dg-methods for microlocalization

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Abstract. For a complex manifold X the ring of microdifferential operators $\mathcal{E}_X$ acts on the microlocalization $\mu hom(F, \mathcal{O}_X)$, for F in the derived category of sheaves on X. Kashiwara, Schapira, Ivorra, Waschkies proved, as a byproduct of their new microlocalization functor for ind-sheaves, μ_X, that $\mu hom(F, \mathcal{O}_X)$ can in fact be defined as an object of $\mathrm{D}(\mathcal{E}_X)$: this follows from the fact that $\mu_X \mathcal{O}_X$ is concentrated in one degree.

In this paper we prove that the tempered microlocalization $T\text{–}\mu hom(F, \mathcal{O}_X)$, or $\mu_X \mathcal{O}_X^t$, also are objects of $\mathrm{D}(\mathcal{E}_X)$. Since we don't know whether $\mu_X \mathcal{O}_X^t$ is concentrated in one degree, we built resolutions, of $\mathcal{E}_X$ and $\mu_X \mathcal{O}_X^t$, such that the action of $\mathcal{E}_X$ is realized in the category of complexes (and not only up to homotopy). To define these resolutions we introduce a version of the de Rham algebra on the subanalytic site which is quasi-injective. We prove that some standard operations in the derived category of sheaves can be lifted to the (non-derived) category of dg-modules over this de Rham algebra. Then we built the microlocalization in this framework, together with a convolution product.

1. Introduction

For a complex analytic manifold the sheaf of microlocal differential operators on its cotangent bundle was introduced by Sato, Kashiwara and Kawai using Sato's microlocalization functor. Let us recall briefly the definition, in the framework of [5]. Let X be a manifold and let $\mathrm{D}^b(\mathbf{C}_X)$ be the bounded derived category of sheaves of $\mathbf{C}$-vector spaces on X. For objects $F, G \in \mathrm{D}^b(\mathbf{C}_X)$, a generalization of Sato's microlocalization functor gives $\mu hom(F, G) \in \mathrm{D}^b(\mathbf{C}_{T^*X})$, and a convolution product is defined in [5] for this functor μhom. When X is a complex analytic manifold of complex dimension d_X, one version of the ring of microlocal operators is defined by $\mathcal{E}_X^{\mathbf{R}} = \mu hom(\mathbf{C}_\Delta, \mathcal{O}_{X\times X}^{(0,d_X)})[d_X]$, where Δ is the diagonal of $X \times X$ and $\mathcal{O}_{X\times X}^{(0,d_X)}$ denotes the holomorphic forms of degree 0 on the first factor and degree d_X on the second factor. It has support on the conormal bundle of Δ, which may be identified with T^*X. The product of $\mathcal{E}_X^{\mathbf{R}}$ is given by the convolution product of μhom.

The convolution product also induces an action of $\mathcal{E}_X^{\mathbf{R}}$ on $\mu hom(F, \mathcal{O}_X)$, for any $F \in \mathrm{D}^b(\mathbf{C}_X)$, i.e. a morphism in $\mathrm{D}^b(\mathbf{C}_{T^*X})$, $\mathcal{E}_X^{\mathbf{R}} \otimes \mu hom(F, \mathcal{O}_X) \to \mu hom(F, \mathcal{O}_X)$, satisfying commutative diagrams which express the properties of an action.

A natural question is then whether $\mu hom(F, \mathcal{O}_X)$ has a natural construction as an object of $\mathrm{D}^b(\mathcal{E}_X^{\mathbf{R}})$. It was answered positively in [9] as a byproduct of the construction of a microlocalization functor for ind-sheaves. The category of ind-sheaves on X, $\mathbf{I}(\mathbf{C}_X)$, is introduced and studied in [7]. It comes with an internal

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Hom functor, $\mathcal{IH}\mathrm{om}$, and contains $\mathrm{Mod}(\mathbf{C}_X)$ as a full subcategory; the embedding of $\mathrm{Mod}(\mathbf{C}_X)$ in $\mathbf{I}(\mathbf{C}_X)$ admits a left adjoint (which corresponds to taking the limit) $\alpha_X\colon \mathbf{I}(\mathbf{C}_X) \to \mathrm{Mod}(\mathbf{C}_X)$ which is exact. In this framework the construction of [9] yields a new microlocalization functor $\mu_X\colon \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*X}))$ such that

$$\mu hom(F,G) \simeq \alpha_{T^*X}\, \mathrm{R}\mathcal{IH}\mathrm{om}(\mu_X F, \mu_X G). \tag{1}$$

In particular μ_X applies to a single object of $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))$ and $\mu hom(F,G)$ takes the form of a usual $\mathcal{H}\mathrm{om}$ functor between objects on T^*X.

The convolution product is also defined in this context, and now it gives an action of $\mathcal{E}^{\mathbf{R}}_X$ on $\mu_X(\mathcal{O}_X)$. Through isomorphism (1) this action on $\mu_X(\mathcal{O}_X)$ induces the action on $\mu hom(F,\mathcal{O}_X)$. Hence it is enough to define $\mu_X(\mathcal{O}_X)$ as an object of $\mathrm{D}^b(\mathcal{E}^{\mathbf{R}}_X)$ to have the answer for all $\mu hom(F,\mathcal{O}_X)$. It turns out that, outside the zero section of T^*X, $\mu_X(\mathcal{O}_X)$ is concentrated in degree $-d_X$. Thus $\mu_X(\mathcal{O}_X) \simeq H^{-d_X}\mu_X(\mathcal{O}_X)[d_X]$ and, since the action of $\mathcal{E}^{\mathbf{R}}_X$ gives an $\mathcal{E}^{\mathbf{R}}_X$-module structure on $H^{-d_X}\mu_X(\mathcal{O}_X)$, we see that $\mu_X(\mathcal{O}_X)$ naturally belongs to $\mathrm{D}^b(\mathcal{E}^{\mathbf{R}}_X)$, as required.

However in many situations differential operators of finite order are more appropriate. In this paper we solve the same problem in the tempered situation, i.e. for the sheaf $\mathcal{E}^{\mathbf{R},f}_X$ of differential operators with bounded degree and for the tempered version of $\mu hom(F,\mathcal{O}_X)$. This tempered microlocalization $\mathit{T}\text{–}\mu hom(F,\mathcal{O}_X)$ is introduced in [1] and also has a reformulation in terms of ind-sheaves. Namely it makes sense to consider the ind-sheaf of tempered $\mathcal{C}^\infty$-functions and the corresponding Dolbeault complex $\mathcal{O}^t_X$ (it is actually a motivation for the theory of ind-sheaves). Then

$$\mathit{T}\text{–}\mu hom(F,\mathcal{O}_X) \simeq \alpha_{T^*X} R\mathcal{IH}\mathrm{om}(\mu_X F, \mu_X \mathcal{O}^t_X).$$

We have as above a natural action of $\mathcal{E}^{\mathbf{R},f}_X$ on $\mu_X(\mathcal{O}^t_X)$. Unfortunately this last complex is a priori not concentrated in one degree and we cannot conclude directly that $\mu_X(\mathcal{O}^t_X)$ is an object of $\mathrm{D}^b(\mathcal{E}^{\mathbf{R},f}_X)$.

We will in fact find resolutions of $\mathcal{E}^{\mathbf{R},f}_X$ and $\mu_X(\mathcal{O}^t_X)$ such that the action corresponds to a dg-module structure over a dg-algebra. More precisely we will define an ind-sheaf of dg-algebras $\mathcal{E}^{\mathcal{A}}_X$ on T^*X (outside the zero section) with cohomology only in degree 0 and such that $H^0(\mathcal{E}^{\mathcal{A}}_X) = \mathcal{E}^{\mathbf{R},f}_X$. We will also find a dg-$\mathcal{E}^{\mathcal{A}}_X$-module, say M, such that $M \simeq \mu_X(\mathcal{O}^t_X)$ in $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*X}))$ and such that the morphism of complexes $\mathcal{E}^{\mathcal{A}}_X \otimes M \to M$ given by the dg-$\mathcal{E}^{\mathcal{A}}_X$-module structure coincides with the action $\mathcal{E}^{\mathbf{R},f}_X \otimes \mu_X(\mathcal{O}^t_X) \to \mu_X(\mathcal{O}^t_X)$. Then, as recalled in section 3, extension and restriction of scalars yield an object $M' \in \mathrm{D}^b(\mathcal{E}^{\mathbf{R},f}_X)$ which represents $\mu_X(\mathcal{O}^t_X)$ with its $\mathcal{E}^{\mathbf{R},f}_X$-action. So we conclude as in the non tempered case.

Now we explain how we construct $\mathcal{E}^{\mathcal{A}}_X$ and M. The main step in the definition of $\mathcal{E}^{\mathbf{R},f}_X$, as well as its action on $\mu hom(F,\mathcal{O}_X)$, is the microlocal convolution product

$$\mu_{X\times X}\mathcal{O}^{t(0,d_X)}_{X\times X} \overset{a}{\circ} \mu_{X\times X}\mathcal{O}^{t(0,d_X)}_{X\times X}[d_X] \to \mu_{X\times X}\mathcal{O}^{t(0,d_X)}_{X\times X}, \tag{2}$$

where $\overset{a}{\circ}$ denotes the composition of kernels. This is a morphism in the derived category. It is obtained from the integration morphism for the Dolbeault complex and the commutation of the functor $\mu_{X\times X}$ with the convolution of sheaves. In order to obtain a true dg-algebra at the end, and not a complex with a product up to homotopy, we will represent the functor μ by a functor between categories of complexes, which satisfies enough functorial properties so that the convolution also corresponds to a morphism of complexes.

Let us be more precise. The first step is the construction of injective resolutions, with some functorial properties. For this we introduce a quasi-injective de Rham algebra, $\mathcal{A}$, below (quasi-injectivity is a property of ind-sheaves weaker than injectivity but sufficient to derive the usual functors). We use the construction of ind-sheaves from sheaves on the "subanalytic site" explained in [7]. For a real analytic manifold X, the subanalytic site, X_{sa}, has for open subsets the subanalytic open subsets of X and for coverings the locally finite coverings. On X_{sa} it makes sense to consider the sheaf of tempered $\mathcal{C}^\infty$ functions, $\mathcal{C}^{\infty,t}_X$.

We consider the embedding $i_X\colon X = X \times \{0\} \to X \times \mathbf{R}$ and define a sheaf of i-forms on X_{sa}, $\mathcal{A}^i_X = i_X^{-1}\Gamma_{X\times\mathbf{R}_{>0}}(\mathcal{C}^{\infty,t(i)}_{X\times\mathbf{R}})$. This gives a de Rham algebra $\mathcal{A}_X$ and it yields a quasi-injective resolution of $\mathbf{C}_{X_{sa}}$. For a morphism of manifolds $f\colon X \to Y$ we have an inverse image $f^*\colon f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$. If f is smooth, with fibers of dimension d, we also have an integration morphism $\int_f\colon f_{!!}\mathcal{A}_X \otimes or_{X|Y}[d] \to \mathcal{A}_Y$, which represents the integration morphism $Rf_{!!}or_{X|Y}[d] \to \mathbf{C}_Y$.

We denote by $\mathrm{Mod}(\mathcal{A}_X)$ the category of sheaves of dg-$\mathcal{A}_X$-modules. We have an obvious forgetful functor $\mathrm{For}'_X\colon \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{D}(\mathbf{C}_{X_{sa}})$. We will prove that the operations needed in the construction of (2) are defined in $\mathrm{Mod}(\mathcal{A}_X)$ and commute with For'_X. For example, for a morphism of manifolds $f\colon X \to Y$ we have functors, f^* and f_*, $f_{!!}$, of inverse and direct images of dg-$\mathcal{A}$-modules. In some cases this gives a way to represent the derived functors f^{-1} and Rf_*, $Rf_{!!}$. For example, since $\mathcal{A}^0_X$ is quasi-injective we can prove, for $F \in \mathrm{Mod}(\mathcal{A}_X)$, $\mathrm{For}'_Y(f_{!!}(F)) \simeq Rf_{!!}(\mathrm{For}'_X(F))$. If f is smooth we also prove, for $G \in \mathrm{Mod}(\mathcal{A}_Y)$, $\mathrm{For}'_X(f^*G) \simeq f^{-1}(\mathrm{For}'_Y(G))$. When X is a complex manifold, we also have a resolution, $\mathbb{O}_X$, of $\mathcal{O}^t_X$ by a dg-$\mathcal{A}_X$-module which is locally free over $\mathcal{A}^0_X$ (it is deduced from the Dolbeault resolution).

Once we have these operations we define a microlocalization functor for dg-$\mathcal{A}$-modules. Let us recall that the functor μ_X is given by composition with a kernel $L_X \in \mathrm{D}^b(\mathbf{C}_{(X\times T^*X)_{sa}})$: for $F \in \mathrm{D}^b(\mathbf{C}_{X_{sa}})$ we have $\mu_X(F) = L_X \circ F = Rp_{2!!}(L_X \otimes p_1^{-1}F)$. We define a corresponding dg-$\mathcal{A}$-module, $L^{\mathcal{A}}_X$, which is quasi-isomorphic to L_X outside the zero section of T^*X, i.e. over $\dot{X} \times \dot{T}^*X$, and we set, for a dg-$\mathcal{A}_X$-module F:

$$\mu^{\mathcal{A}}_X(F) = L^{\mathcal{A}}_X \circ F = p_{2!!}(L^{\mathcal{A}}_X \otimes_{\mathcal{A}} p_1^*F).$$

This functor is defined on the categories of complexes, i.e. it is a functor from $\mathrm{Mod}(\mathcal{A}_X)$ to $\mathrm{Mod}(\mathcal{A}_{T^*X})$. If F is locally free over $\mathcal{A}^0_X$, we show that $\mu^{\mathcal{A}}_X(F)$ is quasi-injective and represents $\mu_X(F)$ over $\dot{T}^*X$: we have $\mathrm{For}'_{T^*X}(\mu^{\mathcal{A}}_X(F)) \simeq \mu_X(\mathrm{For}'_X(F))$. In particular, when X is a complex manifold we obtain the dg-$\mathcal{A}_{T^*X}$-module $\mu^{\mathcal{A}}_X(\mathbb{O}_X)$ which represents $\mu_X(\mathcal{O}^t_X)$ and can be used to compute $\mathrm{R}\mathcal{H}\mathrm{om}(\cdot, \mu_X(\mathcal{O}^t_X))$.

With these tools in hand we define the sheaf $\mathcal{E}^{\mathcal{A}}_X$ mentioned above from $\mu^{\mathcal{A}}$, the same way $\mathcal{E}^{\mathbf{R},f}_X$ was defined from μ. The definition of the product involves a convolution product for $\mu^{\mathcal{A}}$. The kernel $L^{\mathcal{A}}_X$ has indeed the same functorial behavior as L_X, not with respect to all operations, but at least those needed in the composition of kernels. We end up with a dg-$\mathcal{A}_{T^*X}$-module $\mathcal{E}^{\mathcal{A}}_X$, which is a ring object in the category of dg-$\mathcal{A}_{T^*X}$-modules and which represents $\mathcal{E}^{\mathbf{R},f}_X$. In the same way we obtain a structure of $\mathcal{E}^{\mathcal{A}}_X$-module on $\mu^{\mathcal{A}}_X(\mathbb{O}_X)$, as desired. As said above this $\mathcal{E}^{\mathcal{A}}_X$-module gives a $\beta_{T^*X}(\mathcal{E}^{\mathbf{R},f}_X)$-module by extension and restriction of scalars (here β is the functor from sheaves to ind-sheaves which is left adjoint to α). Our result is more precisely stated in Theorem 12.4:

Theorem 1.1. *There exists $\mathcal{O}^\mu_X \in \mathrm{D}(\beta_{T^*X}(\mathcal{E}^{\mathbf{R},f}_X)))$, defined over $\dot{T}^*X$, which is send to $\mu_X\mathcal{O}^t_X$ in $\mathrm{D}(\mathbf{I}(\mathbf{C}_{\dot{T}^*X}))$ by the forgetful functor and satisfies: for $F \in \mathrm{D}^-(\mathbf{I}(\mathbf{C}_X))$ the complex*

$$\alpha_{T^*X}\,\mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}(\pi^{-1}F, \mathcal{O}^\mu_X)$$

which is naturally defined in $\mathrm{D}(\mathcal{E}^{\mathbf{R},f}_X)$, over $\dot{T}^*X$, is isomorphic in $\mathrm{D}(\mathbf{C}_{\dot{T}^*X})$ to T–$\mu hom(F, \mathcal{O}_X)$ endowed with its action of $\mathcal{E}^{\mathbf{R},f}_X$.

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2. Notations

If X is a manifold or a site and R a sheaf of rings on X, we denote by $\mathrm{Mod}(R)$ the category of sheaves of R-modules on X. The corresponding category of complexes is $\mathrm{C}(R)$, and the derived category $\mathrm{D}(R)$; we use superscripts $b, +, -$ for the categories of complexes which are bounded, bounded from below, bounded from above. More generally, if R is a sheaf of dg-algebras on X, $\mathrm{Mod}(R)$ is the category of sheaves of dg-R-modules on X, $D(R)$ its derived category (see section 3). In particular, if X is a real analytic manifold, this applies to the subanalytic site X_{sa} whose definition is recalled in section 4. We denote by ρ_X or ρ the natural morphism of sites $X \to X_{sa}$. We denote by $\mathbf{C}_X$ and $\mathbf{C}_{X_{sa}}$ the constant sheaves with coefficients $\mathbf{C}$ on X and X_{sa}.

If X is a manifold we denote by $\mathbf{I}(\mathbf{C}_X)$ the category of ind-sheaves of $\mathbf{C}_X$-vector spaces on X (see section 4), and $\mathrm{D}(\mathbf{I}(\mathbf{C}_X))$ its derived category. This category comes with a natural functor α_X, or $\alpha\colon \mathbf{I}(\mathbf{C}_X) \to \mathrm{Mod}(\mathbf{C}_X)$ which corresponds to taking the limit. Its left adjoint is denoted β_X, or β.

The dimension of a (real) manifold X is denoted d_X; if X is a complex manifold, its complex dimension is d^c_X.

For a morphism of manifolds $f\colon X \to Y$, we let $\omega_{X|Y} = f^!\mathbf{C}_Y$ be the relative dualizing complex. Hence $\omega_{X|Y}$ is an object of $\mathrm{D}^b(\mathbf{C}_X)$. If Y is a point we simply write ω_X; then $\omega_X \simeq or_X[d_X]$, where or_X is the orientation sheaf of X. In fact, for X connected, $\omega_{X|Y}$ is always concentrated in one degree (since X and Y are manifolds), say i, and we will use the notation $\omega'_{X|Y} = H^i\omega_{X|Y}[-i]$; hence $\omega'_{X|Y}$ is a well-defined object of $\mathrm{C}^b(\mathbf{C}_X)$. For an embedding of manifolds $i_Z\colon Z \hookrightarrow X$ we will often abuse notations and write $\omega_{Z|X}$ for $i_{Z*}\omega_{Z|X}$.

For a manifold X, we let TX and T^*X be the tangent and cotangent bundles. For a submanifold $Z \subset X$ we denote by T_ZX and T^*_ZX the normal and conormal bundle to Z. In particular $T^*_XX \simeq X$ is the zero section of T^*X and we set $\dot{T}^*X = T^*X \setminus T^*_XX$. We denote by $\tilde{X}_Z$ the normal deformation of Z in X (see for example [5]). We recall that it contains T_ZX and comes with a map $\tau\colon \tilde{X}_Z \to \mathbf{R}$ such that $\tau^{-1}(0) = T_ZX$ and $\tau^{-1}(r) \simeq X$ for $r \neq 0$. We also have another map $p\colon \tilde{X}_Z \to X$ such that $p^{-1}(z) = (T_ZX)_z \cup \{z\} \times \mathbf{R}$ for $z \in Z$ and $p^{-1}(x) \simeq \mathbf{R} \setminus \{0\}$ for $x \in X \setminus Z$. We set $\Omega = \tau^{-1}(\mathbf{R}_{>0})$.

For a morphism of manifolds $f\colon X \to Y$, the derivative of f gives the morphisms:

$$T^*X \xleftarrow{f_d} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y.$$

For two manifolds X, Y, $F \in \mathrm{D}^+(\mathbf{C}_X)$, $G \in \mathrm{D}^+(\mathbf{C}_Y)$, we set $F \boxtimes G = p_1^{-1}F \otimes p_2^{-1}G$, where p_i is the projection from $X \times Y$ to the i^{th} factor. For three manifolds X, Y, Z, and "kernels" $K \in \mathrm{D}^+(\mathbf{C}_{X\times Y})$, $L \in \mathrm{D}^+(\mathbf{C}_{Y\times Z})$, we denote the "composition of kernels" by $K \circ L = Rp_{23!}(p_{12}^{-1}K \otimes p_{23}^{-1}L)$, where p_{ij} is the projection from $X \times Y \times Z$ to the $i^{th} \times j^{th}$ factors. We use the same notations for the variants on subanalytic sites or using ind-sheaves.

3. DG-ALGEBRAS

In this section we recall some facts about (sheaves of) dg-algebras and their derived categories. We refer the reader to [2].

A dg-algebra A is a $\mathbf{Z}$-graded algebra with a differential d_A of degree $+1$. A dg-A-module M is a graded A-module with a differential d_M such that, for homogeneous elements $a \in A^i$, $m \in M^j$, $d_M(a \cdot m) = d_A(a) \cdot m + (-1)^i a \cdot d_M m$.

We consider a site X and a sheaf of dg-algebras A_X on X. We denote by $\mathrm{Mod}(A_X)$ the category of (left) dg-A_X-modules. We let $\widetilde{A}_X$ be the graded algebra underlying A_X (i.e. forgetting the differential). A morphism $f\colon M \to N$ in $\mathrm{Mod}(A_X)$ is said to be null homotopic if there exists an $\widetilde{A}_X$-linear morphism $s\colon M \to N[-1]$ such that $f = sd_M + d_N s$. The homotopy category, $\mathrm{K}(A_X)$, has for objects those of $\mathrm{Mod}(A_X)$, and for sets of morphisms those of $\mathrm{Mod}(A_X)$ quotiented by null homotopic morphisms. A morphism in $\mathrm{Mod}(A_X)$ (or $\mathrm{K}(A_X)$) is a quasi-isomorphism if it induces isomorphisms on the cohomology groups. Finally, the derived category $\mathrm{D}(A_X)$ is the localization of $\mathrm{K}(A_X)$ by quasi-isomorphisms.

Derived functors can be defined in this setting, in particular the tensor product $\cdot \otimes^L_{A_X} \cdot$. If $\phi\colon A_X \to B_X$ is a morphism of sheaves of dg-algebras, we obtain the extension of scalars $\phi^*\colon \mathrm{D}(A_X) \to \mathrm{D}(B_X)$, $M \mapsto B_X \otimes^L_{A_X} M$, which is left adjoint to the natural restriction of scalars $\phi_*\colon \mathrm{D}(B_X) \to \mathrm{D}(A_X)$. By [2] (Theorem 10.12.5.1), if ϕ induces an isomorphism $H(A) \xrightarrow{\sim} H(B)$, then these functors of restriction and extension of scalars are mutually inverse equivalences of categories $\mathrm{D}(A_X) \simeq \mathrm{D}(B_X)$.

Some dg-algebras considered in this paper will appear as ring objects in categories of complexes. We recall briefly what it means. We let $\mathcal{C}$ be a tensor category with unit $\underline{\mathbf{C}}$ ($\mathcal{C}$ will be $\mathrm{D}(\mathbf{C}_Y)$, $\mathrm{D}(\mathbf{I}(\mathbf{C}_Y))$ or $\mathrm{Mod}(\mathcal{A}_Y)$ for some manifold Y and the unit is $\underline{\mathbf{C}} = \mathbf{C}_Y$).

Definition 3.1. A ring in $\mathcal{C}$ is a triplet (A, m, ε) where $A \in \mathcal{C}$, $m\colon A \otimes A \to A$ and $\varepsilon\colon \underline{\mathbf{C}} \to A$ are morphisms in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes \underline{\mathbf{C}} & \xrightarrow{A\otimes\varepsilon} & A \otimes A \\ & \searrow & \downarrow m \\ & & A \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{C}} \otimes A & \xrightarrow{\varepsilon\otimes A} & A \otimes A \\ & \searrow & \downarrow m \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m\otimes A} & A \otimes A \\ \downarrow A\otimes m & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

In the same way, for such a "ring" (A, m, ε), an action of A on $M \in \mathcal{C}$ is a morphism, $\alpha\colon A \otimes M \to M$, compatible with m and ε. The pairs (M, α) of this type form a category, where the morphisms from (M, α) to (M', α') are the morphisms from M to M' commuting with the action.

If E_X is a sheaf of (usual) algebras on X, we may consider E_X as a ring object in $\mathrm{D}(\mathbf{C}_X)$ and we denote by $\mathrm{D}_{E_X}(\mathbf{C}_X)$ the category of "objects of $\mathrm{D}(\mathbf{C}_X)$ with E_X-action" as above.

We consider again a sheaf A_X of dg-algebras on X. We assume that its cohomology sheaves are 0 except in degree 0 and we set $E_X = H^0(A_X)$. Hence, if we forget the structures and view A_X, E_X as objects of $\mathrm{D}(\mathbf{C}_X)$, we have isomorphisms $A_X \xleftarrow{\sim} \tau_{\leq 0}A_X \xrightarrow{\sim} E_X$ (where $\tau_{\leq 0}$, $\tau_{\geq 0}$ denote the truncation functors). We note that $\tau_{\leq 0}A_X = \cdots \to A_X^{-1} \to \ker d_0 \to 0$ is sub-dg-algebra of A_X (whereas $\tau_{\geq 0}A_X$ has no obvious structure of dg-algebra). The multiplications of A_X and E_X induce morphisms in $\mathrm{D}(\mathbf{C}_X)$: $A_X \otimes A_X \to A_X$, $E_X \otimes E_X \to E_X$. These morphisms coincide under the identification $A_X \simeq E_X$. Hence A_X and E_X are isomorphic as ring objects in $\mathrm{D}(\mathbf{C}_X)$.

For $M \in \mathrm{D}(A_X)$, the structure of A_X-module induces a morphism in $\mathrm{D}(\mathbf{C}_X)$: $\alpha\colon E_X \otimes M \simeq A_X \otimes M \to M$. Then α is an action of E_X on M. In this way we obtain a forgetful functor $F_{A_X}\colon \mathrm{D}(A_X) \to \mathrm{D}_{E_X}(\mathbf{C}_X)$.

