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# BROWNIAN MOTION CONDITIONED TO STAY IN A CONE

RODOLPHE GARBIT

ABSTRACT. A result of R. Durrett, D. Iglehart and D. Miller states that Brownian meander is Brownian motion conditioned to stay positive for a unit of time, in the sense that it is the weak limit, as  $x$  goes to 0, of Brownian motion started at  $x > 0$  and conditioned to stay positive for a unit of time. We extend this limit theorem to the case of multi-dimensional Brownian motion conditioned to stay in a smooth convex cone. Properties of the limit process are obtained and applications to random walks are given.

## 1. INTRODUCTION

The purpose of this paper is to prove the existence of a process which is, in some sense, multidimensional Brownian motion started at the vertex of a smooth convex cone and conditioned to stay in it for a unit of time.

Let  $\mathcal{C}_\infty$  be the space of continuous functions  $w : [0, +\infty) \rightarrow \mathbb{R}^d$ ,  $d \geq 1$ , endowed with the topology of uniform convergence on compact subsets induced by the distance

$$d(w, w') = \sum_{n>0} 2^{-n} (\max_{t \in [0, n]} \|w(t) - w'(t)\| \wedge 1),$$

and let  $\mathcal{F}$  be the corresponding Borel  $\sigma$ -algebra. We shall use  $\mathcal{C}_\infty$  as a concise notation for  $(\mathcal{C}_\infty, \mathcal{F})$ . If  $(\mu_n), \mu$  are probability measures on  $\mathcal{C}_\infty$ , the sequence  $(\mu_n)$  is said to converge weakly to  $\mu$  if

$$\int f d\mu_n \rightarrow \int f d\mu$$

for all bounded and continuous function  $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ . Weak convergence will be denoted by the symbol  $\Rightarrow$ .

Let  $\{X_t, t \geq 0\}$  be the canonical process on  $\mathcal{C}_\infty$  for which  $X_t(w) = w(t)$  for any  $w \in \mathcal{C}_\infty$ . Consider an open cone  $C$  with vertex at the origin 0 and let  $\tau_C = \inf\{t > 0 : X_t \notin C\}$  be the first exit time of the canonical process from  $C$ . For any  $x \in C$  we define the law  $\widetilde{W}_{x,1}^C$  of the Brownian motion started at  $x$  and conditioned to stay in  $C$  for a unit of time by the formula

$$\widetilde{W}_{x,1}^C(*) = W_x(* | \tau_C > 1),$$

where  $W_x$  is the distribution on  $\mathcal{C}_\infty$  of the standard  $d$ -dimensional Brownian motion started at  $x$ .

Our main result is the following theorem which states that the Brownian motion started at  $x$  and conditioned to stay in  $C$  for unit of time converges in law to a limit process as  $x \in C$  tends to 0, when  $C$  is a *nice* cone. The precise definition of a nice cone is given in Section 5; for example, any circular cone is nice.

**Theorem 1.1.** *Suppose  $C$  is a nice cone. As  $x \in C$  goes to 0, the law  $\widetilde{W}_{x,1}^C$  converges weakly on  $\mathcal{C}_\infty$  to a limit  $\widetilde{W}_{0,1}^C$ .*

*For any  $t \in (0, 1]$ , the entrance law  $\widetilde{W}_{0,1}^C(X_t \in dy)$  has the density  $e(t, y)$  (w.r.t. Lebesgue measure) given by the formula (23).*

Theorem 1.1 is the multidimensional analog of Durrett, Iglehart and Miller result ([5], Theorem 2.1) in which they consider Brownian motion conditioned to stay positive for a unit of time and identify the limit as the Brownian meander. In the case of two-dimensional Brownian motion, Theorem 1.1 is due to Shimura [8]. Our proof of Theorem 1.1 is in the same spirit as Shimura's one. We first prove convergence of the finite-dimensional distributions by using an explicit formula for the heat kernel of a cone given by Bañuelos and Smits in [1]. Then, we prove tightness of the laws  $\widetilde{W}_{x,1}^C$  as  $x \in C \rightarrow 0$  using Shimura's principle: If  $\widetilde{W}_{x,1}^C$  converges weakly as  $x$  tends to any point  $x_0 \in \partial C \setminus \{0\}$ , then the tightness as  $x \in C \rightarrow 0$  follows. For a two-dimensional cone  $C$ , proving weak convergence of  $\widetilde{W}_{x,1}^C$  as  $x \rightarrow x_0 \in \partial C \setminus \{0\}$  is essentially a one-dimensional problem since the boundary of  $C$  is locally linear at  $x_0$ . But in higher dimension, the problem becomes harder. It is the reason why the major part of this paper is intended to provide tools for proving this fact.

In Section 2, we consider the general problem of Brownian motion conditioned to stay in an open set  $U$  and give some useful properties of the conditioned laws  $\widetilde{W}_{x,1}^U$ , such as the Markov property and a form of continuity with respect to the variable  $x$ . In Section 3, we recall Durret-Iglehart-Miller result on Brownian motion conditioned to stay positive and we extend it to the case of Brownian motion conditioned to stay in a half-plane. From the half-space case, we derive in Section 4 a convergence theorem for  $\widetilde{W}_{x,1}^U$  as  $x \rightarrow x_0 \in \partial U$  when  $U$  is nice at  $x_0$ . Finally, in Section 5 we present a complete proof of Theorem 1.1, and we give some important properties of the limit process.

Our interest in such a limit theorem is related to random walks. Bolthausen proved in [3] that a one-dimensional square integrable centered random walk conditioned to stay positive until time  $n$  converges to a Brownian meander, that is, a Brownian motion conditioned to stay positive for a unit of time. By analogy, the limit process given in Theorem 1.1 is very likely to be the limit in law of a multidimensional random walk conditioned to stay in a cone. Under the assumption that such a functional Central Limit Theorem holds, we derive in 5.3 information on the tail probability of the first exit

time of the random walk from a cone. The functional CLT in itself will be the subject of a forthcoming article.

*Unusual notations.* If  $\mu$  is a probability measure on a space  $(X, \mathcal{A})$ , we will denote by  $\mu(f)$  the expectation of a measurable function with respect to  $\mu$ . For a set  $A \in \mathcal{A}$  and a measurable function  $f$ , the notation  $\mu(A; f)$  stands for  $\mu(\mathbb{1}_A \times f)$ , where  $\mathbb{1}_A$  is the characteristic function of the set  $A$ . For consistency,  $\mu(A; B)$  will often be preferred to  $\mu(A \cap B)$ .

## 2. BASIC FACTS ABOUT THE CONDITIONED LAWS

**2.1. Markov property.** Let  $U$  be an open subset of  $\mathbb{R}^d$  and let  $\tau_U$  be the first exit time from  $U$ . For any  $x \in U$  and  $t > 0$ , we set

$$\widetilde{W}_{x,t}^U(*) = W_x(* | \tau_U > t) = \frac{W_x(*; \tau_U > t)}{W_x(\tau_U > t)}.$$

For convenience, we will also use the notation  $\widetilde{W}_{x,t}^U := W_x$  for any  $t \leq 0$  and  $x \in \mathbb{R}^d$ .

Let  $\mathcal{F}_t$  be the  $\sigma$ -algebra generated by the random variables  $\{X_s, s \leq t\}$  and let  $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ . The shift operator  $\theta_t$  on  $\mathcal{C}_\infty$  is defined by  $\theta_t(w)(s) = w(t+s)$ .

The laws  $\widetilde{W}^U$  inherit a Markov property from Brownian motion:

**Proposition 2.1** (Markov property). *Let  $x \in U$  and  $t > 0$ . For all  $s \geq 0$ ,  $A \in \mathcal{F}_{s+}$  and  $B \in \mathcal{F}$ ,*

$$\widetilde{W}_{x,t}^U(A; \theta_s^{-1}B) = \widetilde{W}_{x,t}^U\left(A; \widetilde{W}_{X(s),t-s}^U(B)\right).$$

*Proof.* Suppose first that  $s \in [0, t)$ . By the Markov property of Brownian motion, we get

$$\begin{aligned} & W_x(A; \theta_s^{-1}B; \tau_U > t) \\ &= W_x(A; \tau_U > s; W_{X(s)}(B; \tau_U > t-s)) \\ &= W_x\left(A; \tau_U > s; \widetilde{W}_{X(s),t-s}^U(B) W_{X(s)}(\tau_U > t-s)\right). \end{aligned}$$

Moreover,  $\widetilde{W}_{X(s),t-s}^U(B)$  is  $\mathcal{F}_s$ -measurable, and the Markov property also gives

$$\begin{aligned} & W_x\left(A; \widetilde{W}_{X(s),t-s}^U(B); \tau_U > t\right) \\ &= W_x\left(A; \widetilde{W}_{X(s),t-s}^U(B); \tau_U > s; W_{X(s)}(\tau_U > t-s)\right). \end{aligned}$$

Thus we have

$$W_x(A; \theta_s^{-1}B; \tau_U > t) = W_x\left(A; \widetilde{W}_{X(s),t-s}^U(B); \tau_U > t\right),$$

and the result follows.

Now suppose  $s \geq t$ . Since  $\{\tau_U > t\}$  is  $\mathcal{F}_s$ -measurable, the Markov property gives

$$W_x(A; \theta_s^{-1}B; \tau_U > t) = W_x(A; W_{X(s)}(B); \tau_U > t) ,$$

which is the desired result.  $\square$

We shall also need the related strong Markov property; the proof is quite similar and is omitted here.

**Proposition 2.2** (Strong Markov property). *Let  $x \in U$  and  $t > 0$ . For any optional time  $\tau$ , any  $A \in \mathcal{F}_{\tau+}$  and any positive measurable function  $f(s, w) : [0, +\infty) \times \mathcal{C}_\infty \rightarrow \mathbb{R}$ , we have*

$$\widetilde{W}_{x,t}^U(A; \tau < t; f(\tau, \theta_\tau)) = \widetilde{W}_{x,t}^U\left(A; \tau < t; \widetilde{W}_{X(\tau), t-\tau}^U(f(s, \cdot))|_{s=\tau}\right) .$$

**2.2. Continuity.** Let  $U$  be an open subset of  $\mathbb{R}^d$ . We will say that  $U$  is co-regular if  $W_x(\tau_{\overline{U}} > 0) = 0$  for every  $x \in \partial U$ ; that is, a Brownian motion started at any point of the boundary of  $U$  visits instantaneously the complement of  $\overline{U}$ . For such a set,  $\tau_U$  and  $\tau_{\overline{U}}$  are almost surely equal.

