, if default did not occur at time T , which will be represented by a \mathcal{F}_T -measurable random variable V (for instance, $V = (S_T - K)_+$ for a call option, but in a more general case, V may depend on the paths of one or several asset prices until time T). If default occurs before time T , the seller has to pay a compensation C_τ at time τ , which is in the financial case generally a constant, but may depend on the paths of the risky assets strictly before default time : $C_\tau \in \mathcal{F}_{\tau-}$. The final payoff the agent wants to hedge at time $\tau \wedge T$ has the following general form :

$$\xi = V \mathbb{1}_{\tau > T} + C_\tau \mathbb{1}_{T \leq \tau},$$

This kind of payoff may also be used for life insurance hedging problems : in the case of a life insurance contract seller (or the owner of a life insurance portfolio), the default time will be the death time of the policyholder, or the minimum of several death times, in the case of an entire portfolio. The seller of the contract promises to give either a pension in case of death does not occur before maturity of the contract (in that case, the pension can be represented by its present value at time T , which is a conditional expectation with respect to \mathcal{F}_T and so may also be represented by a random variable $V \in \mathcal{F}_T$), and a compensation at death time if death occurs before time T , represented by a random variable $C_\tau \in \mathcal{F}_{\tau-}$, constant in the most simple case, or which may depend on some financial aspects strictly until death time.

Finally, in both applications, payoff ξ consists in a \mathcal{F}_T -measurable random variable V , to hedge at maturity T if τ has not occurred at time T , and a compensation C_τ , payed at hit (at default/death time) in case of default (or death) occurs before T . The hedging terminal time is $T \wedge \tau$.

Default times, as well as death times, are random variable that do not depend entirely on the paths of some financial risky assets. They may have a financial component, but have an exogeneous part, which makes them not adapted to the natural filtration generated by the observations of prices.

Nevertheless, they are observable. So we suppose that at any time, the agent can observe if default τ has occurred or not, which is quite natural to suppose for default times as for death times. So the information of an agent is not the filtration generated by the price processes \mathcal{F} , but is defined by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathbb{1}_{\tau \leq t}),$$

which is the smallest filtration that contains filtration \mathcal{F}_t and that makes τ a stopping time. So the previous payoff belongs to the following space :

$$\xi \in \mathcal{F}_{T \wedge \tau} \vee \sigma(\tau) = \mathcal{G}_{T \wedge \tau}.$$

The problem is to find a hedging admissible strategy for this defaultable contingent claim, i.e. a strategy that leads to a terminal wealth $X_{T \wedge \tau} = \xi$. An admissible hedging strategy will be a self-financing strategy based on the non risky asset, the risky asset, and the defaultable zero-coupon.

Let us write what this means. Following the notations of C. Blanchet-Scalliet and M. Jeanblanc [7], let us first introduce the following conditional probability denoted by F_t :

$$F_t = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\tau \leq t} | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_t). \quad (1)$$

We will always consider the right-continuous version of this $(\mathcal{F}, \mathbb{P})$ -submartingale.

Suppose that the dynamics of the price S_t at time t of the risky asset is the following (for simplicity reasons, we will only consider the case where there is only one risky financial asset) :

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad 0 \leq t \leq T, \quad (2)$$

where W_t is a one-dimensional Brownian motion under \mathbb{P} , and $(\mathcal{F}_t)_{0 \leq t \leq T}$ is its natural filtration.

We will suppose, as in [7] that there exists at least one probability equivalent to \mathbb{P} on \mathcal{F} under which the discounted price process is a martingale (there exists an equivalent martingale measure), so that the default-free market is arbitrage free. We will also consider here that this probability is unique, so that the default-free market is complete (σ invertible and $\mathcal{E}(\sigma^{-1}W)$ integrable). We denoted by $\tilde{\mathbb{P}}$ this unique equivalent martingale measure (e.m.m.) on \mathcal{F}

The riskless asset follows the dynamics :

$$dS_t^0 = r_t S_t^0 dt, \quad 0 \leq t \leq T. \quad (3)$$

Let R_t be the discount factor :

$$R_t = \exp\left(-\int_0^t r_s ds\right), \quad 0 \leq t \leq T.$$

Then the price of the riskless asset has the following form :

$$S_t^0 = \exp\left(\int_0^t r_s ds\right), \quad 0 \leq t \leq T.$$

Even if the default-free market is complete under filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, it is not complete under $(\mathcal{G}_t)_{0 \leq t \leq T}$ any more (the martingale representation property does

not held in the same way, we need another martingale representation theorem). In short, to be able to hedge against the random time, another asset will be needed, that will appear in the martingale representation property.

In fact, in financial defaultable markets there is often another tradable asset (or at least attainable) : the defaultable zero-coupon bond with maturity T , whose value at time t is $\rho(t, T)$: this asset gives its owner the payment 1 if default did not occur before T , and nothing otherwise. For more convenience, we will always denote $\rho(t, T)$ by ρ_t , without specifying maturity T .

Two different cases will appear : whether this defaultable zero-coupon bond is tradable on the market or not. For instance in an insurance problem, there aren't any defaultable zero-coupon bond that is possibly traded, whereas in the financial settings, it may be attainable.

2 BSDE under \mathcal{G}

Let \mathbb{P} denote a general probability measure. Consider a standard one-dimensional Brownian motion W , and its natural filtration \mathcal{F} . Define the \mathcal{F} -predictable, right-continuous nondecreasing process $(\hat{F}_t)_{t \geq 0}$ such that the process $F - \hat{F}$ is a $(\mathcal{F}, \mathbb{P})$ -martingale.

Denote by $(H_t = \mathbb{1}_{\tau \leq t})_{t \geq 0}$ the defaultable process and define the \mathcal{F} -predictable, right-continuous nondecreasing process $(\Lambda_t)_{t \geq 0}$ such that the process $M_t = H_t - \Lambda_{t \wedge \tau}$ is a $(\mathcal{G}, \mathbb{P})$ -martingale. Process M is called the compensated process of H .

Proposition 2.1 *Process Λ exists, is well defined and satisfies*

$$d\Lambda_t = \frac{d\hat{F}_t}{1 - F_{t-}}, \quad t \geq 0.$$

Proof. See Lemma 8.7 in [6] for the proof. \square

2.1 Representation theorem

For simplicity reasons, we consider a one-dimensional Brownian motion, but all proofs can be adapted to the multi-dimensional case, and all results remain true.

In M. Yor [28], the decomposition of $(\mathcal{F}, \mathbb{P})$ -martingales in filtration \mathcal{G} is given by the following proposition.

Proposition 2.2 *If X is a $(\mathcal{F}, \mathbb{P})$ -martingale, there exists a $(\mathcal{G}, \mathbb{P})$ -martingale \bar{X} such that*

$$X_{t \wedge \tau} = \bar{X}_{t \wedge \tau} - \int_0^{t \wedge \tau} \frac{d\langle X, F \rangle_s}{1 - F_{s-}}, \quad 0 \leq t \leq T.$$

In the case where filtration \mathcal{F} is generated by a Brownian motion W , M. Yor et al established in [1] a representation theorem for \mathcal{G} -martingales.