Lemma 3.2. *Let A_X be a sheaf of dg-algebras, with cohomology sheaves concentrated in degree 0 and $E_X = H^0(A_X)$. Let $\phi\colon A_X \to B_X$ be a morphism of sheaves of dg-algebras such that ϕ induces an isomorphism $H(A) \xrightarrow{\sim} H(B)$. Then we have isomorphisms of functors $F_{A_X} \circ \phi_* \simeq F_{B_X}$ and $F_{B_X} \circ \phi^* \simeq F_{A_X}$.*

Proof. The first isomorphism is obvious and the second one follows because ϕ_* and ϕ^* are inverse equivalences of categories. □

Applying this lemma to the morphisms $A_X \xleftarrow{\phi_{\leq 0}} \tau_{\leq 0}A_X \xrightarrow{\phi_0} E_X$, we obtain:

Corollary 3.3. *With the hypothesis of the above lemma, we have the commutative diagram:*

$$\begin{array}{ccc} \mathrm{D}(A_X) & \xrightarrow{F_{A_X}} & \\ \downarrow{\scriptstyle \phi_0^* \circ \phi_{\leq 0 *}} & & \mathrm{D}_{E_X}(\mathbf{C}_X). \\ \mathrm{D}(E_X) & \xrightarrow{F_{E_X}} & \end{array}$$

In particular, for $M \in \mathrm{D}_{E_X}(\mathbf{C}_X)$, if there exists $N \in \mathrm{D}(A_X)$ such that $F_{A_X}(N) \simeq M$ then there exists $N' \in \mathrm{D}(E_X)$ such that $F_{E_X}(N') \simeq M$

4. Ind-sheaves and subanalytic site

We recall briefly some definitions and results of [7] about ind-sheaves. To define the ind-sheaves we are interested in we will use the "subanalytic site" as in [7], where it is introduced to deal with tempered $\mathcal{C}^\infty$ functions. It is studied in more details in [10].

4.1. **Ind-sheaves.** For a category $\mathcal{C}$ we denote by $\mathcal{C}^\wedge$ the category of functors from $\mathcal{C}^{op}$ to the category of sets. It comes with the "Yoneda embedding", $h\colon \mathcal{C} \to \mathcal{C}^\wedge$, $X \to \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$. The category $\mathcal{C}^\wedge$ admits small inductive limits but, in general, even if $\mathcal{C}$ also admits such limits, the functor h may not commute with inductive limits. We denote by "$\varinjlim$" the inductive limit taken in the category $\mathcal{C}^\wedge$.

An ind-object in $\mathcal{C}$ is an object of $\mathcal{C}^\wedge$ which is isomorphic to "$\varinjlim$" i for some functor $i\colon I \to \mathcal{C}$, with I a small filtrant category. We denote by $\mathrm{Ind}(\mathcal{C})$ the full subcategory of $\mathcal{C}^\wedge$ of ind-objects.

We are interested in two cases. Let X be a real analytic manifold, $\mathrm{Mod}(\mathbf{C}_X)$ the category of sheaves of $\mathbf{C}$-vector spaces on X, $\mathrm{Mod}_{\mathbf{R}-c}(\mathbf{C}_X)$ the subcategory of

$\mathbf{R}$-constructible sheaves, $\mathrm{Mod}^c(\mathbf{C}_X)$ and $\mathrm{Mod}^c_{\mathbf{R}-c}(\mathbf{C}_X)$ their respective full subcategories of objects with compact support. We define as in [7]:

$$\mathbf{I}(\mathbf{C}_X) = \mathrm{Ind}(\mathrm{Mod}^c(\mathbf{C}_X)) \quad \text{and} \quad \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) = \mathrm{Ind}(\mathrm{Mod}^c_{\mathbf{R}-c}(\mathbf{C}_X)).$$

There are natural exact embeddings $I_\tau\colon \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) \to \mathbf{I}(\mathbf{C}_X)$ and $\iota_X\colon \mathrm{Mod}(\mathbf{C}_X) \to \mathbf{I}(\mathbf{C}_X)$, $F \mapsto$ "$\varinjlim$" F_U, U running over relatively compact open sets. Then ι_X sends $\mathrm{Mod}_{\mathbf{R}-c}(\mathbf{C}_X)$ into $\mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X)$.

The functor ι_X admits an exact left adjoint functor $\alpha_X\colon \mathbf{I}(\mathbf{C}_X) \to \mathrm{Mod}(\mathbf{C}_X)$, "$\varinjlim$"$_{i\in I} F_i \mapsto \varinjlim_{i\in I} F_i$. Since ι_X is fully faithful, we have $\alpha_X \circ \iota_X \simeq \mathrm{id}$.

The functor α_X admits an exact fully faithful left adjoint $\beta_X\colon \mathrm{Mod}(\mathbf{C}_X) \to \mathbf{I}(\mathbf{C}_X)$. Since β_X is fully faithful, we have $\alpha_X \circ \beta_X \simeq \mathrm{id}$. For $Z \subset X$ a closed subset, we have

$$\beta_X(\mathbf{C}_Z) \simeq \text{"}\varinjlim_{W,\, Z\subset W}\text{"}\ \mathbf{C}_{\overline{W}}, \qquad W \subset X \text{ open subset.} \tag{3}$$

We write α, β for α_X, β_X when the context is clear. The machinery of Grothendieck's six operations also applies to this context. There are not enough injectives in $\mathbf{I}(\mathbf{C}_X)$, but enough "quasi-injectives" (see [7] and [8]): $F \in \mathbf{I}(\mathbf{C}_X)$ is quasi-injective if the functor $\mathrm{Hom}(\cdot, F)$ is exact on $\mathrm{Mod}^c(\mathbf{C}_X)$. The quasi-injective objects are sufficient to derive the usual functors. In particular, for a morphism of manifolds $f\colon X \to Y$ we have the functors:

$$\begin{aligned} f^{-1},\ f^!\colon & \mathrm{D}^b(\mathbf{I}(\mathbf{C}_Y)) \to \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)),\\ Rf_*,\ Rf_{!!}\colon & \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^b(\mathbf{I}(\mathbf{C}_Y)),\\ \mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}\colon & \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))^{op} \times \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^+(\mathbf{I}(\mathbf{C}_X)),\\ \otimes\colon & \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \times \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)), \end{aligned}$$

and also $\mathrm{R}\mathcal{H}\mathrm{om} = \alpha\,\mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}\colon \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))^{op} \times \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^+(\mathbf{C}_X)$.

It will be convenient for us to use the equivalence of categories given in [7] between $\mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X)$ and sheaves on the subanalytic site, defined below.

4.2. **Subanalytic site.** In this paragraph X is a real analytic manifold. The open sets of the site X_{sa} are the subanalytic open subsets of X. A family $\bigcup_{i\in I} U_i$ of such open sets is a covering of U if and only if, for any compact subset K, there exists a finite subfamily of $J \subset I$ with $K \cap \bigcup_{i\in J} U_i = K \cap U$. We denote by $\mathrm{Mod}(\mathbf{C}_{X_{sa}})$ the category of sheaves of $\mathbf{C}$-vector on X_{sa}.

We have a morphism of sites $\rho_X\colon X \to X_{sa}$ (where X also denotes the site naturally associated to the topological space X). We just write ρ if there is no risk of confusion. In particular we have adjoint functors $\rho_*\colon \mathrm{Mod}(\mathbf{C}_X) \to \mathrm{Mod}(\mathbf{C}_{X_{sa}})$ and $\rho^{-1}\colon \mathrm{Mod}(\mathbf{C}_{X_{sa}}) \to \mathrm{Mod}(\mathbf{C}_X)$.

The functor ρ^{-1} is exact and ρ_* is left exact and fully faithful (hence $\rho^{-1} \circ \rho_* = \mathrm{id}$). We denote by ρ_{c*} the restriction of ρ_* to $\mathrm{Mod}_{\mathbf{R}-c}(\mathbf{C}_X)$. Then ρ_{c*} is exact and, for $F \in \mathrm{Mod}_{\mathbf{R}-c}(\mathbf{C}_X)$, we usually write F instead of $\rho_{c*}F$. The functor ρ_{c*} induces an equivalence of categories (see [7], Theorem 6.3.5):

$$\begin{aligned} \lambda\colon \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) &\to \mathrm{Mod}(\mathbf{C}_{X_{sa}})\\ \text{"}\varinjlim_i\text{"}\ F_i &\mapsto \varinjlim_i \rho_{c*}(F_i). \end{aligned}$$

Through this equivalence, the functor ρ^{-1} corresponds to α and it also admits an exact left adjoint functor, corresponding to β. When dealing with the analytic

site we will use the notation $\rho_!\colon \mathrm{Mod}(\mathbf{C}_X) \to \mathrm{Mod}(\mathbf{C}_{X_{sa}})$ for this functor. For example (3) becomes $\rho_!\mathbf{C}_Z \simeq \varinjlim_{Z\subset W} \mathbf{C}_{\overline{W}}$, where W runs over the subanalytic open subsets of X. We note the commutative diagrams:

$$\begin{array}{ccc} & & \mathrm{Mod}(\mathbf{C}_X) \\ & \overset{\rho_!}{\swarrow} & \downarrow{\scriptstyle \beta_X} \\ \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) \simeq \mathrm{Mod}(\mathbf{C}_{X_{sa}}) & \xrightarrow[I_\tau]{} & \mathbf{I}(\mathbf{C}_X) \end{array} \qquad \begin{array}{ccc} \mathrm{Mod}_{\mathbf{R}-c}(\mathbf{C}_X) & \longrightarrow & \mathrm{Mod}(\mathbf{C}_X) \\ \downarrow{\scriptstyle \rho_{c*}} & & \downarrow{\scriptstyle \iota_X} \\ \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) \simeq \mathrm{Mod}(\mathbf{C}_{X_{sa}}) & \xrightarrow[I_\tau]{} & \mathbf{I}(\mathbf{C}_X) \end{array}$$

The functors appearing in these diagrams are exact and induce similar commutative diagrams at the level of derived categories.

The functor $\mathcal{H}\mathrm{om}$ is defined on $\mathrm{Mod}(\mathbf{C}_{X_{sa}})$ as on every site and we set, for $Z \subset X$ a locally closed subanalytic subset:

$$\Gamma_Z(F) = \mathcal{H}\mathrm{om}(\rho_*\mathbf{C}_Z, F), \qquad F_Z = F \otimes \rho_*\mathbf{C}_Z. \tag{4}$$

The functors ρ_* and $\mathcal{H}\mathrm{om}$ commute, hence ρ_* and Γ_Z also commute. For subanalytic open subsets $U, V \subset X$ we have $\Gamma_U(F)(V) = F(U \cap V)$.

By analogy with ind-sheaves, a notion weaker than injective is introduced in [10]: $F \in \mathrm{Mod}(\mathbf{C}_{X_{sa}})$ is quasi-injective if $\mathrm{Hom}(\cdot, F)$ is exact on $\rho_*\mathrm{Mod}^c_{\mathbf{R}-c}(\mathbf{C}_X)$. In fact, since we consider coefficients in a field, it is equivalent to ask that for any subanalytic open subsets $U \subset V$ with compact closure $\Gamma(V; F) \to \Gamma(U; F)$ is surjective. Quasi-injective sheaves are sufficient to derive usual left exact functors. In particular we obtain $\mathrm{R}\mathcal{H}\mathrm{om}$, $R\Gamma_Z$, and they commute with $R\rho_*$. We note the following identity (which has no equivalent on the classical site): for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbf{R}-\mathrm{c}}(\mathbf{C}_X)$, $H \in \mathrm{D}^+(\mathbf{C}_X)$, $G \in \mathrm{D}^+(\mathbf{C}_{X_{sa}})$,

$$\mathrm{R}\mathcal{H}\mathrm{om}(R\rho_*F, G) \otimes \rho_!H \simeq \mathrm{R}\mathcal{H}\mathrm{om}(R\rho_*F, G \otimes \rho_!H) \qquad \text{in } \mathrm{D}^+(\mathbf{C}_{X_{sa}}). \tag{5}$$

We also have another related result (see [10], Proposition 1.1.3): for $\{F_i\}_{i\in I}$ a filtrant inductive system in $\mathrm{Mod}(\mathbf{C}_{X_{sa}})$ and $U \subset X$ an analytic open subset

$$\varinjlim_i R\Gamma_U(F_i) \xrightarrow{\sim} R\Gamma_U(\varinjlim_i F_i). \tag{6}$$

For a morphism $f\colon X \to Y$ there are the usual direct and inverse image functors on the analytic sites f_*, f^{-1}, but also, as in the case of ind-sheaves, a notion of proper direct image $f_{!!}$, with a behavior slightly different from the behavior of $f_!$ on the classical site. The functor f^{-1} and f_*, $f_{!!}$ admit derived functors. We quote in particular: for $F \in \mathrm{D}^+(\mathbf{C}_{X_{sa}})$, $G \in \mathrm{D}^b_{\mathbf{R}-c}(\mathbf{C}_Y)$ (we identify G with ρ_*G)

$$f_{!!}F = \varinjlim_U f_*(F_U), \quad U \subset X \text{ relatively compact open subanalytic}, \tag{7}$$

$$f_{!!}F = \varinjlim_K f_*(\Gamma_K F), \quad K \subset X \text{ compact subanalytic}, \tag{8}$$

$$Rf_{!!}\,\mathrm{R}\mathcal{H}\mathrm{om}(f^{-1}G, F) \xrightarrow{\sim} \mathrm{R}\mathcal{H}\mathrm{om}(G, Rf_{!!}F), \tag{9}$$

$$Rf_{!!}R\Gamma_{f^{-1}U}F \xrightarrow{\sim} R\Gamma_U Rf_{!!}F. \tag{10}$$

The derived functor $Rf_{!!}\colon \mathrm{D}^+(\mathbf{C}_{X_{sa}}) \to \mathrm{D}^+(\mathbf{C}_{Y_{sa}})$ admits a right adjoint $f^!$. The notation is the same as in the classical case because of the commutation relation $f^! \circ R\rho_* \simeq R\rho_* \circ f^!$. Hence $f^!\mathbf{C}_{Y_{sa}} \simeq \rho_*\omega_{X|Y}$ and we will usually write $\omega_{X|Y}$ for $\rho_*\omega_{X|Y}$. The adjunction morphism between $f_{!!}$ and $f^!$ induces the integration

morphism

$$\mathrm{int}_f \colon Rf_{!!}(\omega_{X|Y}) \to \mathbf{C}_{Y_{sa}}. \tag{11}$$

4.3. **"Soft" sheaves.** In this paragraph X is a real analytic manifold and X_{sa} is the corresponding subanalytic site. Though we are not in a framework of sheaves on a locally compact space, we may introduce a notion of soft sheaves on the subanalytic site which are acyclic for the direct image functors.

Definition 4.1. A sheaf $F \in \mathrm{Mod}(\mathbf{C}_{X_{sa}})$ is soft if for any closed subanalytic subset $Z \subset X$ and any open subanalytic subset $U \subset X$ the natural morphism $\Gamma(U; F) \to \Gamma(U; F_Z)$ is surjective.

We note the following isomorphism, as in the case of sheaves on a reasonable topological space:

$$\Gamma(U; F_Z) \simeq \varinjlim_{U \cap Z \subset W \subset U} \Gamma(W; F), \qquad W \subset X \text{ subanalytic open set.} \tag{12}$$

From this description of sections it follows that quasi-injective sheaves are soft. We also note that if F is soft and $Z \subset X$ is a closed subanalytic subset then F_Z is soft.

Before we prove that soft sheaves are acyclic for functors of direct image we need a lemma on coverings.

Lemma 4.2. *Let $U = \bigcup_{i\in\mathbf{N}} U_i$ be a locally finite covering by subanalytic open subsets of X. There exist subanalytic open subsets of X, $V_i \subset U_i$, $i \in \mathbf{N}$, such that $U = \bigcup_{i\in\mathbf{N}} V_i$ and $(U \cap \overline{V_i}) \subset U_i$.*

Proof. We choose an analytic distance d on X and we define V_n inductively as follows. If V_i, $i < n$, is built we set $W_n = U_n \setminus (\bigcup_{i<n} V_i \cup \bigcup_{j>n} U_j)$ and

$$V_n = \{x \in U_n;\ d(x, W_n) < d(x, \partial U_n)\}.$$

We note that W_n is subanalytic because the covering is locally finite. Since d is analytic the functions $d(\cdot, Z)$, $Z \subset X$ subanalytic, are continuous subanalytic functions (see [3] for the notion of subanalytic function). It follows that V_n is a subanalytic open subset of X and $V_n \subset U_n$.

By construction $W_n \subset V_n$ and we deduce by induction that $U = \bigcup_{i\leq n} V_i \cup \bigcup_{j>n} U_j$. Since the covering is locally finite this gives $U = \bigcup_{i\in\mathbf{N}} V_i$.

It remains to prove that $(U \cap \overline{V_n}) \subset U_n$. If this is false there exists $x_0 \in U \cap \overline{V_n} \cap \partial U_n$. Since W_n is closed in U, we have $\delta = d(x_0, W_n) > 0$, and the ball $B(x_0, \delta/2)$ doesn't meet V_n. In particular $x_0 \notin \overline{V_n}$ which is a contradiction. □

Proposition 4.3. *Let $0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$ be an exact sequence in $\mathrm{Mod}(\mathbf{C}_{X_{sa}})$ with F' soft. Then for any open subanalytic subset $U \subset X$ the morphisms*

$$\Gamma(U; F) \to \Gamma(U; F'') \quad \text{and} \quad \varinjlim_K \Gamma_K(U; F) \to \varinjlim_K \Gamma_K(U; F''),$$

where K runs over the compact subanalytic subsets of X, are surjective.

Proof. We first consider a section $s \in \Gamma(U; F'')$. We may find a locally finite covering $U = \bigcup_{i\in\mathbf{N}} U_i$ and $s_i \in \Gamma(U_i; F)$ such that $\alpha(s_i) = s|_{U_i}$. By Lemma 4.2 there exists a subcovering $U = \bigcup_{i\in\mathbf{N}} V_i$ with $(U \cap \overline{V_i}) \subset U_i$.

We set $Z_n = \bigcup_{i=0}^n \overline{V_i}$ and prove by induction on n that there exists a section $\tilde{s}_n \in \Gamma(U; F_{Z_n})$ such that $\beta(\tilde{s}_n) = s|_{Z_n}$ and $\tilde{s}_n|_{Z_{n-1}} = \tilde{s}_{n-1}$.

This is clear for $n = 0$ and we assume it is proved for n. We set $t_n = (\tilde{s}_n - s_{n+1})|_{Z_n \cap \overline{V_{n+1}}}$. Then $\beta(t_n) = 0$ so that t_n belongs to $\Gamma(U; F'_{Z_n \cap \overline{V_{n+1}}})$ and by hypothesis we may extend it to $t \in \Gamma(U; F')$. Now we define $\tilde{s}_{n+1} \in \Gamma(U; F_{Z_{n+1}})$ by $\tilde{s}_{n+1}|_{Z_n} = \tilde{s}_n$ and $\tilde{s}_{n+1}|_{\overline{V_{n+1}}} = s_{n+1} + \alpha(t)$. The $\tilde{s}_n$ glue together into a section $\tilde{s} \in \Gamma(U; F)$ such that $\beta(\tilde{s}) = s$, which proves the surjectivity of the first morphism.

Now we consider a compact K and $s \in \Gamma_K(U; F'')$. We choose an open subanalytic subset V such that $K \subset V$ and $K' = \overline{V}$ is compact. We set $Z = X \setminus V$. We just have seen that we may find $\tilde{s} \in \Gamma(U; F)$ such that $\beta(\tilde{s}) = s$. Hence $\beta(\tilde{s}|_Z) = 0$ so that $\tilde{s}|_Z \in \Gamma(U; F'_Z)$ and we may extend $\tilde{s}|_Z$ to $t \in \Gamma(U; F')$. Then $\hat{s} = \tilde{s} - \alpha(t)$ satisfies $\operatorname{supp} \hat{s} \subset K'$ and $\beta(\hat{s}) = s$. □

Corollary 4.4. *If $0 \to F' \to F \to F'' \to 0$ is an exact sequence in $\operatorname{Mod}(\mathbf{C}_{X_{sa}})$ with F' and F soft, then F'' also is soft.*

Proof. For $Z \subset X$ a subanalytic closed subset we have the exact sequence $0 \to F'_Z \to F_Z \to F''_Z \to 0$ and F'_Z, F_Z still are soft. Hence Proposition 4.3 implies that, for any subanalytic open subset $U \subset X$, the morphisms $\Gamma(U; F) \to \Gamma(U; F'')$ and $\Gamma(U; F_Z) \to \Gamma(U; F''_Z)$ are surjective. Now it follows from the definition that F'' is soft. □

Corollary 4.5. *Let $f\colon X \to Y$ be a morphism of analytic manifolds, $U \subset X$ an open subanalytic subset. Then soft sheaves in $\operatorname{Mod}(\mathbf{C}_{X_{sa}})$ are acyclic for the functors $\Gamma(U; \cdot)$, $\varinjlim_K \Gamma_K(U; \cdot)$, K running over the compact subsets of X, Γ_U, f_* and $f_{!!}$.*

Proof. For the first two functors this follows from Proposition 4.3 and Corollary 4.4 by usual homological algebra arguments. This implies the result for the other functors. □

4.4. **Tempered functions.** Here we recall the definition of tempered $\mathcal{C}^\infty$ functions. We also state a tempered de Rham lemma on the subanalytic site, which is actually a reformulation of results of [4]. In this paragraph, X is a real analytic manifold.

Definition 4.6. A $\mathcal{C}^\infty$ function f defined on an open set U has "polynomial growth at $p \in X$" if there exist a compact neighborhood K of p and $C, N > 0$ such that $\forall x \in K \cap U$, $|f(x)| < C\, d(x, K \setminus U)^{-N}$, for a distance d defined through some coordinate system around p.

We say that f is tempered if all its derivatives have polynomial growth at any point. In [7] it is proved, using results of Łojasiewicz, that these functions define a subsheaf $\mathcal{C}^{\infty,t}_X$ of $\rho_*\mathcal{C}^\infty_X$ on X_{sa}.

We denote by $\Omega^{t,i}_X$ the sheaf on X_{sa} of forms of degree i with tempered coefficients. We obtain as usual a sheaf of dg-algebras on X_{sa}, the "tempered de Rham algebra" $\Omega^t_X = 0 \to \Omega^{t,0}_X \to \cdots \to \Omega^{t,n}_X \to 0$.

Lemma 4.7. *The tempered de Rham algebra is a resolution of the constant sheaf on the subanalytic site, i.e. we have an exact sequence on X_{sa}:*

$$0 \to \mathbf{C}_{X_{sa}} \to \Omega^{t,0}_X \to \cdots \to \Omega^{t,n}_X \to 0.$$

Proof. In other words we have to prove that the morphism $\mathbf{C}_{X_{sa}} \to \Omega^t_X$ in $\mathrm{D}^b(\mathbf{C}_{X_{sa}})$ is an isomorphism. For this it is enough to see that, for any $F \in \mathrm{D}^b_{\mathbf{R}-c}(\mathbf{C}_X)$ we have

$$\mathrm{RHom}(\rho_* F, \mathbf{C}_{X_{sa}}) \simeq \mathrm{RHom}(\rho_* F, \Omega^t_X). \tag{13}$$

Indeed for any $G \in \mathrm{D}^+(\mathbf{C}_{X_{sa}})$, $H^k(G)$ is the sheaf associated to the presheaf $U \mapsto R^k\Gamma(U; G) = H^k \mathrm{RHom}(\rho_* \mathbf{C}_U, G)$; hence (13) applied to $F = \mathbf{C}_U$ gives the result.

Now we prove (13). Actually this is Proposition 4.6 of [4], except that it is not stated in this language, and that it is given for tempered distributions instead of tempered $\mathcal{C}^\infty$ functions. We let $\mathcal{C}^\omega_X$ be the sheaf of real analytic functions and $\mathcal{D}_X$ the sheaf of linear differential operators with coefficients in $\mathcal{C}^\omega_X$. Using a Koszul resolution of $\mathcal{C}^\omega_X$ we have the standard isomorphism $\mathrm{R}\mathcal{H}\mathrm{om}_{\rho^!\mathcal{D}_X}(\rho^!\mathcal{C}^\omega_X, \mathcal{C}^{\infty\,t}_X) \simeq \Omega^t_X$. In [4] a functor $RTH_X(F)$ is defined (now denoted $T\,\mathcal{H}\mathrm{om}(F, \mathcal{D}b_X)$) and Proposition 4.6 reads:

$$\mathrm{RHom}(F, \mathbf{C}_X) \simeq \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{C}^\omega_X, T\,\mathcal{H}\mathrm{om}(F, \mathcal{D}b_X)).$$

To replace distributions by $\mathcal{C}^\infty$ functions, we have an analog of $T\,\mathcal{H}\mathrm{om}(F, \mathcal{D}b_X)$ for $\mathcal{C}^\infty$ functions, introduced in [6] and [7]. By [6], Theorem 10.5, we have the comparison isomorphism

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{C}^\omega_X, T\,\mathcal{H}\mathrm{om}(F, \mathcal{C}^\infty_X)) \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{C}^\omega_X, T\,\mathcal{H}\mathrm{om}(F, \mathcal{D}b_X)).$$

Actually, in [6] X is a complex manifold and the result is stated for the sheaf of anti-holomorphic functions instead of $\mathcal{C}^\omega_X$, but the proof also works in our case. Following [7], Proposition 7.2.6 or [10], Proposition 3.3.5, we may express the functor $T\,\mathcal{H}\mathrm{om}$ using the analytic site: $T\,\mathcal{H}\mathrm{om}(F, \mathcal{C}^\infty_X) \simeq \rho^{-1}\,\mathrm{R}\mathcal{H}\mathrm{om}(\rho_* F, \mathcal{C}^{\infty\,t}_X)$.

Putting these isomorphisms together we obtain (13):

$$\begin{aligned}
\mathrm{RHom}(\rho_* F, \Omega^t_X) &\simeq \mathrm{RHom}(\rho_* F, \mathrm{R}\mathcal{H}\mathrm{om}_{\rho^!\mathcal{D}_X}(\rho^!\mathcal{C}^\omega_X, \mathcal{C}^{\infty\,t}_X)) \\
&\simeq \mathrm{RHom}_{\rho^!\mathcal{D}_X}(\rho^!\mathcal{C}^\omega_X, \mathrm{R}\mathcal{H}\mathrm{om}(\rho_* F, \mathcal{C}^{\infty\,t}_X)) \\
&\simeq \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{C}^\omega_X, T\,\mathcal{H}\mathrm{om}(F, \mathcal{D}b_X)) \\
&\simeq \mathrm{RHom}(F, \mathbf{C}_X),
\end{aligned}$$

where we have used adjunction morphisms between $\otimes$, $\mathcal{H}\mathrm{om}$ and $\rho_!, \rho^{-1}$. □

The integration of forms also makes sense in the tempered case: we let $f\colon X \to Y$ be a submersion with fibers of dimension d, $V \subset Y$ a constructible open subset and we consider a form $\omega \in \Gamma(f^{-1}(V); \Omega^{t,i+d}_X \otimes or_{X|Y})$ such that the closure (in X) of $\operatorname{supp}\omega$ is compact. Then $\int_f \omega \in \Gamma(V; \Omega^{t,i}_Y)$. We deduce the morphism of complexes

$$\int_f : f_{!!}(\Omega^t_X \otimes \omega'_{X|Y}) \to \Omega^t_Y. \tag{14}$$

Its image in $\mathrm{D}^b(\mathbf{C}_{Y_{sa}})$ coincides with the morphism int_f of (11).