**Proposition 2.3.** *Suppose  $U$  is co-regular. For every bounded continuous function  $f$  on  $\mathcal{C}_\infty$ , the mapping  $(x, t) \mapsto \widetilde{W}_{x,t}^U(f)$  is continuous on  $U \times (0, +\infty)$ .*

*Proof.* Since  $W_x(\tau_U > t) > 0$  for any  $(x, t) \in U \times (0, +\infty)$ , it suffices to prove that the mapping  $(x, t) \mapsto W_x(f; \tau_U > t)$  is continuous on  $U \times (0, +\infty)$ . Suppose  $x_n \rightarrow x \in U$  and  $t_n \rightarrow t > 0$ , and set

$$\phi_n(w) = f(x_n - x + w) \mathbb{1}_{\{\tau_U > t_n\}}(x_n - x + w)$$

and

$$\phi(w) = f(w) \mathbb{1}_{\{\tau_U > t\}}(w) .$$

Since  $W_{x_n}(f; \tau_U > t_n) = W_x(\phi_n)$  and  $W_x(f; \tau_U > t) = W_x(\phi)$ , it is enough to show that  $\phi_n(w) \rightarrow \phi(w)$  for  $W_x$ -almost every  $w$ .

Set  $\Omega = \{X_0 = x; \tau_U = \tau_{\overline{U}} \neq t\}$  and choose a path  $w \in \Omega$  and a sequence  $(w_n)$  that converges to  $w$  on  $\mathcal{C}_\infty$ .

If  $\tau_U(w) > t$ , then there exists  $\epsilon > 0$  such that  $w(s) \in U$  for all  $s \in [0, t + \epsilon]$ . Since  $U$  is open and  $w_n \rightarrow w$  uniformly on  $[0, t + \epsilon]$ , for  $n$  large enough we have  $w_n(s) \in U$  for all  $s \in [0, t + \epsilon]$  and we have  $t_n < t + \epsilon$ , thus  $\tau_U(w_n) > t_n$ .

On the other hand, if  $\tau_U(w) = \tau_{\overline{U}}(w) < t$ , then there exists  $s < t$  such that  $w(s) \notin \overline{U}$ . If  $n$  is sufficiently large, then  $w_n(s) \notin \overline{U}$  and  $s < t_n$ , so that  $\tau_U(w_n) < t_n$ . Hence, in each case,

$$\mathbb{1}_{\{\tau_U > t\}}(w_n) \rightarrow \mathbb{1}_{\{\tau_U > t\}}(w) .$$

Together with  $W_x(\Omega) = 1$ , this proves that  $\phi_n(w) \rightarrow \phi(w)$  for  $W_x$ -almost every  $w$ .  $\square$

**2.3. Finite-dimensional distributions.** Let  $U \subset \mathbb{R}^d$  be a co-regular open set and  $x_0$  a boundary point of  $U$ . Thanks to the Markov property and the continuity, it is now easy to find a sufficient condition for the weak convergence of the finite-dimensional distributions of  $\widetilde{W}_{x,1}^U$ , as  $x$  tends to  $x_0$ , that only involves the first transitions.

**Proposition 2.4.** *Suppose that for any  $t \in (0, 1)$ , the first transition law  $\widetilde{W}_{x,1}^U(X_t \in dy)$  converges weakly as  $x \rightarrow x_0$  to a probability measure for which  $\partial U$  is a null set. Then, the finite-dimensional distributions of  $\widetilde{W}_{x,1}^U$  converge weakly as  $x \rightarrow x_0$ .*

*Proof.* For all  $t \in (0, 1)$ , let us denote by  $\mu_t$  the limit of  $\widetilde{W}_{x,1}^U(X_t \in dy)$  as  $x \rightarrow x_0$ . By Portemanteau Theorem,

$$\mu_t(\overline{U}) \geq \limsup_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(X_t \in U) = 1.$$

Since  $\mu_t(\partial U) = 0$ , we have  $\mu_t(U) = 1$ .

Now, consider times  $0 < t_1 < t_2 < \dots < t_n$  and a bounded continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and set  $F = f(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ . Fix  $0 < t < \min(t_1, 1)$  and observe that for all  $x \in U$  we have

$$\begin{aligned} \widetilde{W}_{x,1}^U(F) &= \widetilde{W}_{x,1}^U\left(\widetilde{W}_{X_t,1-t}^U(f(X_{t_1-t}, X_{t_2-t}, \dots, X_{t_n-t}))\right) \\ &= \widetilde{W}_{x,1}^U(H_t(X_t)), \end{aligned}$$

with  $H_t(y) = \widetilde{W}_{y,1-t}^U(f(X_{t_1-t}, X_{t_2-t}, \dots, X_{t_n-t}))$ . By Proposition 2.3, the function  $y \mapsto H_t(y)$  is continuous on  $U$ . Since  $\widetilde{W}_{x,1}^U(X_t \in dy)$  converges weakly to  $\mu_t(dy)$  as  $x \rightarrow x_0$ , it follows from the continuous mapping theorem that

$$\lim_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(H_t(X_t)) = \mu_t(H_t).$$

That is

$$\lim_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(F) = \int \widetilde{W}_{y,1-t}^U(f(X_{t_1-t}, X_{t_2-t}, \dots, X_{t_n-t})) \mu_t(dy).$$

The expression on the right side of this equation clearly defines a probability measure on  $\mathbb{R}^n$ , thus Proposition 2.4 is proved.  $\square$

Let  $x \in U$  and  $t > 0$ . The Markov property of Brownian motion gives

$$\begin{aligned} \widetilde{W}_{x,1}^U(X_t \in dy) &= \frac{W_x(X_t \in dy; \tau_U > 1)}{W_x(\tau_U > 1)} \\ &= \frac{W_x(X_t \in dy; \tau_U > t; W_{X_t}(\tau_U > 1 - t))}{W_x(\tau_U > 1)}. \end{aligned}$$

Since the transitions  $W_x(X_t \in dy; \tau_U > t)$  of Brownian motion killed on the boundary of  $U$  have densities  $p^U(t, x, y)$  with respect to Lebesgue measure

$dy$ , we get

$$\widetilde{W}_{x,1}^U(X_t \in dy) = \frac{p^U(t, x, y)}{W_x(\tau_U > 1)} W_y(\tau_U > 1 - t) dy .$$

Hence, proving convergence of the finite-dimensional distributions of  $\widetilde{W}_{x,1}^U$  consists essentially in finding an equivalent of the heat kernel  $p^U(t, x, y)$  as  $x \rightarrow x_0$ .

**2.4. Neat convergence.** Let  $U \subset \mathbb{R}^d$  be a co-regular open set and let  $x_0$  be a boundary point of  $U$ . We shall now take the opposite view to the initial problem and exhibit conditions ensuring that some properties of the conditional distributions are preserved by weak limit.

Suppose that there exists a law  $\widetilde{W}_{x_0,1}^U$  on  $\mathcal{C}_\infty$  such that  $\widetilde{W}_{x,1}^U \Rightarrow \widetilde{W}_{x_0,1}^U$  as  $x \in U$  tends to  $x_0$ . We will say that the convergence is neat (or that  $\widetilde{W}_{x,1}^U$  converges neatly to  $\widetilde{W}_{x_0,1}^U$  as  $x \in U \rightarrow x_0$ ) if the limit process does not leave  $U$  before time 1, *i.e.*  $\widetilde{W}_{x_0,1}^U(\tau_U > 1) = 1$ . The next proposition gives a sufficient condition for neat convergence and states that the Markov property then holds for the limit process.

**Proposition 2.5.** *Suppose  $\widetilde{W}_{x,1}^U \Rightarrow \widetilde{W}_{x_0,1}^U$  as  $x \in U \rightarrow x_0$ .*

*If  $\widetilde{W}_{x_0,1}^U(X_t \in \partial U) = 0$  for all  $0 < t < 1$ , then the convergence is neat, and the limit process  $\widetilde{W}_{x_0,1}^U$  has the Markov property:*

*For all  $t > 0$ ,  $A \in \mathcal{F}_{t+}$  and  $B \in \mathcal{F}$ ,*

$$(1) \quad \widetilde{W}_{x_0,1}^U(A; \theta_t^{-1}B) = \widetilde{W}_{x_0,1}^U\left(A; \widetilde{W}_{X_t,1-t}^U(B)\right) .$$

*Proof.* Let  $t \in (0, 1)$ . By Portemanteau Theorem, we get

$$\widetilde{W}_{x_0,1}^U(X_t \in \bar{U}) \geq \limsup_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(X_t \in U) = 1 .$$

Since  $\widetilde{W}_{x_0,1}^U(X_t \in \partial U) = 0$ , we see that  $\widetilde{W}_{x_0,1}^U(X_t \in U) = 1$ . Note that it is enough for the right hand side of equation (1) to be correctly defined. We will now prove that the Markov property holds for  $\widetilde{W}_{x_0,1}^U$  and then use it to show that the convergence is neat.

Consider a finite sequence  $0 < s_1 < s_2 < \dots < s_n$  and a function  $G = g(X_{s_1}, X_{s_2}, \dots, X_{s_n})$ , where  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded and continuous. Choose  $\epsilon \in (0, s_1)$ , and consider another finite sequence  $0 \leq t_1 < t_2 < \dots < t_m \leq t + \epsilon$  and a function  $F = f(X_{t_1}, X_{t_2}, \dots, X_{t_m})$ , where  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is bounded and continuous. For every  $x \in U$ , we have

$$\begin{aligned} \widetilde{W}_{x,1}^U(F; G \circ \theta_t) &= \widetilde{W}_{x,1}^U(F; g(X_{t+s_1}, \dots, X_{t+s_n})) \\ &= \widetilde{W}_{x,1}^U\left(F; \widetilde{W}_{X(t+\epsilon),1-t-\epsilon}^U(g(X_{s_1-\epsilon}, \dots, X_{s_n-\epsilon}))\right) \\ &= \widetilde{W}_{x,1}^U(F; H_\epsilon(X_{t+\epsilon})) , \end{aligned}$$

where  $H_\epsilon(y) = \widetilde{W}_{y,1-t-\epsilon}^U(g(X_{s_1-\epsilon}, \dots, X_{s_n-\epsilon}))$ .

If  $t + \epsilon \geq 1$ , then  $H_\epsilon$  is defined on  $\mathbb{R}^d$  by the formula

$$H_\epsilon(y) = W_y(g(X_{s_1-\epsilon}, \dots, X_{s_n-\epsilon}))$$

and is therefore continuous everywhere; thus the function  $H_\epsilon(X_{t+\epsilon})$  is continuous on  $\mathcal{C}_\infty$ .

If  $t + \epsilon < 1$ , then  $H_\epsilon$  is defined and continuous on  $U$  by Proposition 2.3. Since the coordinate mapping  $X_{t+\epsilon}$  is continuous on  $\mathcal{C}_\infty$  and  $\widetilde{W}_{x_0,1}^U$ -almost surely takes its values in  $U$ , we see that the mapping  $H_\epsilon(X_{t+\epsilon})$  is almost-surely continuous with respect to  $\widetilde{W}_{x_0,1}^U$ .

