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CONVERGENCE OF MULTIPLE ERGODIC AVERAGES ALONG CUBES FOR SEVERAL COMMUTING TRANSFORMATIONS

QING CHU

ABSTRACT. In this paper, we give the convergence result of multiple ergodic averages along cubes for several commuting transformations, and the correspondant combinatorial results. The main tools we use are the seminorms and “magic” extension introduced by Host recently.

1. INTRODUCTION

In this paper, we give the following convergence result of multiple ergodic averages along cubes for several commuting transformations :

Theorem 1.1. *Let $(X, \mu, T_1, \dots, T_d)$ be a probability system, where T_1, \dots, T_d are commuting measure preserving transformations. Then for all $f_\epsilon \in L^\infty(\mu)$, we have the averages*

$$(1) \quad \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i] \\ i=1, \dots, d}} \prod_{\substack{\epsilon \in \{0,1\}^d \\ \epsilon \neq 0}} T_1^{n_1 \epsilon_1} \dots T_d^{n_d \epsilon_d} f_\epsilon$$

converge in $L^2(\mu)$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ ($1 \leq i \leq d$) tend to ∞ .

For better understanding, we take $d = 2$ for example, then the formula (1) is

$$\frac{1}{(N_1 - M_1) \times (N_2 - M_2)} \sum_{\substack{n_1 \in [M_1, N_1) \\ n_2 \in [M_2, N_2)}} T_1^{n_1} f_{10} \cdot T_2^{n_2} f_{01} \cdot T_1^{n_1} T_2^{n_2} f_{11}.$$

The case of all the transformations T_i are same was proved by Host and Kra in [4].

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When restricting Theorem 1.1 to the indicator function of a measurable set, we have the following lower bound for these averages:

Theorem 1.2. *Let $(X, \mu, T_1, \dots, T_d)$ be a probability system, where T_1, \dots, T_d are commuting measure preserving transformations, let $A \in \mathcal{B}$. Then the limit of the averages*

$$(2) \quad \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i) \\ i=1, \dots, d}} \mu \left(\bigcap_{\epsilon \in \{0,1\}^d} T_1^{-n_1 \epsilon_1} \dots T_d^{-n_d \epsilon_d} A \right)$$

exists and is greater than or equal to $\mu(A)^{2^d}$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ ($1 \leq i \leq d$) tend to ∞ .

We recall that the upper density $d^*(A)$ of a set $A \in \mathbb{Z}^d$ is defined to be

$$d^*(A) = \limsup_{\substack{N_i \rightarrow \infty \\ 1 \leq i \leq d}} \prod_{i=1}^d \frac{1}{N_i} |A \cap [1, N_1) \times \dots \times [1, N_d)|.$$

We also have the following correspondent combinatorial result:

Theorem 1.3. *Let $A \in \mathbb{Z}^d$ with $d^*(A) > \delta > 0$. Then the set of $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, so that*

$$d^* \left(\bigcap_{\epsilon \in \{0,1\}^d} \{A + (\epsilon_1 n_1, \dots, \epsilon_d n_d)\} \right) \geq \delta^{2^d}$$

is syndetic (i.e. intersects nontrivially with all rectangles whose area arbitrarily large in \mathbb{Z}^d).

We remark that the convergence of the multiple averages with several commuting transformations of the form

$$\frac{1}{N} \sum_{n=1}^N T_1^n f_1 \dots T_d^n f_d$$

was proved by Conze and Lesigne [2] for two transformations. And the general case was proved by Tao [3], Austin [1], Host [3] and Towsner [6].

We have to mention that, even though our result was covered by the extended results by Austin and Host for \mathbb{Z}^d -action, but our paper present a more explicit understanding about the convergence and the characteristic factor in its “magic” extension system.

The main tools we use in this paper are the seminorms and “magic” extension introduced by Host [3].

2. SEMINORM AND UPPER BOUND

2.1. Notation and definitions. We write $[d] = \{1, 2, \dots, d\}$, and identify $\{0, 1\}^d$ with the family of subsets of $[d]$. Therefore, the assertion “ $i \in \epsilon$ ” is equivalent to $\epsilon_i = 1$. Especially \emptyset is equivalent to $00 \cdots 0 \in \{0, 1\}^d$. We write $|\epsilon| = \sum_i \epsilon_i$ for the number of elements of ϵ .