5. Resolution

In this section we consider real analytic manifolds and sheaves on their associated subanalytic sites.

Definition 5.1. For a real manifold X we introduce the notations, $\widehat{X} = X \times \mathbf{R}$, $i_X \colon X \to \widehat{X}$, $x \mapsto (x, 0)$ and $X^+ = X \times \mathbf{R}_{>0}$. We consider the tempered de Rham algebra on the site $\widehat{X}_{sa}$,

$$\Omega^t_{\widehat{X}} = 0 \to \Omega^{t,0}_{\widehat{X}} \to \cdots \to \Omega^{t,n+1}_{\widehat{X}} \to 0,$$

and we define a sheaf of anti-commutative dg-algebras on X_{sa}: $\mathcal{A}_X = i_X^{-1}\Gamma_{X^+}(\Omega^t_{\widehat{X}})$.

We denote by $\tau_{X,1} \colon \widehat{X} \to X$ and $\tau_{X,2} \colon \widehat{X} \to \mathbf{R}$ the projections, and by t the coordinate on $\mathbf{R}$. This gives a canonical element $dt \in \mathcal{A}^1_X$. The decomposition $\widehat{X} = X \times \mathbf{R}$ induces a decomposition of the differential $d = d_1 + d_2$ in anti-commuting differentials, where we set $d_2(\omega) = (\partial\omega/\partial t)dt$.

The algebra $\mathcal{A}_X$ comes with natural morphisms related to inverse image and direct image by a smooth map. Let $f \colon X \to Y$ be a morphism of manifolds. It induces $\widehat{f} = f \times \mathrm{id}$ and f^+ in the following diagram, whose squares are Cartesian:

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & \widehat{X} & \hookleftarrow & X^+ \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle \widehat{f}} & \square & \downarrow{\scriptstyle f^+} \\ Y & \xrightarrow{i_Y} & \widehat{Y} & \hookleftarrow & Y^+ \end{array}$$

We note that $X^+ = \widehat{f}^{-1}(Y^+)$ and this gives a morphism of functors $\widehat{f}^{-1}\Gamma_{Y^+} \to \Gamma_{X^+}\widehat{f}^{-1}$. Thus we obtain a morphism of dg-algebras:

$$\widehat{f}^{-1}\Gamma_{Y^+}(\Omega^t_{\widehat{Y}}) \to \Gamma_{X^+}\widehat{f}^{-1}(\Omega^t_{\widehat{Y}}) \to \Gamma_{X^+}(\Omega^t_{\widehat{X}}).$$

Definition 5.2. We denote by $f^\sharp \colon f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$ the image of the above morphism by the restriction functor i_X^{-1}. It is a morphism of dg-algebras.

Now we assume that f is smooth. Hence $\widehat{f}$ is also smooth and we have the integration morphism (14) $\int_{\widehat{f}} \colon \widehat{f}_{!!}(\Omega^t_{\widehat{X}} \otimes \omega'_{X|Y}) \to \Omega^t_{\widehat{Y}}$. We apply the functor $i_Y^{-1}\Gamma_{Y^+}$ to this morphism. We note the base change isomorphism $f_{!!}i_X^{-1} \simeq i_Y^{-1}\widehat{f}_{!!}$ and the morphism $\widehat{f}_{!!}\Gamma_{X^+} \to \Gamma_{Y^+}\widehat{f}_{!!}$. They give the sequence of morphisms:

$$\begin{aligned} f_{!!}(\mathcal{A}_X \otimes \omega'_{X|Y}) &= f_{!!}i_X^{-1}\Gamma_{X^+}(\Omega^t_{\widehat{X}} \otimes \omega'_{\widehat{X}|\widehat{Y}}) \simeq i_Y^{-1}\widehat{f}_{!!}\Gamma_{X^+}(\Omega^t_{\widehat{X}} \otimes \omega'_{\widehat{X}|\widehat{Y}}) \\ &\to i_Y^{-1}\Gamma_{Y^+}\widehat{f}_{!!}(\Omega^t_{\widehat{X}} \otimes \omega'_{\widehat{X}|\widehat{Y}}) \to i_Y^{-1}\Gamma_{Y^+}\Omega^t_{\widehat{Y}} = \mathcal{A}_Y \end{aligned} \tag{15}$$

Definition 5.3. For a smooth map $f \colon X \to Y$, we call morphism (15) the integration morphism and denote it $\int_f \colon f_{!!}(\mathcal{A}_X \otimes \omega'_{X|Y}) \to \mathcal{A}_Y$.

The main result of this section is the following theorem. It is proved in the remaining part of the section: the quasi-injectivity of the $\mathcal{A}^i_X$ is proved in Proposition 5.9 and the fact that $\mathcal{A}_X$ is a resolution is Corollary 5.12.

Theorem 5.4. *Let X be a real analytic manifold. The sheaf of dg-algebras $\mathcal{A}_X$ is a quasi-injective resolution of $\mathbf{C}_{X_{sa}}$.*

Remark 5.5. By this theorem we have $f_{!!}(\mathcal{A}_X \otimes \omega'_{X|Y}) \simeq Rf_{!!}(\omega_{X|Y})$. Hence the morphism $\int_f$ of Definition 5.3 induces a morphism in the derived category $Rf_{!!}\omega_{X|Y} \to \mathbf{C}_{Y_{sa}}$. It coincides with the usual integration morphism int_f of (11)

because this holds for the de Rham complex (morphism (14) applied to $\widehat{f}$), and we have the commutative diagram:

$$\begin{array}{ccccc} Rf_{!!}\omega_{X|Y} & \xrightarrow{\sim} & Rf_{!!}(i_X^{-1}R\Gamma_{X^+}\omega_{\widehat{X}|\widehat{Y}}) & \longrightarrow & i_Y^{-1}R\Gamma_{Y^+}R\widehat{f}_{!!}\omega_{\widehat{X}|\widehat{Y}} \\ \downarrow{\scriptstyle \mathrm{int}_f} & & & & \downarrow{\scriptstyle \mathrm{int}_{\widehat{f}}} \\ \mathbf{C}_{Y_{sa}} & & \xrightarrow{\qquad\sim\qquad} & & i_Y^{-1}R\Gamma_{Y^+}\mathbf{C}_{\widehat{Y}_{sa}}. \end{array}$$

For the proof of the theorem we need some lemmas on tempered functions. We refer to [3] for results on subanalytic sets. We recall that a function is subanalytic if its graph is a subanalytic set. We introduce the following notation, for $U \subset X$ an open subset, and $\varphi\colon U \to \mathbf{R}$ a positive continuous function on U:

$$U_\varphi = \{(x,t) \in \widehat{X};\ x \in U,\ |t| < \varphi(x)\}, \qquad U_\varphi^+ = U_\varphi \cap X^+.$$

Lemma 5.6. *Let $U \subset X$ be a subanalytic open subset and $V \subset \widehat{X}$ be a subanalytic open neighborhood of U in $\widehat{X}$. Then there exists a subanalytic continuous function φ defined on $\overline{U}$ such that $\varphi = 0$ on the boundary of U and $U_\varphi \subset V$.*

Proof. We set $V' = V \cap (U \times \mathbf{R})$, $Z = \widehat{X} \setminus V'$ and let φ be the distance function to Z: $\varphi(x) = d(x,Z)$. By [3], Remark 3.11, this is a subanalytic function on $\widehat{X}$ and its restriction to $\overline{U}$ satisfies the required property. □

The following result is similar to a division property for flat $\mathcal{C}^\infty$ functions, which can be found for example in [12], Lemma V.2.4.

Lemma 5.7. *Let $U \subset X$ be a subanalytic open subset and $\varphi\colon \overline{U} \to \mathbf{R}$ a subanalytic continuous function on U, such that $\varphi = 0$ on the boundary of U and $\varphi > 0$ on U. Then there exists a $\mathcal{C}^\infty$ function $\psi\colon U \to \mathbf{R}$ such that*

(i) *$\forall x \in U$, $0 < \psi(x) < \varphi(x)$,*
(ii) *ψ and $1/\psi$ are tempered.*

Proof. We first note that it is enough to find a ψ such that ψ is tempered, $0 < \psi < \varphi$ and $1/\psi$ has polynomial growth along ∂U. We may also work locally: assuming the result is true on local charts, we choose

- locally finite coverings of X by subanalytic open subsets, (U_i), (V_i), together with a partition of unity $\mu_i\colon X \to \mathbf{R}$ such that $\overline{U}_i \subset V_i$, $0 \le \mu_i$, $\sum \mu_i = 1$, $\mu_i = 1$ on U_i and $\mu_i = 0$ on a neighborhood of $X \setminus V_i$,
- $\mathcal{C}^\infty$ functions $\psi_i\colon U \cap V_i \to \mathbf{R}$ such that $0 < \psi_i < \varphi$ on $U \cap V_i$, ψ_i is tempered and $1/\psi_i$ has polynomial growth along $\partial(U \cap V_i)$

and we set $\psi = \sum_i \mu_i\psi_i$. Then ψ satisfies the conclusion of the lemma. Indeed, each $\mu_i\psi_i$ is defined and tempered on U, and so is ψ since the sum is locally finite, and, for $x \in \partial U$, i such that $x \in U_i$, $1/\psi \le 1/\psi_i$ has polynomial growth at x.

Hence we assume $X = \mathbf{R}^n$ and U is bounded. By [12], Lemma IV.3.3, there exist constants C_k, $k \in \mathbf{N}^n$, such that, for any compact $K \subset \mathbf{R}^n$ and any $\varepsilon > 0$, there exists a $\mathcal{C}^\infty$ function α on $\mathbf{R}^n$ such that

$$0 \le \alpha \le 1, \qquad \alpha(x) = 0 \text{ if } d(x,K) \ge \varepsilon, \qquad \alpha(x) = 1 \text{ if } x \in K,$$
$$\forall k \in \mathbf{N}^n, \quad |D^k\alpha| \le C_k\varepsilon^{-|k|}.$$

(The function α is the convolution of the characteristic function of $\{x; d(x,K) \leq \varepsilon/2\}$ with a suitable test function.)

We set $K_i = \{x \in U; 2^{-i-1} \leq d(x,\partial U) \leq 2^{-i}\}$ and we let α_i be the function associated to $K = K_i$ and $\varepsilon = 2^{-i-2}$ by the above result. In particular $\alpha_i = 1$ on K_i, $\operatorname{supp}\alpha_i \subset S_i$, where we set $S_i = K_{i-1} \cup K_i \cup K_{i+1}$, and $|D^k\alpha_i| \leq C'_k 2^{ik}$, for some $C'_k \in \mathbf{R}$. This implies: $\forall x \in U$, $|D^k\alpha_i(x)| \leq C''_k d(x,\partial U)^{-k}$, for some other constants $C''_k \in \mathbf{R}$.

Łojasiewicz's inequality gives, for $x \in U$, $c\, d(x,\partial U)^r \leq \varphi(x) \leq c' d(x,\partial U)^{r'}$, for some constants $c, r, c', r' > 0$ (see [3], Theorem 6.4). We set $\lambda_i = \min\{\varphi(x); x \in S_i\}$. We note that for $x, x' \in S_i$, we have $1/8 \leq d(x,\partial U)/d(x',\partial U) \leq 8$. Hence, for $x \in S_i$, we have $Cd(x,\partial U)^r \leq \lambda_i \leq C' d(x,\partial U)^{r'}$, for some $C, C' > 0$. Since $\operatorname{supp}\alpha_i \subset S_i$, we also have $\forall i$, $\lambda_i\alpha_i \leq \varphi$.

We note that an $x \in U$ belongs to at most three sets S_i and we define $\psi = (1/3)\sum_i \lambda_i\alpha_i$. The above inequalities give, for $x \in U$, $0 < \psi(x) \leq \varphi(x)$ and

$$|D^k\psi(x)| \leq C''_k\, C'\, d(x,\partial U)^{r'-k}, \qquad \frac{1}{\psi(x)} \leq 3\, C^{-1} d(x,\partial U)^{-r},$$

so that ψ and $1/\psi$ are tempered. □

Lemma 5.8. *Let $U \subset X$ and $\varphi\colon \overline{U} \to \mathbf{R}$ be as in Lemma 5.7. There exist another subanalytic continuous function $\varphi'\colon \overline{U} \to \mathbf{R}$ and a tempered fuction $\alpha \in \Gamma(X^+; \mathcal{C}^{\infty,t}_{\hat{X}})$ such that $\forall x \in U$, $0 < \varphi'(x) < \varphi(x)$ and*

$$\forall (x,t) \in X^+ \quad 0 \leq \alpha(x,t) \leq 1, \qquad \alpha(x,t) = \begin{cases} 1 & \text{for } (x,t) \in U^+_{\varphi'}, \\ 0 & \text{for } (x,t) \notin U^+_{\varphi}. \end{cases}$$

Proof. We choose a $\mathcal{C}^\infty$ function $\psi\colon U \to]0,+\infty[$ satisfying the conclusion of Lemma 5.7 and another $\mathcal{C}^\infty$ function $h\colon \mathbf{R} \to \mathbf{R}$ such that $\forall t \in \mathbf{R}$, $0 \leq h(t) \leq 1$, $h(t) = 1$ for $t \leq 1/2$ and $h(t) = 0$ for $t \geq 1$. We define our function α on X^+ by

$$\alpha(x,t) = \begin{cases} h(\frac{t}{\psi(x)}) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

We first see that α is $\mathcal{C}^\infty$. This is clear except at points (x_0,t_0) with $x_0 \in \partial U$. For such a point, by continuity of φ, we may find a neighborhood V of x_0 in X such that $\forall x \in V$, $\varphi(x) < t_0/2$. Thus, on the neighborhood $V \times]t_0/2; +\infty[$ of (x_0,t_0), α is identically 0, and certainly $\mathcal{C}^\infty$.

Let us check that α is tempered. We only have to check growth conditions at points $(x,0) \in \partial X^+$. We note that $d((x,t),\partial X^+) = t$ so that we have to bound the $D^k\alpha(x,t)$ by powers of t. Since $D^k\alpha = 0$ outside U^+_φ, we assume $(x,t) \in U^+_\varphi$. The $D^k\alpha$ are polynomial expressions in t, the derivatives of h and the derivatives of $1/\psi$. The derivatives of h to a given order are bounded, hence it just remains to bound $D^l(1/\psi)(x)$, with $(x,t) \in U^+_\varphi$, by a power of t. Since $1/\psi$ is tempered $D^l(1/\psi)(x)$ has a bound of the type $C\, d(x,\partial U)^{-N}$. By Łojasiewicz's inequality we have $\varphi(x) \leq C' d(x,\partial U)^r$ and, since $(x,t) \in U^+_\varphi$, we have $t \leq \varphi(x)$. Hence $D^l(1/\psi)(x) \leq C'' t^{-N/r}$, for some $C'' > 0$, which is the desired bound.

By definition $\alpha = 1$ on $U^+_{\psi/2}$ and $\alpha = 0$ outside U^+_φ. Hence we just have to find a subanalytic continuous function φ' such that $\varphi' \leq \psi/2$. Since $1/\psi$ is tempered, there exist constants D, M such that $\psi^{-1}(x) \leq Dd(x,\partial U)^{-M}$, and we may take $\varphi'(x) = \frac{1}{2D} d(x,\partial U)^M$. □

Proposition 5.9. *Let F be a $\mathcal{C}^{\infty,t}_{\widehat{X}}$-module and set $G = i_X^{-1}\Gamma_{X^+}F$. Let $U \subset X$ be a subanalytic open subset.*

Then the natural map $\Gamma(X^+;F) \to \Gamma(U;G)$ is surjective. In particular, the sheaf G is quasi-injective.

Proof. We consider $s \in \Gamma(U;G)$. As in the case of sheaves on manifolds we have, for $H \in \mathrm{Mod}(\mathbf{C}_{\widehat{X}_{sa}})$ and $U \subset X$, $\Gamma(U;i_X^{-1}H) \simeq \varinjlim_V \Gamma(V;H)$ where V runs over the subanalytic open subsets of $\widehat{X}$ containing U. Hence, by Lemma 5.6, we may represent s by a section $\tilde{s} \in \Gamma(U^+_\varphi;F)$, for some subanalytic continuous function φ defined on $\overline{U}$ such that $\varphi = 0$ on the boundary of U.

We apply Lemma 5.8 to the function $\varphi/2\colon \overline{U} \to \mathbf{R}$ and obtain $\varphi'\colon \overline{U} \to \mathbf{R}$ and $\alpha \in \Gamma(X^+;\mathcal{C}^{\infty,t}_{\widehat{X}})$ such that $0 < \varphi' < \varphi/2$, $\alpha = 1$ on $U^+_{\varphi'}$ and $\alpha = 0$ outside $U^+_{\varphi/2}$. We set $\hat{s} = \alpha\tilde{s}$. Then $\hat{s} \in \Gamma(U^+_\varphi;F)$ extends by 0 to a section $\hat{s} \in \Gamma(X^+;F)$ and we have $\hat{s}|_{U^+_{\varphi'}} = \tilde{s}|_{U^+_{\varphi'}}$ so that $\hat{s}$ also represents s. This shows the surjectivity of $\Gamma(X^+;F) \to \Gamma(U;G)$. □

We have the following resolution of $\mathcal{C}^{\infty,t}_X$ as an $\mathcal{A}_X$-module. Let I_X be the ideal of $\mathcal{A}_X$ generated by $\Omega^{t,1}_X \subset \mathcal{A}^1_X$. In local coordinates $(x_1,\dots,x_n,t)$, I_X consists of the forms involving one of the dx_i and we obtain the isomorphism $\mathcal{A}_X/I_X \simeq 0 \to \mathcal{A}^0_X \xrightarrow{\partial/\partial t} \mathcal{A}^0_X \to 0$, where the differential is given by $f(x,t) \mapsto \frac{\partial f}{\partial t}(x,t)$. The following result implies that the complex $\mathcal{A}_X/I_X$ is a resolution of $\mathcal{C}^{\infty,t}_X$.

Corollary 5.10. *For any subanalytic open set $U \subset X$ we have the exact sequence:*

$$0 \to \Gamma(U;\mathcal{C}^{\infty,t}_X) \to \Gamma(U;\mathcal{A}^0_X) \xrightarrow{\partial/\partial t} \Gamma(U;\mathcal{A}^0_X) \to 0.$$

Proof. The less obvious point is the surjectivity. We have the restriction maps

$$\begin{array}{ccc} \Gamma(U\times\mathbf{R};\mathcal{C}^{\infty,t}_{\widehat{X}}) & \xrightarrow{\partial/\partial t} & \Gamma(U\times\mathbf{R};\mathcal{C}^{\infty,t}_{\widehat{X}}) \\ \downarrow & & \downarrow \\ \Gamma(U;\mathcal{A}^0_X) & \xrightarrow{\partial/\partial t} & \Gamma(U;\mathcal{A}^0_X). \end{array}$$

The vertical arrows are surjective by Proposition 5.9, and so is the top horizontal arrow: we integrate with respect to t with starting points on $X\times\{1\}$, which insures that the resulting function is tempered. □

Corollary 5.11. *For any subanalytic open set $U \subset X$, the sheaf $\mathcal{C}^{\infty,t}_X$ is acyclic with respect to the functor Γ_U.*

Proof. We have to prove that $R^i\Gamma_U(\mathcal{C}^{\infty,t}_X) = 0$ for $i > 0$. By Proposition 5.9 $\mathcal{A}^0_X$ is quasi-injective and we may use the resolution $\mathcal{C}^{\infty,t}_X \to \mathcal{A}^0_X \xrightarrow{\partial/\partial t} \mathcal{A}^0_X$ to compute $R^i\Gamma_U(\mathcal{C}^{\infty,t}_X)$. We are thus reduced to proving the surjectivity of the morphism $\partial/\partial t\colon \Gamma_U(\mathcal{A}^0_X) \to \Gamma_U(\mathcal{A}^0_X)$. This follows from Corollary 5.10 since $\Gamma_U(\mathcal{A}^0_X)(V) = \Gamma(U\cap V;\mathcal{A}^0_X)$. □

Corollary 5.12. *The sheaf of dg-algebras $\mathcal{A}_X$ is quasi-isomorphic to $\mathbf{C}_{X_{sa}}$, i.e. we have the exact sequence:*

$$0 \to \mathbf{C}_{X_{sa}} \to \mathcal{A}^0_X \to \mathcal{A}^1_X \to \cdots \to \mathcal{A}^{n+1}_X \to 0.$$

Proof. By Lemma 4.7, we have the exact sequence on $\widehat{X}$:

$$0 \to \mathbf{C}_{\widehat{X}_{sa}} \to \Omega^{t,0}_{\widehat{X}} \to \cdots \to \Omega^{t,n+1}_{\widehat{X}} \to 0. \tag{16}$$

By the previous corollary the sheaves $\Omega^{t,i}_{\widehat{X}}$ are Γ_{X^+}-acyclic. The constant sheaf $\mathbf{C}_{\widehat{X}_{sa}}$ also is Γ_{X^+}-acyclic because $R\Gamma_{X^+}(\mathbf{C}_{\widehat{X}_{sa}}) \simeq \mathbf{C}_{\overline{X^+_{sa}}}$ (recall that ρ_* commutes with $R\Gamma_{X^+}$). Hence we still have an exact sequence when we apply Γ_{X^+} to (16), and applying the exact functor i_X^{-1} gives the corollary. □

6. $\mathcal{A}$-MODULES

For a real analytic manifold X, we denote by $\mathrm{Mod}(\mathcal{A}_X)$ the category of sheaves of *bounded below* dg-$\mathcal{A}_X$-modules on X_{sa}. We have an obvious forgetful functor and its composition with the localization:

$$\mathrm{For}_X \colon \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{C}^+(\mathbf{C}_{X_{sa}}), \qquad \mathrm{For}'_X \colon \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{D}^+(\mathbf{C}_{X_{sa}}). \tag{17}$$

We will usually write F instead of $\mathrm{For}_X(F)$ or $\mathrm{For}'_X(F)$ when the context is clear. We still write For_X, For'_X for the compositions of these forgetful functors with the exact functor $I_\tau \colon \mathrm{C}(\mathbf{C}_{X_{sa}}) \to \mathrm{C}(\mathbf{I}(\mathbf{C}_X))$.

In this section we define operations on $\mathrm{Mod}(\mathcal{A}_X)$ and check usual formulas in this framework, as well as some compatibility with the corresponding operations in $\mathrm{C}(\mathbf{C}_{X_{sa}})$ or $\mathrm{D}(\mathbf{C}_{X_{sa}})$ (hence also in $\mathrm{C}(\mathbf{I}(\mathbf{C}_X))$ or $\mathrm{D}(\mathbf{I}(\mathbf{C}_X))$, because I_τ commutes with the standard operations).

6.1. **Tensor product.** For $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{A}_X)$, the tensor product $\mathcal{M} \otimes_{\mathcal{A}_X} \mathcal{N} \in \mathrm{Mod}(\mathcal{A}_X)$ is defined as usual by taking the tensor product of the underlying sheaves of graded modules over the underlying sheaf of graded algebras and defining the differential by $d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn$ (for m homogeneous). We have an exact sequence in $\mathrm{C}^+(\mathbf{C}_{X_{sa}})$:

$$\mathcal{M} \otimes \mathcal{A}_X \otimes \mathcal{N} \xrightarrow{\delta} \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}_X} \mathcal{N} \to 0, \tag{18}$$

where $\delta(m \otimes a \otimes n) = (-1)^{\deg a \deg m} am \otimes n - m \otimes an$, for homogeneous a, m, n.

For two real analytic manifolds X, Y and $\mathcal{M} \in \mathrm{Mod}(\mathcal{A}_X)$, $\mathcal{N} \in \mathrm{Mod}(\mathcal{A}_Y)$, we denote by $\underline{\boxtimes}$ the external tensor product in the category of $\mathcal{A}$-modules:

$$\mathcal{M} \underline{\boxtimes} \mathcal{N} = \mathcal{A}_{X\times Y} \otimes_{(\mathcal{A}_X \boxtimes \mathcal{A}_Y)} (\mathcal{M} \boxtimes \mathcal{N}).$$

6.2. **Inverse image and direct image.** Let $f \colon X \to Y$ be a morphism of real analytic manifolds. Recall the morphism of dg-algebras $f^\sharp \colon f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$ introduced in Definition 5.2.

Definition 6.1. For $\mathcal{N} \in \mathrm{Mod}(\mathcal{A}_Y)$ we define its inverse image in $\mathrm{Mod}(\mathcal{A}_X)$:

$$f^*\mathcal{N} = \mathcal{A}_X \otimes_{f^{-1}\mathcal{A}_Y} f^{-1}\mathcal{N}.$$

By adjunction $f^\sharp$ gives a morphism $\mathcal{A}_Y \to f_*\mathcal{A}_X$. Hence, for $\mathcal{M} \in \mathrm{Mod}(\mathcal{A}_X)$, $f_*\mathcal{M}$ has a natural structure of dg-$\mathcal{A}_Y$-module, as well as $f_{!!}\mathcal{M}$, through the natural morphism $f_*\mathcal{A}_X \otimes f_{!!}\mathcal{M} \to f_{!!}(\mathcal{A}_X \otimes \mathcal{M}) \to f_{!!}\mathcal{M}$.

We have a natural morphism $f^{-1}\mathcal{N} \to f^*\mathcal{N}$ in $\mathrm{C}(\mathbf{C}_{X_{sa}})$ (with the notations of Remark 17, it could be written more exactly $f^{-1}(\mathrm{For}_Y \mathcal{N}) \to \mathrm{For}_X f^*\mathcal{N}$). We show in Proposition 6.3 that it is a quasi-isomorphism when f is smooth. We first consider a particular case.

Lemma 6.2. *We set $X = \mathbf{R}^{m+1}$, $Y = \mathbf{R}^m$ and we let $f\colon X \to Y$ be the projection. We consider coordinates $(y_1, \dots, y_m, u)$ on X. For $\mathcal{N} \in \mathrm{Mod}(\mathcal{A}^0_Y)$ we have an exact sequence in $\mathrm{Mod}(\mathbf{C}_{X_{sa}})$:*

$$0 \longrightarrow f^{-1}\mathcal{N} \longrightarrow \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N} \xrightarrow{\ d\ } \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N} \longrightarrow 0,$$

where d is defined by $d(a \otimes n) = \dfrac{\partial a}{\partial u} \otimes n$, for $a \in \mathcal{A}^0_X$, $n \in \mathcal{N}$.

Proof. We have the exact sequence $0 \to f^{-1}\mathcal{A}^0_Y \to \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^0_X \to 0$, where $d(a) = \frac{\partial a}{\partial u}$. The tensor product with $f^{-1}\mathcal{N}$ gives the exactness of the sequence of the lemma except at the first term. It just remains to check that $\iota\colon f^{-1}\mathcal{N} \to \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}$, $n \mapsto 1 \otimes n$, is injective.