Thus, in each case, the continuous mapping theorem gives

$$\lim_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(F; H_\epsilon(X_{t+\epsilon})) = \widetilde{W}_{x_0,1}^U(F; H_\epsilon(X_{t+\epsilon})).$$

We have also, by hypothesis

$$\lim_{x \rightarrow x_0} \widetilde{W}_{x,1}^U(F; G \circ \theta_t) = \widetilde{W}_{x_0,1}^U(F; G \circ \theta_t).$$

Hence

$$(2) \quad \widetilde{W}_{x_0,1}^U(F; G \circ \theta_t) = \widetilde{W}_{x_0,1}^U(F; H_\epsilon(X_{t+\epsilon})).$$

By density, relation (2) holds for all  $\mathcal{F}_{s+\epsilon}$ -measurable function  $F$ , and *a fortiori* for all  $F \in \mathcal{F}_{s+}$ . Now suppose  $F \in \mathcal{F}_{s+}$ . We shall let  $\epsilon \rightarrow 0$  in (2).

If  $t \geq 1$ , then  $H_\epsilon(X_{t+\epsilon}) = W_{X(t+\epsilon)}(g(X_{s_1-\epsilon}, \dots, X_{s_n-\epsilon}))$ , and it is clear that  $H_\epsilon(X_{t+\epsilon})(w) \rightarrow H_0(X_t)(w)$  as  $\epsilon \rightarrow 0$  for all  $w \in \mathcal{C}_\infty$ .

If  $t < 1$ , then by Proposition 2.3, we have  $\widetilde{W}_{w(t+\epsilon),1-t-\epsilon}^U \Rightarrow \widetilde{W}_{w(t),1-t}^U$  as  $\epsilon \rightarrow 0$  for all  $w \in \mathcal{C}_\infty$  such that  $w(t) \in U$ . Since the collection of those  $w$  forms a set of full  $\widetilde{W}_{x_0,1}^U$ -measure, we see that  $H_\epsilon(X_{t+\epsilon})$  converges  $\widetilde{W}_{x_0,1}^U$ -almost surely to  $H_0(X_t)$  as  $\epsilon \rightarrow 0$ .

In each case, the dominated convergence theorem gives

$$\lim_{\epsilon \rightarrow 0} \widetilde{W}_{x_0,1}^U(F; H_\epsilon(X_{t+\epsilon})) = \widetilde{W}_{x_0,1}^U(F; H_0(X_t)).$$

Hence

$$\widetilde{W}_{x_0,1}^U(F; G \circ \theta_t) = \widetilde{W}_{x_0,1}^U(F; H_0(X_t)) = \widetilde{W}_{x_0,1}^U(F; \widetilde{W}_{X_t,1-t}^U(G)).$$

Here again, by density, the result holds for all  $G \in \mathcal{F}$ .

It remains to prove that  $\widetilde{W}_{x_0,1}^U(\tau_U > 1) = 1$ . By the Markov property of  $\widetilde{W}_{x_0,1}^U$ , we have

$$\widetilde{W}_{x_0,1}^U(\forall t \in (\epsilon, 1], X_t \in U) = \widetilde{W}_{x_0,1}^U\left(\widetilde{W}_{X_{\epsilon,1-\epsilon}}^U(\tau_U > 1 - \epsilon)\right) = 1$$

for all  $\epsilon \in (0, 1)$ . The expected result follows by letting  $\epsilon \rightarrow 0$ .  $\square$

**Remark 2.6.** Letting  $t \rightarrow 0$  in (1) would give a zero-one law for  $\widetilde{W}_{x_0,1}^U$  (i.e.  $\widetilde{W}_{x_0,1}^U(A) = 0$  or  $1$  if  $A \in \mathcal{F}_{0+}$ ) if we had the stronger assumption that  $\widetilde{W}_{x,t}^U \Rightarrow \widetilde{W}_{x_0,1}^U$  as  $(x, t) \rightarrow (x_0, 1)$ . Note that in the special case where  $U = \mathcal{C}$



is a cone, the last convergence follows from the hypothesis  $\widetilde{W}_{x,1}^C \Rightarrow \widetilde{W}_{x_0,1}^C$  because of the scaling property of Brownian motion. More precisely, let  $K_t$  be the scaling operator defined for all  $w \in \mathcal{C}_\infty$  by

$$K_t(w)(s) = \sqrt{t}w(s/t) .$$

Recall that  $W_0$  is  $K_t$ -invariant. From the scaling invariance of the cone  $C$ , it is easily checked that

$$\widetilde{W}_{x,t}^C = \widetilde{W}_{x/\sqrt{t},1}^C \circ K_t^{-1} .$$

If  $(x, t) \rightarrow (x_0, 1)$ , then  $x/\sqrt{t} \rightarrow x_0$  and, by the continuous mapping theorem, we get

$$\widetilde{W}_{x,t}^C \Rightarrow \widetilde{W}_{x_0,1}^C \circ K_1^{-1} = \widetilde{W}_{x_0,1}^C .$$

Therefore, the zero-one law follows under the hypothesis of Proposition 2.5.

### 3. THE HALF-SPACE CASE

**3.1. Brownian motion conditioned to stay positive.** We will now recall the one dimensional theorem of Durrett, Iglehart and Miller ([5], Theorem 2.1) and give a sketch of their proof. Auxiliary results such as Propositions 3.2 and 3.3 shall also be used in Section 4. Throughout this section we set  $d = 1$  and we denote by  $\tau_+ = \inf\{t > 0 : X_t \leq 0\}$  the first exit time from the half-line  $(0, +\infty)$ . The related conditional laws will be denoted by  $\widetilde{W}_{x,1}^+$ .

The Brownian meander is an inhomogeneous Markov process with continuous path that is obtained from Brownian motion by the following path transformation:

Let  $\sigma = \max\{t < 1 : X_t = 0\}$  be the time of the last zero before time 1, and

$$\widetilde{X}_t = \frac{1}{\sqrt{1-\sigma}} |X(\sigma + t(1-\sigma))| .$$

Then, with respect to Wiener measure  $W_0$ , the process  $(\widetilde{X}_t)_{t \geq 0}$  is the Brownian meander. Let  $\widetilde{W}_{0,1}^+$  be the law of the Brownian meander on  $\mathcal{C}_\infty$ . We have the following theorem.

**Theorem 3.1** (Durrett, Iglehart, Miller). *As  $x > 0$  tends to 0,  $\widetilde{W}_{x,1}^+ \Rightarrow \widetilde{W}_{0,1}^+$ .*

Note that Theorem 3.1 gives a precise meaning to the statement that Brownian meander is Brownian motion conditioned to stay positive until time 1.

The idea of the proof of Theorem 3.1 is to turn the conditioned laws into unconditioned ones by the mean of well-chosen sections of the original process. Let us give some details.

For all  $x \geq 0$ , introduce the random time

$$T_x = \inf\{t \geq 0 : X_t = x \text{ and } X_s > 0 \text{ for all } s \in (t, t+1]\} .$$

These times are  $W_0$ -almost surely finite. To see this, suppose first  $x > 0$ , and let  $h = \inf\{t > 0 : X_t = x\}$  be the first hitting time of  $x$  and  $g = \inf\{t > h : X_t = 0\}$  be the time of the first return to 0 after time  $h$ . Note that they are both stopping times. By the Markov property of Brownian motion, we have

$$W_0(g - h > 1) = W_x(\tau_+ > 1) > 0.$$

We construct an increasing sequence of stopping times  $h_1 < g_1 < h_2 < g_2 < \dots$ , setting  $h_1 = h$ ,  $g_1 = g$  and for all  $i \geq 1$ ,  $h_{i+1} = g_i + h \circ \theta_{g_i}$  and  $g_{i+1} = g_i + g \circ \theta_{g_i}$ . Then  $(g_i - h_i)_{i \geq 1}$  is a sequence of i.i.d. random variables. Since  $\sum W_0(g_i - h_i > 1) = +\infty$ , we get  $W_0(g_i - h_i > 1 \text{ i.o.}) = 1$  by the Borel-Cantelli lemma. In particular  $W_0(T_x < +\infty) = 1$ . Finally, we remark that  $W_0(T_0 \leq T_x) = 1$  for every  $x > 0$ , so that  $T_0$  is also  $W_0$ -almost surely finite.

To each time  $T_x$  we associate the shift operator  $\phi_x := \theta_{T_x}$  acting on  $\mathcal{C}_\infty$ . We then have:

**Proposition 3.2.** *For every  $x > 0$ ,  $\widetilde{W}_{x,1}^+ = W_0 \circ \phi_x^{-1}$ .*

*Proof.* Consider a finite sequence of times  $0 \leq t_1 < t_2 < \dots < t_n$ , a sequence  $B_1, B_2, \dots, B_n$  of Borel subsets of  $\mathbb{R}$ , and set  $B = \{w \in \mathcal{C}_\infty : w(t_i) \in B_i, \forall i = 1 \dots n\}$ .

Let  $h_x = \inf\{t > 0 : X_t = x\}$  be the hitting time of  $x$  and write

$$(3) \quad W_0(\phi_x \in B) = W_0(\phi_x \in B; T_x = h_x) + W_0(\phi_x \in B; T_x > h_x).$$

By the Strong Markov property, we have

$$(4) \quad W_0(\phi_x \in B; T_x = h_x) = W_0(\theta_{h_x}^{-1}\{B; \tau_+ > 1\}) = W_x(B; \tau_+ > 1).$$

In order to compute the second term of the right hand side of (3), observe that

$$\{\phi_x \in B; T_x > h_x\} = \{\tau_+ \circ \theta_{h_x} \leq 1; \theta_{T_x} \circ \theta_{\tau_+} \circ \theta_{h_x} \in B\},$$

so that applying successively the strong Markov property to  $h_x$  and then to  $\tau_+$  gives

$$(5) \quad \begin{aligned} W_0(\phi_x \in B; T_x > h_x) &= W_x(\tau_+ \leq 1; \theta_{T_x} \circ \theta_{\tau_+} \in B) \\ &= W_x(\tau_+ \leq 1; W_0(\theta_{T_x} \in B)) \\ &= W_x(\tau_+ \leq 1)W_0(\phi_x \in B). \end{aligned}$$

Putting (4) and (5) in equation (3) gives

$$W_0(\phi_x \in B) = W_x(B; \tau_+ > 1) + \{1 - W_x(\tau_+ > 1)\}W_0(\phi_x \in B),$$

Therefore

$$W_0(\phi_x \in B) = \frac{W_x(B; \tau_+ > 1)}{W_x(\tau_+ > 1)} = \widetilde{W}_{x,1}^+(B).$$

□

Proposition 3.2 gives an “unconditioned” representation of the laws  $\widetilde{W}_{x,1}^+$ ,  $x > 0$ . It is noteworthy that  $W_0 \circ \phi_x^{-1}$  also make sense for  $x = 0$  whereas the definition of  $\widetilde{W}_{x,1}^+$  does not.