For $n = (n_1, \dots, n_d)$ and $\epsilon = \{i_1, \dots, i_k\} \subset [d]$ for some integer $1 \leq k \leq d$, we write

$$T_\epsilon^n = T_{i_1}^{n_{i_1}} \cdots T_{i_k}^{n_{i_k}}.$$

For any transformation S of some system, we denote $\mathcal{I}(S)$ the σ -algebra of sets invariant under S .

We define a measure μ_1 on X^2 by

$$\mu_1 = \mu \times_{\mathcal{I}(T_1)} \mu_1.$$

For $2 \leq k \leq d$, we define a measure μ_k on X^{2^k} by

$$\mu_k = \mu_{k-1} \times_{\mathcal{I}(T_k^\Delta)} \mu_{k-1}.$$

where $T_k^\Delta := \underbrace{T_k \times \cdots \times T_k}_{2^{k-1}}$.

We write $X^* = X^{2^d}$, any points of X^* are written as $x = (x_\epsilon : \epsilon \subset [d])$. we denote $\mu^* := \mu_d$.

For $f \in L^\infty(\mu)$, define

$$\|f\|_{T_1, \dots, T_d} := \left(\int \prod_{\epsilon \in \{0, 1\}^d} f(x_\epsilon) d\mu^*(x) \right)^{1/2^d}.$$

It was shown by Proposition 3 in [3], $\|\cdot\|_{T_1, \dots, T_d}$ is a seminorm on $L^\infty(\mu)$. We call it the box seminorm associated to T_1, \dots, T_d .

For $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, we write $\|\cdot\|_\epsilon$ for the seminorm on $L^\infty(\mu)$ associated to the transformations T_i , $i \in \epsilon$. For example, $\|\cdot\|_{110 \cdots 00}$ is the seminorm associated to T_1, T_2 because $\epsilon = 110 \cdots 0 \in \{0, 1\}^d$ is identified with $\{1, 2\} \subset [d]$.

By Proposition 2 in [3], we know that if we exchange the ordering of digits in $[d]$, the measure of each step in our construction will be changed by permutation, but the seminorm $\|\cdot\|_\epsilon$ will only depend on transformations T_i , $i \in \epsilon$, not on their ordering.

2.2. Upper bound. In the following, we assume that all functions f_ϵ , $\epsilon \subset [d]$, are real valued and satisfy $|f_\epsilon| \leq 1$.

Proposition 2.1.

(3)

$$\limsup_{\substack{N_i - M_i \rightarrow \infty \\ i=1, \dots, d}} \left\| \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \dots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} T_\epsilon^n f_\epsilon \right\|_{L^2(\mu)} \leq \min_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} \|f_\epsilon\|_{T_1 \dots T_d}.$$

Proof. We proceed by induction on d . For $d = 1$, we have

$$\left\| \frac{1}{N_1 - M_1} \sum_{n_1 \in [M_1, N_1]} T_1^{n_1} f_1 \right\|_{L^2(\mu)}^2 \rightarrow \int \mathbb{E}(f_1 | \mathcal{I}(T_1))^2 d\mu = \|f_1\|_{T_1}^2.$$

Let $d \geq 2$ and assume that (3) is established for $d-1$ transformations.

We show that for every given $\alpha \subset [d], \alpha \neq \emptyset$, the limsup of the the left side of (3) is $\leq \|f_\alpha\|$. By exchanging the digits if needed, we can assume that $\alpha \neq \underbrace{0 \dots 0}_{d-1} 1$. We have the square of the norm in the left side of (3) is equal to