We consider a section $n \in \Gamma(U; f^{-1}\mathcal{N})$ such that $\iota(n) = 0$. This means that there exist a locally finite covering $U = \bigcup_{i\in I} U_i$ and sections, setting $V_i = f(U_i)$,

$$n_i, n_{ij} \in \Gamma(V_i; \mathcal{N}), \quad a_{ij} \in \Gamma(U_i; \mathcal{A}^0_X), \quad b_{ij} \in \Gamma(V_i; \mathcal{A}^0_Y),$$

such that for each $i \in I$, $n|_{U_i} = f^* n_i$, j runs over a finite set J_i, and we have the identity in $\Gamma(U_i; \mathcal{A}^0_X) \otimes \Gamma(V_i; \mathcal{N})$:

$$1 \otimes n_i = \sum_{j\in J_i} (a_{ij}(b_{ij} \circ f) \otimes n_{ij} - a_{ij} \otimes b_{ij} n_{ij}). \tag{19}$$

We may as well assume that the $\overline{U_i}$ are compact. We show in this case that $n_i = 0$, which will prove $n = 0$, hence the injectivity of ι.

By Proposition 5.9 we may represent the a_{ij}, b_{ij} by tempered $\mathcal{C}^\infty$ functions defined on X^+, Y^+. We choose continuous subanalytic functions $\varphi_i\colon \overline{U_i} \to \mathbf{R}$, $\varphi_i > 0$ on U_i, such that the identities (19) hold in $\Gamma(U^+_{\varphi_i}; \mathcal{C}^{\infty,t}_{\widehat{X}}) \otimes \Gamma(V_i; \mathcal{N})$.

We apply Lemma 5.8 to the function $\varphi_i/2\colon \overline{U_i} \to \mathbf{R}$ and obtain $\varphi_i'\colon \overline{U_i} \to \mathbf{R}$ and $\alpha_i \in \Gamma(X^+; \mathcal{C}^{\infty,t}_{\widehat{X}})$ such that $0 < \varphi_i' < \varphi_i/2$, $0 \le \alpha_i \le 1$, $\alpha_i = 1$ in $U^+_{\varphi_i'}$ and $\alpha_i = 0$ outside $U^+_{\varphi_i/2}$. Multiplying both sides of (19) by α_i we obtain identities which now hold on $\Gamma(X^+; \mathcal{C}^{\infty,t}_{\widehat{X}}) \otimes \Gamma(V_i; \mathcal{N})$. These identities imply:

$$\alpha_i \otimes n_i = 0 \qquad \text{in} \qquad \Gamma(X^+; \mathcal{C}^{\infty,t}_{\widehat{X}}) \otimes_{\Gamma(Y^+; \mathcal{C}^{\infty,t}_{\widehat{Y}})} \Gamma(V_i; \mathcal{N}).$$

We note that α_i has compact support and we set $\beta_i = \int \alpha_i du$. We have $\beta_i \in \Gamma(Y^+; \mathcal{C}^{\infty,t}_{\widehat{Y}})$ and the last identity gives $\beta_i n_i = 0$. Now $\Gamma(V_i; \mathcal{N})$ is a $\Gamma(V_i; \mathcal{A}^0_Y)$-module and to conclude that $n_i = 0$ it just remains to prove that $\beta_i|_{V_i}$ is invertible in $\Gamma(V_i; \mathcal{A}^0_Y)$.

Since β_i is a tempered $\mathcal{C}^\infty$ function on Y^+ it is enough to check that β_i^{-1} has polynomial growth along the boundary of $W_i = f(U^+_{\varphi_i'})$. We set $Z_i = X^+ \setminus U^+_{\varphi_i'}$ and for $(x,t) \in X^+$, $d_i(x,t) = d((x,t), \partial Z_i)$. We obtain the bound, for $(y,t) \in W_i$:

$$\beta_i(y,t) \ \ge \int_{U^+_{\varphi_i'} \cap (\{(y,t)\} \times \mathbf{R})} 1 \cdot du \ \ge \ 2 \max_{u\in\mathbf{R}} d_i(y,u,t)$$

The function $m_i(y,t) = \max_{u\in\mathbf{R}} d_i(y,u,t)$ is subanalytic since the max can be taken for u running on a compact set. We have $m_i(y,t) > 0$ for $(y,t) \in W_i$. Hence, by Łojasiewicz's inequality we have $m_i(y,t) > C' d((y,t), \partial W_i)^{-N'}$ for some $C', N' \in \mathbf{R}$ and it follows that β_i^{-1} has polynomial growth along ∂W_i. □

Proposition 6.3. *Let $f\colon X \to Y$ be a smooth morphism and $\mathcal{N} \in \mathrm{Mod}(\mathcal{A}_Y)$.*

(i) The morphism in $\mathrm{C}(\mathbf{C}_{X_{sa}})$, $f^{-1}\mathcal{N} \to f^*\mathcal{N}$, is a quasi-isomorphism.

(ii) If $\mathcal{N}$ is locally free as an $\mathcal{A}^0_Y$-module, then $f^*\mathcal{N}$ is locally free as an $\mathcal{A}^0_X$-module.

(iii) If $\mathcal{N}$ is flat over $\mathcal{A}^0_Y$ and we have an exact sequence in $\mathrm{Mod}(\mathcal{A}_Y)$, $0 \to \mathcal{N}'' \to \mathcal{N}' \to \mathcal{N} \to 0$, then the sequence $0 \to f^*\mathcal{N}'' \to f^*\mathcal{N}' \to f^*\mathcal{N} \to 0$ is exact.

Proof. The statements are local on X, so that, up to restriction to open subsets, we may assume $X = Y \times \mathbf{R}^n$ and f is the projection. Then we factorize f as a composition of projections with fiber dimension 1, so that we may even assume $X = Y \times \mathbf{R}$ (and $\widehat{X} = Y \times \mathbf{R} \times \mathbf{R}$). We take coordinates $(y_1, \dots, y_m, u, t)$ on $\widehat{X}$ (u is the coordinate in the fiber of f).

With this decomposition of X we define the $\mathcal{A}^0_X$-module $\mathcal{A}_{vert} = \mathcal{A}^0_X \oplus \mathcal{A}^0_X du$. This is a sub-$\mathcal{A}^0_X$-algebra of $\mathcal{A}_X$ (not a sub-dg-algebra); $f^{-1}\mathcal{A}_Y$ is another sub-algebra and the multiplication, $\mathcal{A}_{vert} \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$, is an isomorphism of $\mathcal{A}^0_X$-algebras. This shows that we have an isomorphism of $\mathcal{A}^0_X$-modules, for any dg-$\mathcal{A}_Y$-module $\mathcal{N}'$:

$$\mathcal{A}_{vert} \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}' \xrightarrow{\sim} f^*\mathcal{N}'. \tag{20}$$

Since $\mathcal{A}_{vert}$ is free over $\mathcal{A}^0_X$, this implies (ii). To check that the sequence in (iii) is exact, we consider it as a sequence of $\mathcal{A}^0_X$-modules. Since $\mathcal{N}$ is flat over $\mathcal{A}^0_Y$, isomorphism (20) gives the exactness.

Now we prove (i). By (20) again, $f^*\mathcal{N}$ is identified with the total complex of the double complex with two rows:

$$\begin{array}{ccccc}
\mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i-1} & \longrightarrow & \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i} & \xrightarrow{d_h^{1,i}} & \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i+1} \\
\downarrow & & \downarrow{\scriptstyle d_v^i} & & \downarrow \\
\mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i-1} & \longrightarrow & \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i} & \xrightarrow{d_h^{2,i}} & \mathcal{A}^0_X \otimes_{f^{-1}\mathcal{A}^0_Y} f^{-1}\mathcal{N}^{i+1}
\end{array} \tag{21}$$

where $d_v^i(a \otimes n) = \dfrac{\partial a}{\partial u} \otimes n$, $\quad d_h^{1,i}(a \otimes n) = \sum_k \dfrac{\partial a}{\partial y_k} \otimes dy_k \cdot n + \dfrac{\partial a}{\partial t} \otimes dt \cdot n$ and $d_h^{2,i} = -d_h^{1,i}$. By Lemma 6.2 the i^{th} column is a resolution of $f^{-1}\mathcal{N}^i$. The induced differential on the cohomology of the columns is easily seen to be the differential of $f^{-1}\mathcal{N}$ and (i) follows. □

Lemma 6.4. *Any sheaf of $\mathcal{A}^0_X$-module is soft in the sense of Definition 4.1*

Proof. Let U and Z be respectively open and closed subanalytic subsets of X. Let F be an $\mathcal{A}^0_X$-module and $s \in \Gamma(U; F_Z)$. We may assume $s \in \Gamma(W; F)$ for a subanalytic open set W with $(U \cap Z) \subset W \subset U$. We choose two subanalytic open sets W_1, W_2 such that $(U \cap Z) \subset W_1 \subset \overline{W_1} \subset W_2 \subset \overline{W_2} \subset W$. Since $\mathcal{A}^0_X$ is quasi-injective we may find $\alpha \in \Gamma(X; \mathcal{A}^0_X)$ such that $\alpha = 1$ on W_1 and $\alpha = 0$ on $X \setminus \overline{W_2}$. Then $\alpha s \in \Gamma(W; F)$ extends by 0 on U and $\alpha s = s$ in $\Gamma(U; F_Z)$. It follows that $\Gamma(U; F) \to \Gamma(U; F_Z)$ is surjective, as required. □

Proposition 6.5. *Let $f\colon X \to Y$ be a morphism of real analytic manifolds. For any $\mathcal{M} \in \mathrm{Mod}(\mathcal{A}_X)$, $\mathrm{For}(\mathcal{M}) \in \mathrm{C}^+(\mathbf{C}_{X_{sa}})$ is acyclic with respect to f_* and $f_{!!}$. In particular we have isomorphisms in $\mathrm{D}^+(\mathbf{C}_{Y_{sa}})$, $\mathrm{For}'(f_*(\mathcal{M})) \simeq Rf_*(\mathrm{For}'(\mathcal{M}))$ and $\mathrm{For}'(f_{!!}(\mathcal{M})) \simeq Rf_{!!}(\mathrm{For}'(\mathcal{M}))$.*

Proof. This follows from Lemma 6.4 and Corollary 4.5. □

6.3. **Projection formula.**

Lemma 6.6. *Let $f\colon X \to Y$ be a morphism of analytic manifolds, $\mathcal{M} \in \mathrm{Mod}(\mathcal{A}_X)$, $\mathcal{N} \in \mathrm{Mod}(\mathcal{A}_Y)$. There exists a natural isomorphism in* $\mathrm{Mod}(\mathcal{A}_Y)$*:*

$$\mathcal{N} \otimes_{\mathcal{A}_Y} f_{!!}\mathcal{M} \xrightarrow{\sim} f_{!!}(f^*\mathcal{N} \otimes_{\mathcal{A}_X} \mathcal{M}),$$

whose image in $\mathrm{C}^+(Y_{sa})$ *gives a commutative diagram:*

$$\begin{array}{ccc} \mathcal{N} \otimes_{\mathcal{A}_Y} f_{!!}\mathcal{M} & \xrightarrow{\sim} & f_{!!}(f^*\mathcal{N} \otimes_{\mathcal{A}_X} \mathcal{M}) \\ \uparrow & & \uparrow \\ \mathcal{N} \otimes f_{!!}\mathcal{M} & \xrightarrow{\sim} & f_{!!}(f^{-1}\mathcal{N} \otimes \mathcal{M}), \end{array}$$

where the bottom arrow is the usual projection formula.

Proof. Using (18) and $f^*\mathcal{N} \otimes_{\mathcal{A}_X} \mathcal{M} \simeq f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{A}_Y} \mathcal{M}$ we have the commutative diagram (extending the diagram of the lemma):

$$\begin{array}{ccccccc} \mathcal{N}\otimes\mathcal{A}_Y\otimes f_{!!}\mathcal{M} & \longrightarrow & \mathcal{N}\otimes f_{!!}\mathcal{M} & \longrightarrow & \mathcal{N}\otimes_{\mathcal{A}_Y} f_{!!}\mathcal{M} & \longrightarrow & 0 \\ \downarrow a & & \downarrow b & & \downarrow & & \\ f_{!!}(f^{-1}\mathcal{N}\otimes f^{-1}\mathcal{A}_Y\otimes\mathcal{M}) & \longrightarrow & f_{!!}(f^{-1}\mathcal{N}\otimes\mathcal{M}) & \longrightarrow & f_{!!}(f^{-1}\mathcal{N}\otimes_{f^{-1}\mathcal{A}_Y}\mathcal{M}) & \longrightarrow & 0 \end{array}$$

The top row of this diagram is exact by definition of the tensor product, as well as the bottom row, before we take the image by $f_{!!}$. But any complex of the type $\mathcal{P} \otimes \mathcal{M}$ is an $\mathcal{A}^0_X$-module, because $\mathcal{M}$ is; hence it is $f_{!!}$-acyclic by Lemma 6.4 and Corollary 4.5. It follows that the bottom row is exact. Now, the vertical arrows a and b are isomorphisms in view of the classical projection formula. Hence so is the morphism of the lemma. □

6.4. **Base change.** We consider a Cartesian square of real analytic manifolds

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

We have the usual base change formula in $\mathrm{Mod}(\mathbf{C}_{Y'_{sa}})$ or $\mathrm{C}^+(\mathbf{C}_{Y'_{sa}})$, $f^{-1}g_{!!} \simeq g'_{!!}f'^{-1}$ (and its derived version in $\mathrm{D}^+(\mathbf{C}_{Y'_{sa}})$, $f^{-1}Rg_{!!} \simeq Rg'_{!!}f'^{-1}$).

Lemma 6.7. *Let $\mathcal{N}$ a dg-$\mathcal{A}_{Y'}$-module. There exists a natural morphism*

$$f^*g_{!!}\mathcal{N} \to g'_{!!}f'^*\mathcal{N} \tag{22}$$

of dg-$\mathcal{A}_X$-modules, whose image in the category of complexes $\mathrm{C}^+(X_{sa})$ *gives a commutative diagram:*

$$\begin{array}{ccc} f^*g_{!!}\mathcal{N} & \longrightarrow & g'_{!!}f'^*\mathcal{N} \\ \uparrow & & \uparrow \\ f^{-1}g_{!!}\mathcal{N} & \xrightarrow{\sim} & g'_{!!}f'^{-1}\mathcal{N}, \end{array}$$

where the bottom arrow is the usual base change isomorphism.

Moreover, if f is an immersion and g is smooth, then (22) *is an isomorphism.*

Proof. The morphism is defined by the following composition:

$$f^*g_{!!}\mathcal{N} \simeq \mathcal{A}_X \otimes_{f^{-1}\mathcal{A}_Y} g'_{!!}f'^{-1}\mathcal{N} \simeq g'_{!!}(g'^{-1}\mathcal{A}_X \otimes_{g'^{-1}f^{-1}\mathcal{A}_Y} f'^{-1}\mathcal{N}) \xrightarrow{\varphi} g'_{!!}(\mathcal{A}_{X'} \otimes_{f'^{-1}\mathcal{A}_{Y'}} f'^{-1}\mathcal{N}) = g'_{!!}f'^*\mathcal{N},$$

where the first isomorphism uses the classical base change formula (for complexes), and the second one the classical projection formula. Morphism φ is induced by $g'^\sharp$.

Now we show that φ is an isomorphism when f is an immersion and g is smooth. It is enough to show that

$$g'^{-1}\mathcal{A}_X \otimes_{g'^{-1}f^{-1}\mathcal{A}_Y} f'^{-1}\mathcal{N} \simeq \mathcal{A}_{X'} \otimes_{f'^{-1}\mathcal{A}_{Y'}} f'^{-1}\mathcal{N}. \tag{23}$$

This is a local statement on X' so that we may as well assume that f is an embedding and $X' = X \times Z$, $Y' = Y \times Z$ for some manifold Z. We may also assume that X is given by equations $y_i = 0$, $i = 1 \dots, d$ in Y. Then $\mathcal{A}_X$ is the quotient of $f^{-1}\mathcal{A}_Y$ by the ideal generated by y_i, dy_i, $i = 1 \dots, d$. The same holds for X' and we have the presentations:

$$f^{-1}(\mathcal{A}_Y)^{2d} \xrightarrow{(y_1,\dots,dy_d)} f^{-1}(\mathcal{A}_Y) \to \mathcal{A}_X \to 0,$$
$$f'^{-1}(\mathcal{A}_{Y'})^{2d} \xrightarrow{(y_1,\dots,dy_d)} f'^{-1}(\mathcal{A}_{Y'}) \to \mathcal{A}_{X'} \to 0.$$

Since the tensor product is right exact, the images of these exact sequences by $g'^{-1}(\cdot) \otimes_{g'^{-1}f^{-1}\mathcal{A}_Y} f'^{-1}\mathcal{N}$ and $(\cdot) \otimes_{f'^{-1}\mathcal{A}_{Y'}} f'^{-1}\mathcal{N}$ give the same presentations of both sides of (23), which shows that they are isomorphic. □

6.5. **Complex manifolds.** Now we assume that X is a complex analytic manifold, of dimension d_X^c over $\mathbf{C}$; we denote by $\overline{X}$ the complex conjugate manifold and $X_{\mathbf{R}}$ the underlying real analytic manifold. We recall that t is the coordinate on $\widehat{X}_{\mathbf{R}}$ given by the projection $\tau_{X_{\mathbf{R}},2}\colon \widehat{X}_{\mathbf{R}} \to \mathbf{R}$, and that we have the decomposition $d = d_1 + d_2$ of the differential of $\mathcal{A}_{X_{\mathbf{R}}}$ ($d_2(\omega) = \partial\omega/\partial t\, dt$). We consider the complex of "tempered holomorphic functions", $\mathcal{O}_X^t \in \mathrm{D}^b(\mathbf{C}_{X_{sa}})$, defined as the Dolbeault complex with tempered coefficients:

$$\mathcal{O}_X^t = \mathrm{R}\mathcal{H}\mathrm{om}_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X_{\mathbf{R}}}) \simeq 0 \to \Omega^{t,0,0}_{X_{\mathbf{R}}} \xrightarrow{\bar\partial} \Omega^{t,0,1}_{X_{\mathbf{R}}} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{t,0,d_X^c}_{X_{\mathbf{R}}}, \tag{24}$$

where $\Omega^{t,i,j}_{X_{\mathbf{R}}}$ denotes as usual the forms of type (i,j). The product of forms induces a morphism $\mathcal{O}_X^t \otimes \mathcal{O}_X^t \to \mathcal{O}_X^t$ in $\mathrm{D}^b(\mathbf{C}_{X_{sa}})$. In degree 0, $H^0(\mathcal{O}_X^t)$ is a subalgebra of $\rho_*\mathcal{O}_X$.

Definition 6.8. We let $\Omega^{t,i,j}_{\widehat{X_{\mathbf{R}}}} = \mathcal{C}^{\infty,t}_{\widehat{X_{\mathbf{R}}}} \tau^*_{X_{\mathbf{R}},1}(\Omega^{t,i,j}_{X_{\mathbf{R}}})$ be the sub-$\mathcal{C}^{\infty,t}_{\widehat{X_{\mathbf{R}}}}$-module of $\Omega^{t,i+j}_{\widehat{X_{\mathbf{R}}}}$ generated by the forms of type (i,j) coming from $X_{\mathbf{R}}$.

We define $\mathcal{A}^{i,j}_{X_{\mathbf{R}}} = i^{-1}_{X_{\mathbf{R}}}\Gamma_{X^+_{\mathbf{R}}}\Omega^{t,i,j}_{\widehat{X_{\mathbf{R}}}}$. This is a sub-$\mathcal{A}^0_{X_{\mathbf{R}}}$-module of $\mathcal{A}^{i+j}_{X_{\mathbf{R}}}$ and we have the decomposition $\mathcal{A}^k_{X_{\mathbf{R}}} = \bigoplus_{i+j=k} \mathcal{A}^{i,j}_{X_{\mathbf{R}}} \oplus \bigoplus_{i+j=k-1} \mathcal{A}^{i,j}_{X_{\mathbf{R}}} dt$. The operators $\partial, \bar\partial$ on $\Omega^t_{X_{\mathbf{R}}}$ induce a decomposition of the differential of $\mathcal{A}_{X_{\mathbf{R}}}$, $d = \partial + \bar\partial + d_2$.

We let $J_X < \mathcal{A}_{X_{\mathbf{R}}}$ be the differential ideal generated by $\mathcal{A}^{1,0}_{X_{\mathbf{R}}}$ and introduce the dg-$\mathcal{A}_{X_{\mathbf{R}}}$-module $\mathbb{O}_X = \mathcal{A}_{X_{\mathbf{R}}}/J_X$. As a quotient by a differential ideal, $\mathbb{O}_X$ inherits a structure of dg-algebra. We note the obvious inclusions $\rho_!\mathcal{O}_X \subset \rho_!\mathcal{C}^{\infty,t}_X \subset \mathcal{A}^0_{X_{\mathbf{R}}}$ and we define, for two complex analytic manifolds, X, Y:

$$\mathbb{O}_X^{(i)} = \mathbb{O}_X \otimes_{\rho_!\mathcal{O}_X} \rho_!\mathcal{O}_X^{(i)}, \qquad \mathbb{O}^{(p,q)}_{X\times Y} = \mathbb{O}_{X\times Y} \otimes_{\rho_!\mathcal{O}_{X\times Y}} \rho_!\mathcal{O}^{(p,q)}_{X\times Y},$$

where $\mathcal{O}_X^{(i)}$ denotes the holomorphic i-forms on X and $\mathcal{O}_{X\times Y}^{(p,q)} = \mathcal{O}_{X\times Y} \otimes_{(\mathcal{O}_X \boxtimes \mathcal{O}_Y)} (\mathcal{O}_X^{(p)} \boxtimes \mathcal{O}_Y^{(q)})$.

Proposition 6.9. *(i) We have an isomorphism of complexes between $\mathbb{O}_X$ and*

$$\mathcal{A}^{0,0}_{X_{\mathbf{R}}} \to \big(\mathcal{A}^{0,1}_{X_{\mathbf{R}}} \oplus \mathcal{A}^{0,0}_{X_{\mathbf{R}}} dt\big) \to \big(\mathcal{A}^{0,2}_{X_{\mathbf{R}}} \oplus \mathcal{A}^{0,1}_{X_{\mathbf{R}}} dt\big) \to \cdots \to \mathcal{A}^{0,d_X^c}_{X_{\mathbf{R}}} dt,$$

with differential $\bar\partial + d_2$.

(ii) $\mathbb{O}_X^{(d_X^c)}[-d_X^c]$ is isomorphic to the differential ideal of $\mathcal{A}_{X_{\mathbf{R}}}$:

$$\mathcal{A}^{d_X^c,0}_{X_{\mathbf{R}}} \to \big(\mathcal{A}^{d_X^c,1}_{X_{\mathbf{R}}} \oplus \mathcal{A}^{d_X^c,0}_{X_{\mathbf{R}}} dt\big) \to \cdots \to \mathcal{A}^{d_X^c,d_X^c}_{X_{\mathbf{R}}} dt.$$

Moreover we have a decomposition $\mathcal{A}_{X_{\mathbf{R}}} \simeq \mathbb{O}_X^{(d_X^c)}[-d_X^c] \oplus M_X$ in free $\mathcal{A}^0_{X_{\mathbf{R}}}$-modules.

(iii) There exist a natural isomorphism $\mathcal{O}_X^t \simeq \mathbb{O}_X$, in $\mathrm{D}^b(\mathbf{C}_{(X_{\mathbf{R}})_{sa}})$, which commutes with the products $\mathcal{O}_X^t \otimes \mathcal{O}_X^t \to \mathcal{O}_X^t$ and $\mathbb{O}_X \otimes \mathbb{O}_X \to \mathbb{O}_X$. We also have $\mathcal{O}_{X\times Y}^{t(p,q)} \simeq \mathbb{O}_{X\times Y}^{(p,q)}$, in $\mathrm{D}^b(\mathbf{C}_{(X_{\mathbf{R}}\times Y_{\mathbf{R}})_{sa}})$.

Proof. (i), (ii) The decomposition of $\mathcal{A}^k_{X_{\mathbf{R}}}$ given in Definition 6.8 yields projections $\mathcal{A}^k_{X_{\mathbf{R}}} \to \mathcal{A}^{0,k}_{X_{\mathbf{R}}} \oplus \mathcal{A}^{0,k-1}_{X_{\mathbf{R}}} dt$. The sum of these projections is a surjective morphism from $\mathcal{A}_{X_{\mathbf{R}}}$ to the complex of the proposition and we see that its kernel is J_X. Assertion (ii) follows from (i).

(iii) We use the isomorphism $\mathcal{O}_X^t \simeq 0 \to \Omega^{t,0,0}_{X_{\mathbf{R}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{t,0,d_X^c}_{X_{\mathbf{R}}} \to 0$. The exact sequences

$$0 \to \Omega^{t,0,j}_{X_{\mathbf{R}}} \to \mathcal{A}^{0,j}_{X_{\mathbf{R}}} \xrightarrow{\alpha \mapsto (\partial\alpha/\partial t)dt} \mathcal{A}^{0,j}_{X_{\mathbf{R}}} dt \to 0,$$

combine into an isomorphism between $\mathcal{O}_X^t$ and the complex given in (i). This proves the first isomorphism. The second one follows from the first and the definitions. □

For a morphism of complex analytic manifolds $f\colon X \to Y$, we have an integration morphism in the derived category $Rf_!\mathcal{O}_X^{(d_X^c)}[d_X^c] \to \mathcal{O}_Y^{(d_Y^c)}[d_Y^c]$ and its tempered version $Rf_{!!}\mathcal{O}_X^{t(d_X^c)}[d_X^c] \to \mathcal{O}_Y^{t(d_Y^c)}[d_Y^c]$. By adjunction between $Rf_{!!}$ and $f^!$ we obtain $\mathcal{O}_X^{t(d_X^c)}[d_X^c] \to f^!\mathcal{O}_Y^{t(d_Y^c)}[d_Y^c]$.

When f is a submersion we have $f^! \simeq f^{-1}[2(d_X^c - d_Y^c)]$ (note that the manifolds are complex, hence oriented) and our last morphism becomes:

$$\mathcal{O}_X^{t(d_X^c)}[-d_X^c] \to f^{-1}\mathcal{O}_Y^{t(d_Y^c)}[-d_Y^c]. \tag{25}$$

Proposition 6.10. *For a submersion of complex analytic manifolds $f\colon X \to Y$, the embeddings, for $Z = X, Y$, $\mathbb{O}_Z^{(d_Z^c)}[-d_Z^c] \subset \mathcal{A}_{Z_{\mathbf{R}}}$ of Proposition 6.9 (ii) induce a morphism of dg-$\mathcal{A}_{X_{\mathbf{R}}}$-modules*

$$\mathbb{O}_X^{(d_X^c)}[-d_X^c] \to f^*\mathbb{O}_Y^{(d_Y^c)}[-d_Y^c], \tag{26}$$

which represents (25) *through the isomorphism of Proposition 6.9 (iii).*

Proof. By Proposition 6.9 we have a decomposition $\mathcal{A}_{Y_{\mathbf{R}}} \simeq \mathbb{O}_Y^{(d_Y^c)}[-d_Y^c] \oplus M_Y$ in free $\mathcal{A}^0_{Y_{\mathbf{R}}}$-modules; hence the quotient $\mathcal{A}_{Y_{\mathbf{R}}}/\mathbb{O}_Y^{(d_Y^c)}[-d_Y^c]$ is free over $\mathcal{A}^0_{Y_{\mathbf{R}}}$ and Proposition 6.3 implies that the morphism $f^*\mathbb{O}_Y^{(d_Y^c)}[-d_Y^c] \to f^*\mathcal{A}_{Y_{\mathbf{R}}} \simeq \mathcal{A}_{X_{\mathbf{R}}}$ is injective. Hence we just have to check the inclusion of ideals of $\mathcal{A}_{X_{\mathbf{R}}}$: $\mathbb{O}_X^{(d_X^c)}[-d_X^c] \subset f^*\mathbb{O}_Y^{(d_Y^c)}[-d_Y^c]$.