For any $\gamma \in \mathbb{R}$, let us define $\mathcal{B}_\gamma^2 = \mathcal{S}_\gamma^2 \times \mathcal{L}_\gamma^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_\gamma^2(M, \mathbb{P})$ where we denote by :

- \mathcal{S}_γ^2 the set of 1-dimensional \mathcal{G} -adapted càdlàg processes $(Y_t)_{0 \leq t \leq T}$ such that $\|Y\|_{\mathcal{S}_\gamma^2}^2 = \mathbb{E}_{\mathbb{P}} \left(\sup_{0 \leq t \leq T} e^{\gamma(t \wedge \tau)} |Y_{t \wedge \tau}|^2 \right) < \infty$,
- $\mathcal{L}_\gamma^2(\bar{W}, \mathbb{P})$ the set of all m -dimensional \mathcal{G} -predictable processes $(Z_t)_{0 \leq t \leq T}$ such that $\|Z\|_{\mathcal{L}_\gamma^2(\bar{W}, \mathbb{P})}^2 = \mathbb{E}_{\mathbb{P}} \left(\int_0^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds \right) < \infty$,
- $\mathcal{L}_\gamma^2(M, \mathbb{P})$ the set of all 1-dimensional \mathcal{G} -predictable processes $(U_t)_{0 \leq t \leq T}$ such that $\|U\|_{\mathcal{L}_\gamma^2(M, \mathbb{P})}^2 = \mathbb{E}_{\mathbb{P}} \left(\int_0^{T \wedge \tau} e^{\gamma s} |U_s|^2 d\Lambda_s \right) < \infty$.

Theorem 2.3 *Any $(\mathcal{G}, \mathbb{P})$ square-integrable martingale stopped in τ can be written as a sum of two $(\mathcal{G}, \mathbb{P})$ square-integrable orthogonal martingales*

$$\bar{X}_{t \wedge \tau} = \int_0^{t \wedge \tau} Z_s d\bar{W}_s + \int_0^{t \wedge \tau} U_s dM_s, \quad 0 \leq t \leq T,$$

where $\mathbb{E} \left(\int_0^\infty \|Z_s\|^2 ds \right) < \infty$ and $\mathbb{E} \left(\int_0^\infty |U_s|^2 d\Lambda_s \right) < \infty$.

This decomposition is unique in $\mathcal{L}_0^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$.

Proof.

- Existence : The proof is stated in [1]. In this paper, the integrability condition on U is

$$\mathbb{E} \left(\int_0^\infty |U_s|^2 d\hat{F}_s \right) < \infty.$$

But one has

$$\mathbb{E} \left(\int_0^\infty |U_s|^2 d\Lambda_s \right) \leq \mathbb{E} \left(\int_0^\infty |U_s|^2 d\hat{F}_s \right) < \infty.$$

- Uniqueness : Let X be a square-integrable \mathcal{G} -martingale stopped in τ . Suppose that there exist (Z, U) and (\bar{Z}, \bar{U}) in $\mathcal{L}_0^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$ such that

$$\begin{aligned} X_{t \wedge \tau} &= \int_0^{t \wedge \tau} Z_s d\bar{W}_s + \int_0^{t \wedge \tau} U_s dM_s \\ &= \int_0^{t \wedge \tau} \bar{Z}_s d\bar{W}_s + \int_0^{t \wedge \tau} \bar{U}_s dM_s. \end{aligned}$$

So

$$0 = \int_0^{t \wedge \tau} \hat{Z}_s d\bar{W}_s + \int_0^{t \wedge \tau} \hat{U}_s dM_s,$$

where $\hat{Z} = Z - \bar{Z}$ and $\hat{U} = U - \bar{U}$.

$$\begin{aligned} 0 &= \mathbb{E} \left[\left(\int_0^{t \wedge \tau} \hat{Z}_s d\bar{W}_s \right)^2 + \left(\int_0^{t \wedge \tau} \hat{U}_s dM_s \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \tau} \|\hat{Z}_s\|^2 ds + \sum_{0 \leq s \leq t} |\hat{U}_s|^2 \Delta H_s \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \tau} \|\hat{Z}_s\|^2 ds + \int_0^{t \wedge \tau} |\hat{U}_s|^2 dH_s \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \tau} \|\hat{Z}_s\|^2 ds + \int_0^{t \wedge \tau} |\hat{U}_s|^2 dM_s + \int_0^{t \wedge \tau} |\hat{U}_s|^2 d\Lambda_s \right]. \end{aligned}$$

This leads to

$$0 = \mathbb{E} \left(\int_0^{t \wedge \tau} \|\hat{Z}_s\|^2 ds \right) + \mathbb{E} \left(\int_0^{t \wedge \tau} |\hat{U}_s|^2 d\Lambda_s \right),$$

and uniqueness is clear in $\mathcal{L}_0^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$.

□

2.2 Existence theorem

Fix $T > 0$ and $\xi \in \mathcal{L}^2(\mathcal{G}_{T \wedge \tau})$. $(W_t)_{t \geq 0}$ is a Brownian motion in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by \bar{W} the associated Brownian motion under $(\Omega, \mathcal{G}, \mathbb{P})$.

Consider process Λ as defined in Proposition 2.1. In addition we assume that it is a continuous process, not necessarily absolutely continuous. So we suppose that

$$d\Lambda_t = \lambda_t dt + d\Lambda_t^e.$$

The BSDE to be solved is the following :

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \quad 0 \leq t \leq T. \quad (4)$$

The aim of this section is to prove an existence and uniqueness result for this BSDE stopped at \mathcal{G} -stopping time $T \wedge \tau$. In the previous financial interpretation, this unique \mathcal{G} -adapted solution (Y, Z, U) , stopped at time τ , will represent the unique portfolio that hedges the defaultable contingent claim.

Hypotheses on f and λ :

- f is a Lipschitz function such that there exist a constant K satisfying

$$|f(s, y, z, u) - f(s, y', z', u')| \leq K(|y - y'| + \|z - z'\|) + \lambda_s |u - u'|, \quad (5)$$

where $0 \leq \lambda_s \leq K$.