$$\begin{aligned} & \left\| \frac{1}{N_d - M_d} \sum_{n_d \in [M_d, N_d]} T_d^{n_d} f_{0 \dots 0 1} \cdot \prod_{i=1}^{d-1} \frac{1}{N_i - M_i} \sum_{m \in [M_1, N_1] \times \dots \times [M_{d-1}, N_{d-1}]} \right. \\ & \quad \left. \prod_{\substack{\eta \subset [d-1] \\ \eta \neq \emptyset}} T_\eta^m (f_{\eta 0} \cdot T_d^{n_d} f_{\eta 1}) \right\|_{L^2(\mu)}^2 \\ & \leq \frac{1}{N_d - M_d} \sum_{n_d \in [M_d, N_d]} \left\| \prod_{i=1}^{d-1} \frac{1}{N_i - M_i} \sum_{m \in [M_1, N_1] \times \dots \times [M_{d-1}, N_{d-1}]} \right. \\ & \quad \left. \prod_{\substack{\eta \subset [d-1] \\ \eta \neq \emptyset}} T_\eta^m (f_{\eta 0} \cdot T_d^{n_d} f_{\eta 1}) \right\|_{L^2(\mu)}^2. \end{aligned}$$

By the induction hypothesis, when $N_i - M_i \rightarrow \infty, i = 1, \dots, d-1$, we have the limsup of the square of the norm in the above formula is less than

$$\min_{\substack{\eta \subset [d-1] \\ \eta \neq \emptyset}} \|f_{\eta 0} \cdot T_d^{n_d} f_{\eta 1}\|_{T_1, \dots, T_{d-1}}^2.$$

where $\|\cdot\|_{T_1, \dots, T_{d-1}}$ is the seminorm associated to the $d-1$ transformations T_1, \dots, T_{d-1} .

Note that α is equal to $\eta 0$ or $\eta 1$ for some $\eta \subset [d-1]$, and by the Cauchy-Schwartz Inequality, we have

$$\begin{aligned}
 & \lim_{N_d - M_d \rightarrow \infty} \frac{1}{N_d - M_d} \sum_{n_d \in [M_d, N_d]} \|f_{\eta 0} \cdot T_d^{n_d} f_{\eta 1}\|_{T_1, \dots, T_{d-1}}^{2^{d-1}} \\
 &= \lim_{N_d - M_d \rightarrow \infty} \frac{1}{N_d - M_d} \sum_{n_d \in [M_d, N_d]} \int_{\eta \subset [d-1]} \bigotimes (f_{\eta 0} \cdot T_d^{n_d} f_{\eta 1}) d\mu_{d-1} \\
 &= \int \mathbb{E} \left(\bigotimes_{\eta \subset [d-1]} f_{\eta 0} \middle| \mathcal{I}(T_d^\Delta) \right) \cdot \mathbb{E} \left(\bigotimes_{\eta \subset [d-1]} f_{\eta 1} \middle| \mathcal{I}(T_d^\Delta) \right) d\mu_{d-1} \\
 &\leq \left(\int_{\alpha \subset [d]} |\mathbb{E}(\bigotimes_{\alpha \subset [d]} f_\alpha | \mathcal{I}(T_d^\Delta))|^2 d\mu_{d-1} \right)^{1/2} \\
 &= \left(\int_{\alpha \subset [d]} \bigotimes f_\alpha d\mu^* \right)^{1/2} = \|f_\alpha\|_{T_1, \dots, T_d}^{2^{d-1}}.
 \end{aligned}$$

We are done. \square

The following proposition can be considered as a generality of Proposition 2.1, but its proof uses Proposition 2.1.

Proposition 2.2. Let r be an integer with $1 \leq r \leq d$. Then

$$(4) \quad \limsup_{\substack{N_i - M_i \rightarrow \infty \\ i=1, \dots, d}} \left\| \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \dots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset [d] \\ 0 < |\epsilon| \leq r}} T_\epsilon^n f_\epsilon \right\|_{L^2(\mu)} \leq \min_{\substack{\epsilon \subset [d] \\ |\epsilon|=r}} \|f_\epsilon\|_\epsilon$$

Proof. For $\alpha \subset [d]$ with $|\alpha| = r$ we want to show that the above lim sup is $\leq \|f_\alpha\|_\alpha$.