This is a local problem on X and we may assume $X = Y \times Z$. As an $\mathcal{A}^0_{X_{\mathbf{R}}}$-module, $\mathbb{O}_X^{(d^c_X)}[-d^c_X]$ decomposes into summands $\mathcal{A}^{d^c_X,i}_{X_{\mathbf{R}}}$ and $\mathcal{A}^{d^c_X,i}_{X_{\mathbf{R}}}dt$. Now any form of type (d^c_X, i) on $X = Y \times Z$ is a sum of products of forms of types (d^c_Y, j) and (d^c_Z, k), with $j + k = i$. In particular $\mathbb{O}_X^{(d^c_X)}[-d^c_X]$ is in the image of $(\mathcal{A}_{X_{\mathbf{R}}} \otimes f^{-1}\mathbb{O}_Y^{(d^c_Y)}[-d^c_Y]) \to \mathcal{A}_{X_{\mathbf{R}}}$. □

Corollary 6.11. *With the hypothesis of Proposition 6.10, the integration morphism of Definition 5.3 induces a morphism of dg-$\mathcal{A}_{Y_{\mathbf{R}}}$-modules*

$$f_{!!}\mathbb{O}_X^{(d^c_X)}[d^c_X] \to \mathbb{O}_Y^{(d^c_Y)}[d^c_Y], \tag{27}$$

which represents the integration morphism $Rf_{!!}\mathcal{O}_X^{t(d^c_X)}[d^c_X] \to \mathcal{O}_Y^{t(d^c_Y)}[d^c_Y]$.

Proof. Morphism (26), the projection formula and the integration morphism give:

$$\begin{aligned} f_{!!}\mathbb{O}_X^{(d^c_X)}[d^c_X] &\to f_{!!}f^*\mathbb{O}_Y^{(d^c_Y)}[2d^c_X - d^c_Y] \\ &\simeq \mathbb{O}_Y^{(d^c_Y)}[d^c_Y] \otimes_{\mathcal{A}_{Y_{\mathbf{R}}}} f_{!!}\mathcal{A}_{X_{\mathbf{R}}}[2(d^c_X - d^c_Y)] \\ &\to \mathbb{O}_Y^{(d^c_Y)}[d^c_Y]. \end{aligned}$$

We define (27) as the composition of these arrows. The integration morphism for $\mathcal{O}^t_X$ is also defined by integration of forms using the Dolbeault complex. It is nothing but the restriction of the integration morphism for $\mathcal{A}_{X_{\mathbf{R}}}$ to a subcomplex, so that it coincides with (27). □

In section 10 we need the following composition of kernels. Let X, Y, Z be three complex analytic manifolds and q_{ij} the projection from their product to the $i^{th} \times j^{th}$ factors. The product of $\mathcal{O}_Y$ and the integration morphism give a convolution product: $Rq_{13!}(q_{12}^{-1}\mathcal{O}_{X\times Y}^{(0,d^c_Y)}[d^c_Y] \otimes q_{23}^{-1}\mathcal{O}_{Y\times Z}^{(0,d^c_Z)}[d^c_Z]) \to \mathcal{O}_{X\times Z}^{(0,d^c_Z)}[d^c_Z]$. We can also define a tempered version of this convolution, and in fact we can even realize this tempered convolution product at the level of complexes, using the above sheaf $\mathbb{O}_{X\times Y}^{(0,d^c_Y)}$. As in Proposition 6.10, we rather define its "adjoint" morphism as the following composition:

$$\begin{aligned} q_{12}^*\mathbb{O}_{X\times Y}^{(0,d^c_Y)}[-d^c_Y] \otimes_{\mathcal{A}} q_{23}^*\mathbb{O}_{Y\times Z}^{(0,d^c_Z)}[-d^c_Z] &\to \mathbb{O}_{X\times Y\times Z}^{(0,d^c_Y,d^c_Z)}[-d^c_Y - d^c_Z] \\ &\to q_{13}^*\mathbb{O}_{X\times Z}^{(0,d^c_Z)}[-d^c_Z], \end{aligned} \tag{28}$$

where the first morphism is induced by the product $\mathbb{O}_Y \otimes \mathbb{O}_Y \otimes_{\rho_!\mathcal{O}_Y} \rho_!\mathcal{O}_Y^{(d^c_Y)} \to \mathbb{O}_Y \otimes_{\rho_!\mathcal{O}_Y} \rho_!\mathcal{O}_Y^{(d^c_Y)}$ and the second morphism is induced by morphism (26).

7. Microlocalization functor

In this section we recall the definition of the microlocalization functor μ introduced in [9]. For a manifold X this is a functor, μ_X, from $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))$ to $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*X}))$ given by a kernel $L_X \in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{X\times T^*X}))$.

We define analogs of this kernel and of the microlocalization functor in the framework of $\mathcal{A}$-modules. We check that, in the case we are interested in, this gives a resolution of $\mu_X F$, and that it has a functorial behavior with respect to the usual operations.

In fact, with the definition of [9], the construction of the external tensor product is not so straightforward. For this reason we define another kernel for which the

tensor product is easy and which coincides with the kernel of [9] outside the zero section.

7.1. **Microlocalization functor in the derived category.** In [9] the authors define a kernel associated to the following data: let X be a manifold, $Z \subset X$ a closed submanifold and σ a 1-form defined on Z, i.e. σ is a section of the bundle $Z \times_X T^*X \to Z$. To simplify the exposition, we make the following assumption which will be satisfied in our case:

$$\forall z \in Z,\ \sigma_z \text{ vanishes on } T_z Z. \tag{29}$$

Hence σ induces a section of $Z \times_X T^*_Z X \to Z$ and we may define:

$$P_\sigma = \{(x,v) \in T_Z X;\ \langle v, \sigma(x)\rangle \geq 0\}.$$

Hence P_σ is a subset of $T_Z X$, viewed itself as a subset of the normal deformation of Z in X, $\tilde{X}_Z$. We recall that $\tilde{X}_Z$ and the projection $p\colon \tilde{X}_Z \to X$ are given in local coordinates as follows. We choose coordinates $(x_1, \dots, x_n)$ on X such that Z is given by $x_i = 0$, $i = 1, \dots, d$. This gives coordinates (x_i, τ) on $\tilde{X}_Z$ and $p(x_i, \tau) = (\tau x_1, \dots, \tau x_d, x_{d+1}, \dots, x_n)$. The normal bundle $T_Z X$ is embedded in $\tilde{X}_Z$ as the submanifold $\{\tau = 0\}$ and we define $\Omega = \{\tau > 0\}$.

$$\begin{array}{ccccccc} P_\sigma & \hookrightarrow & T_Z X & \hookrightarrow & \tilde{X}_Z & \hookleftarrow & \Omega \\ & & \downarrow & & \downarrow p & \swarrow & \\ & & Z & \hookrightarrow & X & & \end{array}$$

We will often restrict ourself outside the zero set of σ and we set $T_\sigma = \{z \in Z;\ \sigma_z$ vanishes on $T_z X\}$.

Definition 7.1. Under hypothesis (29), the kernel associated to these data is the object of $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))$ (recall that, for $i\colon Z \hookrightarrow X$, we write $\omega_{Z|X}$ instead of $i_*\omega_{Z|X}$):

$$\mathcal{L}_\sigma = \mathcal{L}_\sigma(Z, X) = Rp_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}}) \otimes \beta_X(\omega_{Z|X}^{\otimes -1}).$$

We recall that $\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) = \text{“}\varinjlim\text{”}_W \mathbf{C}_{\overline{W}}$, W running over the open neighborhoods of P_σ in $\tilde{X}_Z$. Since “$\varinjlim$” commutes with $\otimes$ we obtain

$$\mathcal{L}_\sigma = Rp_{!!}(\text{“}\varinjlim_W\text{”}\, \mathbf{C}_{\overline{W} \cap \overline{\Omega}}) \otimes \beta_X(\omega_{Z|X}^{\otimes -1}), \qquad W \text{ open in } \tilde{X}_Z,\ P_\sigma \subset W.$$

We also notice that $Rp_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}})$ is supported on Z (i.e. its restriction outside Z is 0). Hence taking the tensor product with $\beta_X(\omega_{Z|X}^{\otimes -1})$ reduces locally to a shift by the codimension of Z.

In Proposition 1.2.11 of [9] we also have a description of $\mathcal{L}_\sigma$ outside the zero set of σ, T_σ:

$$Rp_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}})|_{X \setminus T_\sigma} \simeq Rp_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\Omega})|_{X \setminus T_\sigma} \simeq \text{“}\varinjlim_U\text{”}\, \mathbf{C}_U, \tag{30}$$

where U runs over the open subsets of $X \setminus T_\sigma$ such that the cone of U along $Z \setminus T_\sigma$ doesn't intersect P_σ outside the zero section. In particular the complexes in (30) are concentrated in degree 0:

$$Rp_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}})|_{X \setminus T_\sigma} \simeq p_{!!}(\beta_{\tilde{X}_Z}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}})|_{X \setminus T_\sigma}. \tag{31}$$

When considering resolutions of $\mathcal{L}_\sigma$ by $\mathcal{A}$-modules, it will be convenient to use the following different formulation, which is equivalent outside the zero set of σ. First,

using the embedding of categories $\mathbf{I}_\tau\colon \mathrm{Mod}(\mathbf{C}_{X_{sa}}) \simeq \mathbf{I}_{\mathbf{R}-c}(\mathbf{C}_X) \to \mathbf{I}(\mathbf{C}_X)$ we have $\mathcal{L}_\sigma \simeq \mathbf{I}_\tau(\mathcal{L}^{sa}_\sigma)$, where $\mathcal{L}^{sa}_\sigma \in \mathrm{D}^b(\mathbf{C}_{X_{sa}})$ is given by

$$\mathcal{L}^{sa}_\sigma = Rp_{!!}(\rho_{\tilde{X}_Z!}(\mathbf{C}_{P_\sigma}) \otimes \mathbf{C}_{\overline{\Omega}}) \otimes \rho_{X!}(\omega^{\otimes -1}_{Z|X}) \tag{32}$$

$$\simeq Rp_{!!}(R\Gamma_\Omega(\rho_{\tilde{X}_Z!}(\mathbf{C}_{P_\sigma}))) \otimes \rho_{X!}(\omega^{\otimes -1}_{Z|X}), \tag{33}$$

where the second isomorphism follows from (5) and $\mathbf{C}_{\overline{\Omega}} \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathbf{C}_\Omega, \mathbf{C}_{\tilde{X}_Z})$.

Definition 7.2. For a real analytic manifold Y and $T \subset Y$ a locally closed analytic subset we introduce the notation $K_T = \rho_{Y!}\mathbf{C}_{\overline{T}} \otimes \varinjlim_W \mathbf{C}_{Y\setminus\overline{W}}$, where W runs over the open neighborhoods of T in Y. We note that K_T has support in the boundary $\overline{T} \setminus T$.

We let P^0_σ be the relative interior of P_σ, i.e. $P^0_\sigma = \{(x,v) \in T_Z X;\ \langle v, \sigma(x)\rangle > 0\}$ and we define $\mathcal{L}^0_\sigma \in \mathrm{D}^b(\mathbf{C}_{X_{sa}})$ by:

$$\mathcal{L}^0_\sigma = Rp_{!!}(R\Gamma_\Omega K_{P^0_\sigma}) \otimes \rho_{X!}(\omega^{\otimes -1}_{Z|X}).$$

Lemma 7.3. *We let (X, Z, σ) be a kernel data satisfying hypothesis* (29) *and we assume that σ doesn't vanish.*

(i) We have $R\Gamma_\Omega K_{P^0_\sigma} \simeq \mathbf{C}_{\overline{\Omega}} \otimes K_{P^0_\sigma}$.

(ii) The natural morphism $K_{P^0_\sigma} \to \rho_{\tilde{X}_Z!}(\mathbf{C}_{P_\sigma})$ induces an isomorphism $\mathcal{L}^0_\sigma \to \mathcal{L}^{sa}_\sigma$ in $\mathrm{D}^b(\mathbf{C}_{X_{sa}})$.

Proof. (i) By definition $K_{P^0_\sigma} \simeq \varinjlim_{W,W^0} \mathbf{C}_{\overline{W}\setminus\overline{W^0}}$, where W and W^0 run over the open neighborhoods of P_σ and P^0_σ in $\tilde{X}_Z$. By formula (6) we may commute the limit with $R\Gamma_\Omega$ so that $R\Gamma_\Omega K_{P^0_\sigma} \simeq \varinjlim_{W,W^0} R\Gamma_\Omega \mathbf{C}_{\overline{W}\setminus\overline{W}^0}$. Our situation is locally isomorphic to $\tilde{X}_Z \simeq \mathbf{R}^n$, $\Omega \simeq \mathbf{R}^{n-1} \times \mathbf{R}_{>0}$ and $P_\sigma \simeq \mathbf{R}^{n-2} \times \mathbf{R}_{\geq 0} \times \{0\}$. Hence, choosing for example

$$W = \{|x_n| < \varepsilon,\ x_{n-1} > -\varepsilon\}, \qquad W^0 = \{|x_n| < \varphi(x_1, \dots, x_{n-1})\},$$

for $\varepsilon > 0$ and subanalytic continuous functions φ on $\mathbf{R}^{n-2} \times \mathbf{R}_{>0}$, we may assume that our W, W^0 satisfy $R\Gamma_\Omega(\mathbf{C}_{\overline{W}}) \simeq \mathbf{C}_{\overline{W}\cap\overline{\Omega}}$ (and the same with W^0 instead of W) and this gives the desired isomorphism.

(ii) We define $F = \varinjlim_{W^0} \mathbf{C}_{\overline{W^0}}$, where W^0 runs over the open neighborhoods of P^0_σ in $\tilde{X}_Z$. Hence we have an exact sequence $0 \to K_{P_\sigma} \to \rho_{\tilde{X}_Z!}(\mathbf{C}_{P_\sigma}) \to F \to 0$ and it is enough to show that $Rp_{!!}(R\Gamma_\Omega F) = 0$.

As in (i) we have $R\Gamma_\Omega F \simeq \varinjlim_{W^0} \mathbf{C}_{\overline{W^0}\cap\overline{\Omega}}$. We deduce that $Rp_{!!}(R\Gamma_\Omega F) \simeq \varinjlim_{W^0,U} Rp_* \mathbf{C}_{\overline{W^0}\cap\overline{\Omega}\cap U}$, where W^0 runs on the same set as above and U runs over the open subsets of $\tilde{X}_Z$ with compact closure. Since p_* commutes with ρ_* we are reduced to a computation with sheaves on topological spaces.

For $x \in X \setminus Z$, x near Z, and U big enough, $p^{-1}(x) \cap \overline{W^0} \cap \overline{\Omega} \cap U$ is a union of intervals of the line, all of them compact except at most one which is homeomorphic to $[0,1[$. When we take the limit over W^0 and U only the last one has a non-zero contribution in the morphisms $\mathbf{C}_{p^{-1}(x)\cap\overline{W^0}\cap\overline{\Omega}\cap U} \to \mathbf{C}_{p^{-1}(x)\cap\overline{W'^0}\cap\overline{\Omega}\cap U'}$. In the same way, for $x \in Z$, since $P_\sigma \subset T_Z X$ is locally homeomorphic to a closed half plane, we may assume that $p^{-1}(x) \cap \overline{W^0} \cap \overline{\Omega} \cap U$ is homeomorphic to an half ball $\{|x| < 1, x_1 \geq 0\}$.

Since $R\Gamma(\mathbf{R}; \mathbf{C}_{[0,1[}) = 0$ and $R\Gamma(\mathbf{R}^{n-1}; \mathbf{C}_{\{|x|<1, x_1\geq 0\}}) = 0$, we deduce that our direct image vanishes. □

Now for any manifold X, the cotangent bundle T^*X is endowed with a canonical 1-form, say ω_X. We set $\mathfrak{X} = X \times T^*X$ and $\mathfrak{Z} = X \times_X T^*X \simeq T^*X$ and consider the section $\sigma_X \colon X \times_X T^*X \to T^*X \times T^*(T^*X)$ defined by $\sigma_X = (-\mathrm{id}, \omega_X)$, i.e. in local coordinates

$$\sigma_X(x, x, \xi) = ((x; -\xi), \omega_X(x, \xi)) = ((x; -\xi), (x, \xi; \xi, 0)).$$

Hence hypothesis (29) is satisfied for the data $(\mathfrak{X}, \mathfrak{Z}, \sigma_X)$.

Definition 7.4. With the above notations, we set $L_X = \mathcal{L}_{\sigma_X}(\mathfrak{Z}, \mathfrak{X})$ so that $L_X \in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{X \times T^*X}))$. We denote by $p_1 \colon X \times T^*X \to X$, $p_2 \colon X \times T^*X \to T^*X$ the projections. The microlocalization is the functor

$$\mu_X \colon \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X)) \to \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*X})), \qquad F \mapsto L_X \circ F = Rp_{2!!}(L_X \otimes p_1^{-1}F).$$

We note that σ_X doesn't vanish outside the zero section of T^*X so that we can use $\mathcal{L}^0_{\sigma_X}(\mathfrak{Z}, \mathfrak{X})$ instead of L_X when we consider $\mu_X F|_{\dot{T}^*X}$.

7.2. Microlocalization functor for $\mathcal{A}$-modules.

Definition 7.5. For a real analytic manifold Y and $T \subset Y$ a locally closed subset we introduce the notation $\mathcal{B}_T = \mathcal{A}_Y \otimes K_T$, where K_T is given in Definition 7.2. Let (X, Z, σ) be a kernel data satisfying hypothesis (29). We define $\mathcal{L}^{\mathcal{A}}_\sigma \in \mathrm{Mod}(\mathcal{A}_X)$ by

$$\mathcal{L}^{\mathcal{A}}_\sigma = \mathcal{L}^{\mathcal{A}}_\sigma(Z, X) = p_{!!}(\Gamma_\Omega(\mathcal{B}_{P^0_\sigma})) \otimes \rho_{X!}(\omega^{\otimes -1}_{Z|X}).$$

Remark 7.6. For $U \subset \tilde{X}_Z$ a subanalytic open subset, a section of $\Gamma_\Omega \mathcal{B}^i_{P^0_\sigma}$ on U is given by the following data: open neighborhoods W of P_σ and W^0 of P^0_σ in $\tilde{X}_Z$, and a section $s \in \mathcal{A}^i_{\tilde{X}_Z}(\Omega \cap W \cap U)$ such that $s|_{\Omega \cap W^0 \cap U} = 0$. Actually the definition would require that s be defined on a neighborhood of $\overline{W}$ and that $(\mathrm{supp}\, s) \cap \overline{W^0} = \emptyset$. But, up to shrinking W and W^0, this amounts to the above statement.

Lemma 7.7. *The complex $\mathcal{L}^{\mathcal{A}}_\sigma$ consists of quasi-injective sheaves of $\mathbf{C}_{X_{sa}}$-vector spaces. We have a natural isomorphism $\mathcal{L}^0_\sigma \simeq \mathcal{L}^{\mathcal{A}}_\sigma$ in $\mathrm{D}^+(\mathbf{C}_{X_{sa}})$. Hence, if σ doesn't vanish, $\mathcal{L}_\sigma \simeq \mathcal{L}^{\mathcal{A}}_\sigma$ in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_X))$.*

Proof. We recall the definition $\mathcal{L}^0_\sigma = Rp_{!!}(R\Gamma_\Omega K_{P^0_\sigma}) \otimes \rho_{X!}(\omega^{\otimes -1}_{Z|X})$.

Since $\mathcal{A}_{\tilde{X}_Z}$ is a quasi-injective resolution of $\mathbf{C}_{(\tilde{X}_Z)_{sa}}$, we have $\mathcal{B}_{P^0_\sigma} \simeq K_{P^0_\sigma}$ in $\mathrm{D}^+(\mathbf{C}_{X_{sa}})$. The complex $\mathcal{B}_{P^0_\sigma}$ consists of $\mathcal{A}^0$-modules, hence soft sheaves. It follows from Corollary 4.5 that $R\Gamma_\Omega \mathcal{B}_{P^0_\sigma} \simeq \Gamma_\Omega \mathcal{B}_{P^0_\sigma}$. This last complex also is formed by $\mathcal{A}^0$-modules, hence $p_{!!}$-acyclic sheaves, and we deduce the isomorphism of the lemma.

Let us now check that $\Gamma_\Omega \mathcal{B}_{P^0_\sigma}$ consists of quasi-injective sheaves. Let $U \subset \tilde{X}_Z$ be a subanalytic open subset; a section of $\Gamma_\Omega \mathcal{B}^i_{P^0_\sigma}$ on U is given by W, W^0, $s \in \mathcal{A}^i_{\tilde{X}_Z}(\Omega \cap W \cap U)$ as in Remark 7.6. The condition on s says that we may extend s to a section s' of $\mathcal{A}^i_{\tilde{X}_Z}$ on $(\Omega \cap W \cap U) \cup W^0$, with $s'|_{W^0} = 0$. By Proposition 5.9 we may extend s' to $\tilde{X}_Z$, and this gives the quasi-injectivity of $\Gamma_\Omega \mathcal{B}^i_{P^0_\sigma}$. Since $p_{!!}$ sends quasi-injective sheaves to quasi-injective sheaves, we obtain the first assertion.

The last assertion follows from Lemma 7.3. □

Now we can define the microlocalization functor for $\mathcal{A}$-modules. We keep the notations introduced before Definition 7.4: for a manifold X we have the kernel data $(\mathfrak{X}, \mathfrak{Z}, \sigma_X)$.

Definition 7.8. With the above notations, we set $L_X^{\mathcal{A}} = \mathcal{L}^{\mathcal{A}}_{\sigma_X}(\mathfrak{Z}, \mathfrak{X})$ so that $L_X^{\mathcal{A}} \in \mathrm{Mod}(\mathcal{A}_{X\times T^*X})$. The microlocalization is the functor

$$\mu_X^{\mathcal{A}} \colon \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(\mathcal{A}_{T^*X}), \qquad F \mapsto L_X^{\mathcal{A}} \circ F = p_{2!!}(L_X^{\mathcal{A}} \otimes_{\mathcal{A}_{\mathfrak{X}}} p_1^* F).$$

For $F \in \mathrm{Mod}(\mathcal{A}_X)$ we have a natural morphism in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*X}))$:

$$\mu_X(\mathrm{For}'_X(F)) \to \mathrm{For}'_{T^*X}(\mu_X^{\mathcal{A}}(F)), \tag{34}$$

defined by the composition of morphisms in the derived category (we don't write the functors For) on $\dot{T}^*X$:

$$\mu_X F \simeq Rp_{2!!}(L_X^{\mathcal{A}} \otimes p_1^{-1}F) \to Rp_{2!!}(L_X^{\mathcal{A}} \otimes_{\mathcal{A}_{\mathfrak{X}}} p_1^* F) \leftharpoonup p_{2!!}(L_X^{\mathcal{A}} \otimes_{\mathcal{A}_{\mathfrak{X}}} p_1^* F), \tag{35}$$

where the first isomorphism is given by Lemma 7.7, the second morphism is given by the morphisms $p_1^{-1}F \to p_1^*F$ and $\otimes \to \otimes_{\mathcal{A}_{\mathfrak{X}}}$, and the third arrow is an isomorphism by Proposition 6.5.

Lemma 7.9. *Let (X, Z, σ) be a kernel data satisfying hypothesis* (29) *and consider $F \in \mathrm{Mod}(\mathcal{A}_X)$. We assume that F is locally free as an $\mathcal{A}_X^0$-module. Then:*

(i) $\mathcal{L}_\sigma^{\mathcal{A}} \otimes_{\mathcal{A}_X} F$ is a complex of quasi-injective sheaves on X_{sa}.

(ii) Let $T_\sigma \subset Z$ be the zero set of σ. The natural morphism, in $\mathrm{C}^+(\mathbf{C}_{X_{sa}\setminus T_\sigma})$,

$$\mathcal{L}_\sigma^{\mathcal{A}} \otimes F \to \mathcal{L}_\sigma^{\mathcal{A}} \otimes_{\mathcal{A}_X} F$$

is a quasi-isomorphism.

Proof. The proof is similar to the proof of Proposition 6.3. Both statements are local on X. We choose coordinates $(x_1, \dots, x_d, z_1, \dots, z_m)$ on X such that Z is given by $x_i = 0$, $i = 1, \dots, d$. This gives coordinates (x, z, τ) on $\tilde{X}_Z$ such that $p(x, z, \tau) = (\tau x, z)$. On Ω we take the coordinates (x', z, τ), where $x' = \tau x$, so that $p(x', z, \tau) = (x', z)$. With these coordinates we argue as in the proof of Proposition 6.3 to see that $p_{!!}\Gamma_\Omega(\mathcal{B}_{P_\sigma^0}) \otimes_{\mathcal{A}_X} F$ is isomorphic to a complex

$$G = p_{!!}(\Gamma_\Omega(\mathcal{B}^0_{P_\sigma^0}) \oplus \Gamma_\Omega(\mathcal{B}^0_{P_\sigma^0}) d\tau) \otimes_{\mathcal{A}_X^0} F,$$

with a differential defined as in (21).

(i) Since F is locally free over $\mathcal{A}_X^0$ and $p_{!!}\Gamma_\Omega(\mathcal{B}^0_{P_\sigma})$ is quasi-injective, by Lemma 7.7, G also is quasi-injective.

(ii) We will see the exactness of the sequence:

$$0 \to p_{!!}(\Gamma_\Omega(K_{P_\sigma^0})) \otimes \mathcal{A}_X^0 \to p_{!!}\Gamma_\Omega(\mathcal{B}^0_{P_\sigma^0}) \xrightarrow{\frac{\partial}{\partial\tau}} p_{!!}\Gamma_\Omega(\mathcal{B}^0_{P_\sigma^0}) \to 0. \tag{36}$$

Thus G is quasi-isomorphic to $p_{!!}(\Gamma_\Omega(K_{P_\sigma^0})) \otimes F$ and this implies (ii) because we already know that $\mathcal{L}_\sigma^{\mathcal{A}}$ is quasi-isomorphic to $\mathcal{L}_\sigma^0$.