Definition 2.4

A $(\Omega, \mathcal{G}, \mathbb{P})$ -solution (or a solution on $(\Omega, \mathcal{G}, \mathbb{P})$) to equation (4) is a triple of $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$ -valued $(Y_t, Z_t, U_t)_{t \geq 0}$ processes such that

1. Y is a \mathcal{G} -adapted càdlàg process and $(Z, U) \in \mathcal{L}_0^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$,
2. On the set $\{t \geq \tau\}$, we have $Y_t = \xi$, $Z_t = 0$ and $U_t = 0$,
3. $\forall r \geq 0, \forall t \in [0, r]$, we have

$$Y_{t \wedge \tau} = Y_{r \wedge \tau} + \int_{t \wedge \tau}^{r \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{r \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{r \wedge \tau} U_s dM_s.$$

Corollary 2.5

Let $\xi \in \mathcal{L}^2(\Omega, \mathcal{G}_{T \wedge \tau}, \mathbb{P})$. If $(Y_t, Z_t, U_t)_{0 \leq t \leq T}$ is a $(\Omega, \mathcal{G}, \mathbb{P})$ -solution of the BDSE (4) as defined in the Definition 2.4, with f satisfying hypothesis (5) and

$$\mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) < +\infty,$$

then

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) < +\infty.$$

Proof. Let (Y_t) be a solution of (4) :

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \quad 0 \leq t \leq T.$$

Let us consider $\gamma \in \mathbb{R}$. Apply Itô's formula to the process $(e^{\gamma t} Y_t^2)$ between $t \wedge \tau$ and $T \wedge \tau$.

$$\begin{aligned} e^{\gamma(t \wedge \tau)} Y_{t \wedge \tau}^2 &= e^{\gamma(T \wedge \tau)} \xi^2 - \gamma \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s^2 ds - 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s dY_s \\ &\quad - \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} d[Y, Y]_s^c - \sum_{t \wedge \tau \leq s \leq T \wedge \tau} e^{\gamma s} (Y_s^2 - Y_{s-}^2 - 2Y_{s-} \Delta Y_{s-}) \\ &= e^{\gamma(T \wedge \tau)} \xi^2 - \gamma \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s^2 ds - \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds \\ &\quad - \sum_{t \wedge \tau \leq s \leq T \wedge \tau} e^{\gamma s} U_s^2 \Delta H_s - 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s Z_s d\bar{W}_s \\ &\quad - 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s U_s dM_s + 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s f(s, Y_s, Z_s, U_s) ds. \end{aligned}$$

Then

$$\begin{aligned}
& e^{\gamma(t\wedge\tau)} Y_{t\wedge\tau}^2 + \gamma \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|Z_s\|^2 ds + \sum_{t\wedge\tau \leq s \leq T\wedge\tau} e^{\gamma s} U_s^2 \Delta H_s \\
&= e^{\gamma(T\wedge\tau)} \xi^2 - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s U_s dM_s \\
&\quad + 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s f(s, Y_s, Z_s, U_s) ds.
\end{aligned}$$

Hence, for any $\eta, \varepsilon, \delta \in \mathbb{R}_+^*$,

$$\begin{aligned}
& e^{\gamma(t\wedge\tau)} Y_{t\wedge\tau}^2 + \gamma \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|Z_s\|^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 (dM_s + d\Lambda_s) \\
&\leq e^{\gamma(T\wedge\tau)} \xi^2 - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s U_s dM_s \\
&\quad + 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |Y_s| |f(s, 0, 0, 0)| ds + 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |Y_s| (K |Y_s| + K \|Z_s\| + \lambda_s |U_s|) ds \\
&\leq e^{\gamma(T\wedge\tau)} \xi^2 - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s U_s dM_s + \frac{1}{\delta} \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |f(s, 0, 0, 0)|^2 ds \\
&\quad + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |Y_s|^2 (\delta + 2K + \varepsilon K^2 + \eta K) + \frac{1}{\varepsilon} \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|Z_s\|^2 ds + \frac{1}{\eta} \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 \lambda_s ds.
\end{aligned}$$

This yields to

$$\begin{aligned}
& e^{\gamma(t\wedge\tau)} Y_{t\wedge\tau}^2 + (\gamma - \delta - 2K - \varepsilon K^2 - \eta K) \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s^2 ds + \left(1 - \frac{1}{\varepsilon}\right) \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|Z_s\|^2 ds \\
&\quad + \left(1 - \frac{1}{\eta}\right) \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 \lambda_s ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 d\Lambda_s^e + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 dM_s \\
&\leq e^{\gamma(T\wedge\tau)} \xi^2 + \frac{1}{\delta} \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |f(s, 0, 0, 0)|^2 ds - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} Y_s U_s dM_s.
\end{aligned}$$

Choose $\varepsilon = \eta = \delta = 1$, and $\gamma > 1 + 3K + K^2$. Taking the supremum under 0 and T and the expectation, we obtain

$$\begin{aligned}
& \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\gamma(t\wedge\tau)} Y_{t\wedge\tau}^2 \right) + \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} U_s^2 dM_s \right) \\
&\leq \mathbb{E} \left(e^{\gamma(T\wedge\tau)} \xi^2 \right) + \mathbb{E} \left(\int_0^{T\wedge\tau} e^{\gamma s} |f(s, 0, 0, 0)|^2 ds \right) \\
&\quad + 2 \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^{t\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s \right| \right) + 2 \mathbb{E} \left(\left| \int_0^{T\wedge\tau} e^{\gamma s} Y_s Z_s d\bar{W}_s \right| \right)
\end{aligned}$$

$$+ 2 \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^{t \wedge \tau} e^{\gamma s} Y_s U_s dM_s \right| \right) + 2 \mathbb{E} \left(\left| \int_0^{T \wedge \tau} e^{\gamma s} Y_s U_s dM_s \right| \right).$$

Moreover,

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} Y_t^2 \right) &\leq \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\gamma(t \wedge \tau)} Y_{t \wedge \tau}^2 \right) + \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} U_s^2 dM_s \right) \\ &\leq e^{\gamma T} \mathbb{E} (\xi^2) + e^{\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) \\ &\quad + 4 C_{BDG} \mathbb{E} \left(\left(\int_0^{T \wedge \tau} e^{2\gamma s} Y_s^2 \|Z_s\|^2 ds \right)^{1/2} \right) \\ &\quad + 4 C_{BDG} \mathbb{E} \left(\left(\int_0^{T \wedge \tau} e^{2\gamma s} Y_s^2 U_s^2 d[M, M]_s \right)^{1/2} \right) \\ &\leq e^{\gamma T} \mathbb{E} (\xi^2) + e^{\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) \\ &\quad + 4 C_{BDG} e^{2\gamma T} \mathbb{E} \left(\left(\sup_{0 \leq t \leq T} |Y_t|^2 \int_0^{T \wedge \tau} \|Z_s\|^2 ds \right)^{1/2} \right) \\ &\quad + 4 C_{BDG} e^{2\gamma T} \mathbb{E} \left(\left(\sup_{0 \leq t \leq T} |Y_t|^2 \int_0^{T \wedge \tau} U_s^2 d[M, M]_s \right)^{1/2} \right) \\ &\leq e^{\gamma T} \mathbb{E} (\xi^2) + e^{\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) \\ &\quad + 4(\varepsilon' + \eta') C_{BDG} e^{2\gamma T} \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) \\ &\quad + \frac{4}{\varepsilon'} e^{2\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} \|Z_s\|^2 ds \right) + \frac{4}{\eta'} e^{2\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} U_s^2 d[M, M]_s \right), \end{aligned}$$

for any $\eta', \varepsilon' \in \mathbb{R}_+^*$.