**Proposition 3.3.** *As  $x \rightarrow 0$ ,  $\phi_x$  converges almost surely to  $\phi_0$  with respect to  $W_0$ .*

*Proof.* It suffices to prove that  $T_x$  almost surely converges to  $T_0$  with respect to  $W_0$ . Let  $w$  be a continuous path such that  $w(0) = 0$  and  $T_0 := T_0(w) < +\infty$ . By definition of  $T_0$  and continuity of  $w$ , if  $\epsilon$  is small enough, then  $w(t) > 0$  for all  $t \in (T_0, T_0 + 1 + \epsilon]$ . Put  $\eta = w(T_0 + \epsilon) > 0$ . For all  $x \in [0, \eta]$ , there exists  $t \in [T_0, T_0 + \epsilon]$  such that  $w(t) = x$ . Since  $w(s) > 0$  for all  $s \in (t, t + 1]$ , we see that  $T_x(w) \leq T_0 + \epsilon$ . On the other hand it is clear that  $T_0 \leq T_x(w)$ . Hence  $T_x(w) \rightarrow T_0$  as  $x \rightarrow 0$ .  $\square$

From Propositions 3.2 and 3.3, it follows by the dominated convergence theorem that

$$\widetilde{W}_{x,1}^+ = W_0 \circ \phi_x^{-1} \Rightarrow W_0 \circ \phi_0^{-1}$$

as  $x \rightarrow 0$ . Note that the limit law clearly satisfies  $W_0 \circ \phi_0^{-1}(\tau_+ > 1) = 1$ ; hence the convergence is neat. In order to prove Theorem 3.1, it remains to identify the limit with the Brownian meander. This can be done by computing the limit of the finite-dimensional distributions of the laws  $\widetilde{W}_{x,1}^+$  which are easily derived from a classical formula for the joint distribution of Brownian motion and its minimum. We do not give further detail since no expression of these finite-dimensional distributions will be needed in what follows.

**3.2. Brownian motion conditioned to stay in a half-space.** Theorem 3.1 can easily be extended to multidimensional Brownian motion conditioned to stay in a half-space. Let  $d \geq 2$ . Because of invariance properties of  $d$ -dimensional Brownian motion we need only to study the case of the half-space  $D = \{x \in \mathbb{R}^d : x_1 > 0\}$ . Let  $BM$  be a Brownian meander and  $B_2, \dots, B_d$  be one-dimensional Brownian motions such that  $BM, B_2, \dots, B_d$  are mutually independent. The  $d$ -dimensional process  $(BM, B_2, \dots, B_d)$  will be called  $D$ -Brownian meander and its law will be denoted by  $\widetilde{W}_{0,1}^D$ .

**Corollary 3.4.** *As  $x \in D \rightarrow 0$ ,  $\widetilde{W}_{x,1}^D \Rightarrow \widetilde{W}_{0,1}^D$ .*

*Proof.* A Brownian motion conditioned to stay in the half-space  $D$  is a Brownian motion whose first coordinate is conditioned to stay positive. Since the coordinates are independent one-dimensional Brownian motions, the result follows immediately from Theorem 3.1.  $\square$

**Remark 3.5.** It is clear from the definition of the  $D$ -Brownian meander that it satisfies  $\widetilde{W}_{0,1}^D(\tau_D > 1) = 1$ ; thus the convergence in Theorem 3.4 is neat. Therefore,  $\widetilde{W}_{0,1}^D$  has the Markov property of Proposition 2.5. Moreover, since  $D$  is a cone, it follows from Remark 2.6 that we also have a zero-one law with respect to  $\widetilde{W}_{0,1}^D$ .

**Remark 3.6.** For  $x = (x_1, x_2, \dots, x_d) \in \overline{D}$ , define a process  $Z_x$  by

$$\forall t \geq 0, \quad Z_x(t) = x + X(T_{x_1}(X_1) + t) - X(T_{x_1}(X_1)).$$

It is not difficult to verify that, under the Brownian distribution, the law of the random process  $Z_x$  is precisely  $\widetilde{W}_{x,1}^D$ . This alternative representation of the laws  $\widetilde{W}_{x,1}^D$  will be used in the next section.

#### 4. PRECONDITIONING

We shall now use the results of Section 3 in order to obtain a convergence theorem for the Brownian motion conditioned to stay in a set satisfying some regularity and convexity assumptions (Theorem 4.11). Section 4.1 introduces the idea of preconditioning and explains how it could be applied to the convergence problem. The proposed method requires two estimates that are studied in Sections 4.2 and 4.3. This finally lead us to introduce a class of *nice sets* for which we solve the convergence problem in Section 4.4.

**4.1. Changing laws for the convergence problem.** Let  $U \subset \mathbb{R}^d$  be a co-regular open set with  $0 \in \partial U$ . Recall that the definition of  $\widetilde{W}_{x,1}^U$  by the formula

$$\widetilde{W}_{x,1}^U(*) = \frac{W_x(*; \tau_U > 1)}{W_x(\tau_U > 1)},$$

does not make any sense for  $x = 0$  since  $W_0(\tau_U > 1) = 0$ .

Now suppose that  $U$  is contained in the half-space  $D$ . Then a Brownian motion conditioned to stay in  $U$  is also a Brownian motion conditioned to stay in  $D$  and then conditioned to stay in  $U$ , that is:

$$(6) \quad \widetilde{W}_{x,1}^U(*) = \widetilde{W}_{x,1}^D(* | \tau_U > 1) = \frac{\widetilde{W}_{x,1}^D(*; \tau_U > 1)}{\widetilde{W}_{x,1}^D(\tau_U > 1)}.$$

This simple identity is what we call *preconditioning*, for if we take it as a definition, it is the same as before except we have changed the initial law of the paths ( $W_x \leftrightarrow \widetilde{W}_{x,1}^D$ ) which are now preconditioned to stay in  $D$ . The gain is that, although  $W_0(\tau_U > 1) = 0$ , we might have  $\widetilde{W}_{0,1}^D(\tau_U > 1) > 0$  if the boundary of  $U$  is smooth enough at 0. Proposition 4.3 in Section 4.2 gives a sufficient condition on  $U$  which ensures that  $\widetilde{W}_{0,1}^D(\tau_U > 1) > 0$ . If so, we will set  $\widetilde{W}_{0,1}^U(*) := \widetilde{W}_{0,1}^D(* | \tau_U > 1)$ .

At this point, we have to note that  $\widetilde{W}_{0,1}^D(\tau_U > 1) > 0$  is equivalent to  $\widetilde{W}_{0,1}^D(\tau_U > 0) = 1$ . The last condition is necessary for if  $\widetilde{W}_{0,1}^D(\tau_U > 0) < 1$ , then it would be 0 by the zero-one law, and  $\widetilde{W}_{0,1}^D(\tau_U > 1)$  would also be equal to 0. The condition is also sufficient: It implies that there exists an  $\epsilon > 0$  such that  $\widetilde{W}_{0,1}^D(\tau_U > \epsilon) > 0$ , and by the Markov property we then have

$$\widetilde{W}_{0,1}^D(\tau_U > 1) = \widetilde{W}_{0,1}^D(\tau_U > \epsilon; \widetilde{W}_{X_{\epsilon,1-\epsilon}}^D(\tau_U > 1 - \epsilon)) > 0$$

since  $W_x(\tau_U > 1 - \epsilon) > 0$  for all  $x \in U$ .

The problem we then have to solve is the following: Since  $\widetilde{W}_{x,1}^D \Rightarrow \widetilde{W}_{0,1}^D$  and  $\widetilde{W}_{0,1}^D(\tau_U > 1) > 0$ , do we have

$$\widetilde{W}_{x,1}^D(* | \tau_U > 1) \Rightarrow \widetilde{W}_{0,1}^D(* | \tau_U > 1) ?$$

The next lemma shows that the answer is positive when  $U$  is locally a half-space at 0.

**Lemma 4.1.** *Let  $V \subset \mathbb{R}^d$  be an open co-regular set such that  $0 \in V$ . For every bounded continuous function  $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ , we have*

$$\lim_{x \rightarrow 0} \widetilde{W}_{x,1}^D(f; \tau_V > 1) = \widetilde{W}_{0,1}^D(f; \tau_V > 1) .$$

In addition, we have  $\widetilde{W}_{0,1}^D(\tau_V > 1) > 0$ ; thus, as  $x \rightarrow 0$ ,

$$\widetilde{W}_{x,1}^{D \cap V} \Rightarrow \widetilde{W}_{0,1}^{D \cap V} .$$

*Proof.* Set  $\Omega = \{X_0 = 0, \tau_V = \tau_{\overline{V}} \neq 1\}$ .

We first prove that  $\widetilde{W}_{0,1}^D(\Omega) = 1$ . For all  $\epsilon \in (0, 1)$ , we have

$$\widetilde{W}_{0,1}^D(\epsilon < \tau_V < \tau_{\overline{V}}) = \widetilde{W}_{0,1}^D(\widetilde{W}_{X_{\epsilon,1-\epsilon}}^D(\tau_V < \tau_{\overline{V}})) = 0$$

since  $W_x(\tau_V < \tau_{\overline{V}}) = 0$  for all  $x \in \mathbb{R}^d$ . Hence, letting  $\epsilon \rightarrow 0$  gives

$$\widetilde{W}_{0,1}^D(\tau_V < \tau_{\overline{V}}) = 0 .$$

To see that  $\widetilde{W}_{0,1}^D(\tau_V = 1) = 0$ , fix  $\epsilon \in (0, 1)$  and write

$$\widetilde{W}_{0,1}^D(\tau_V = 1) = \widetilde{W}_{0,1}^D(\widetilde{W}_{X_{\epsilon,1-\epsilon}}^D(\tau_V = 1 - \epsilon)) .$$

The result follows immediatly because  $W_x(\tau_V = 1 - \epsilon) = 0$  for all  $x \in \mathbb{R}^d$ .

Then, we prove the following statement: For all  $w \in \Omega$  and every sequence  $(w_n) \in C_\infty$  such that  $w_n \rightarrow w$ , we have

$$(7) \quad \mathbb{1}_{\{\tau_V > 1\}}(w_n) \rightarrow \mathbb{1}_{\{\tau_V > 1\}}(w) .$$

If  $\tau_V(w) > 1$ , then  $w(t) \in U$  for all  $t \in [0, 1]$  and for  $n$  large enough  $w_n(t) \in U$  for all  $t \in [0, 1]$ ; thus  $\tau_V(w_n) > 1$ .