Now we prove (36). We have the exact sequence on Ω: $0 \to K_{P_\sigma^0} \otimes p^{-1}\mathcal{A}_X^0 \to \mathcal{B}^0_{P_\sigma^0} \xrightarrow{\frac{\partial}{\partial\tau}} \mathcal{B}^0_{P_\sigma^0} \to 0$. Since $\mathcal{A}^0$-modules are soft this gives (36) if we prove that

$$u \colon p_{!!}(\Gamma_\Omega(K_{P_\sigma^0})) \otimes \mathcal{A}_X^0 \to p_{!!}\Gamma_\Omega(K_{P_\sigma^0} \otimes p^{-1}\mathcal{A}_X^0)$$

is an isomorphism. Let s be a section of $p_{!!}(\Gamma_\Omega(K_{P_\sigma^0})) \otimes \mathcal{A}_X^0$ over some open set U. Up to shrinking U we may assume that s is of the form $1 \otimes a$ where $a \in \mathcal{A}_X^0(U)$ and $1 \in \mathbf{C}_{\overline{W}\setminus\overline{W^0}}(\Omega \cap p^{-1}(U))$, for some open neighborhoods W and W^0 of P_σ and P_σ^0 in $\tilde{X}_Z$. In the same way a section s' of $p_{!!}\Gamma_\Omega(K_{P_\sigma^0} \otimes p^{-1}\mathcal{A}_X^0)$ over U is given by $1 \otimes b$ with $b \in \Gamma((\overline{W'} \setminus \overline{W'^0}) \cap \Omega \cap p^{-1}(U); p^{-1}(\mathcal{A}_X^0))$ for some other neighborhoods of P_σ and P_σ^0 in $\tilde{X}_Z$.

In the coordinates (x', z, τ) on Ω we define $i_\varepsilon\colon X \to \Omega$, $(x, z) \mapsto (x, z, \varepsilon)$. Then $i_\varepsilon^{-1}((\overline{W'} \setminus \overline{W'^0}) \cap \Omega \cap p^{-1}(U))$ is a neighborhood of Z in U for ε small enough. The inverse to morphism u is then given by $a = i_\varepsilon^*(b)$. □

Proposition 7.10. *We consider $F \in \mathrm{Mod}(\mathcal{A}_X)$ and we assume that it is locally free as an $\mathcal{A}_X^0$-module. Then:*

(i) $\mu_X^{\mathcal{A}}(F)$ is a complex of quasi-injective sheaves on T^*X_{sa}.

(ii) The natural morphism (34) *in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*X}))$, $\mu_X(F) \to \mu_X^{\mathcal{A}}(F)$, is an isomorphism.*

Proof. (i) Since $p_{2!!}$ sends quasi-injective sheaves to quasi-injective sheaves, it is enough to prove that $L_X^{\mathcal{A}} \otimes_{\mathcal{A}_{\mathfrak{X}}} p_1^* F$ is quasi-injective. By Proposition 6.3 $p_1^* F$ is locally free over $\mathcal{A}_{\mathfrak{X}}^0$ and we conclude by Lemma 7.9 (i).

(ii) We have to prove that the second arrow in (35) is an isomorphism over $\mathfrak{X} \setminus (X \times T^*_X X)$. By Proposition 6.3 again, $p_1^{-1}F \xrightarrow{\sim} p_1^* F$ in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\mathfrak{X}}))$, and we conclude by Lemma 7.9 (ii). □

8. Functorial behavior of the kernel

We will use the functorial properties of $\mathcal{L}_\sigma$ given in Propositions 1.3.1, 1.3.3 and 1.3.4 of [9], and recalled in Proposition 8.2 below. In fact we state these properties on the site X_{sa}, using the kernel $\mathcal{L}_\sigma^0 \in \mathrm{Mod}(\mathbf{C}_{X_{sa}})$, and our formulas are equivalent to those of [9] when σ doesn't vanish, by Lemma 7.3. We give slightly different proofs than in [9] so that we can translate them easily in the framework of $\mathcal{A}$-modules in Proposition 8.3. In this section (X_1, Z_1, σ_1) and (X_2, Z_2, σ_2) are two sets of data as above, satisfying hypothesis (29). We set for short $\tilde{X}_i = (\tilde{X}_i)_{Z_i}$.

8.1. **Direct and inverse images.** We assume to be given a morphism $f\colon X_1 \to X_2$ is a morphism such that $f(Z_1) \subset Z_2$ and $\sigma_1 = f^*\sigma_2$. The morphism f induces $\tilde{f}\colon \tilde{X}_1 \to \tilde{X}_2$, decomposed as $\tilde{f} = h \circ g$ in the following diagram, where the square is Cartesian:

(37)
$$\begin{array}{ccccccccc}
\Omega_1 & \hookrightarrow & \tilde{X}_1 & \xrightarrow{g} & X_1 \times_{X_2} \tilde{X}_2 & \xrightarrow{h} & \tilde{X}_2 & \hookleftarrow & \Omega_2 \\
& \nwarrow & & \searrow^{p_1} & \downarrow{\scriptstyle q} & \square & \downarrow{\scriptstyle p_2} & \nwarrow & \\
P_{\sigma_1} & & & & X_1 & \xrightarrow{f} & X_2 & & P_{\sigma_2}
\end{array}$$

We have $\Omega_1 = \tilde{f}^{-1}\Omega_2$, $T_{Z_1}X_1 = \tilde{f}^{-1}T_{Z_2}X_2$, $P_{\sigma_1} = \tilde{f}^{-1}P_{\sigma_2}$. We note that $X_1 \times_{X_2} \tilde{X}_2$ is in general not a manifold and may have components of different dimensions. When f is clean with respect to Z_2 and $Z_1 = f^{-1}(Z_2)$ (clean then means that $g'\colon T_{Z_1}X_1 \to X_1 \times_{X_2} T_{Z_2}X_2$ is injective), g is a closed embedding. When f is transversal to Z_2 and $Z_1 = f^{-1}(Z_2)$, g is an isomorphism.

Lemma 8.1. *Let $f\colon X \to Y$ be a morphism of real analytic manifolds, $T \subset Y$ a locally closed subset and $Z = f^{-1}T$.*

(i) There exists a natural isomorphism $f^{-1}K_T \simeq K_Z$.

(ii) Let $V \subset Y$ be an open subset and $U = f^{-1}(V)$ and let $G \in \mathrm{C}^+(\mathbf{C}_{Y_{sa}})$. We assume that the restriction $f|_U\colon U \to V$ is smooth. Then the integration of forms induces a morphism of complexes:

$$f_{!!}\Gamma_U(\mathcal{A}_X \otimes f^{-1}G \otimes \omega'_{X|Y}) \to \Gamma_V(\mathcal{A}_Y \otimes G), \tag{38}$$

whose image in $\mathrm{D}^b(\mathbf{C}_{X_{sa}})$ *is the natural morphism* $Rf_{!!}R\Gamma_U(f^{-1}G\otimes\omega_{X|Y})\to R\Gamma_V G$.

Proof. (i) By definition $K_T=\varinjlim_{W_1,W_2}\mathbf{C}_{\overline{W_2}\setminus\overline{W_1}}$, where W_1, W_2 run over the open neighborhoods of T, $\overline{T}$ in Y. For any compact $M\subset X$, the $f^{-1}(W_i)\cap M$ give fundamental systems of neighborhoods of $Z\cap M$ and $\overline{Z}\cap M$ in M. Since the inductive limit commutes with f^{-1} we deduce the isomorphism.

(ii) We first reduce the statement to $G=\mathbf{C}_Y$. Indeed, for $F\in\mathrm{C}^+(\mathbf{C}_{X_{sa}})$ and $F'\in\mathrm{C}^+(\mathbf{C}_{Y_{sa}})$ with a morphism $f_{!!}\Gamma_U(F)\to\Gamma_V F'$, we have the sequence of morphisms

$$\begin{aligned}f_{!!}\Gamma_U(F\otimes f^{-1}G)&\xrightarrow{\sim} f_{!!}\Gamma_U(\Gamma_U(F)\otimes f^{-1}G)\\&\xrightarrow{\sim}\Gamma_V(f_{!!}(\Gamma_U(F)\otimes f^{-1}G))\\&\to\Gamma_V(\Gamma_V(F')\otimes G)\\&\xleftarrow{\sim}\Gamma_V(F'\otimes G),\end{aligned}$$

where the first one and the last one are induced by $F\to\Gamma_U(F)$ and $F'\to\Gamma_V(F')$ (they are isomorphisms because $F|_U\simeq\Gamma_U(F)|_U$ and $F'|_V\simeq\Gamma_V(F')|_V$), the second one is morphism (10) and the third one is given by the projection formula and the given morphism $f_{!!}\Gamma_U(F)\to\Gamma_V F'$.

Hence it is enough to define $f_{!!}\Gamma_U(\mathcal{A}_X\otimes\omega'_{X|Y})\to\Gamma_V\mathcal{A}_V$. By definition a section of $f_{!!}(\Gamma_U\mathcal{A}_X\otimes\omega'_{X|Y})$ over $W\subset Y$ is represented by a section $\omega\in\Gamma(U\cap f^{-1}W;\mathcal{A}_X\otimes\omega'_{X|Y})$ whose support has compact closure in X. Since f is smooth on U we may define $\int_f\omega$, and it is tempered on V, i.e. it gives an element of $\Gamma(V\cap W;\mathcal{A}_Y)$. This gives morphism (38). □

Proposition 8.2. *(i) There exists a natural morphism in* $\mathrm{D}^b(\mathbf{C}_{(X_2)_{sa}})$*:*

$$Rf_{!!}(\mathcal{L}^0_{\sigma_1}\otimes\rho_{X_1!}(\omega_{Z_1|Z_2}))\to\mathcal{L}^0_{\sigma_2}. \tag{39}$$

(ii) We assume moreover that $Z_1=f^{-1}(Z_2)$ *and* f *is clean with respect to* Z_2*. Then there exists a natural morphism in* $\mathrm{D}^b(\mathbf{C}_{(X_1)_{sa}})$*:*

$$f^{-1}\mathcal{L}^0_{\sigma_2}\to\mathcal{L}^0_{\sigma_1}\otimes\rho_{X_1!}(\omega_{Z_1|Z_2})\otimes\omega^{-1}_{X_1|X_2}. \tag{40}$$

If f *is transversal to* Z_2 *it reduces to:*

$$f^{-1}\mathcal{L}^0_{\sigma_2}\to\mathcal{L}^0_{\sigma_1}. \tag{41}$$

Proof. (i) We note that $p_1^{-1}\omega_{X_1|X_2}\simeq\omega_{\tilde{X}_1|\tilde{X}_2}$. We have the morphisms:

$$\begin{aligned}Rf_{!!}(Rp_{1!!}R\Gamma_{\Omega_1}(K_{P^0_{\sigma_1}})\otimes\omega_{X_1|X_2})&\simeq Rp_{2!!}R\tilde{f}_{!!}R\Gamma_{\Omega_1}(K_{P^0_{\sigma_1}}\otimes\omega_{\tilde{X}_1|\tilde{X}_2})\\&\xrightarrow{\sim}Rp_{2!!}R\Gamma_{\Omega_2}R\tilde{f}_{!!}(\tilde{f}^{-1}K_{P^0_{\sigma_2}}\otimes\omega_{\tilde{X}_1|\tilde{X}_2})\\&\to Rp_{2!!}R\Gamma_{\Omega_2}K_{P^0_{\sigma_2}},\end{aligned}$$

where in the first line we use the projection formula for p_1 and $f\,p_1=p_2\,\tilde{f}$ (we note that $\omega_{\tilde{X}_1|\tilde{X}_2}$ enters the parenthesis because it is locally constant). In the second line we use formula (10) and Lemma 8.1, (i). In the third line we use the projection formula for $\tilde{f}$ and the integration morphism.

Now we take the tensor product with $\omega^{\otimes-1}_{Z_2|X_2}$ and we obtain (39).

(ii) Since f is clean with respect to Z_2 and $Z_1 = f^{-1}Z_2$, the morphism g in diagram (37) is an embedding. Hence $g_* = g_{!!}$ and we have the adjunction morphism $\mathrm{id} \to g_{!!}\, g^{-1}$. We deduce a morphism of functors

$$f^{-1}Rp_{2!!} \to Rp_{1!!}\tilde{f}^{-1} \tag{42}$$

as the composition of the base change $f^{-1}Rp_{2!!} \to Rq_{!!}h^{-1}$ and the adjunction morphism $Rq_{!!}h^{-1} \to Rq_{!!}g_{!!}\, g^{-1}h^{-1} = Rp_{1!!}\tilde{f}^{-1}$. Now we define (40) by the sequence of morphisms:

$$\begin{aligned} f^{-1}\mathcal{L}^0_{\sigma_2} &= f^{-1}(Rp_{2!!}(R\Gamma_{\Omega_2}K_{P^0_{\sigma_2}}) \otimes \rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2}) \\ &\to (Rp_{1!!}\tilde{f}^{-1}(R\Gamma_{\Omega_2}K_{P^0_{\sigma_2}})) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2} \\ &\to Rp_{1!!}(R\Gamma_{\Omega_1}K_{P^0_{\sigma_1}}) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2} \\ &= \mathcal{L}^0_{\sigma_1} \otimes \rho_{X_1!}(\omega_{Z_1|X_1}) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2}, \end{aligned}$$

where the second line is given by (42) and in the third line we use the morphism $\tilde{f}^{-1}R\Gamma_{\Omega_2}K_{P^0_{\sigma_2}} \to R\Gamma_{\Omega_1}K_{P^0_{\sigma_1}}$, obtained from the morphism of functor $\tilde{f}^{-1}R\Gamma_{\Omega_2} \to R\Gamma_{\Omega_1}\tilde{f}^{-1}$ and Lemma 8.1, (i).

If f is transversal to Z_2 we have $\omega_{Z_1|X_1} \simeq f^{-1}\omega_{Z_2|X_2}$. □

Now we have the following analog of Proposition 8.2 for $\mathcal{A}$-modules, with the additional hypothesis that f is smooth, for the case of direct image.

Proposition 8.3. *(i) Assume that f is smooth. Then there exists a natural morphism of dg-$\mathcal{A}_{X_2}$-modules:*

$$f_{!!}(\mathcal{L}^{\mathcal{A}}_{\sigma_1} \otimes \rho_{X_1!}(\omega_{Z_1|Z_2})) \to \mathcal{L}^{\mathcal{A}}_{\sigma_2}, \tag{43}$$

whose image in $\mathrm{D}^b(\mathbf{C}_{(X_2)_{sa}})$ is morphism (39).

(ii) Assume that $Z_1 = f^{-1}(Z_2)$ and f is clean with respect to Z_2. Then there exists a natural morphism of dg-$\mathcal{A}_{X_1}$-modules:

$$f^*\mathcal{L}^{\mathcal{A}}_{\sigma_2} \to \mathcal{L}^{\mathcal{A}}_{\sigma_1} \otimes \rho_{X_1!}(\omega_{Z_1|Z_2}) \otimes \omega^{-1}_{X_1|X_2}, \tag{44}$$

whose image in $\mathrm{D}^b(\mathbf{C}_{(X_1)_{sa}})$ is morphism (40). *If f is transversal to Z_2 it becomes:*

$$f^*\mathcal{L}^{\mathcal{A}}_{\sigma_2} \to \mathcal{L}^{\mathcal{A}}_{\sigma_1}. \tag{45}$$

Proof. The proof is similar to the proof of Proposition 8.2. We keep the same notations and we just point out the changes.

(i) We note that $\tilde{f}$ is smooth on Ω_1 and apply Lemma 8.1. This gives the morphisms:

$$\begin{aligned} f_{!!}(p_{1!!}(\Gamma_{\Omega_1}\mathcal{B}_{P^0_{\sigma_1}}) \otimes \omega_{X_1|X_2}) &\simeq p_{2!!}\tilde{f}_{!!}\Gamma_{\Omega_1}(\mathcal{A}_{\tilde{X}_1} \otimes \tilde{f}^{-1}K_{P^0_{\sigma_2}} \otimes \omega'_{\tilde{X}_1|\tilde{X}_2}) \\ &\to p_{2!!}(\Gamma_{\Omega_2}\mathcal{B}_{P^0_{\sigma_2}}), \end{aligned}$$

and the tensor product with $\omega^{\otimes -1}_{Z_2|X_2}$ gives (43).

(ii) Morphism (42) has a non derived version $f^{-1}p_{2!!} \to p_{1!!}\tilde{f}^{-1}$. Taking the tensor product $\mathcal{A}_{X_1} \otimes_{f^{-1}\mathcal{A}_{X_2}} \cdot$ and using the projection formula we obtain:

$$f^*p_{2!!} \to p_{1!!}\tilde{f}^* \tag{46}$$

Now we define (44) by the sequence of morphisms:

$$\begin{aligned} f^*\mathcal{L}^{\mathcal{A}}_{\sigma_2} &= f^*(p_{2!!}(\Gamma_{\Omega_2}\mathcal{B}_{P^0_{\sigma_2}}) \otimes \rho_{X_2!}(\omega^{\otimes -1}_{Z_2|X_2})) \\ &\to (p_{1!!}\tilde{f}^*(\Gamma_{\Omega_2}\mathcal{B}_{P^0_{\sigma_2}})) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2} \\ &\to p_{1!!}(\Gamma_{\Omega_1}\mathcal{B}_{P^0_{\sigma_1}}) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2} \\ &= \mathcal{L}^{\mathcal{A}}_{\sigma_1} \otimes \rho_{X_1!}(\omega_{Z_1|X_1}) \otimes f^{-1}\rho_{X_2!}\omega^{\otimes -1}_{Z_2|X_2}, \end{aligned}$$

where the second line is given by (46) and the third line is the composition

$$\begin{aligned} \tilde{f}^*(\Gamma_{\Omega_2}\mathcal{B}_{P^0_{\sigma_2}}) = \mathcal{A}_{\tilde{X}_1} \otimes_{f^{-1}\mathcal{A}_{\tilde{X}_2}} f^{-1}\Gamma_{\Omega_2}(\mathcal{A}_{\tilde{X}_2} \otimes K_{P^0_{\sigma_2}}) \\ \to \mathcal{A}_{\tilde{X}_1} \otimes_{f^{-1}\mathcal{A}_{\tilde{X}_2}} \Gamma_{\Omega_1}(f^{-1}\mathcal{A}_{\tilde{X}_2} \otimes K_{P^0_{\sigma_1}}) \to \Gamma_{\Omega_1}\mathcal{B}_{P_{\sigma_1}} \end{aligned}$$

of standard morphisms of sheaves and the isomorphism of Lemma 8.1, (i). □

8.2. **External tensor product.** The external tensor product is a consequence of Proposition 1.3.8 of [9]. We give a different proof here, using the kernel $\mathcal{L}^0_\sigma$ (hence our morphism coincides with the one in [9] for a non-vanishing σ) and check that it works for $\mathcal{A}$-modules. We still consider (X_1, Z_1, σ_1) and (X_2, Z_2, σ_2) as in the beginning of this section. We set $X = X_1 \times X_2$, $Z = Z_1 \times Z_2$, $\sigma = \sigma_1 + \sigma_2$. Then (X, Z, σ) also is a kernel data satisfying (29). We keep the notations of diagram (37) and let $p\colon \tilde{X}_Z \to X$ be the projection. We also have a natural embedding $k\colon \tilde{X}_Z \to \tilde{X}_1 \times \tilde{X}_2$. We set $p' = p_1 \times p_2\colon \tilde{X}_1 \times \tilde{X}_2 \to X$.

Proposition 8.4. *There exists a morphism $\mathcal{L}^0_{\sigma_1} \boxtimes \mathcal{L}^0_{\sigma_2} \to \mathcal{L}^0_\sigma$ in $\mathrm{D}^b(\mathbf{C}_{(X_1\times X_2)_{sa}})$.*

Proof. The kernel $\mathcal{L}^0_{\sigma_i}$ is the tensor product of $Rp_{i!!}(R\Gamma_{\Omega_i}K_{P^0_{\sigma_i}})$ and $\rho_{X_i!}(\omega^{\otimes -1}_{Z_i|X_i})$. The external product for the second term is straightforward:

$$\rho_{X_1!}(\omega^{\otimes -1}_{Z_1|X_1}) \boxtimes \rho_{X_2!}(\omega^{\otimes -1}_{Z_2|X_2}) \simeq \rho_{X!}(\omega^{\otimes -1}_{Z|X})$$

and now we only take care of the first term. We have the sequence of morphisms:

$$\begin{aligned} (Rp_{1!!}R\Gamma_{\Omega_1}K_{P^0_{\sigma_1}}) \boxtimes (Rp_{2!!}R\Gamma_{\Omega_2}K_{P^0_{\sigma_2}}) &\to Rp'_{!!}R\Gamma_{\Omega_1\times\Omega_2}(K_{P^0_{\sigma_1}} \boxtimes K_{P^0_{\sigma_2}}) \\ &\to Rp'_{!!}k_{!!}k^{-1}R\Gamma_{\Omega_1\times\Omega_2}(K_{P^0_{\sigma_1}} \boxtimes K_{P^0_{\sigma_2}}) \\ &\to Rp_{!!}R\Gamma_\Omega(k^{-1}(K_{P^0_{\sigma_1}} \boxtimes K_{P^0_{\sigma_2}})) \\ &\to Rp_{!!}R\Gamma_\Omega(K_{P^0_\sigma}), \end{aligned}$$

where the first three arrows are standard morphisms of sheaves and the last one is defined as follows. We recall that $K_{P^0_{\sigma_i}} \simeq \varinjlim_{W_i, W_i^0} \mathbf{C}_{\overline{W_i}\setminus\overline{W_i^0}}$, where W_i, W_i^0 run over the open neighborhoods of P_{σ_i}, $P^0_{\sigma_i}$ in $\tilde{X}_i$. For such W_i, W_i^0 we have

$$(\overline{W_1} \setminus \overline{W_1^0}) \times (\overline{W_2} \setminus \overline{W_2^0}) = (\overline{W_1} \times \overline{W_2}) \setminus ((\overline{W_1} \times \overline{W_2}) \cap \overline{W^0}) = \overline{W} \setminus \overline{W^0},$$
$$\text{where} \quad W^0 = W_1^0 \times \tilde{X}_2 \cup \tilde{X}_1 \times W_2^0 \quad \text{and} \quad W = (W_1 \times W_2) \cup W^0.$$

Now W and W^0 are open neighborhoods of P_σ and P^0_σ in $\tilde{X}_1 \times \tilde{X}_2$ (note that $P_\sigma \subset T_Z X$ and $T_Z X$ can be viewed as a subset of $\tilde{X}_1 \times \tilde{X}_2$). This defines a natural morphism $K_{P^0_{\sigma_1}} \boxtimes K_{P^0_{\sigma_2}} \to \varinjlim_{W,W^0} \mathbf{C}_{\overline{W}\setminus\overline{W^0}}$, where W and W^0 run over the open neighborhoods of P_σ and P^0_σ in $\tilde{X}_1 \times \tilde{X}_2$. The inverse image by k gives the required morphism $k^{-1}(K_{P^0_{\sigma_1}} \boxtimes K_{P^0_{\sigma_2}}) \to K_{P^0_\sigma}$. □

Proposition 8.5. *There exists a morphism* $\mathcal{L}^{\mathcal{A}}_{\sigma_1}\underline{\boxtimes}\mathcal{L}^{\mathcal{A}}_{\sigma_2} \to \mathcal{L}^{\mathcal{A}}_{\sigma}$ *in* $\mathrm{Mod}(\mathcal{A}_{X_1\times X_2})$*, whose image in* $\mathrm{D}^b(\mathbf{C}_{(X_1\times X_2)_{sa}})$ *is the morphism of Proposition 8.4.*

Proof. The proof of the previous proposition adapts immediately, with the following modifications in the sequence of morphisms:

$$\begin{aligned}
&(p_{1!!}\Gamma_{\Omega_1}(\mathcal{A}_{\tilde{X}_1}\otimes K_{P^0_{\sigma_1}}))\boxtimes(p_{2!!}\Gamma_{\Omega_2}(\mathcal{A}_{\tilde{X}_2}\otimes K_{P^0_{\sigma_2}}))\\
&\quad\to p'_{!!}\Gamma_{\Omega_1\times\Omega_2}(\mathcal{A}_{\tilde{X}_1\times\tilde{X}_2}\otimes(K_{P^0_{\sigma_1}}\boxtimes K_{P^0_{\sigma_2}}))\\
&\quad\to p'_{!!}k_{!!}k^{-1}\Gamma_{\Omega_1\times\Omega_2}(\mathcal{A}_{\tilde{X}_1\times\tilde{X}_2}\otimes(K_{P^0_{\sigma_1}}\boxtimes K_{P^0_{\sigma_2}}))\\
&\quad\to p_{!!}\Gamma_{\Omega}(\mathcal{A}_{\tilde{X}_Z}\otimes k^{-1}(K_{P^0_{\sigma_1}}\boxtimes K_{P^0_{\sigma_2}}))\\
&\quad\to p_{!!}\Gamma_{\Omega}(\mathcal{A}_{\tilde{X}_Z}\otimes K_{P^0_{\sigma}}).
\end{aligned}$$

□

9. Functorial properties of microlocalization

In this section $f\colon X\to Y$ is a morphism of real analytic manifolds. We recall the functorial behavior of microlocalization with respect to inverse image, in case f is an embedding, and to direct image. We check that the constructions make sense for dg-$\mathcal{A}$-modules (restricting to the case of a smooth map for the direct image).

We define the submanifold $Z = X\times_Y T^*Y$ diagonally embedded in $X\times(X\times_Y T^*Y)$. We have the morphisms of kernel data

$$\begin{array}{ccccc}
X\times T^*X & \xleftarrow{\mathrm{id}\times f_d} & X\times(X\times_Y T^*Y) & \xrightarrow{f\times f_\pi} & Y\times T^*Y\\
\cup & & \cup & & \cup\\
X\times_X T^*X & \longleftarrow & Z & \longrightarrow & Y\times_Y T^*Y\\
\| & & \| & & \|\\
T^*X & \xleftarrow{f_d} & X\times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y,
\end{array}$$

where the 1-form for the kernel corresponding to the middle column is

$$\sigma_{Y\leftarrow X} = (\mathrm{id}\times f_d)^*(\sigma_X) = (f\times f_\pi)^*(\sigma_Y).$$

This equality follows from $f_d^*(\omega_X) = f_\pi^*(\omega_Y)$. We note that $Z = (\mathrm{id}\times f_d)^{-1}(X\times_X T^*X)$ and $Z\subset(f\times f_\pi)^{-1}(Y\times_Y T^*Y)$, with equality if f is an embedding. This implies that hypothesis (29) is satisfied for $(X\times(X\times_Y T^*Y), Z, \sigma_{Y\leftarrow X})$. We denote the corresponding kernel by $L_{Y\leftarrow X} = \mathcal{L}_{\sigma_{Y\leftarrow X}}$.