Notice that $d[M, M]_s = (\Delta M_s)^2 = (\Delta H_s)^2 = \Delta H_s = dH_s = dM_s + d\Lambda_s$, so

$$\mathbb{E} \left(\int_0^{T \wedge \tau} U_s^2 d[M, M]_s \right) = \mathbb{E} \left(\int_0^{T \wedge \tau} U_s^2 d\Lambda_s \right)$$

Finally, taking $\varepsilon' = \eta' = \frac{1}{16C_{BDG}}$,

$$\frac{1}{2} \mathbb{E} \left(\sup_{0 \leq t \leq T} Y_t^2 \right) \leq e^{\gamma T} \mathbb{E} (\xi^2) + e^{\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right)$$

$$\begin{aligned}
& + 64 C_{BDG}^2 e^{2\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} \|Z_s\|^2 ds \right) \\
& + 64 C_{BDG}^2 e^{2\gamma T} \mathbb{E} \left(\int_0^{T \wedge \tau} U_s^2 d\Lambda_s \right) < \infty.
\end{aligned}$$

□

Theorem 2.6

Let $\xi \in \mathcal{L}^2(\Omega, \mathcal{G}_{T \wedge \tau}, \mathbb{P})$. If $(Y_t, Z_t, U_t)_{0 \leq t \leq T}$ is a $(\Omega, \mathcal{G}, \mathbb{P})$ -solution of BDSE (4) as defined in Definition 2.4, with f satisfying hypothesis (5) and

$$\begin{aligned}
& \mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) < +\infty, \text{ then} \\
& \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_{t \wedge \tau}|^2 + \int_0^{T \wedge \tau} \|Z_s\|^2 ds + \int_0^{T \wedge \tau} U_s^2 d\Lambda_s \right) \\
& \leq 2 e^{CT} \mathbb{E} \left(\xi^2 + \int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right),
\end{aligned}$$

where C is a constant only depending on K .

Proof. Let $\gamma \in \mathbb{R}$. We apply Itô's formula to process $e^{\gamma t} Y_t^2$ between $t \wedge \tau$ and $T \wedge \tau$. Hence, for any $\eta, \varepsilon, \delta \in \mathbb{R}_+^*$, we have

$$\begin{aligned}
& e^{\gamma(t \wedge \tau)} Y_{t \wedge \tau}^2 + (\gamma - \delta - 2K - \varepsilon K^2 - \eta K) \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s^2 ds + \left(1 - \frac{1}{\varepsilon}\right) \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds \\
& + \left(1 - \frac{1}{\eta}\right) \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} U_s^2 \lambda_s ds + \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} U_s^2 d\Lambda_s^e + \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} U_s^2 dM_s \\
& = e^{\gamma(T \wedge \tau)} \xi^2 + \frac{1}{\delta} \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} |f(s, 0, 0, 0)|^2 ds - 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s Z_s d\bar{W}_s - 2 \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} Y_s U_s dM_s.
\end{aligned}$$

Choose $\delta = 1$. Taking the supremum under 0 and T and the expectation, we have

$$\begin{aligned}
& \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\gamma(t \wedge \tau)} Y_{t \wedge \tau}^2 \right) + (\gamma - 1 - 2K - \varepsilon K^2 - \eta K) \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} Y_s^2 ds \right) \\
& + \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} U_s^2 dM_s \right| \right) + \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} U_s^2 d\Lambda_s^e \right) \\
& + \left(1 - \frac{1}{\varepsilon}\right) \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds \right) + \left(1 - \frac{1}{\eta}\right) \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} U_s^2 \lambda_s ds \right)
\end{aligned}$$

$$\begin{aligned}
&\leq e^{\gamma T} \mathbb{E} \left(\xi^2 + \int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) + 4C_{BDG} \mathbb{E} \left(\int_0^{T \wedge \tau} e^{2\gamma s} Y_s^2 \|Z_s\|^2 ds \right)^{1/2} \\
&\quad + 4C_{BDG} \mathbb{E} \left(\int_0^{T \wedge \tau} e^{2\gamma s} Y_s^2 U_s^2 d[M, M]_s \right)^{1/2} \\
&\leq e^{\gamma T} \mathbb{E} \left(\xi^2 + \int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) + 2C_{BDG} (\varepsilon' + \eta') \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} Y_s^2 ds \right) \\
&\quad + 2C_{BDG} \mathbb{E} \left(\frac{1}{\varepsilon'} \int_0^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds + \frac{1}{\eta'} \int_0^{T \wedge \tau} e^{\gamma s} U_s^2 d\Lambda_s \right)
\end{aligned}$$

for any $\eta', \varepsilon' \in \mathbb{R}_+^*$, noticing that $[M, M]_s = dM_s + d\Lambda_s$.

Finally, taking $\varepsilon' = \eta' = 8C_{BDG}$, $\varepsilon = \eta = 4$ and $\gamma > 1 + 6K + 4K^2 + 32C_{BDG}^2$,

$$\begin{aligned}
&\mathbb{E} \left(\sup_{0 \leq t \leq T} Y_t^2 + \frac{1}{2} \int_0^{T \wedge \tau} \|Z_s\|^2 ds + \frac{1}{2} \int_0^{T \wedge \tau} U_s^2 d\Lambda_s \right) \\
&\leq \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\gamma(t \wedge \tau)} Y_{t \wedge \tau}^2 + \frac{1}{2} \int_0^{T \wedge \tau} e^{\gamma s} \|Z_s\|^2 ds + \frac{1}{2} \int_0^{T \wedge \tau} e^{\gamma s} U_s^2 (d\Lambda_s + d\Lambda_s^c) \right) \\
&\leq e^{\gamma T} \mathbb{E} \left(\xi^2 + \int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right).
\end{aligned}$$

By choosing $C = 2 + 6K + 4K^2 + 32C_{BDG}^2$, the result follows. \square

Corollary 2.7 *Let $\xi \in \mathcal{L}^2(\Omega, \mathcal{G}_{T \wedge \tau}, \mathbb{P})$. If $(Y_t, Z_t, U_t)_{0 \leq t \leq T}$ is a $(\Omega, \mathcal{G}, \mathbb{P})$ -solution of BDSE (4) as defined in Definition 2.4, with f satisfying hypothesis (5) and*

$$\begin{aligned}
&\mathbb{E} \left(\int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right) < +\infty, \text{ then} \\
&\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_{t \wedge \tau}|^2 + \int_0^{T \wedge \tau} (1 - F_s)^2 \|Z_s\|^2 ds + \int_0^{T \wedge \tau} U_s^2 d\Lambda_s \right) \\
&\leq 2e^{CT} \mathbb{E} \left(\xi^2 + \int_0^{T \wedge \tau} |f(s, 0, 0, 0)|^2 ds \right),
\end{aligned}$$

where C is a constant only depending on K .