Up to a permutation of digits we can restrict to the case that

$$\alpha = \underbrace{11 \dots 1}_r 00 \dots 0.$$

We show that,

$$(5) \quad \limsup_{\substack{N_i - M_i \rightarrow \infty \\ i=1, \dots, d}} \left\| \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \dots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset [d] \\ 0 < |\epsilon| \leq r}} T_\epsilon^n f_\epsilon \right\|_{L^2(\mu)} \leq \|f_\alpha\|_\alpha.$$

The norm in (5) is equal to

$$\begin{aligned} & \left\| \prod_{i=r+1}^d \frac{1}{N_i - M_i} \sum_{m \in [M_{r+1}, N_{r+1}] \times \cdots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset \{r+1, \dots, d\} \\ \epsilon \neq \emptyset}} T_\epsilon^m f_\epsilon \cdot \prod_{i=1}^r \frac{1}{N_i - M_i} \right. \\ & \left. \sum_{n \in [M_1, N_1] \times \cdots \times [M_r, N_r]} \left(\prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n \prod_{\substack{\theta \subset [d-r] \\ |\eta\theta| \leq r}} T_{r+\theta} f_{\eta\theta} \right) \cdot (T_1^{n_1} \cdots T_r^{n_r} f_\alpha) \right\|_{L^2(\mu)}. \end{aligned}$$

where $r + \theta = \{r + i, i \in \theta\}$.

Let

$$g_\eta = \begin{cases} \prod_{\substack{\theta \subset [d-r] \\ |\eta\theta| \leq r}} T_{r+\theta} f_{\eta\theta} & 0 < |\eta| < r \\ \underbrace{f_1 \dots 1}_{r} 0 \dots 0 & |\eta| = r \end{cases}$$

Then the above is equal to

$$\begin{aligned} & \left\| \prod_{i=r+1}^d \frac{1}{N_i - M_i} \cdot \sum_{m \in [M_{r+1}, N_{r+1}] \times \cdots \times [M_d, N_d]} \left(\prod_{\substack{\epsilon \subset \{r+1, \dots, d\} \\ \epsilon \neq \emptyset}} T_\epsilon^m f_\epsilon \right) \right. \\ & \left. \cdot \prod_{i=1}^r \frac{1}{N_i - M_i} \cdot \sum_{n \in [M_1, N_1] \times \cdots \times [M_r, N_r]} \left(\prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n g_\eta \right) \right\|_{L^2(\mu)}. \end{aligned}$$

By the Cauchy-Schwartz Inequality, the square of the norm in (5) is less than

$$\prod_{i=r+1}^d \frac{1}{N_i - M_i} \sum_{m \in [M_{r+1}, N_{r+1}] \times \cdots \times [M_d, N_d]} \left\| \prod_{i=1}^r \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \cdots \times [M_r, N_r]} \prod_{\substack{\eta \subset [r] \\ \eta \neq \emptyset}} T_\eta^n g_\eta \right\|_{L^2(\mu)}^2.$$

By Proposition 2.1, let $N_i - M_i \rightarrow \infty$, $i = 1, \dots, r$, this is bounded by

$$\prod_{i=r+1}^d \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i] \\ i=r+1, \dots, d}} \| \underbrace{f_{11 \dots 1}}_r 00 \dots 0 \|_{T_1, \dots, T_r}^2 = \| \underbrace{f_{11 \dots 1}}_r 00 \dots 0 \|_{T_1, \dots, T_r}^2.$$

This finishes our proof. \square

3. THE CASE OF THE MAGIC EXTENSION

In the sequel, $(X^*, \mu^*, T_1^*, \dots, T_d^*)$ denotes the “magic” system associated to $(X, \mu, T_1, \dots, T_d)$ (see [3]). Recall that X^* and μ^* are defined in Section 2.1. We denote T_i^* the *side transformations* of X^* , which are given by

$$\text{for every } \epsilon \in \{0, 1\}^d, \quad (T_i^* x)_\epsilon = \begin{cases} T_i x_\epsilon & \text{if } \epsilon_i = 0; \\ x_\epsilon & \text{if } \epsilon_i = 1. \end{cases}$$

We use notation similar to the notation introduced above.

For $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, we write $\|\cdot\|_\epsilon^*$ for the seminorm on $L^\infty(\mu^*)$ associated to the transformations T_i^* , $i \in \epsilon$. Moreover, we define the σ -algebra

$$\mathcal{Z}_\epsilon^* := \bigvee_{i \in \epsilon} \mathcal{I}(T_i^*).$$

of (X^*, μ^*) . For example, $\mathcal{Z}_{\{1,2,d\}}^* = \mathcal{I}(T_1^*) \vee \mathcal{I}(T_2^*) \vee \mathcal{I}(T_d^*)$.