9.1. **Microlocalization and inverse image.** For the next two propositions we assume that $f\colon X\to Y$ is an embedding. For $G\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_Y))$ we have a morphism $Rf_{d!!}f_\pi^{-1}\mu_Y(G)\to\mu_X(f^{-1}G)$, defined in Theorem 2.4.4 of [9]. We recall its construction below. The notations are introduced in the diagram:

$$\begin{array}{ccccccc}
X & = & X & & \xrightarrow{\quad f\quad} & & Y\\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle p} & & & & \uparrow{\scriptstyle q_1}\\
X\times T^*X & \xleftarrow{\mathrm{id}\times f_d} & X\times(X\times_Y T^*Y) & & \xrightarrow{f\times f_\pi} & & Y\times T^*Y\\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle r} & & & & \downarrow{\scriptstyle q_2}\\
T^*X & \xleftarrow{f_d} & X\times_Y T^*Y & & \xrightarrow{f_\pi} & & T^*Y
\end{array}\tag{47}$$

Proposition 9.1 ([9], Theorem 2.4.4)**.** *We have a natural morphism, for an embedding $f\colon X \to Y$ and $G \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_Y))$:*

$$Rf_{d!!}f_\pi^{-1}\mu_Y(G) \to \mu_X(f^{-1}G). \tag{48}$$

Proof. We first note the morphism of functors $f_\pi^{-1}Rq_{2!!} \to Rr_{!!}(f\times f_\pi)^{-1}$. It is obtained by the following composition of adjunction morphisms, where we use the fact that f, hence f_π and $f\times f_\pi$, are embeddings, so that direct and proper direct images coincide:

$$\begin{aligned} f_\pi^{-1}Rq_{2!!} &\to f_\pi^{-1}Rq_{2!!}(f\times f_\pi)_*(f\times f_\pi)^{-1} \\ &\simeq f_\pi^{-1}Rf_{\pi*}Rr_{!!}(f\times f_\pi)^{-1} \to Rr_{!!}(f\times f_\pi)^{-1}. \end{aligned} \tag{49}$$

We also note the morphisms of kernels:

$$(f\times f_\pi)^{-1}L_Y \to L_{Y\leftarrow X}\otimes \omega_{X|Y}^{-1} \qquad R(\mathrm{id}\times f_d)_{!!}(L_{Y\leftarrow X}\otimes\omega_{X|Y}^{-1}) \to L_X. \tag{50}$$

The first one is morphism (40) of Proposition 8.2 (for $\mathcal{L}_\sigma$ instead of $\mathcal{L}^0_\sigma$), applied to $f\times f_\pi$: we note that $f\times f_\pi$ is clean with respect to $Y\times_Y T^*Y$ and $X\times_Y T^*Y = (f\times f_\pi)^{-1}(Y\times_Y T^*Y)$. The second one is morphism (39) (for $\mathcal{L}_\sigma$ instead of $\mathcal{L}^0_\sigma$), applied to $\mathrm{id}\times f_d$.

Now the morphism of the lemma is defined by the succession of morphisms:

$$Rf_{d!!}f_\pi^{-1}\mu_Y(G) = Rf_{d!!}f_\pi^{-1}Rq_{2!!}(L_Y\otimes q_1^{-1}G) \tag{51}$$

$$\to Rf_{d!!}Rr_{!!}((f\times f_\pi)^{-1}L_Y\otimes p^{-1}f^{-1}G) \tag{52}$$

$$\leftharpoonup Rp_{2!!}(R(\mathrm{id}\times f_d)_{!!}(f\times f_\pi)^{-1}L_Y\otimes p_1^{-1}f^{-1}G) \tag{53}$$

$$\to Rp_{2!!}(L_X\otimes p_1^{-1}f^{-1}G), \tag{54}$$

where in line (52) we used morphism (49) and the commutativity of inverse image and tensor product, and in line (53) the identities $f_d r = p_2(\mathrm{id}\times f_d)$, $p = p_1(\mathrm{id}\times f_d)$ and the projection formula for $(\mathrm{id}\times f_d)$. The last morphism is the composition of the morphisms in (50). □

Proposition 9.2. *For an embedding $f\colon X\to Y$ and $G\in \mathrm{Mod}(\mathcal{A}_Y)$, we have a morphism of $\mathcal{A}_{T^*X}$-modules:*

$$f_{d!!}f_\pi^*\mu_Y^{\mathcal{A}}(G) \to \mu_X^{\mathcal{A}}(f^*G), \tag{55}$$

which makes a commutative diagram in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*X}))$ with morphism (48):

$$\begin{array}{ccc} Rf_{d!!}f_\pi^{-1}\mu_Y(G) & \longrightarrow & \mu_X(f^{-1}G) \\ \downarrow & & \downarrow \\ Rf_{d!!}f_\pi^*\mu_Y^{\mathcal{A}}(G) & \leftharpoonup & f_{d!!}f_\pi^*\mu_Y^{\mathcal{A}}(G) \longrightarrow \mu_X^{\mathcal{A}}(f^*G). \end{array}$$

Proof. We follow the construction of morphism (48), replacing each morphism by its analog for $\mathcal{A}$-modules. We have the analogs of morphisms (49) and (50):

$$\begin{aligned} f_\pi^*q_{2!!} &\to f_\pi^*q_{2!!}(f\times f_\pi)_*(f\times f_\pi)^* \\ &\simeq f_\pi^*f_{\pi*}r_{!!}(f\times f_\pi)^* \to r_{!!}(f\times f_\pi)^*, \end{aligned} \tag{56}$$

$$(f\times f_\pi)^*L_Y^{\mathcal{A}} \to L_{Y\leftarrow X}^{\mathcal{A}}\otimes\omega_{X|Y}^{-1}, \qquad (\mathrm{id}\times f_d)_{!!}(L_{Y\leftarrow X}^{\mathcal{A}}\otimes\omega_{X|Y}^{-1}) \to L_X^{\mathcal{A}}. \tag{57}$$

Morphism (56) is defined with the same adjunction properties as morphism (49). The morphisms in line (57) are defined the same way as (50), using Proposition 8.3 instead of Proposition 8.2. We deduce the succession of morphisms:

$$f_{d!!}f_\pi^*\mu_Y^{\mathcal{A}}(G) = f_{d!!}f_\pi^* q_{2!!}(L_Y^{\mathcal{A}} \otimes_{\mathcal{A}} q_1^* G) \tag{58}$$

$$\to f_{d!!}r_{!!}((f\times f_\pi)^* L_Y^{\mathcal{A}} \otimes_{\mathcal{A}} p^* f^* G) \tag{59}$$

$$\leftharpoonup p_{2!!}((\mathrm{id}\times f_d)_{!!}(f\times f_\pi)^* L_Y^{\mathcal{A}} \otimes_{\mathcal{A}} p_1^* f^* G) \tag{60}$$

$$\to p_{2!!}(L_X^{\mathcal{A}} \otimes_{\mathcal{A}} p_1^* f^* G), \tag{61}$$

where in line (59) we used morphism (56) and the commutativity of inverse image and tensor product, and in line (60) the identities $f_d r = p_2(\mathrm{id}\times f_d)$, $p = p_1(\mathrm{id}\times f_d)$ and the projection formula for $(\mathrm{id}\times f_d)$ (Lemma 6.6). The last morphism is the composition of the morphisms in (57).

The vertical arrows in the diagram are the compositions of $g^{-1}\to g^*$ respectively for $g = f_\pi$, $g = f$, and $\mu_Z \to \mu_Z^{\mathcal{A}}$, respectively for $Z = Y$, $Z = X$. This last morphism is defined only on $\dot{T}^*Z$. The diagram commutes because it is obtained by morphisms of functors. The isomorphism between the direct image and the derived direct image by f_d follows from the softness of $\mathcal{A}$-modules (Proposition 6.5). □

9.2. **Microlocalization and direct image.** In Proposition 9.3 below we recall a weak version of the direct image morphism, defined in Theorem 2.4.2 of [9]. This theorem gives a morphism, for $F\in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))$, $Rf_{\pi!!}f_d^{-1}\mu_X(F)\to \mu_Y(f_{!!}F)$. We consider the case where $F = f^{-1}G\otimes\omega_{X|Y}$ which is sufficient for our purpose, and we give an easier proof in this case. This proof also works for the resolutions by $\mathcal{A}$-modules, assuming moreover that f is smooth (see Proposition 9.4). We use the notations of diagram (47).

Proposition 9.3 (special case of [9], Theorem 2.4.2)**.** *There exists a natural morphism, for $f\colon X\to Y$ and $G\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_Y))$:*

$$Rf_{\pi!!}f_d^{-1}\mu_X(f^{-1}G\otimes\omega_{X|Y}) \to \mu_Y(G). \tag{62}$$

Proof. We set $F = f^{-1}G\otimes\omega_{X|Y}$ and obtain the sequence of morphisms:

$$Rf_{\pi!!}f_d^{-1}\mu_X(F) \tag{63}$$

$$= Rf_{\pi!!}f_d^{-1}Rp_{2!!}(L_X\otimes p_1^{-1}F) \tag{64}$$

$$\xrightarrow{\sim} Rf_{\pi!!}Rr_{!!}((\mathrm{id}\times f_d)^{-1}L_X\otimes p^{-1}F) \tag{65}$$

$$\simeq Rq_{2!!}R(f\times f_\pi)_{!!}((\mathrm{id}\times f_d)^{-1}L_X\otimes p^{-1}\omega_{X|Y}\otimes (f\times f_\pi)^{-1}q_1^{-1}G) \tag{66}$$

$$\leftharpoonup Rq_{2!!}(R(f\times f_\pi)_{!!}((\mathrm{id}\times f_d)^{-1}L_X\otimes p^{-1}\omega_{X|Y})\otimes q_1^{-1}G) \tag{67}$$

$$\to Rq_{2!!}(L_Y\otimes q_1^{-1}G), \tag{68}$$

where in line (65) we used the base change formula $f_d^{-1}Rp_{2!!}\xrightarrow{\sim} Rr_{!!}(\mathrm{id}\times f_d)^{-1}$ and the identity $p = p_1(\mathrm{id}\times f_d)$, in line (66) the identities $f_\pi r = q_2(f\times f_\pi)$ and $fp = q_1(f\times f_\pi)$, and in line (67) the projection formula for $(f\times f_\pi)$. The last line is given by the composition of

$$(\mathrm{id}\times f_d)^{-1}L_X\to L_{Y\leftarrow X} \quad\text{and}\quad R(f\times f_\pi)_{!!}(L_{Y\leftarrow X}\otimes p^{-1}\omega_{X|Y})\to L_Y,$$

which are respectively given by (ii) and (i) of Proposition 8.2 (for the first morphism we note that $(\mathrm{id}\times f_d)$ is transversal to $X\times_X T^*X$ and for the second one we note that the restriction of $p^{-1}\omega_{X|Y}$ to $X\times_Y T^*Y$ is isomorphic to $\omega_{X\times_Y T^*Y|Y\times_Y T^*Y}$). □

The following proposition gives a realization of morphism (62) by $\mathcal{A}$-modules. We restrict to the case where f is a submersion because we only have an integration morphism in this case.

Proposition 9.4. *There exists a natural morphism of $\mathcal{A}_{T^*Y}$-modules, for a submersion $f\colon X \to Y$ and for $G \in \mathrm{Mod}(\mathcal{A}_Y)$:*

$$f_{\pi!!} f_d^* \mu_X^{\mathcal{A}}(f^*G \otimes \omega'_{X|Y}) \to \mu_Y^{\mathcal{A}}(G), \tag{69}$$

which makes a commutative diagram in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*Y}))$ with morphism (62):

$$\begin{array}{ccc} Rf_{\pi!!} f_d^{-1} \mu_X(f^{-1}G \otimes \omega_{X|Y}) & \longrightarrow & \mu_Y(G) \\ \downarrow & & \downarrow \\ Rf_{\pi!!} f_d^* \mu_X^{\mathcal{A}}(f^*G \otimes \omega'_{X|Y}) \leftharpoonup f_{\pi!!} f_d^* \mu_X^{\mathcal{A}}(f^*G \otimes \omega'_{X|Y}) & \longrightarrow & \mu_Y^{\mathcal{A}}(G). \end{array}$$

Proof. We follow the proof of Proposition 9.3, but now we consider morphisms of $\mathcal{A}$-modules. We set $F = f^*G \otimes \omega'_{X|Y}$ and obtain the sequence of morphisms:

$$f_{\pi!!} f_d^* \mu_X^{\mathcal{A}}(F) \tag{70}$$

$$= f_{\pi!!} f_d^* p_{2!!}(L_X^{\mathcal{A}} \otimes_{\mathcal{A}} p_1^* F) \tag{71}$$

$$\to f_{\pi!!} r_{!!}((\mathrm{id} \times f_d)^* L_X^{\mathcal{A}} \otimes p^* F) \tag{72}$$

$$\simeq q_{2!!}(f \times f_\pi)_{!!}((\mathrm{id} \times f_d)^* L_X^{\mathcal{A}} \otimes p^{-1}\omega'_{X|Y} \otimes_{\mathcal{A}} (f \times f_\pi)^* q_1^* G) \tag{73}$$

$$\leftharpoonup q_{2!!}((f \times f_\pi)_{!!}((\mathrm{id} \times f_d)^* L_X^{\mathcal{A}} \otimes p^{-1}\omega'_{X|Y}) \otimes_{\mathcal{A}} q_1^* G) \tag{74}$$

$$\to q_{2!!}(L_Y^{\mathcal{A}} \otimes_{\mathcal{A}} q_1^* G), \tag{75}$$

where in line (72) we used the base change formula $f_d^* p_{2!!} \to r_{!!}(\mathrm{id} \times f_d)^*$ and the identity $p = p_1(\mathrm{id} \times f_d)$, in line (73) the identities $f_\pi r = q_2(f \times f_\pi)$ and $fp = q_1(f \times f_\pi)$, and in line (74) the projection formula for $(f \times f_\pi)$. The last line is given by the composition of

$$(\mathrm{id} \times f_d)^* L_X^{\mathcal{A}} \to L_{Y \leftarrow X}^{\mathcal{A}} \quad \text{and} \quad (f \times f_\pi)_{!!}(L_{Y \leftarrow X}^{\mathcal{A}} \otimes p^{-1}\omega'_{X|Y}) \to L_Y^{\mathcal{A}},$$

which are given by (ii) and (i) of Proposition 8.3.

The diagram is defined as in Proposition 9.2. □

9.3. **External tensor product.** We consider X, Y as above and $F \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_X))$, $G \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_Y))$. Proposition 2.1.14 of [9] implies the existence of a natural morphism:

$$\mu_X F \boxtimes \mu_Y G \to \mu_{X \times Y}(F \boxtimes G). \tag{76}$$

Proposition 9.5. *For $F \in \mathrm{Mod}(\mathcal{A}_X)$ and $G \in \mathrm{Mod}(\mathcal{A}_Y)$ there exists a natural morphism*

$$\mu_X^{\mathcal{A}} F \underline{\boxtimes} \mu_Y^{\mathcal{A}} G \to \mu_{X \times Y}^{\mathcal{A}}(F \boxtimes G).$$

Its restriction to $\dot{T}^*X \times \dot{T}^*Y$ makes a commutative diagram with morphism (76) in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*X \times \dot{T}^*Y}))$:

$$\begin{array}{ccc} \mu_X F \boxtimes \mu_Y G & \longrightarrow & \mu_{X \times Y}(F \boxtimes G) \\ \downarrow & & \downarrow \\ \mu_X^{\mathcal{A}} F \boxtimes \mu_Y^{\mathcal{A}} G & \longrightarrow & \mu_{X \times Y}^{\mathcal{A}}(F \boxtimes G). \end{array}$$

Proof. The existence of the morphisms follows from the Künneth formula and Proposition 8.5. It coincides with the already known construction outside the zero section by Proposition 7.10. □

10. Composition of kernels

We recall the microlocal composition of kernels defined in [9], Theorem 2.5.1, and we check that a similar construction also works for $\mathcal{A}$-modules. This construction is a composition of the operations recalled in section 9, and we just have to check that the restrictive hypothesis assumed in the case of $\mathcal{A}$-modules are satisfied.

We first recall some standard notations and definitions. We consider three analytic manifolds X, Y, Z and we let q_{ij} be the (i,j)-th projection from $X\times Y\times Z$ and p_{ij} the (i,j)-th projection from $T^*X\times T^*Y\times T^*Z$. We also denote by $a\colon T^*Y\to T^*Y$ the antipodal map and we set $p^a_{12}=(\mathrm{id}\times a)\circ p_{12}$. For $F\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Y}))$, $G\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{Y\times Z}))$ and $\mathfrak{F}\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{T^*X\times T^*Y}))$, $\mathfrak{G}\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{T^*Y\times T^*Z}))$ we define:

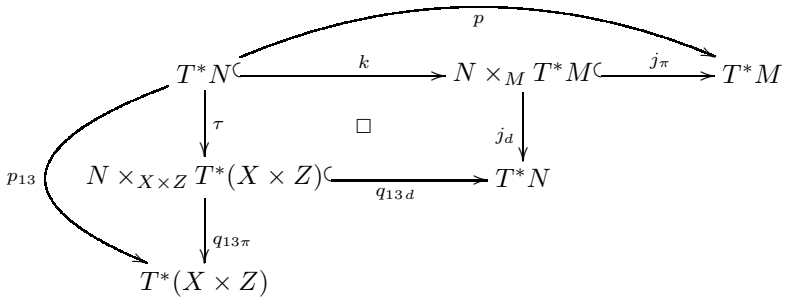
$$F\circ G = Rq_{13!!}(q_{12}^{-1}F\otimes q_{23}^{-1}G),\qquad \mathfrak{F}\overset{a}{\circ}\mathfrak{G} = Rp_{13!!}(p_{12}^{a\,-1}\mathfrak{F}\otimes p_{23}^{-1}\mathfrak{G}). \tag{77}$$

We set for short $M=X\times Y\times Y\times Z$, $N=X\times Y\times Z$ and let $j\colon N\to M$ be the diagonal embedding. We define the maps:

$$\begin{aligned}
&k\colon T^*N\hookrightarrow N\times_M T^*M, && (x,y,z;\xi,\eta,\zeta)\mapsto(x,y,y,z;\xi,-\eta,\eta,\zeta)\\
&\tau\colon T^*N\to N\times_{X\times Z} T^*(X\times Z), && (x,y,z;\xi,\eta,\zeta)\mapsto(x,y,z;\xi,\zeta)\\
&p=j_\pi\circ k
\end{aligned}$$

and obtain the following commutative diagram, with a Cartesian square:

(78)



We note that $\mathfrak{F}\overset{a}{\circ}\mathfrak{G}\simeq Rp_{13!!}\,p^{-1}(\mathfrak{F}\boxtimes\mathfrak{G})$. Theorem 2.5.1 of [9] gives a natural morphism, the composition of kernels:

$$\mu_{X\times Y}K_1\overset{a}{\circ}\mu_{Y\times Z}K_2\to\mu_{X\times Z}(K_1\circ K_2), \tag{79}$$

for $K_1\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Y}))$, $K_2\in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{Y\times Z}))$. Since the commutation of microlocalization and direct image has a weaker statement in the case of $\mathcal{A}$-modules than in the case of ind-sheaves of vector spaces, we also give a weaker statement than (79) for the composition of kernels.

In fact, for ind-sheaves, morphism (80) below is equivalent to (79): indeed using the adjunction between $Rq_{13!!}$ and $q_{13}^!$ we may apply (80) to $K_3=K_1\circ K_2$ and recover (79). But for $\mathcal{A}$-modules we don't have this adjunction and the statement of Proposition 10.2 is actually weaker than an $\mathcal{A}$-module analog of (79).

Proposition 10.1. *For complexes $K_1 \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Y}))$, $K_2 \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{Y\times Z}))$ and $K_3 \in \mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Z}))$, with a morphism $q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2 \to q_{13}^{-1}K_3 \otimes \omega_Y$, there exists a natural morphism*

$$\mu_{X\times Y}K_1 \overset{a}{\circ} \mu_{Y\times Z}K_2 \to \mu_{X\times Z}K_3. \tag{80}$$

Proof. By definition $\mu_{X\times Y}K_1 \overset{a}{\circ} \mu_{Y\times Z}K_2 = Rp_{13!!}p^{-1}(\mu_{X\times Y}K_1 \boxtimes \mu_{Y\times Z}K_2)$. The external tensor product (76) gives $\mu_{X\times Y}K_1 \boxtimes \mu_{Y\times Z}K_2 \to \mu_M(K_1 \boxtimes K_2)$ and the base change formula gives $Rp_{13!!}p^{-1} = Rq_{13\pi!!}R\tau_{!!}k^{-1}\, j_\pi^{-1} \simeq Rq_{13\pi!!}q_{13d}^{-1}\, Rj_{d!!}j_\pi^{-1}$. We obtain the morphisms

$$\begin{aligned}
\mu_{X\times Y}K_1 \overset{a}{\circ} \mu_{Y\times Z}K_2 &\to Rq_{13\pi!!}q_{13d}^{-1}\, Rj_{d!!}j_\pi^{-1}(\mu_M(K_1 \boxtimes K_2)) \\
&\to Rq_{13\pi!!}q_{13d}^{-1}\ \mu_N\, j^{-1}(K_1 \boxtimes K_2) \\
&\to Rq_{13\pi!!}q_{13d}^{-1}\, \mu_N(q_{13}^{-1}K_3 \otimes \omega_Y) \\
&\to \mu_{X\times Z}K_3,
\end{aligned}$$

where in the second line we have applied Proposition 9.1, in the third the hypothesis and in the fourth Proposition 9.3. □

Now we give the $\mathcal{A}$-module analog of the above result. For $\mathfrak{F} \in \mathrm{Mod}(\mathcal{A}_{T^*X\times T^*Y})$ and $\mathfrak{G} \in \mathrm{Mod}(\mathcal{A}_{T^*Y\times T^*Z})$ we set

$$\mathfrak{F} \overset{a\mathcal{A}}{\circ} \mathfrak{G} = p_{13!!}(p_{12}^{a*}\mathfrak{F} \otimes_{\mathcal{A}_N} p_{23}^*\mathfrak{G}) \simeq p_{13!!}p^*(\mathfrak{F}\underline{\boxtimes}\mathfrak{G}).$$

We note the morphisms in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{T^*X\times T^*Z}))$:

$$\mathfrak{F} \overset{a}{\circ} \mathfrak{G} \to Rp_{13!!}(p_{12}^{a*}\mathfrak{F} \otimes_{\mathcal{A}_N} p_{23}^*\mathfrak{G}) \leftarrow\!\sim \mathfrak{F} \overset{a\mathcal{A}}{\circ} \mathfrak{G}, \tag{81}$$

where the second arrow is an isomorphism by Proposition 6.5.

Proposition 10.2. *For $\mathcal{A}$-modules $K_1 \in \mathrm{Mod}(\mathcal{A}_{X\times Y})$, $K_2 \in \mathrm{Mod}(\mathcal{A}_{Y\times Z})$ and $K_3 \in \mathrm{Mod}(\mathcal{A}_{X\times Z})$ with a morphism $q_{12}^*K_1 \otimes q_{23}^*K_2 \to q_{13}^*K_3 \otimes \omega'_Y$ there exists a natural morphism*

$$\mu^{\mathcal{A}}_{X\times Y}K_1 \overset{a\mathcal{A}}{\circ} \mu^{\mathcal{A}}_{Y\times Z}K_2 \to \mu^{\mathcal{A}}_{X\times Z}K_3, \tag{82}$$

with the following property. Setting $U = \dot{T}^*X \times \dot{T}^*Y$, $V = \dot{T}^*Y \times \dot{T}^*Z$, the restrictions of morphisms (80) and (82) outside the zero section make a commutative diagram in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{\dot{T}^*X\times\dot{T}^*Z}))$:

$$\begin{array}{ccc}
(\mu_{X\times Y}K_1)_U \overset{a}{\circ} (\mu_{Y\times Z}K_2)_V & \longrightarrow & \mu_{X\times Z}K_3 \\
\downarrow & & \downarrow \\
\mu^{\mathcal{A}}_{X\times Y}K_1 \overset{a\mathcal{A}}{\circ} \mu^{\mathcal{A}}_{Y\times Z}K_2 & \longrightarrow & \mu^{\mathcal{A}}_{X\times Z}K_3.
\end{array}$$

Proof. The proof is similar to the proof of Proposition 10.1, replacing operations in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_\cdot))$ by the same operations in $\mathrm{Mod}(\mathcal{A}_\cdot)$. In particular Proposition 6.7 gives the base change $p_{13!!}p^* = q_{13\pi!!}\tau_{!!}k^*\, j_\pi^* \leftarrow\!\sim q_{13\pi!!}q_{13d}^*\, j_{d!!}j_\pi^*$, which is an isomorphism because q_{13d} is an embedding and j_d is smooth. Then we use Propositions 9.2 and 9.4 instead of Propositions 9.1 and 9.3.

By Proposition 6.3 we have $q_{12}^*K_1 \otimes q_{23}^*K_2 \simeq q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2$ and $q_{13}^*K_3 \simeq q_{13}^{-1}K_3$ in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Y\times Z}))$. Hence the morphism in the hypothesis of the proposition yields a morphism $q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2 \to q_{13}^{-1}K_3 \otimes \omega_Y$ in $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{X\times Y\times Z}))$ and we

may apply Proposition 10.1. The vertical arrows in the diagram are given by the morphisms of functors $\mu \to \mu^{\mathcal{A}}$ and (81). □

We are in fact only interested in the following example. We assume now that X, Y, Z are complex analytic manifolds. We use the $\mathcal{A}$-module $\mathbb{O}_X$ and its variants introduced in Definition 6.8. We set $K_1 = \mathbb{O}^{(0,d^c_Y)}_{X\times Y}[d^c_Y]$, which gives a resolution of $\mathcal{O}^{t(0,d^c_Y)}_{X\times Y}[d^c_Y]$, $K_2 = \mathbb{O}^{(0,d^c_Z)}_{Y\times Z}[d^c_Z]$, $K_3 = \mathbb{O}^{(0,d^c_Z)}_{X\times Z}[d^c_Z]$. With these notations morphism (28) yields a morphism $q_{12}^*K_1 \otimes q_{23}^*K_2 \to q_{13}^*K_3 \otimes \omega'_Y$ and Proposition 10.2 gives the microlocal convolution:

$$\mu^{\mathcal{A}}_{X\times Y}\mathbb{O}^{(0,d^c_Y)}_{X\times Y}[d^c_Y] \overset{a\mathcal{A}}{\circ} \mu^{\mathcal{A}}_{Y\times Z}\mathbb{O}^{(0,d^c_Z)}_{Y\times Z}[d^c_Z] \to \mu^{\mathcal{A}}_{X\times Z}\mathbb{O}^{(0,d^c_Z)}_{X\times Z}[d^c_Z]. \tag{83}$$

This convolution product is associative, because the composition of kernels $\overset{a\mathcal{A}}{\circ}$ is associative, as well as the integration morphism, by Fubini.