Proof. As $\int_0^{T \wedge \tau} (1 - F_s)^2 \|Z_s\|^2 ds \leq \int_0^{T \wedge \tau} \|Z_s\|^2 ds$, the result follows from Theorem 2.6. \square

We can now state the following theorem :

Theorem 2.8

Let $\xi \in \mathcal{L}^2(\Omega, \mathcal{G}_{T \wedge \tau}, \mathbb{P})$ and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ be \mathcal{G} -measurable.

If $\mathbb{E} \left(\int_0^T |f(s, 0, 0, 0)|^2 ds \right) < \infty$ and if f satisfies condition (5), there exists a unique \mathcal{G} -adapted triple $(Y, Z, U) \in \mathcal{B}_0^2$ solution of the BSDE:

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \quad 0 \leq t \leq T.$$

Proof.

We can adopt the usual contraction method using representation Theorem 2.3.

Let $\gamma \in \mathbb{R}$. Recall that $\mathcal{B}_\gamma^2 = \mathcal{S}_\gamma^2 \times \mathcal{L}_\gamma^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_\gamma^2(M, \mathbb{P})$. We define a function $\Phi : \mathcal{B}_0^2 \rightarrow \mathcal{B}_0^2$ such that $(Y, Z, U) \in \mathcal{B}_0^2$ is a solution of our BSDE if it is a fixed point of Φ .

Let $(\bar{Y}, \bar{Z}, \bar{U}) \in \mathcal{B}_0^2$. Define $(Y, Z, U) = \Phi(\bar{Y}, \bar{Z}, \bar{U})$ with :

$$Y_t = \mathbb{E} \left(\xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds \mid \mathcal{G}_t \right), \quad 0 \leq t \leq T,$$

and processes $(Z_t)_{0 \leq t \leq T}$ and $(U_t)_{0 \leq t \leq T}$ obtained by using martingale representation Theorem 2.3 applied to square integrable $(\mathcal{G}, \mathbb{P})$ -martingale $(N_t)_{0 \leq t \leq T}$ where $N_t = \mathbb{E} \left(\xi + \int_0^{T \wedge \tau} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds \mid \mathcal{G}_t \right)$.

Moreover, $Z \in \mathcal{L}_0^2(\bar{W}, \mathbb{P})$ and $U \in \mathcal{L}_0^2(M, \mathbb{P})$.

Hence

$$\begin{aligned} N_{t \wedge \tau} &= N_{T \wedge \tau} - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \\ Y_{t \wedge \tau} + \int_0^{t \wedge \tau} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds &= \xi + \int_0^{T \wedge \tau} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds \\ &\quad - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s. \end{aligned}$$

Consequently

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s.$$

This means that (Y, Z, U) is a $(\Omega, \mathcal{G}, \mathbb{P})$ -solution to Equation (4) with particular generator $s \mapsto g(s) = f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s)$, which implies thanks to Theorem 2.6 that the triple (Y, Z, U) belongs to the convenient space \mathcal{B}_0^2 and consequently map Φ is well defined.

Next, for $(\bar{Y}_1, \bar{Z}_1, \bar{U}_1)$ and $(\bar{Y}_2, \bar{Z}_2, \bar{U}_2)$ in \mathcal{B}_0^2 , we define $(Y_1, Z_1, U_1) = \Phi(\bar{Y}_1, \bar{Z}_1, \bar{U}_1)$ and $(Y_2, Z_2, U_2) = \Phi(\bar{Y}_2, \bar{Z}_2, \bar{U}_2)$. We denote $(\bar{Y}, \bar{Z}, \bar{U}) = (\bar{Y}_1 - \bar{Y}_2, \bar{Z}_1 - \bar{Z}_2, \bar{U}_1 - \bar{U}_2)$ and $(\hat{Y}, \hat{Z}, \hat{U}) = (Y_1 - Y_2, Z_1 - Z_2, U_1 - U_2)$.

So

$$\begin{aligned}\hat{Y}_{t\wedge\tau} &= \int_{t\wedge\tau}^{T\wedge\tau} (f(s, \bar{Y}_{1s}, \bar{Z}_{1s}, \bar{U}_{1s}) - f(s, \bar{Y}_{2s}, \bar{Z}_{2s}, \bar{U}_{2s})) ds \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} \hat{Z}_s d\bar{W}_s - \int_{t\wedge\tau}^{T\wedge\tau} \hat{U}_s dM_s.\end{aligned}$$

Then, from Itô's formula,

$$\begin{aligned}e^{\gamma(t\wedge\tau)} \hat{Y}_{t\wedge\tau}^2 &= -\gamma \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s^2 ds - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s d\hat{Y}_s - \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} d[\hat{Y}, \hat{Y}]_s^c \\ &\quad - \sum_{t\wedge\tau \leq s \leq T\wedge\tau} e^{\gamma s} \left(\hat{Y}_s^2 - \hat{Y}_{s-}^2 - 2\hat{Y}_{s-} \Delta\hat{Y}_{s-} \right) \\ &= -\gamma \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s^2 ds - \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|\hat{Z}_s\|^2 ds - \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{U}_s^2 (dM_s + d\Lambda_s) \\ &\quad - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s \hat{Z}_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s \hat{U}_s dM_s \\ &\quad + 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s (f(s, \bar{Y}_{1s}, \bar{Z}_{1s}, \bar{U}_{1s}) - f(s, \bar{Y}_{2s}, \bar{Z}_{2s}, \bar{U}_{2s})) ds.\end{aligned}$$

Then

$$\begin{aligned}e^{\gamma(t\wedge\tau)} \hat{Y}_{t\wedge\tau}^2 &+ \gamma \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|\hat{Z}_s\|^2 ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{U}_s^2 d\Lambda_s \\ &= - \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{U}_s^2 dM_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s \hat{Z}_s d\bar{W}_s - 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s \hat{U}_s dM_s \\ &\quad + 2 \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{Y}_s (f(s, \bar{Y}_{1s}, \bar{Z}_{1s}, \bar{U}_{1s}) - f(s, \bar{Y}_{2s}, \bar{Z}_{2s}, \bar{U}_{2s})) ds.\end{aligned}$$

Taking expectation, it yields to the following equation, for any $\varepsilon'', \eta'', \delta'' \in \mathbb{R}_+^*$:

$$\begin{aligned}\mathbb{E} \left(e^{\gamma(t\wedge\tau)} |\hat{Y}_{t\wedge\tau}|^2 + \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} (\gamma |\hat{Y}_s|^2 + \|\hat{Z}_s\|^2) ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \hat{U}_s^2 d\Lambda_s \right) \right) \\ \leq 2 \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |\hat{Y}_s| (K |\bar{Y}_s| + K \|\bar{Z}_s\| + \lambda_s |\bar{U}_s|) ds \right) \\ \leq \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} (K^2 \delta'' + K^2 \varepsilon'' + \eta'' \lambda_s) e^{\gamma s} |\hat{Y}_s|^2 ds \right) + \frac{1}{\delta''} \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} |\bar{Y}_s|^2 ds \right) \\ + \frac{1}{\varepsilon''} \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \|\bar{Z}_s\|^2 ds \right) + \frac{1}{\eta''} \mathbb{E} \left(\int_{t\wedge\tau}^{T\wedge\tau} e^{\gamma s} \bar{U}_s^2 \lambda_s ds \right).\end{aligned}$$