If  $\tau_V(w) = \tau_{\overline{V}}(w) < 1$ , then there exists  $t \in (0, 1)$  such that  $w(t) \notin \overline{V}$ . For  $n$  large enough we then have  $w_n(t) \notin \overline{V}$  and so  $\tau_V(w_n) < 1$ . This proves (7).

We can now apply the continuous mapping theorem which gives

$$\widetilde{W}_{x,1}^D(f; \tau_V > 1) \rightarrow \widetilde{W}_{0,1}^D(f; \tau_V > 1), \quad \text{as } x \rightarrow 0 ,$$

for any bounded continuous function  $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ , and the lemma is proved.  $\square$

If  $U$  is not locally a half-space at 0, then  $\tau_U$  is everywhere discontinuous. A way to get round this difficulty is given by the next lemma which will be the basic tool for applying preconditioning to the convergence problem.

**Lemma 4.2.** *Suppose  $\widetilde{W}_{0,1}^D(\tau_U > 1) > 0$  and let  $(x_n)$  be a sequence of points in  $U$  such that  $x_n \rightarrow 0$ .*

*If we have*

$$\lim_{s \rightarrow 0} \limsup_n \widetilde{W}_{x_n,1}^D(\tau_U \leq s) = 0,$$

*then*

$$\lim_{n \rightarrow 0} \widetilde{W}_{x_n,1}^D(f; \tau_U > 1) = \widetilde{W}_{0,1}^D(f; \tau_U > 1)$$

*for every bounded continuous function  $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ .*

*As a consequence  $\widetilde{W}_{x_n,1}^U \Rightarrow \widetilde{W}_{0,1}^U$ .*

*Proof.* We follow Shimura's proof ([7], Lemma 4.1). Fix  $s \in (0, 1)$  and introduce the random times  $\tau_U^s = \inf\{t \geq s : X_t \notin U\}$  and  $\tau_{\overline{U}}^s = \inf\{t \geq s : X_t \notin \overline{U}\}$ . Set  $\Omega = \{\tau_U^s = \tau_{\overline{U}}^s \neq 1\}$ .

We shall prove that  $\widetilde{W}_{0,1}^D(\Omega) = 1$ . By the Markov property we have

$$\widetilde{W}_{0,1}^D(\tau_U^s = 1) = \widetilde{W}_{0,1}^D\left(X_s \in U; \widetilde{W}_{X_s,1-s}^D(\tau_U = 1 - s)\right) = 0$$

because  $W_x(\tau_U = 1 - s) = 0$  for all  $x \in \mathbb{R}^d$ . We have also

$$\begin{aligned} & \widetilde{W}_{0,1}^D(\tau_U^s < \tau_{\overline{U}}^s) \\ &= \widetilde{W}_{0,1}^D(X_s \in U; \tau_U^s < \tau_{\overline{U}}^s) + \widetilde{W}_{0,1}^D(X_s \notin U; s < \tau_{\overline{U}}^s) \\ &= \widetilde{W}_{0,1}^D\left(X_s \in U; \widetilde{W}_{X_s,1-s}^D(\tau_U < \tau_{\overline{U}})\right) + \widetilde{W}_{0,1}^D\left(X_s \in \partial U; \widetilde{W}_{X_s,1-s}^D(\tau_{\overline{U}} > 0)\right) \\ &= 0 \end{aligned}$$

since  $W_x(\tau_U < \tau_{\overline{U}}) = 0$  for all  $x \in \mathbb{R}^d$  and  $W_x(\tau_{\overline{U}} > 0) = 0$  for all  $x \in \partial U$  (remember that  $U$  is co-regular). Hence  $\widetilde{W}_{0,1}^D(\Omega) = 1$ .

Then, we shall prove the following statement: For all  $w \in \Omega$  and every sequence  $(w_n) \in \mathcal{C}_\infty$  such that  $w_n \rightarrow w$ , we have

$$(8) \quad \mathbb{1}_{\{\tau_U^s > 1\}}(w_n) \rightarrow \mathbb{1}_{\{\tau_U^s > 1\}}(w).$$

If  $\tau_U^s(w) > 1$ , then  $w(t) \in U$  for all  $t \in [s, 1]$  and for  $n$  large enough  $w_n(t) \in U$  for all  $t \in [s, 1]$ ; thus  $\tau_U^s(w_n) > 1$ .

If  $\tau_U^s(w) = \tau_{\overline{U}}^s(w) < 1$ , then there exists  $t \in [s, 1]$  such that  $w(t) \notin \overline{U}$ . For  $n$  large enough we then have  $w_n(t) \notin \overline{U}$  and so  $\tau_{\overline{U}}^s(w_n) < 1$ . This proves (8).

For any bounded continuous function  $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ , the continuous mapping theorem gives

$$\widetilde{W}_{x_n,1}^D(f; \tau_U^s > 1) \rightarrow \widetilde{W}_{0,1}^D(f; \tau_U^s > 1).$$

If  $B$  is a bound for  $|f|$ , then

$$\begin{aligned} & |\widetilde{W}_{x_n,1}^D(f; \tau_U > 1) - \widetilde{W}_{0,1}^D(f; \tau_U > 1)| \\ & \leq |\widetilde{W}_{x_n,1}^D(f; \tau_U^s > 1) - \widetilde{W}_{0,1}^D(f; \tau_U^s > 1)| \\ & \quad + B \left( \widetilde{W}_{x_n,1}^D(\tau_U \leq s) + \widetilde{W}_{0,1}^D(\tau_U \leq s) \right). \end{aligned}$$

Hence

$$\begin{aligned} & \limsup_n |\widetilde{W}_{x_n,1}^D(f; \tau_U > 1) - \widetilde{W}_{0,1}^D(f; \tau_U > 1)| \\ & \leq B \left( \limsup_n \widetilde{W}_{x_n,1}^D(\tau_U \leq s) + \widetilde{W}_{0,1}^D(\tau_U \leq s) \right). \end{aligned}$$

Therefore, using the hypothesis of the Lemma and the fact that  $\widetilde{W}_{0,1}^D(\tau_U = 0) = 0$ , we obtain the announced result by letting  $s$  go to 0.  $\square$

**4.2. An irregularity criterion.** In order to apply the ideas of preconditioning to a set  $U$ , we have to know whether  $\widetilde{W}_{0,1}^D(\tau_U > 0) = 1$  or not. Such a criterion was discovered independently by Shimura ([7], Lemma 3.1) in the two dimensional case, and by Burdzy ([4], Corollary 3.1) when studying excursions from hyperplanes and smooth surfaces.

Suppose  $d \geq 2$ . Let  $h : [0, +\infty) \rightarrow [0, +\infty)$  be a continuous function and set

$$U_h = \left\{ x \in \mathbb{R}^d : x_1 > h \left( \sqrt{x_2^2 + \cdots + x_d^2} \right) \right\}.$$

We then have :

**Proposition 4.3.** *If  $h(r)/r$ ,  $r > 0$ , is non-decreasing in a neighborhood of 0, then  $\widetilde{W}_{0,1}^D(\tau_{U_h} > 0) = 1$  if and only if  $\int_0^1 h(r)r^{-2} dr < \infty$ .*

For the reader convenience, we will now sketch a proof of Proposition 4.3. We go back first to dimension  $d = 1$ . Let  $n > 0$  and let  $h_n = \inf\{t > 0 : X_t = 1/n\}$  be the first hitting time of  $1/n$ . Let  $a_{1,n} = \max\{t < h_n : X_t = 0\}$  be the time of the last return to 0 before time  $h_n$  and let  $b_{1,n} = \inf\{t > h_n : X_t = 0\}$  be the first hitting time of 0 after time  $h_n$ . Note that  $b_{1,n}$  is a stopping time. We construct an increasing sequence of random time  $a_{1,n} < b_{1,n} < a_{2,n} < b_{2,n} < \cdots$  by setting  $a_{j+1,n} = a_{1,n} \circ \theta_{b_{j,n}} + b_{j,n}$  and  $b_{j+1,n} = b_{1,n} \circ \theta_{b_{j,n}} + b_{j,n}$ . Since the Brownian motion starts afresh at  $b_{j,n}$ , the distribution of the process  $X(a_{j,n} + \cdot)$  (w.r.t.  $W_0$ ) does not depend on  $j$ .

Now recall that Brownian meander is the section of Brownian motion which starts at time

$$T_0 = \inf\{t > 0 : X_t = 0 \text{ and } X_s > 0 \text{ for all } s \in (t, t+1]\}.$$

If  $w \in$

$C_\infty$  is such that  $T_0(w) < +\infty$ , then there exists an integer  $n > 0$  such that, after time  $T_0(w)$ , the path  $w$  hits  $1/n$  before hitting 0. Hence,  $T_0(w)$  is one of the times  $a_{j,n}(w)$ .

Let  $d \geq 2$  and consider the  $d$ -dimensional processes  $Z_{j,n}$  and  $Z_0$  defined by

$$Z_{j,n}(t) = (X_1(a_{j,n}(X_1) + t), X_2(t), \dots, X_d(t))$$

and

$$Z_0(t) = (X_1(T_0(X_1) + t), X_2(t), \dots, X_d(t)) .$$

We know that  $Z_0$  has the law of a  $D$ -Brownian meander with respect to  $W_0$ . Hence, for an open set  $U$ , we have

$$\widetilde{W}_{0,1}^D(\tau_U > 0) = W_0(\tau_U(Z_0) > 0) .$$

Since there almost surely exists  $j, n > 0$  such that  $T_0 = a_{j,n}$ , the condition  $W_0(\tau_U(Z_0) > 0) = 1$  is implied by the condition

$$W_0(\tau_U(Z_{j,n}) > 0 \text{ for all } j, n > 0) = 1$$

which is equivalent to

$$\forall j, n > 0, \quad W_0(\tau_U(Z_{j,n}) > 0) = 1 .$$

As the distribution of the processes  $Z_{j,n}$  does not depend on  $j$ , this is also equivalent to

$$\forall n > 0, \quad W_0(\tau_U(Z_{1,n}) > 0) = 1 .$$

Recall that  $a_{1,n}(X_1)$  is the time of the last return to 0 of  $X_1$  before it hits  $1/n$ . We know from Williams (see [10]) that a Brownian motion taken between those two times is a 3-dimensional Bessel process taken between time 0 and its first hitting time of  $1/n$ . Therefore

$$W_0(\tau_U(Z_{1,n}) > 0) = W_0(\tau_U(BS, X_2, \dots, X_d) > 0) ,$$

where  $BS$  is a 3-dimensional Bessel process independent from  $X_2, \dots, X_d$ . Since  $BS$  is the norm of a 3-dimensional Brownian motion, we can rewrite the last equality as

$$W_0(\tau_U(Z_{1,n}) > 0) = W_0(\tau_U(\sqrt{X_1^2 + X_2^2 + X_3^2}, X_4, \dots, X_{d+2}) > 0) .$$

Now, for any set  $U \subset \mathbb{R}^d$ , put

$$U^* = \{x \in \mathbb{R}^{d+2} : (\sqrt{x_1^2 + x_2^2 + x_3^2}, x_4, \dots, x_{d+2}) \in U\} .$$

We have just proved that  $W_0(\tau_{U^*} > 0) = 1$  implies  $\widetilde{W}_{0,1}^D(\tau_U > 1) = 1$ . The same arguments hold if the event  $\{\tau_U > 0\}$  is replaced by  $\{\tau_U = 0\}$ , and we have in fact an equivalence:

$$W_0(\tau_{U^*} > 0) = 1 \Leftrightarrow \widetilde{W}_{0,1}^D(\tau_U > 1) = 1 .$$

Thanks to this duality, Proposition 4.3 simply follows from a classical irregularity criterion for Brownian motion (e.g. Port and Stone [6], Proposition 3.5).