11. Sheaves of morphisms

We will in fact use the morphisms of the previous section in a slightly more general situation, namely for complexes of the type $\mathcal{H}\mathrm{om}(\pi^{-1}F, \mu G)$, rather than μG. For this we use the following proposition. Once again we recall the convolution for sheaves and then build it for $\mathcal{A}$-modules. To compare them we use the convolution products for complexes F, G, $\mathfrak{F}, \mathfrak{G}$:

$$F \overset{0}{\circ} G = q_{13!!}(q_{12}^{-1}F \otimes q_{23}^{-1}G), \qquad \mathfrak{F} \overset{a0}{\circ} \mathfrak{G} = p_{13!!}(p_{12}^{a-1}\mathfrak{F} \otimes p_{23}^{-1}\mathfrak{G}).$$

Proposition 11.1. *We consider* $F \in \mathrm{C}^+(\mathbf{I}(\mathbf{C}_{X\times Y}))$, $G \in \mathrm{C}^+(\mathbf{I}(\mathbf{C}_{Y\times Z}))$, $\mathfrak{F} \in \mathrm{C}^+(\mathbf{I}(\mathbf{C}_{T^*(X\times Y)}))$ *and* $\mathfrak{G} \in \mathrm{C}^+(\mathbf{I}(\mathbf{C}_{T^*(Y\times Z)}))$, *there exists natural morphisms, respectively in* $\mathrm{D}^+(\mathbf{I}(\mathbf{C}_{T^*(X\times Z)}))$ *and* $\mathrm{C}^+(\mathbf{I}(\mathbf{C}_{T^*(X\times Z)}))$*:*

$$\mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a}{\circ} \mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) \to \mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F\circ G), \mathfrak{F} \overset{a}{\circ} \mathfrak{G}), \tag{84}$$

$$\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a0}{\circ} \mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) \to \mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F \overset{0}{\circ} G), \mathfrak{F} \overset{a0}{\circ} \mathfrak{G}). \tag{85}$$

For $F \in \mathrm{C}^+(\mathbf{C}_{(X\times Y)_{sa}})$, $G \in \mathrm{C}^+(\mathbf{C}_{(Y\times Z)_{sa}})$, $\mathfrak{F} \in \mathrm{Mod}(\mathcal{A}_{T^*(X\times Y)})$ *and* $\mathfrak{G} \in \mathrm{Mod}(\mathcal{A}_{T^*(Y\times Z)})$, *we also have the natural morphism*

$$\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a\mathcal{A}}{\circ} \mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) \to \mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F \overset{0}{\circ} G), \mathfrak{F} \overset{a\mathcal{A}}{\circ} \mathfrak{G}). \tag{86}$$

These morphisms fit into the commutative diagram:

$$\begin{array}{ccc}
\mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a}{\circ} \mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) & \longrightarrow & \mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F\circ G), \mathfrak{F} \overset{a}{\circ} \mathfrak{G}) \\
\uparrow & & \downarrow \\
 & & \mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F \overset{0}{\circ} G), \mathfrak{F} \overset{a}{\circ} \mathfrak{G}) \\
 & & \uparrow \\
\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a0}{\circ} \mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) & \longrightarrow & \mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F \overset{0}{\circ} G), \mathfrak{F} \overset{a0}{\circ} \mathfrak{G}) \\
\downarrow & & \downarrow \\
\mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Y}F, \mathfrak{F}) \overset{a\mathcal{A}}{\circ} \mathcal{H}\mathrm{om}(\pi^{-1}_{Y\times Z}G, \mathfrak{G}) & \longrightarrow & \mathcal{H}\mathrm{om}(\pi^{-1}_{X\times Z}(F \overset{0}{\circ} G), \mathfrak{F} \overset{a\mathcal{A}}{\circ} \mathfrak{G}).
\end{array} \tag{87}$$

Proof. We first build morphism (84), in the derived category. We keep the notations of section 10, in particular diagram (78). To simplify the notations we suppress some subscripts on π^{-1}. Let us denote by LHS the left hand side of (84). We have

$$\begin{aligned} LHS &= Rp_{13!!}p^{-1}(\mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}F,\mathfrak{F})\boxtimes \mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}G,\mathfrak{G})) \\ &\simeq Rp_{13!!}p^{-1}\,\mathrm{R}\mathcal{H}\mathrm{om}(\pi^{-1}(F\boxtimes G),\mathfrak{F}\boxtimes\mathfrak{G}). \end{aligned}$$

We can enter the functor p^{-1} inside the $\mathrm{R}\mathcal{H}\mathrm{om}$, and use the morphism of functors $Rp_{13!!}\,\mathrm{R}\mathcal{H}\mathrm{om}(\cdot,\cdot)\to \mathrm{R}\mathcal{H}\mathrm{om}(Rp_{13*}(\cdot),Rp_{13!!}(\cdot))$. Thus we obtain a morphism:

$$LHS \to \mathrm{R}\mathcal{H}\mathrm{om}(Rp_{13*}p^{-1}\pi^{-1}(F\boxtimes G),\mathfrak{F}\overset{a}{\circ}\mathfrak{G}). \tag{88}$$

We let $\sigma\colon N\times_{X\times Z}T^*(X\times Z)\to T^*N$ be induced by the inclusion of the zero section of Y and we let $\pi'_N\colon N\times_{X\times Z}T^*(X\times Z)\to N$ be the projection. Then $\pi_M\circ p\circ\sigma = j\circ\pi'_N$. Moreover, since we deal with conic sheaves, we have the isomorphism of functors $R\tau_*\simeq\sigma^{-1}$. We also have a morphism $Rq_{13\pi!!}\to Rq_{13\pi*}$. We deduce the sequence of morphisms:

$$\begin{aligned} Rp_{13*}p^{-1}\pi_M^{-1} &\simeq Rq_{13\pi*}R\tau_*p^{-1}\pi_M^{-1} \\ &\leftarrow Rq_{13\pi!!}\sigma^{-1}p^{-1}\pi_M^{-1} \\ &\simeq Rq_{13\pi!!}{\pi'}_N^{-1}j^{-1} \\ &\simeq \pi_{X\times Z}^{-1}Rq_{13!!}j^{-1}, \end{aligned} \tag{89}$$

where the last isomorphism is a base change. So we obtain $\pi_{X\times Z}^{-1}Rq_{13!!}j^{-1}\to Rp_{13*}p^{-1}\pi_M^{-1}$ and composing this morphism with (88) we deduce (84).

Morphism (85) in the category of complexes is obtained in the same way. In particular the analog of morphism (88) is obtained from the morphism of complexes $p_{13!!}\,\mathcal{H}\mathrm{om}(\cdot,\cdot)\to\mathcal{H}\mathrm{om}(p_{13*}(\cdot),p_{13!!}(\cdot))$. Moreover, τ_* is exact on conic sheaves, so that $\tau_*\simeq R\tau_*\simeq\sigma^{-1}$ and we have a sequence of morphisms in the category of complexes analog to (89) (note that the base change formula is true for complexes).

The top part of diagram (87) is given by the natural morphisms between functors and their derived functors.

The difference between morphism (86) and morphism (85) only concerns the right hand side of the $\mathcal{H}\mathrm{om}$ functors. Namely we replace the functor $\overset{a0}{\circ}$ by $\overset{a\mathcal{A}}{\circ}$ and obtain the same proof. The bottom part of diagram (87) is then given by the morphism of functors (81). □

12. $\mathcal{E}$-MODULES

In this section X is a complex analytic manifold of complex dimension $n=d_X^c$ and Δ denotes the diagonal of $X\times X$. We identify T^*X and $T^*_\Delta(X\times X)$ by the first projection. We denote by $\mathcal{E}_X$ the sheaf of microdifferential operators of finite order. This is a sheaf on T^*X and its restriction to $\dot T^*X$ was interpreted using the tempered microlocalization in [1] (see also [11]), as follows. We let $\gamma\colon\dot T^*X\to\mathbf{P}(X)$ be the projection to the complex projective bundle associated to T^*X. Then $\mathcal{E}_X\simeq\gamma^{-1}\gamma_*(\mathcal{E}_X^{\mathbf{R},f})$, where $\mathcal{E}_X^{\mathbf{R},f}$ is the sheaf on $T^*X\simeq T^*_\Delta(X\times X)$:

$$\mathcal{E}_X^{\mathbf{R},f} = T\text{-}\mu hom(\mathbf{C}_\Delta,\mathcal{O}^{(0,n)}_{X\times X}[n]).$$

The product of $\mathcal{E}_X^{\mathbf{R},f}$ is defined in [1] by the convolution product for tempered microlocalization. This can be defined also in the language of ind-sheaves, following [9]. We first define $\mathcal{E}'^{ind}_X \in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*(X\times X)}))$ by

$$\mathcal{E}'^{ind}_X = \mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}(\pi^{-1}\mathbf{C}_\Delta, \mu_{X\times X}\mathcal{O}^{t(0,n)}_{X\times X}[n]),$$

where $\mathcal{O}^{t(0,n)}_{X\times X}$, defined in (24) as an object of $\mathrm{D}^b(\mathbf{C}_{(X\times X)_{sa}})$, is now considered in $\mathrm{D}^b(\mathbf{I}(\mathbf{C}_{X\times X})$ using the functor I_τ. Thus $\mathcal{E}'^{ind}_X$ has support on $T^*X \simeq T^*_\Delta(X\times X)$ but this doesn't imply that it is the image of an ind-sheaf on T^*X. We recall the notations $p_1, p_2 \colon T^*(X\times X) \to T^*X$ for the projections, $a \colon T^*X \to T^*X$ for the antipodal map and we define the embedding

$$\delta' \colon T^*X \simeq T^*_\Delta(X\times X) \to T^*(X\times X), \quad (x,\xi) \mapsto (x,x,\xi,-\xi).$$

Since $\operatorname{supp}\mathcal{E}'^{ind}_X = T^*_\Delta(X\times X)$ the morphisms of functors $p_{1*} \to p_{1*}\delta'_*\delta'^{-1} = \delta'^{-1}$ and $p_{2*} \to a^{-1}\delta'^{-1}$ induce isomorphisms:

$$\delta'^{-1}\mathcal{E}'^{ind}_X \simeq p_{1*}\mathcal{E}'^{ind}_X \simeq a^{-1}p_{2*}\mathcal{E}'^{ind}_X. \tag{90}$$

We could write the same isomorphisms with $p_{i!!}$ instead of p_{i*} or their derived functors.

Definition 12.1. We let $\mathcal{E}^{ind}_X \in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_{T^*X}))$ be the ind-sheaf on T^*X defined by (90).

Since the functor α from ind-sheaves to sheaves commute with direct image (or inverse image) we have $\mathcal{E}_X^{\mathbf{R},f} \simeq \alpha_{T^*X}(\mathcal{E}^{ind}_X)$.

The complex $\mathcal{E}^{ind}_X$ comes with a product in the sense of Definition 3.1, defined as follows:

(i) Using (90) we see that $\mathcal{E}^{ind}_X \otimes \mathcal{E}^{ind}_X \simeq \delta'^{-1}(\mathcal{E}'^{ind}_X \overset{a}{\circ} \mathcal{E}'^{ind}_X)$.

(ii) We have $\mathbf{C}_\Delta \circ \mathbf{C}_\Delta = \mathbf{C}_\Delta$ and morphism (84), with $X = Y = Z$, gives a morphism $\mathcal{E}^{ind}_X \otimes \mathcal{E}^{ind}_X \to \delta'^{-1}\,\mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}(\pi^{-1}\mathbf{C}_\Delta, \mu\mathcal{O}^{t(0,n)}_{X\times X} \overset{a}{\circ} \mu\mathcal{O}^{t(0,n)}_{X\times X}[2n])$.

(iii) The convolution product $Rq_{13!!}(q_{12}^{-1}\mathcal{O}^{t(0,n)}_{X\times X}[n] \otimes q_{23}^{-1}\mathcal{O}^{t(0,n)}_{X\times X}[n]) \to \mathcal{O}^{t(0,n)}_{X\times X}[n]$ together with Proposition 10.1 gives a morphism $\mu\mathcal{O}^{t(0,n)}_{X\times X} \overset{a}{\circ} \mu\mathcal{O}^{t(0,n)}_{X\times X}[2n] \to \mu\mathcal{O}^{t(0,n)}_{X\times X}[n]$.

The composition of (i)–(iii) defines the product $\mathcal{E}^{ind}_X \otimes \mathcal{E}^{ind}_X \to \mathcal{E}^{ind}_X$. In the same way Propositions 11.1 and 10.1, applied to $X = Y$ and Z a point, give an action of $\mathcal{E}^{ind}_X$ on $\mu\mathcal{O}^t_X$, in the sense of Definition 3.1. We deduce an action of $\mathcal{E}^{ind}_X$ on $\mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}(\pi^{-1}F, \mu\mathcal{O}^t_X)$, for any $F \in \mathrm{D}^b(\mathbf{I}(\mathbf{C}_X))$.

This product and this action are just morphisms in the derived category and do not endow the complex $\mathcal{E}^{ind}_X$ with a structure of algebra. However, when we go back to the derived category of sheaves with the functor α_{T^*X}, the product gives a morphism $\mathcal{E}_X^{\mathbf{R},f} \otimes \mathcal{E}_X^{\mathbf{R},f} \to \mathcal{E}_X^{\mathbf{R},f}$. But $\mathcal{E}_X^{\mathbf{R},f}$ is a sheaf (i.e. concentrated in degree 0) and this morphism really endows $\mathcal{E}_X^{\mathbf{R},f}$ with a structure of sheaf of algebras. But this is not enough to define a structure of $\mathcal{E}_X^{\mathbf{R},f}$-module on $T\text{–}\mu hom(F,\mathcal{O}_X) \simeq \alpha_{T^*X}\,\mathrm{R}\mathcal{I}\mathcal{H}\mathrm{om}(\pi^{-1}F, \mu\mathcal{O}^t_X)$, which is in general not concentrated in degree.

To solve this problem we define a dg-algebra $\mathcal{E}^{\mathcal{A}}_X$ on the site X_{sa} (and not merely an object in the derived category) such that $\mathcal{E}^{ind}_X \simeq I_\tau(\mathcal{E}^{\mathcal{A}}_X)$. We also define in the same way a dg-module over $\mathcal{E}^{\mathcal{A}}_X$ representing $\mu\mathcal{O}^t_X$. In fact our definition is exactly the previous one, but in the categories of $\mathcal{A}$-modules instead of the derived categories.

Definition 12.2. We define a complex of sheaves on $T^*X_{sa} = T^*_\Delta(X \times X)_{sa}$

$$\mathcal{E}^{\mathcal{A}}_X = \delta'^{-1}\,\mathcal{H}\mathrm{om}(\pi^{-1}\mathbf{C}_\Delta, \mu^{\mathcal{A}}_{X\times X}\mathbb{O}^{(0,n)}_{X\times X}[n]),$$

with a product defined as follows:

(i) as in the case of $\mathcal{E}^{ind}_X$, morphism (86) gives a morphism

$$\mathcal{E}^{\mathcal{A}}_X \otimes \mathcal{E}^{\mathcal{A}}_X \to \delta'^{-1}\,\mathcal{H}\mathrm{om}(\pi^{-1}\mathbf{C}_\Delta, \mu^{\mathcal{A}}\mathbb{O}^{(0,n)}_{X\times X} \overset{a,\mathcal{A}}{\circ} \mu^{\mathcal{A}}\mathbb{O}^{(0,n)}_{X\times X}[2n]),$$

(ii) the convolution product (28) together with Proposition 10.2 gives a morphism

$$\mu^{\mathcal{A}}\mathbb{O}^{(0,n)}_{X\times X} \overset{a,\mathcal{A}}{\circ} \mu^{\mathcal{A}}\mathbb{O}^{(0,n)}_{X\times X}[2n] \to \mu^{\mathcal{A}}\mathbb{O}^{(0,n)}_{X\times X}[n].$$

The composition of (i) and (ii) defines the product $\mathcal{E}^{\mathcal{A}}_X \otimes \mathcal{E}^{\mathcal{A}}_X \to \mathcal{E}^{\mathcal{A}}_X$.

In the same way, Propositions 11.1 and 10.2, applied to $X = Y$ and Z a point, give a morphism $\mathcal{E}^{\mathcal{A}}_X \otimes \mu^{\mathcal{A}}_X\mathbb{O}_X \to \mu^{\mathcal{A}}_X\mathbb{O}_X$.

Proposition 12.3. *The morphisms introduced in the previous definition give $\mathcal{E}^{\mathcal{A}}_X$ a structure of dg-algebra and give $\mu^{\mathcal{A}}_X\mathbb{O}_X$ a structure of dg-$\mathcal{E}^{\mathcal{A}}_X$-module.*

Over $\dot{T}^*X$, we have isomorphisms $\mathcal{E}^{ind}_X \simeq I_\tau(\mathcal{E}^{\mathcal{A}}_X)$ and $\mu\mathcal{O}^t_X \simeq I_\tau(\mu^{\mathcal{A}}_X\mathbb{O}_X)$. Through these isomorphisms the product of $\mathcal{E}^{\mathcal{A}}_X$ and its action on $\mu^{\mathcal{A}}_X\mathbb{O}_X$ coincide with the product of $\mathcal{E}^{ind}_X$ and its action on $\mu\mathcal{O}^t_X$ defined above.

Proof. The complex $\mathcal{E}^{\mathcal{A}}_X$ is a dg-algebra and $\mu^{\mathcal{A}}_X\mathbb{O}_X$ is a dg-$\mathcal{E}^{\mathcal{A}}_X$-module because the product and the action are defined in categories of complexes, and not merely up to homotopy.

Let us check that the product of $\mathcal{E}^{\mathcal{A}}_X$ represents the product of $\mathcal{E}^{ind}_X$ and that their action on $\mu^{\mathcal{A}}_X\mathbb{O}_X$ and $\mu\mathcal{O}^t_X$ are the same. This is a consequence of diagram (87) and Proposition 10.2; but in diagram (87) some vertical arrows go in the wrong direction and the commutative diagram in Proposition 10.2 requires a restriction outside the zero section. These problems are solved as follows.

In diagram (87) the vertical arrows are isomorphisms. Indeed, we consider the cases $\mathfrak{F} = \mu^{\mathcal{A}}_X\mathbb{O}^{(0,n)}_{X\times X}[n]$ and $\mathfrak{G} = \mathfrak{F}$ or $\mathfrak{G} = \mu^{\mathcal{A}}_X\mathbb{O}_X$. Hence, by Proposition 7.10, $\mathfrak{F}$ and $\mathfrak{G}$ consist of quasi-injective sheaves (on the site T^*X_{sa}), and so are acyclic for the functors $\mathcal{H}\mathrm{om}(H, \cdot)$, when H is constructible. In our cases the complexes F, G in the diagram are $\mathbf{C}_\Delta$ or $\mathbf{C}_X$, so that the $\mathcal{H}$om sheaves are isomorphic to the R$\mathcal{H}$om. For the composition of kernels $\circ$ we also have to compute a direct image. Since we deal with $\mathcal{A}$-modules, Proposition 6.5 implies that direct images and derived direct images coincide. This proves that the vertical arrows are isomorphisms.

This diagram can be extended to the right, using Proposition 10.2. We can use the commutative diagram of Proposition 10.2 because of the following remark: setting $U = \dot{T}^*X \times \dot{T}^*X$, we have, on $\dot{T}^*X \times T^*X$, $(\mathcal{E}'^{ind}_X)_U \xrightarrow{\sim} \mathcal{E}'^{ind}_X$. Then, for the same reason as above, the right vertical arrows in this extended diagram are isomorphisms. □

We still have to make the link between $\mathcal{E}^{\mathcal{A}}_X$ and $\mathcal{E}^{\mathbf{R},f}_X$. We note that $\rho^{-1}\mathcal{E}^{\mathcal{A}}_X$ is quasi-isomorphic to $\mathcal{E}^{\mathbf{R},f}_X$. In particular $\rho^{-1}\mathcal{E}^{\mathcal{A}}_X$ has its cohomology concentrated in degree 0 and we have isomorphisms of sheaves:

$$\mathcal{E}^{\mathbf{R},f}_X \simeq H^0(\rho^{-1}\mathcal{E}^{\mathcal{A}}_X) \simeq H^0(\alpha I_\tau(\mathcal{E}^{\mathcal{A}}_X)).$$

Moreover the structure of dg-algebra on $\mathcal{E}_X^{\mathcal{A}}$ gives a structure of dg-algebra on $\rho^{-1}\mathcal{E}_X^{\mathcal{A}}$ and a structure of algebra on $H^0(\rho^{-1}\mathcal{E}_X^{\mathcal{A}})$. The above proposition implies that this product induced on $\mathcal{E}_X^{\mathbf{R},f}$ coincides with the one defined previously.

We also have a structure of dg-$\mathcal{E}_X^{\mathcal{A}}$-module on $\mu_X^{\mathcal{A}}\mathbb{O}_X$; in particular it defines an object $I_\tau(\mu_X^{\mathcal{A}}\mathbb{O}_X) \in \mathrm{D}(I_\tau(\mathcal{E}_X^{\mathcal{A}}))$. For any $G \in \mathrm{D}^-(\mathbf{I}(\mathbf{C}_{T^*X}))$ the complex $\mathrm{R}\mathcal{IH}\mathrm{om}(G, I_\tau(\mu_X^{\mathcal{A}}\mathbb{O}_X))$ is thus also naturally defined as an object of $\mathrm{D}(I_\tau(\mathcal{E}_X^{\mathcal{A}}))$. For $G = \pi^{-1}F$, $F \in \mathrm{D}^-(\mathbf{I}(\mathbf{C}_X))$, we deduce that

$$T\text{–}\mu hom(F, \mathcal{O}_X) = \alpha\, \mathrm{R}\mathcal{IH}\mathrm{om}(\pi^{-1}F, I_\tau(\mu_X^{\mathcal{A}}\mathbb{O}_X)) \quad \in \mathrm{D}(\rho^{-1}\mathcal{E}_X^{\mathcal{A}}),$$

and, by construction, the corresponding action in $\mathrm{D}(\mathbf{C}_{T^*X})$

$$\rho^{-1}\mathcal{E}_X^{\mathcal{A}} \otimes T\text{–}\mu hom(F, \mathcal{O}_X) \to T\text{–}\mu hom(F, \mathcal{O}_X)$$

coincides with the action of $\mathcal{E}_X^{\mathbf{R},f}$ on $T\text{–}\mu hom(F, \mathcal{O}_X)$ defined above. Thus we are almost done, except that $T\text{–}\mu hom(F, \mathcal{O}_X)$ is defined as an object of $\mathrm{D}(\rho^{-1}\mathcal{E}_X^{\mathcal{A}})$ rather than $\mathrm{D}(\mathcal{E}_X^{\mathbf{R},f})$. But the dg-algebra $\rho^{-1}\mathcal{E}_X^{\mathcal{A}}$ is quasi-isomorphic to $\mathcal{E}_X^{\mathbf{R},f}$ and it just remains to apply Corollary 3.3, as follows.

We have the quasi-isomorphisms of dg-algebras on $\dot{T}^*X$

$$\rho^{-1}\mathcal{E}_X^{\mathcal{A}} \xleftarrow{\phi_{\leq 0}} \tau_{\leq 0}\rho^{-1}\mathcal{E}_X^{\mathcal{A}} \xrightarrow{\phi_0} \mathcal{E}_X^{\mathbf{R},f},$$

and the equivalence of categories $\phi_0^* \circ \phi_{\leq 0*} \colon \mathrm{D}(\rho^{-1}\mathcal{E}_X^{\mathcal{A}}) \xrightarrow{\sim} \mathrm{D}(\mathcal{E}_X^{\mathbf{R},f})$. We set $\mathcal{E}'_X = \beta_{T^*X}(\rho^{-1}\mathcal{E}_X^{\mathcal{A}})$ so that we have an adjunction morphism $\mathcal{E}'_X \to I_\tau(\mathcal{E}_X^{\mathcal{A}})$. This morphism induces a functor of restriction of scalars, and $\phi_0^* \circ \phi_{\leq 0*}$ induces an equivalence of categories:

$$r \colon \mathrm{D}(I_\tau(\mathcal{E}_X^{\mathcal{A}})) \to \mathrm{D}(\mathcal{E}'_X), \qquad \Phi \colon \mathrm{D}(\mathcal{E}'_X) \xrightarrow{\sim} \mathrm{D}(\beta_{T^*X}(\mathcal{E}_X^{\mathbf{R},f})).$$

Hence we obtain an object $\mathcal{O}_X^\mu = \Phi(r(I_\tau(\mu_X^{\mathcal{A}}\mathbb{O}_X))) \in \mathrm{D}(\beta_{T^*X}(\mathcal{E}_X^{\mathbf{R},f})))$, representing $\mu\mathcal{O}_X^t$ and we can state the final result:

Theorem 12.4. *The object $\mathcal{O}_X^\mu \in \mathrm{D}(\beta_{T^*X}(\mathcal{E}_X^{\mathbf{R},f})))$ defined above, over $\dot{T}^*X$, is send to $\mu_X\mathcal{O}_X^t$ in $\mathrm{D}(\mathbf{I}(\mathbf{C}_{\dot{T}^*X}))$ by the forgetful functor. It satisfies moreover: for $F \in \mathrm{D}^-(\mathbf{I}(\mathbf{C}_X))$ the complex*

$$\alpha_{T^*X}\, \mathrm{R}\mathcal{IH}\mathrm{om}(\pi^{-1}F, \mathcal{O}_X^\mu)$$

which is naturally defined in $\mathrm{D}(\mathcal{E}_X^{\mathbf{R},f})$, over $\dot{T}^*X$, is isomorphic in $\mathrm{D}(\mathbf{C}_{\dot{T}^*X})$ to $T\text{–}\mu hom(F, \mathcal{O}_X)$ endowed with its action of $\mathcal{E}_X^{\mathbf{R},f}$.

References

[1] E. Andronikof, *Microlocalisation tempérée.* (French) Mém. Soc. Math. France (N.S.) No. **57** (1994).

[2] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, **1578**, Springer-Verlag, Berlin, (1994).

[3] E. Bierstone and P. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. No. **67** (1988), 5–42.

[4] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 2, 319–365.

[5] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, **292**, Springer-Verlag, Berlin, (1990).

[6] ———, *Moderate and formal cohomology associated with constructible sheaves*, Mém. Soc. Math. France (N.S.) No. **64** (1996).

[7] ———, *Ind-sheaves*, Astérisque, **271**, (2001).

[8] ———, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften, **332**, Springer-Verlag, Berlin, (2006).
[9] M. Kashiwara, P. Schapira, F. Ivorra and I. Waschkies, *Microlocalization of ind-sheaves*, in: Studies in Lie theory Progr. Math. **243**, Birkhäuser (2006) 171–221.
[10] L. Prelli, *Sheaves on subanalytic sites*, arXiv:math/0505498
[11] ———, *Microlocalization of subanalytic sheaves.* C. R. Math. Acad. Sci. Paris **345** (2007), no. 3, 127–132.
[12] J.-C. Tougeron, *Idéaux de fonctions différentiables*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **71**. Springer-Verlag, Berlin-New York, (1972).

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