We prove Theorem 1.1 for the magic system $(X^{2^d}, \mu^*, T_1^*, \dots, T_d^*)$.

Theorem 3.1. *Let f_ϵ , $\epsilon \subset [d]$, be functions on X^{2^d} with $\|f_\epsilon\|_{L^\infty(\mu^*)} \leq 1$ for every ϵ . Then*

$$(6) \quad \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \dots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset [d] \\ \epsilon \neq \emptyset}} T_\epsilon^{*n} f_\epsilon$$

converge in $L^2(\mu^*)$ for all sequences of intervals $[M_1, N_1], \dots, [M_d, N_d]$ whose lengths $N_i - M_i$ ($1 \leq i \leq d$) tend to ∞ .

Theorem 3.2. *For every $\epsilon \subset [d]$, $\epsilon \neq \emptyset$, and every function $f \in L^\infty(\mu^*)$ we have:*

$$(7) \quad \text{If } \mathbb{E}_{\mu^*}(f \mid \mathcal{Z}_\epsilon^*) = 0 \text{ then } \|f\|_\epsilon^* = 0.$$

Proof. Assume $|\epsilon| = r > 0$. By a permutation of the coordinates we can assume that

$$\epsilon = \{d - r + 1, d - r + 2, \dots, d\}$$

for some r with $1 \leq r \leq d$.

We define a new system $(Y, \nu, S_1, \dots, S_r)$, where $Y = X^{2^{d-r}}$, $\nu = \mu_{d-r}$, the $d - r$ step measure associated to T_1^*, \dots, T_{d-r}^* , and $S_i = \underbrace{T_{d-r+i}^* \times \dots \times T_{d-r+i}^*}_{2^{d-r}}$. Note that by definition, $Y^* = X^{2^d} = X^*$, $S_i^* =$

T_{d-r+i}^* , $i = 1, \dots, r$, $S_i^\Delta = T_{d-r+i}^\Delta$, $i = 1, \dots, r$. Moreover $\nu_1 = \nu \times_{\mathcal{I}(S_1)} \nu = \mu_{d-r} \times_{\mathcal{I}(T_{d-r+1}^\Delta)} \mu_{d-r} = \mu_{d-r+1}$, and by induction, $\nu_{i+1} = \nu_i \times_{\mathcal{I}(S_{i+1}^\Delta)}$

$\nu_i = \mu_{d-r+i} \times_{\mathcal{I}(S_{i+1}^\Delta)} \mu_{d-r+i} = \mu_{d-r+i} \times_{\mathcal{I}(T_{d-r+i+1}^\Delta)} \mu_{d-r+i} = \mu_{d-r+i+1}$,
 $i = 1, \dots, d-1$. So we get its magic extension $(Y^*, \nu^*, S_1^*, \dots, S_r^*)$ is
just $(X^*, \mu^*, T_{d-r+1}^*, \dots, T_d^*)$. So

$$\mathcal{W}_Y^* := \bigvee_{i=1}^r \mathcal{I}(S_i^*) = \bigvee_{i \in \epsilon} \mathcal{I}(T_i^*) = \mathcal{Z}_\epsilon^*.$$

If $f \in L^\infty(\mu^*)$, $\mathbb{E}_{\mu^*}(f | \mathcal{Z}_\epsilon^*) = 0$, this is equivalent to $\mathbb{E}_{\mu^*}(f | \mathcal{W}_Y^*) = 0$, and by Theorem 1 in [3], we have $\|f\|_{S_1^*, \dots, S_r^*}^* = 0$. Thus $\|f\|_\epsilon^* = \|f\|_{S_1^*, \dots, S_r^*}^* = 0$ \square

Proposition 3.3. Let $f_\epsilon, \epsilon \subset [d]$, be functions on X^{2^d} with $\|f_\epsilon\|_{L^\infty(\mu^*)} \leq 1$ for every ϵ . Let r be an integer with $1 \leq r \leq d$. Then the averages

$$(8) \quad \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{n \in [M_1, N_1] \times \dots \times [M_d, N_d]} \prod_{\substack{\epsilon \subset [d] \\ 0 < |\epsilon| \leq r}} T_\epsilon^{*n} f_\epsilon$$

converge in $L^2(\mu^*)$ for all sequences of intervals $[M_1, N_1), \dots, [M_d, N_d)$ whose lengths $N_i - M_i$ ($1 \leq i \leq d$) tend to ∞ .