**Remark 4.4.** Although the criterion given in Proposition 4.3 is designed for the sets  $U_h$ , it can be applied to any set  $U$  that coincides with a set  $U_h$  in a neighborhood of 0: If  $V$  is a neighborhood of 0 then  $\widetilde{W}_{0,1}^D(\tau_V > 0) = 1$  and by consequence

$$\widetilde{W}_{0,1}^D(\tau_U > 0) = \widetilde{W}_{0,1}^D(\tau_{U \cap V} > 0) .$$



Thus, if  $U \cap V = U_h \cap V$ ,

$$\widetilde{W}_{0,1}^D(\tau_U > 0) = \widetilde{W}_{0,1}^D(\tau_{U_h} > 0).$$

As an example of application, the ball  $B$  with center at  $(1, 0, \dots, 0)$  and radius 1 verifies  $\widetilde{W}_{0,1}^D(\tau_B > 0) = 1$ . On the contrary, for a proper cone  $C \subset D$  with vertex at the origin, Proposition 4.3 shows that  $\widetilde{W}_{0,1}^D(\tau_C > 0) = 0$ , hence a law  $\widetilde{W}_{0,1}^C$  can not be defined directly.

**4.3. The ball estimate.** In this section we will prove a fundamental estimate (Lemma 4.6) which solves the convergence problem for the Brownian motion conditioned to stay in a ball. This estimate will also play a key role in the study of Brownian motion conditioned to stay in nice sets (see Section 4.4).

Fix  $d \geq 2$ . Let  $D$  be the half-space  $\{x \in \mathbb{R}^d : x_1 > 0\}$  and  $B$  the open ball with center at  $e_1 = (1, 0, \dots, 0)$  and radius 1. As we have already seen, Proposition 4.3 shows that  $\widetilde{W}_{0,1}^D(\tau_B > 0) = 1$ .

Set  $\widetilde{W}_{0,1}^B(*) = \widetilde{W}_{0,1}^D(* | \tau_B > 1)$ . According to Lemma 4.2, in order to prove that  $\widetilde{W}_{x,1}^B \Rightarrow \widetilde{W}_{0,1}^B$  as  $x \in B \rightarrow 0$ , it would be sufficient that

$$(9) \quad \lim_{s \rightarrow 0} \limsup_{n \rightarrow \infty} \widetilde{W}_{x_n,1}^D(\tau_B \leq s) = 0.$$

for every sequence  $(x_n) \in B$  such that  $x_n \rightarrow 0$ . But (9) does not hold in general: for any  $x \in \partial B \setminus \{0\}$  we have  $\widetilde{W}_{x,1}^D(\tau_B = 0) = 1$ , thus it is possible to find sequences  $(x_n) \in B$  and  $(s_n)$  such that  $x_n \rightarrow 0$ ,  $s_n \rightarrow 0$  and  $\widetilde{W}_{x_n,1}^D(\tau_B \leq s_n) \rightarrow 1$ .

To overcome this difficulty, we will only prove that (9) holds for every sequence  $(\lambda_n e_1)$  with  $\lambda_n \rightarrow 0$  and  $e_1 = (1, 0, \dots, 0)$ , and then deduce the convergence of  $\widetilde{W}_{x_n,1}^B$  -for a general sequence  $(x_n)$ - from the convergence of  $\widetilde{W}_{\lambda_n e_1,1}^B$  thanks to invariance properties of Brownian motion.

We start with elementary geometry. Let  $E$  be the set of all  $(d-1)$ -uples  $(\epsilon_2, \dots, \epsilon_d)$  with  $\epsilon_i = \pm 1$ . For all  $\epsilon = (\epsilon_2, \dots, \epsilon_d) \in E$ , let  $\bar{\epsilon} = (-\epsilon_2, \dots, -\epsilon_d)$  be the opposite of  $\epsilon$ . We define a family of  $2^{d-1}$  disjoint subsets of  $D$  indexed by  $E$  by setting

$$\forall \epsilon \in E, \quad D_\epsilon = \{x \in D : \epsilon_2 x_2, \dots, \epsilon_d x_d > 0\}.$$

Let  $H$  be the hyperplane  $\{x_1 = 1\}$  and, for all  $x \in \mathbb{R}^d$ , let  $B(x)$  be the open ball with center at  $x$  and radius 1.

**Lemma 4.5.** *If  $x \in D_\epsilon \cap H$ , then  $B(x) \cap B^c \cap D_{\bar{\epsilon}} = \emptyset$ .*

*Proof.* Let  $x \in D_\epsilon \cap H$ . For any  $y \in B(x) \cap D_{\bar{\epsilon}}$  we have

$$\begin{aligned} 1 &> (y_1 - 1)^2 + (y_2 - x_2)^2 + \cdots + (y_d - x_d)^2 \\ &= (y_1 - 1)^2 + y_2^2 + \cdots + y_d^2 + \sum_{i=2}^d (x_i^2 - 2y_i x_i) \\ &> (y_1 - 1)^2 + y_2^2 + \cdots + y_d^2 \end{aligned}$$

since  $x \in D_\epsilon$  and  $y \in D_{\bar{\epsilon}}$  imply  $-x_i y_i > 0$  for all  $i = 2 \dots d$ . Hence  $y$  belongs to  $B$ .  $\square$

We now come to the estimate which is the heart of this section:

**Lemma 4.6.**

$$\lim_{s \rightarrow 0} \limsup_{\lambda \rightarrow 0} \widetilde{W}_{\lambda e_1, 1}^D(\tau_B \leq s) = 0.$$

*Proof.* We will show that

$$(10) \quad \limsup_{\lambda \rightarrow 0} \widetilde{W}_{\lambda e_1, 1}^D(\tau_B \leq s) \leq 2^{d-1} \widetilde{W}_{0,1}^D(\tau_B \leq s)$$

for all  $s > 0$ , and the lemma will then follow by letting  $s \rightarrow 0$  since  $\widetilde{W}_{0,1}^D(\tau_B = 0) = 0$ .

For  $\lambda \geq 0$ , set

$$T_\lambda = \inf\{t \geq 0 : X_1(t) = \lambda \text{ and } X_1(s) > 0, \forall s \in (t, t+1]\}$$

and consider the process  $Z_\lambda$  defined by

$$\forall t \geq 0, \quad Z_\lambda(t) = X(T_\lambda + t) - X(T_\lambda) + \lambda e_1.$$

Remember that  $Z_\lambda$  has the distribution  $\widetilde{W}_{\lambda e_1, 1}^D$  with respect to  $W_0$  (see Lemma 3.6).

Write

$$(11) \quad W_0(\tau_B(Z_\lambda) \leq s) \leq W_0(\tau_B(Z_0) \leq s + T_\lambda - T_0) + W_0(\tau_B(Z_0) > s + T_\lambda - T_0; \tau_B(Z_\lambda) \leq s).$$

For convenience, we set  $u = \tau_B(Z_\lambda)$ . If  $\tau_B(Z_0) > s + T_\lambda - T_0$  and  $u \leq s$ , then  $Z_0(u + T_\lambda - T_0) = X(T_\lambda + u) - X(T_0)$  belongs to  $B$ ; this means that  $Z_\lambda(u)$  belongs to  $B(Y_\lambda)$ , where we have put

$$\begin{aligned} Y_\lambda &= X(T_0) - X(T_\lambda) + (1 + \lambda)e_1 \\ &= (1, X_2(T_0) - X_2(T_\lambda), \dots, X_d(T_0) - X_d(T_\lambda)). \end{aligned}$$

Note that  $Y_\lambda \in H$ . Since  $Z_\lambda(u) \notin B$ , we see by Lemma 4.5 that  $Z_\lambda(u) \notin D_{\bar{\epsilon}}$  as soon as  $Y_\lambda \in D_\epsilon$ . Therefore

$$\begin{aligned} &W_0(\tau_B(Z_0) > s + T_\lambda - T_0; \tau_B(Z_\lambda) \leq s) \\ &\leq \sum_{\epsilon \in E} W_0(Y_\lambda \in D_\epsilon; \tau_B(Z_\lambda) \leq s; Z_\lambda(u) \notin D_{\bar{\epsilon}}). \end{aligned}$$

Now, it is easily seen that  $Y_\lambda$  is independent of  $Z_\lambda$  conditionally to  $X_1$ . In addition, we have  $W_0(Y_\lambda \in D_\epsilon | X_1) = 1/2^{d-1}$ . Thus

$$\begin{aligned} & W_0(\tau_B(Z_0) > s + T_\lambda - T_0; \tau_B(Z_\lambda) \leq s) \\ & \leq \frac{1}{2^{d-1}} \sum_{\epsilon \in E} W_0(\tau_B(Z_\lambda) \leq s; Z_\lambda(u) \notin D_\epsilon) \\ & = \frac{2^{d-1} - 1}{2^{d-1}} W_0(\tau_B(Z_\lambda) \leq s). \end{aligned}$$

Combining this inequality with equation (11) gives

$$W_0(\tau_B(Z_\lambda) \leq s) \leq 2^{d-1} W_0(\tau_B(Z_0) \leq s + T_\lambda - T_0)$$

and the result follows by letting  $\lambda \rightarrow 0$  since  $\lim_{\lambda \rightarrow 0} T_\lambda = T_0$  almost surely (see Lemma 3.3).  $\square$

**Remark 4.7.** Note that the proof of Lemma 4.6 does not involve the “size” of  $B$ . Hence the result holds if  $B$  is replaced by any open ball  $B'$  that is tangent to  $\partial D$  at 0.