Choosing $\varepsilon'' = \eta'' = \delta'' = 2$ we obtain

$$\begin{aligned} & \mathbb{E} \left(e^{\gamma(t \wedge \tau)} |\hat{Y}_{t \wedge \tau}|^2 \right) + \mathbb{E} \left(\int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} (\gamma |\hat{Y}_s|^2 + \|\hat{Z}_s\|^2) ds + \int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} \hat{U}_s^2 d\Lambda_s \right) \\ & \leq (4K^2 + 2K) \mathbb{E} \left(\int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} |\hat{Y}_s|^2 ds \right) + \frac{1}{2} \mathbb{E} \left(\int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} |\bar{Y}_s|^2 ds \right) \\ & \quad + \frac{1}{2} \mathbb{E} \left(\int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} \|\bar{Z}_s\|^2 ds \right) + \frac{1}{2} \mathbb{E} \left(\int_{t \wedge \tau}^{T \wedge \tau} e^{\gamma s} \bar{U}_s^2 \lambda_s ds \right). \end{aligned}$$

Choosing $\gamma = 4K^2 + 2K + 1$, it gives :

$$\begin{aligned} & \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} (|\hat{Y}_s|^2 + \|\hat{Z}_s\|^2) ds + \int_0^{T \wedge \tau} e^{\gamma s} \hat{U}_s^2 d\Lambda_s \right) \\ & \leq \frac{1}{2} \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} (|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2) ds + \int_0^{T \wedge \tau} e^{\gamma s} \bar{U}_s^2 \lambda_s ds \right) \\ & \quad + \frac{1}{2} \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} \bar{U}_s^2 d\Lambda_s^e \right). \end{aligned}$$

Then Φ is a strict contraction on \mathcal{B}_0^2 with norm

$$\| (Y, Z, U) \|_\gamma = \mathbb{E} \left(\int_0^{T \wedge \tau} e^{\gamma s} (|Y_s|^2 + \|Z_s\|^2) ds + \int_0^{T \wedge \tau} e^{\gamma s} U_s^2 d\Lambda_s \right)^{\frac{1}{2}}.$$

We finally deduce that Φ has a unique fixed point and conclude that the BSDE has a unique solution. \square

2.3 Uniqueness result

The previous theorem yields to uniqueness of the solution in \mathcal{B}_0^2 . But uniqueness is needed in a larger space. Let us define

- $\mathcal{L}_{F,0}^2(\bar{W}, \mathbb{P})$ the set of all $1 \times m$ -dimensional \mathcal{G} -predictable processes $(Z_t)_{0 \leq t \leq T}$ such that $\|Z\|_{\mathcal{L}_{F,0}^2(\bar{W}, \mathbb{P})}^2 = \mathbb{E}_{\mathbb{P}} \left(\int_0^{T \wedge \tau} (1 - F_s)^2 \|Z_s\|^2 ds \right) < \infty$.

Corollary 2.9

BDSE (4) has a unique solution in $\mathcal{S}_0^2 \times \mathcal{L}_{F,0}^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$.

Proof.

Let (Y_t, Z_t, U_t) and $(\bar{Y}_t, \bar{Z}_t, \bar{U}_t)$ in $\mathcal{S}_0^2 \times \mathcal{L}_{F,0}^2(\bar{W}, \mathbb{P}) \times \mathcal{L}_{F,0}^2(M, \mathbb{P})$ be solutions of (4). Define $\hat{Y} = Y - \bar{Y}$, $\hat{Z} = Z - \bar{Z}$ and $\hat{U} = U - \bar{U}$. Then

$$\begin{aligned} \hat{Y}_{t \wedge \tau} &= \int_{t \wedge \tau}^{T \wedge \tau} (f(s, Y_s, Z_s, U_s) - f(s, \bar{Y}_s, \bar{Z}_s, \bar{U}_s)) ds - \int_{t \wedge \tau}^{T \wedge \tau} \hat{Z}_s d\bar{W}_s \\ &\quad - \int_{t \wedge \tau}^{T \wedge \tau} \hat{U}_s dM_s, \quad 0 \leq t \leq T. \end{aligned}$$

Define also

$$\alpha_s = \begin{cases} \frac{f(s, Y_s, Z_s, U_s) - f(s, \bar{Y}_s, Z_s, U_s)}{Y_s - \bar{Y}_s} & \text{if } Y_s \neq \bar{Y}_s, \\ 0 & \text{else.} \end{cases}$$

For any $z, z' \in \mathbb{R}^m$ and $k = 1, 2, \dots, m$ we will use the following usual notations : $(z, z')_k = (z'_1, z'_2, \dots, z'_{k-1}, z_k, z_{k+1}, \dots, z_m)$, and $(z - z')_k = z_k - z'_k$.

Denote by $\Gamma_{k,s}^Z = (Z_s, \bar{Z}_s)_k$ and define

$$\beta_{k,s} = \begin{cases} \frac{f(s, Y_s, \Gamma_{k,s}^Z, U_s) - f(s, Y_s, \Gamma_{k+1,s}^Z, U_s)}{(\hat{Z}_s)_k} & \text{if } (\hat{Z}_s)_k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For the last coordinate, we also introduce

$$\theta_s = \begin{cases} \frac{f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z_s, \bar{U}_s)}{U_s - \bar{U}_s} & \text{if } U_s \neq \bar{U}_s, \\ 0 & \text{else.} \end{cases}$$

Consequently the triple of processes $(\hat{Y}, \hat{Z}, \hat{U})$ is solution of the following BSDE

$$\hat{Y}_{t \wedge \tau} = \int_{t \wedge \tau}^{T \wedge \tau} g(s, \hat{Y}_s, \hat{Z}_s, \hat{U}_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} \hat{Z}_s d\bar{W}_s - \int_{t \wedge \tau}^{T \wedge \tau} \hat{U}_s dM_s,$$

with generator $g(s, y, z, u) = \alpha_s y + \sum_{k=1}^m \beta_{k,s} z_k + \theta_s u$.

The result follows then from Corollary 2.7 \square

Remark. Given that $\mathcal{L}_0^2 \subseteq \mathcal{L}_{F,0}^2$, solutions of BSDE (4) stated in Theorems 2.8 and in Corollary 2.9 are equal.

3 Hedging strategy in the defaultable world with BSDE

3.1 Defaultable zero-coupon

We have to introduce the following hypothesis, which is a standard hypothesis in default risk models :

Hypothesis (H) Any square integrable $(\mathcal{F}, \mathbb{P})$ -martingale is a square integrable $(\mathcal{G}, \mathbb{P})$ -martingale.

Under this hypothesis, Brownian motion W is still a Brownian motion in the enlarged filtration and so results obtained in the previous section hold with W instead of \bar{W} .