We remark that the Theorem 3.1 follows immediately from the case $r = d$ of this proposition.

Proof. We proceed by induction on r . For $r = 1$, the convergence was known (see remarks in Section 1). For $r > 1$, assume that the proposition is true for $r-1$ transformations, we prove it for the system with r transformations.

For $\alpha \subset [d]$, $|\alpha| = r$, if $\mathbb{E}_{\mu^*}(f_\alpha | \mathcal{Z}_\alpha) = 0$, then by Theorem 3.2, we have $\|f_\alpha\|_\alpha^* = 0$. By Proposition 2.1, we have the averages (8) converge to 0. Otherwise, by density, we can assume that

$$f_\alpha = \prod_{i \in \alpha} f_{\alpha, i}$$

where $f_{\alpha, i}$ is T_i^* -invariant. Then

$$T_\alpha^{*n} f_\alpha = \prod_{i \in \alpha} T_{\alpha \setminus \{i\}}^{*n} f_{\alpha, i}.$$

Thus

$$\prod_{\substack{\epsilon \subset [d] \\ 0 < |\epsilon| \leq r}} T_\epsilon^{*n} f_\epsilon = \prod_{\substack{\eta \subset [d] \\ 0 < |\eta| \leq r-1}} T_\eta^{*n} g_\eta$$

where

$$g_\eta = \begin{cases} f_\eta & |\eta| < r-1 \\ f_\eta \prod_{i \notin \eta} f_{\eta \cup i, i} & |\eta| = r-1 \end{cases}$$

So we get (8) converges by the induction hypothesis. \square

4. COMBINATORIAL INTERPRETATION

Proof of Theorem 1.2. By Theorem 1.1, applied to the indicator function $\mathbf{1}_A$, we know that the average

$$(9) \quad \prod_{i=1}^d \frac{1}{N_i - M_i} \sum_{\substack{n_i \in [M_i, N_i] \\ i=1, \dots, d}} \int \prod_{\epsilon \in \{0,1\}^d} T_1^{n_1 \epsilon_1} \dots T_d^{n_d \epsilon_d} \mathbf{1}_A d\mu$$

converges. By Lemma 1 in [3], if we take the limit for $N_1 - M_1 \rightarrow \infty$, then for $N_2 - M_2 \rightarrow \infty, \dots$ and then for $N_d - M_d \rightarrow \infty$, the average (9) converges to $\|\mathbf{1}_A\|_{T_1, \dots, T_d}^{2^d}$, so we have (9) converge to $\|\mathbf{1}_A\|_{T_1, \dots, T_d}^{2^d}$. Since

$$\|f\|_{T_1, \dots, T_d}^{2^d} = \|\mathbb{E}(\bigotimes_{\epsilon \in \{0,1\}^d} f|\mathcal{I}(T_d^\Delta))\|_{L^2(\mu)}^2 \geq \left(\int \bigotimes_{\epsilon \in \{0,1\}^{d-1}} f d\mu_d\right)^2 = \|f\|_{T_1, \dots, T_{d-1}}^{2^d},$$

we have $\|\mathbf{1}_A\|_{T_1, \dots, T_d} \geq \|\mathbf{1}_A\|_{T_1} \geq \int \mathbf{1}_A d\mu = \mu(A)$, and the result follows. \square

Theorem 1.2 has the following corollary:

Corollary 4.1. Let $(X, \mu, T_1, \dots, T_d)$ be a system, where T_1, \dots, T_d are commuting measure preserving transformations, let $A \in \mathcal{B}$. Then for any $c > 0$, the set of $n \in \mathbb{Z}^k$ such that

$$\mu\left(\bigcap_{\epsilon \in \{0,1\}^d} T^{\epsilon \cdot n} A\right) \geq \mu(A)^{2^d} - c$$

is syndetic.

The proof is exactly the same as Corollary 13.8 in [4].

Theorem 1.3 follows by combining Furstenberg's correspondence principle and Corollary 4.1.

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