The following proposition and its proof illustrate how the ball estimate can be combined to invariance properties of Brownian motion in order to solve the convergence problem.

**Proposition 4.8.** *As  $x \in B \rightarrow 0$ ,  $\widetilde{W}_{x,1}^B \Rightarrow \widetilde{W}_{0,1}^B$ .*

*Proof.* It follows from Lemmas 4.2 and 4.6 that

$$\widetilde{W}_{\lambda e_1,1}^B \Rightarrow \widetilde{W}_{0,1}^B \quad \text{as } \lambda \rightarrow 0.$$

Let  $(x_n) \in B$  be a sequence such that  $x_n \rightarrow 0$  and set  $\lambda_n = 1 - \|x_n - e_1\|$ . For each  $n$ , choose a rotation  $R_n$  in the plane  $(0, e_1, x_n)$  with center at  $e_1$  such that  $R_n(\lambda_n e_1) = x_n$ . From the invariance properties of Brownian motion, it is easily seen that

$$\widetilde{W}_{x_n,1}^B = \widetilde{W}_{\lambda_n e_1,1}^B \circ R_n^{-1}.$$

Now, by elementary geometry,

$$\|R_n(y) - y\| \leq \|y - e_1\| \frac{\|x_n - \lambda_n e_1\|}{\|x_n - e_1\|}$$

for all  $y \in \mathbb{R}^d$ . Thus, for any  $w \in \mathcal{C}_\infty$  and any sequence  $(w_n) \in \mathcal{C}_\infty$  such that  $w_n \rightarrow w$ , we get

$$R_n(w_n) \rightarrow w.$$

It therefore follows from the continuous mapping theorem that

$$\widetilde{W}_{x_n,1}^B = \widetilde{W}_{\lambda_n e_1,1}^B \circ R_n^{-1} \Rightarrow \widetilde{W}_{0,1}^B.$$

$\square$

**4.4. Application to nice sets.** In this section, we introduce the notion of *nice* sets and solve the convergence problem for those sets.

4.4.1. *Convergence with variable sets.* For any set  $U \subset \mathbb{R}^d$  and any  $\epsilon > 0$ , put

$$U_{\epsilon^+} = \{x \in \mathbb{R}^d : d(x, U) \leq \epsilon\}$$

and

$$U_{\epsilon^-} = \{x \in U : d(x, U^c) \geq \epsilon\}.$$

If  $(U_n)$  is a sequence of subsets of  $\mathbb{R}^d$ , we will say that  $(U_n)$  converges to  $U$  and write  $U_n \rightarrow U$  if for all  $\epsilon > 0$  there exist a  $n_0$  such that

$$n \geq n_0 \Rightarrow U_{\epsilon^-} \subset U_n \subset U_{\epsilon^+}.$$

Let  $D$  be the half-space  $\{x \in \mathbb{R}^d : x_1 > 0\}$  and let  $B$  be an open ball tangent to  $\partial D$  at 0. Set  $e_1 = (1, 0, \dots, 0)$ .

**Proposition 4.9.** *Let  $U$  be an open co-regular set such that  $B \subset U \subset D$  and let  $(U_n)$  be a sequence of sets such that:*

- (1) *For all  $n$ ,  $B \subset U_n \subset D$ ;*
- (2) *For all  $R > 0$ ,  $U_n \cap B(0, R) \rightarrow U \cap B(0, R)$ .*

*Then, for all sequence  $(\lambda_n)$  of positive numbers converging to 0,*

$$\widetilde{W}_{\lambda_n e_1, 1}^{U_n} \Rightarrow \widetilde{W}_{0, 1}^U.$$

*Proof.* Since  $\widetilde{W}_{0, 1}^D(\tau_U > 1) \geq \widetilde{W}_{0, 1}^D(\tau_B > 1) > 0$ , it suffices to prove that

$$(12) \quad \lim_{n \rightarrow \infty} \widetilde{W}_{\lambda_n e_1, 1}^D(f; \tau_{U_n} > 1) = \widetilde{W}_{0, 1}^D(f; \tau_U > 1)$$

for all bounded continuous function  $f : C_\infty \rightarrow \mathbb{R}$ .

First, suppose that  $U_n \rightarrow U$ . Since each set  $U_n$  contains the ball  $B$ , we have

$$\widetilde{W}_{\lambda_n e_1, 1}^D(\tau_{U_n} \leq s) \leq \widetilde{W}_{\lambda_n e_1, 1}^D(\tau_B \leq s).$$

Hence

$$(13) \quad \lim_{s \rightarrow 0} \limsup_n \widetilde{W}_{\lambda_n e_1, 1}^D(\tau_{U_n} \leq s) = 0$$

by Lemma 4.6. Now, a minor modification of the proof of Lemma 4.2 shows that (12) holds. Let us give some details. Because of (13) it is sufficient to prove that

$$(14) \quad \lim_{n \rightarrow \infty} \widetilde{W}_{\lambda_n e_1, 1}^D(f; \tau_{U_n}^s > 1) = \widetilde{W}_{0, 1}^D(f; \tau_U^s > 1)$$

for all  $s > 0$ , where  $\tau_{U_n}^s = \inf\{t \geq s : X_t \notin U_n\}$  and  $\tau_U^s = \inf\{t \geq s : X_t \notin U\}$ . Recall that  $\Omega = \{\tau_U^s = \tau_U^s \neq 1\}$  satisfies  $\widetilde{W}_{0, 1}^D(\Omega) = 1$ . Let  $w \in \Omega$  be given and let  $(w_n) \in C_\infty$  be a sequence such that  $w_n \rightarrow w$ .

If  $\tau_U^s(w) > 1$ , then  $w(t) \in U$  for all  $t \in [s, 1]$ . Choose  $\epsilon > 0$  such that  $w(t) \in U_{2\epsilon^-}$  for all  $t \in [s, 1]$ . Then, for  $n$  large enough,  $w_n(t) \in U_{\epsilon^-}$  for all  $t \in [s, 1]$  and  $U_n \supset U_{\epsilon^-}$ ; thus  $\tau_{U_n}^s(w_n) > 1$ .

If  $\tau_U^s(w) = \tau_U^s(w) < 1$ , then there exists  $t \in [s, 1)$  such that  $w(t) \notin \overline{U}$ . Pick

$\epsilon > 0$  such that  $w(t) \notin U_{2\epsilon^+}$ . Then, for  $n$  sufficiently large,  $w_n(t) \notin U_{\epsilon^+}$  and  $U_n \subset U_{\epsilon^+}$ ; hence  $\tau_{U_n}^s(w_n) < 1$ . This proves that in each case

$$\mathbb{1}_{\{\tau_U^s > 1\}}(w_n) \rightarrow \mathbb{1}_{\{\tau_U^s > 1\}}(w)$$

and (14) follows from the continuous mapping theorem.

Now we turn to the general case, that is we consider the local convergence hypothesis 2. of Proposition 4.9. Fix  $\epsilon > 0$  and choose  $R > 0$  such that  $\widetilde{W}_{0,1}^D(\tau_{B(0,R)} > 1) \geq 1 - \epsilon$ . By Lemma 4.1, we have

$$\lim_{n \rightarrow \infty} \widetilde{W}_{\lambda_n \epsilon_1, 1}^D(\tau_{B(0,R)} > 1) = \widetilde{W}_{0,1}^D(\tau_{B(0,R)} > 1).$$

Therefore

$$\widetilde{W}_{\lambda_n \epsilon_1, 1}^D(\tau_{B(0,R)} > 1) \geq 1 - 2\epsilon$$

for all large enough  $n$ . Set  $U'_n = U_n \cap B(0, R)$  and  $U' = U \cap B(0, R)$ . Then

$$\begin{aligned} & |\widetilde{W}_{\lambda_n \epsilon_1, 1}^D(f; \tau_{U_n} > 1) - \widetilde{W}_{0,1}^D(f; \tau_U > 1)| \\ & \leq |\widetilde{W}_{\lambda_n \epsilon_1, 1}^D(f; \tau_{U'_n} > 1) - \widetilde{W}_{0,1}^D(f; \tau_{U'} > 1)| + 3M\epsilon \end{aligned}$$

where  $M$  is a bound for  $|f|$ . By hypothesis  $U'_n \rightarrow U'$ , hence

$$\limsup_n |\widetilde{W}_{\lambda_n \epsilon_1, 1}^D(f; \tau_{U'_n} > 1) - \widetilde{W}_{0,1}^D(f; \tau_{U'} > 1)| = 0$$

by the first step of this proof. Therefore

$$\limsup_n |\widetilde{W}_{\lambda_n \epsilon_1, 1}^D(f; \tau_{U_n} > 1) - \widetilde{W}_{0,1}^D(f; \tau_U > 1)| \leq 4B\epsilon$$

and the desired result follows by letting  $\epsilon \rightarrow 0$ .  $\square$

**4.4.2. Nice sets.** Let  $U$  be an open subset of  $\mathbb{R}^d$  and  $x_0$  a boundary point of  $U$ . We will say that  $U$  is *nice* at  $x_0$  if there exist a neighborhood  $V$  of  $x_0$  and a number  $r > 0$  such that the following conditions are satisfied:

- (1) For all  $x \in \partial U \cap V$  there exists a half-space  $D_x \supset U$  such that:
  - $x \in \partial D_x$ ;
  - The ball  $B_x \subset D_x$  with radius  $r$  which is tangent to  $\partial D_x$  at  $x$  is contained in  $U$ ;
  - The application  $c$  which maps  $x$  to the center  $c(x)$  of the ball  $B_x$  is continuous at  $x_0$ .
- (2) For all  $y \in U \cap V$  such that  $d(y, \partial U) \leq r/2$ , there exists a point  $x = p(y) \in \partial U \cap V$  such that:
  - $y \in (x, c(x)]$ ;
  - The mapping  $y \mapsto p(y)$  is continuous.

**Remark 4.10.** It can be verified that regularity and convexity assumptions ensures the property of being a “nice set”. If the open set  $U$  is convex and has a boundary of class  $C^2$  in a neighborhood of  $x_0$  then the set  $U$  is nice at  $x_0$ .

Suppose that  $U$  is nice at  $x_0$ . With the above notations, for any  $x \in \partial U \cap V$ , the point  $x + c(x_0) - c(x)$  belongs to the boundary of the ball  $B_{x_0}$ ; thus we can choose a planar rotation  $R_x$  with center at  $c(x_0)$  and such that  $R_x(x + c(x_0) - c(x)) = x_0$ . Note that the angle of  $R_x$  tends to 0 as  $x \rightarrow x_0$ , since  $c(x) \rightarrow c(x_0)$ . Set  $\phi_x(y) = R_x(y + c(x_0) - c(x))$  and  $U_x = \phi_x(U)$ . Then it can be seen that

$$(15) \quad B_{x_0} \subset U_x \subset D_{x_0}$$

and

$$(16) \quad U_x \cap B(0, R) \rightarrow U_{x_0} \cap B(0, R), \quad \text{as } x \rightarrow x_0,$$

for all  $R > 0$ .