In this section, we assume that process Λ is absolutely continuous with respect to the Lebesgue measure.

As explained in the introduction, we denote by $\tilde{\mathbb{P}}$ the unique e.m.m equivalent to \mathbb{P} on \mathcal{F} .

According to [7], section 3.3, when (H) holds on the historical probability, and when the \mathcal{F} -market is complete, then the defaultable market is arbitrage free. Hence, (H) holds under any \mathcal{G} -equivalent martingale measure \mathbb{P}^ψ such that $\mathbb{P}^\psi|_{\mathcal{G}_t} = K_t^\psi \mathbb{P}|_{\mathcal{G}_t}$ with

$$dK_t^\psi = K_{t-}^\psi (-\theta_t dW_t + \psi_t dM_t), \quad 0 \leq t \leq T,$$

where $\theta = \sigma^{-1}(\mu - r)$ denotes the risk premium and $\psi > -1$.

The equation satisfied by K^ψ is obtained using the representation theorem for all $(\mathcal{G}, \mathbb{P})$ square-integrable martingales established by S. Kusuoka [18] under hypothesis (H).

Let us denote by \mathbb{P}^ψ such a \mathcal{G} -equivalent martingale measure, chosen by the market. Then we have $\mathbb{P}^\psi|_{\mathcal{F}} = \mathbb{P}|_{\mathcal{F}} = \tilde{\mathbb{P}}|_{\mathcal{F}}$. We shall denote by W^0 the Brownian motion obtained using Girsanov's transformation (since the coefficient in the Radon-Nykodym density associated to the Brownian motion is always θ). We also introduce processes $F^\psi, \Lambda^\psi, M^\psi$ constructed in the same way as F, Λ, M but associated to the probability \mathbb{P}^ψ instead of \mathbb{P} . Let us denote by $(\tilde{\rho}_t)_{0 \leq t \leq T}$ the discounted price of the defaultable zero-coupon bond.

We obtain from Proposition 2 in [7] that :

$$d\tilde{\rho}_t = \frac{\mathbb{1}_{\tau > t}}{1 - F_{t-}^\psi} \phi_t^m dW_t^0 - \tilde{\rho}_{t-} dM_t^\psi, \quad 0 \leq t \leq T,$$

where $(\phi_t^m)_{t \geq 0}$ comes from the representation of $(\mathcal{F}, \mathbb{P}^0)$ -martingale $(m_t)_{t \geq 0} = \left(\mathbb{E}_{\mathbb{P}^0}(R_T \mathbb{1}_{\tau > T} | \mathcal{F}_t) \right)_{t \geq 0}$ with respect to $(\mathcal{F}, \mathbb{P}^0)$ -Brownian motion W^0 .

As $\forall t \in]0, T \wedge \tau[\tilde{\rho}_{t-} \neq 0$, we can set $c_t = \frac{\mathbb{1}_{\tau > t}}{1 - F_{t-}^\psi} \frac{\phi_t^m}{\tilde{\rho}_{t-}}$.

Using Girsanov transformation, the dynamics of the defaultable zero-coupon under historical probability is given by

$$d\rho_t = \rho_{t-} (a_t dt + c_t dW_t - dM_t), \quad (6)$$

where a_t is determined by the \mathcal{G} -equivalent martingale measure \mathbb{P}^ψ in the following way :

$$a_t = r_t + \theta_t c_t + (1 - H_{t-}) \psi_t \lambda_t. \quad (7)$$

3.2 Wealth's dynamic

Let Y_t be the wealth at time t of the agent. Suppose that she has α_t parts of the risky asset, δ_t parts of the riskless asset, and β_t parts of the defaultable zero-coupon bond. At any time t , we have :

$$Y_t = \alpha_t S_t + \beta_t \rho_{t-} + \delta_t S_t^0. \quad (8)$$

where α_t, β_t and δ_t are predictable

The self-financing hypothesis can be written :

$$dY_t = \alpha_t dS_t + \beta_t d\rho_t + \delta_t dS_t^0,$$

which can be developed, for any t in $[0, T \wedge \tau]$, using (8) and the dynamics of the three assets (2), (6) and (3). This yields to

$$\begin{aligned} dY_t &= (\alpha_t \mu_t S_t + r_t Y_t - \alpha_t r_t S_t - \beta_t r_t \rho_{t-} + \beta_t a_t \rho_{t-}) dt \\ &\quad + (\alpha_t \sigma_t S_t + \beta_t c_t \rho_{t-}) dW_t - \beta_t \rho_{t-} dM_t. \end{aligned}$$

Then, denoting by $Z_t = \alpha_t \sigma_t S_t + \beta_t c_t \rho_{t-}$ and $U_t = -\beta_t \rho_{t-}$, we obtain a stochastic differential equation for wealth :

$$\begin{cases} dY_t &= -f(t, Y_t, Z_t, U_t) dt + Z_t dW_t + U_t dM_t, \quad 0 \leq t \leq T \wedge \tau \\ Y_{T \wedge \tau} &= \xi \end{cases} \quad (9)$$

with $f(t, y, z, u) = -r_t y - \theta_t z + (a_t - r_t - \theta_t c_t) u$. Using (7), we obtain

$$f(t, y, z, u) = -r_t y - \theta_t z + (1 - H_{t-}) \psi_t \lambda_t u. \quad (10)$$

This provides a stochastic differential equation with \mathcal{G}_t -adapted coefficients. As \mathcal{F} -Brownian motion W remains a Brownian motion under the new filtration \mathcal{G} , the previous stochastic differential equation has a sense.

This equation is a backward stochastic differential equation with uncertain horizon, with terminal condition :

$$Y_{T \wedge \tau} = \xi = V \mathbb{1}_{\tau > T} + C_\tau \mathbb{1}_{T \leq \tau},$$

where $V \in \mathcal{L}^2(\mathcal{F}_T)$ and $(C_t)_{t \geq 0}$ is an \mathcal{F} -adapted process.

As condition (5) holds true, as r, θ and λ are bounded, and as $f(s, 0, 0, 0) = 0$, the integrability condition on f under \mathbb{P} is also satisfied, we can apply Theorem 2.8.

3.3 Explicit solution for the hedging strategy

We consider now the particular case of a hedging strategy against a defaultable contingent claim. Then, the terminal value represents the payoff and has the form $\xi = V \mathbb{1}_{\tau > T} + C_\tau \mathbb{1}_{\tau \leq T}$, where $V \in \mathcal{L}^2(\mathcal{F}_T)$ and C is an \mathcal{F} -adapted process. Previous

Theorem 2.8 guarantees existence and uniqueness of the solution. But we can go further by giving an explicit formulation of processes Z and U that stands for the strategy, thanks to the explicit representation Theorem obtained C. Blanchet-Scalliet and M. Jeanblanc [7], instead of M. Yor representation Theorem previously used.

Theorem 3.1

Let \mathbb{P}^ψ an e.m.m chosen by the market and let $V \in \mathcal{L}^2(\mathcal{F}_T)$ and C be an \mathcal{F} -adapted process. We set $\xi = V \mathbb{1}_{\tau > T} + C_\tau \mathbb{1}_{\tau \leq T}$.

Consider also the \mathcal{G} -measurable generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(t, y, z, u) = -r_t y - \theta_t z + (1 - H_{t-}) \psi_t \lambda_t u,$$

satisfying condition (5).

Suppose that Λ^ψ is absolutely continuous and that F^ψ , defined in (1), is continuous. Then, under hypothesis (H), there exists a unique \mathcal{G} -adapted triple $(Y, Z, U) \in \mathcal{B}_0^2$ solution of the BSDE :

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s, U_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dW_s - \int_{t \wedge \tau}^{T \wedge \tau} U_s dM_s, \quad 0 \leq t \leq T. \quad (11)$$

Moreover, $Z_t = \frac{a_t^C + a_t^Y}{R_t(1 - F_t^\psi)}$ and $U_t = C_t - R_t^{-1} \mathbb{E}_{\mathbb{P}^\psi}(R_\tau C_\tau | \mathcal{G}_{t-}) - R_t^{-1} \mathbb{E}_{\mathbb{P}^\psi}(R_T V \mathbb{1}_{T < \tau} | \mathcal{G}_{t-})$,

where $(a_t^C)_{t \geq 0}$ comes from the representation of $(\mathcal{F}, \mathbb{P}^\psi)$ -martingale

$\left(\mathbb{E}_{\mathbb{P}^\psi} \left(\int_0^\infty R_s C_s dF_s^\psi | \mathcal{F}_t \right) \right)_{t \geq 0}$ and $(a_t^Y)_{t \geq 0}$ from $\left(\mathbb{E}_{\mathbb{P}^\psi}(R_T V \mathbb{1}_{\tau > T} | \mathcal{F}_t) \right)_{t \geq 0}$, where R denotes the discounted factor.

Proof.

Let us consider the discounted process $(R_t Y_t)_{0 \leq t \leq T}$. Given that

$$R_{t \wedge \tau} Y_{t \wedge \tau} = \mathbb{E}_{\mathbb{P}^\psi}(R_{T \wedge \tau} \xi | \mathcal{G}_t),$$

we can compute separately for $R_\tau C_\tau$ and for $R_T V \mathbb{1}_{T < \tau}$.

First case. Let $X_t^C = \mathbb{E}_{\mathbb{P}^\psi}(R_\tau C_\tau | \mathcal{G}_t)$, where C is an \mathcal{F} -predictable process. Then from Proposition 3 in C. Blanchet-Scalliet and M. Jeanblanc [7], we have

$$X_t^C = X_0^C + \int_0^{t \wedge \tau} \frac{1}{\mathbb{P}^\psi(\tau > s | \mathcal{F}_s)} a_s^C dW_s^0 + \int_0^{t \wedge \tau} (R_s C_s - X_{s-}^C) dM_s^\psi, \quad (12)$$

where $(a_t^C)_{t \geq 0}$ comes from the representation of $(\mathcal{F}, \mathbb{P}^\psi)$ -martingale

$\left(\mathbb{E}_{\mathbb{P}^\psi} \left(\int_0^\infty R_s C_s dF_s | \mathcal{F}_t \right) \right)_{t \geq 0}$ with respect to $(\mathcal{F}, \mathbb{P}^\psi)$ -Brownian motion W^0 .

Second case. Let $V \in \mathcal{F}_T$ be an integrable random variable. From Proposition 3 of [7], the $(\mathcal{G}, \mathbb{P}^\psi)$ -martingale $X_t^V = \mathbb{E}_{\mathbb{P}^\psi}(R_T V \mathbb{1}_{T < \tau} | \mathcal{G}_t)$ can be represented as follows

:

$$X_t^V = X_0^V + \int_0^{t \wedge \tau} \frac{1}{\mathbb{P}^\psi(\tau > s | \mathcal{F}_s)} a_s^V dW_s^0 - \int_0^{t \wedge \tau} X_{s-}^V dM_s^\psi, \quad (13)$$

where $(a_t^V)_{t \geq 0}$ comes from the representation of $(\mathcal{F}, \mathbb{P}^\psi)$ -martingale $\left(\mathbb{E}_{\mathbb{P}^\psi}(R_T V \mathbb{1}_{\tau > T} | \mathcal{F}_t) \right)_{t \geq 0}$ with respect to $(\mathcal{F}, \mathbb{P}^\psi)$ -Brownian motion W^0 .

By summing (12) and (13), we obtain $R_s Z_s = \frac{a_s^C + a_s^V}{\mathbb{P}^\psi(\tau > s | \mathcal{F}_s)}$ and $R_s U_s = R_s C_s - X_{s^-}^C - X_{s^-}^V$.

Consequently, using Corollary 2.9, (Y, Z, U) is the unique solution of BSDE (11) in $\mathcal{S}^2 \times \mathcal{L}_{F,0}^2(W, \mathbb{P}) \times \mathcal{L}_0^2(M, \mathbb{P})$. From uniqueness results in Corollary 2.9 and in Theorem 2.8, it is also the unique solution of BSDE (11) in \mathcal{B}_0^2 . \square

The main consequence of Theorems 2.8 and 3.1 is that the BSDE has a unique solution adapted to the enlarged space $(\Omega, \mathcal{G}, \mathbb{P})$. The interest of the second Theorem instead of the first one is that it gives an explicit form of the solution, and so of the hedging portfolio. This particular case can be used in solving hedging problems of defaultable contingent claims and also in problems issued from hedging of non life insurance products. These financial examples are a particular case of Theorem 3.1, applied to BSDE (9) with generator f defined as in (10).

Remark. By means of solving BSDEs, we find the same results as those stated in C. Blanchet-Scalliet and M. Jeanblanc [7], as a special case of the last Theorem.

4 Conclusion

This article has presented a BSDE approach to finding hedging strategies in a defaultable world. Results have been obtained for a large panel of hedging payoffs, and under general assumptions. The hedging portfolios have been expressed in term of a solution of a backward stochastic differential equation. Well known results from T.R. Bielecki, M. Jeanblanc and M. Rutkowski [4, 3] or C. Blanchet and M. Jeanblanc [7] have been found, giving a new way to obtain it.

From a practical point of view, the use of this approach may lead to obtain numerical approximations of the hedging portfolios, in cases where direct computation may not be implemented. One can use numerical schemes for backward stochastic differential equations, such as E. Gobet and C. Labart [13], and even for reflected BSDEs, see J. Mémin, S. Peng and M. Xu [19].

Following the approach of T.R. Bielecki, M. Jeanblanc and M. Rutkowski [5], the market may also be completed by adding a CDS (credit default swap). The method presented in the present article will provide hedging strategies by solving the corresponding BSDE. This method may be useful as soon as the generator of the backward equation can be expressed in a closed-form formula.

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