**Theorem 4.11.** *Suppose that  $U$  is co-regular and nice at  $x_0$ . Then, as  $x \in U \rightarrow x_0$ , we have*

$$\widetilde{W}_{x,1}^U \Rightarrow \widetilde{W}_{x_0,1}^U.$$

*Proof.* For  $y$  close to  $x_0$ , set  $x = p(y)$ . Since  $y$  belongs to  $(x, c(x)]$ , the point  $q(y) = \phi_x(y)$  belongs to  $(x_0, c(x_0)]$ . Moreover,  $q(y)$  tends to  $x_0$  as  $y \rightarrow x_0$ . Thus, from (15) and (16) together with Proposition 4.9, we obtain

$$\widetilde{W}_{q(y),1}^{U_x} \Rightarrow \widetilde{W}_{x_0,1}^U, \quad \text{as } y \rightarrow x_0.$$

Now by the invariance properties of Brownian motion, we have

$$\widetilde{W}_{y,1}^U = \widetilde{W}_{q(y),1}^{U_x} \circ \phi_x.$$

Since  $\phi_x$  tends to the identity mapping as  $x \rightarrow x_0$ , uniformly on compact subsets of  $\mathbb{R}^d$ , it follows from the continuous mapping theorem that

$$\widetilde{W}_{y,1}^U \Rightarrow \widetilde{W}_{x_0,1}^U, \quad \text{as } y \rightarrow x_0.$$

□

## 5. NICE CONES

Let  $d \geq 2$  and let  $C \subset \mathbb{R}^d$  be an open<sup>1</sup> cone with vertex at 0. We will say that  $C$  is a *nice cone* if it is nice (see 4.4.2) at any point of its boundary, excepting 0. For example, any two-dimensional convex cone is nice. In higher dimension, any circular cone or ellipsoidal<sup>2</sup> cone is nice.

We note two important facts about nice cones:

- (1) If  $C$  is a nice cone, then it is a Lipschitz cone;
- (2) If  $C$  is a nice cone,  $\partial C$  is a null set with respect to Lebesgue measure.

<sup>1</sup>We only suppose that  $C$  is open to simplify the notations. But this is really of no importance because we will suppose that  $C$  is co-regular, so  $\tau_C$  and  $\tau_{\overline{C}}$  are almost surely equal.

<sup>2</sup>An ellipsoidal cone in  $\mathbb{R}^d$  is a cone that is obtained by any rotation of the cone with equations  $a_2x_2^2 + \dots + a_dx_d^2 \leq x_1^2$  and  $x_1 > 0$ , where  $a_2, \dots, a_d$  are positive numbers. If  $a_2 = \dots = a_d$ , then the cone is circular.

The proof of the first one is elementary but quite tedious, so we omit it here. Note that the second fact is clearly a consequence of the first one.

**5.1. Proof of Theorem 1.1.** We begin with the following lemma:

**Lemma 5.1.** *Suppose  $C$  is a nice cone. Let  $x_0 \in \partial C \setminus \{0\}$  and  $t_0 > 0$ . As  $(x, t) \rightarrow (x_0, t_0)$ ,*

$$\widetilde{W}_{x,t}^C \Rightarrow \widetilde{W}_{x_0/\sqrt{t_0},1}^C \circ K_{t_0}^{-1}.$$

*Proof.* By the scaling property of  $\widetilde{W}^C$  (see Remark 2.6), we have

$$\widetilde{W}_{x,t}^C = \widetilde{W}_{x/\sqrt{t},1}^C \circ K_t^{-1}.$$

The result simply follows from Theorem 4.11 together with the continuous mapping theorem.  $\square$

**5.1.1. Convergence of the finite-dimensional distributions.** We will prove in this section that the finite-dimensional distributions of  $\widetilde{W}_{x,1}^C$  converge weakly as  $x \in C \rightarrow 0$ . Recall that for any  $t \in (0, 1]$  the law  $\widetilde{W}_{x,1}^C(X_t \in dy)$  has the density  $e_x(t, y)$  given by

$$e_x(t, y) = \frac{p^C(t, x, y)}{W_x(\tau_C > 1)} W_y(\tau_C > 1 - t).$$

By using an expansion of the heat kernel  $p^C(t, x, y)$  of  $C$  that is given by R. Bañuelos and R. Smits in [1], we shall prove that  $e_x(t, y)$  converges to a limit density  $e(t, y)$ , as  $x \in C \rightarrow 0$ .

Before we recall their result, let us introduce some notations. Let  $C_\Sigma$  be the intersection of the cone  $C$  with the unit sphere  $\mathbb{S}^{d-1}$  and suppose that it is a regular set for the Dirichlet problem with respect to the Laplace-Beltrami operator  $L$  on  $\mathbb{S}^{d-1}$ . Then there exists a complete set of orthonormal eigenfunctions  $m_j$  with corresponding eigenvalues  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 < \dots$  satisfying

$$\begin{cases} Lm_j = -\lambda_j m_j & \text{on } C_\Sigma; \\ m_j = 0 & \text{on } \partial C_\Sigma. \end{cases}$$

We set  $\alpha_j = \sqrt{\lambda_j + (\frac{d}{2} - 1)^2}$ . We will use the following facts that are proved in [1]:

- there exists two constants  $0 < c_1 < c_2$  such that

$$(17) \quad \forall j \geq 1, \quad c_1 j^{\frac{1}{d-1}} \leq \alpha_j \leq c_2 j^{\frac{1}{d-1}};$$

- there exists a constant  $c$  such that

$$(18) \quad \forall j \geq 1, \quad \|m_j\|_\infty \leq c \alpha_j^{\frac{d-1}{2}};$$

- if  $C$  is a Lipschitz cone, then there exists a constant  $c'$  such that

$$(19) \quad \forall j \geq 1, \forall \eta \in C_\Sigma, \quad m_j^2(\eta) \leq \frac{c' m_1^2(\eta)}{I_{\alpha_j}(1)},$$

where  $I_\nu$  is the modified Bessel function of order  $\nu$  :

$$(20) \quad \begin{aligned} I_\nu(x) &= \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} (\sin t)^{2\nu} \cosh(x \cos t) dt \\ &= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\nu+2m}}{m!\Gamma(\nu + m + 1)}. \end{aligned}$$

Then we have the following lemma :

**Proposition 5.2** (Bañuelos, Smits). *Write  $x = \rho\theta$ ,  $y = r\eta$ ,  $\rho, r > 0$ ,  $\theta, \eta \in C_\Sigma$ . We have*

$$p^C(t, x, y) = \frac{e^{-\frac{(r^2+\rho^2)}{2t}}}{t(\rho r)^{\frac{d}{2}-1}} \sum_{j=1}^{\infty} I_{\alpha_j} \left( \frac{\rho r}{t} \right) m_j(\theta) m_j(\eta).$$

Together with the expression of  $I_{\alpha_j}$ , this suggests that  $p^C(t, x, y)$  is equivalent at  $x = 0$  to the product  $g(x)h(t, y)$  where

$$g(x) = \rho^{\alpha_1 - (\frac{d}{2}-1)} m_1(\theta)$$

and

$$h(t, y) = \frac{r^{\alpha_1 - (\frac{d}{2}-1)} e^{-\frac{r^2}{2t}}}{2^{\alpha_1} \Gamma(\alpha_1 + 1) t^{\alpha_1+1}} m_1(\eta).$$

In fact, we have the following :

**Lemma 5.3.** *For  $x = \rho\theta$ ,  $y = r\eta$ ,  $\rho, r > 0$ ,  $\theta, \eta \in C_\Sigma$ , we have*

$$\lim_{\rho \rightarrow 0} \frac{p^C(t, x, y)}{g(x)} = h(t, y),$$

*uniformly in  $(t, r, \theta, \eta) \in [T, +\infty) \times [0, R] \times C_\Sigma \times C_\Sigma$ , for any positive constants  $T$  and  $R$ .*

*Proof.* Set  $M = \frac{\rho r}{t}$ . We have

$$\frac{p^C(t, x, y)}{g(x)h(t, y)} = 2^{\alpha_1} \Gamma(\alpha_1 + 1) e^{-\frac{\rho^2}{2t}} \sum_{j=1}^{\infty} \frac{I_{\alpha_j}(M) m_j(\theta) m_j(\eta)}{M^{\alpha_1} m_1(\theta) m_1(\eta)}.$$

Using relation (19), we get

$$\left| \frac{I_{\alpha_j}(M) m_j(\theta) m_j(\eta)}{M^{\alpha_1} m_1(\theta) m_1(\eta)} \right| \leq \frac{\kappa}{M^{\alpha_1}} \frac{I_{\alpha_j}(M)}{I_{\alpha_j}(1)}.$$

Now, using the integral expression for  $I_{\alpha_j}$ , we obtain

$$I_{\alpha_j}(M) \leq \frac{2\left(\frac{M}{2}\right)^{\alpha_j}}{\sqrt{\pi}\Gamma\left(\alpha_j + \frac{1}{2}\right)} \cosh(M) \int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha_j} dt$$

and

$$I_{\alpha_j}(1) \geq \frac{2\left(\frac{1}{2}\right)^{\alpha_j}}{\sqrt{\pi}\Gamma\left(\alpha_j + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha_j} dt.$$



Hence

$$\frac{I_{\alpha_j}(M)}{I_{\alpha_j}(1)} \leq M^{\alpha_j} \cosh M ,$$

and so

$$\left| \frac{I_{\alpha_j}(M)}{M^{\alpha_1}} \frac{m_j(\theta)}{m_1(\theta)} \frac{m_j(\eta)}{m_1(\eta)} \right| \leq \kappa M^{\alpha_j - \alpha_1} \cosh M .$$

From relation (17), it is easily seen that the series  $\sum_j M^{\alpha_j - \alpha_1} \cosh M$  is uniformly convergent on  $[0, 1 - \epsilon]$ . So, the series

$$\sum_j \frac{I_{\alpha_j}(M)}{M^{\alpha_1}} \frac{m_j(\theta)}{m_1(\theta)} \frac{m_j(\eta)}{m_1(\eta)}$$

is uniformly convergent for  $(M, \theta, \eta) \in [0, 1 - \epsilon] \times C_\Sigma \times C_\Sigma$ . Therefore we can take the limit term by term : since

$$\lim_{M \rightarrow 0} \frac{I_{\alpha_j}(M)}{M^{\alpha_1}} \frac{m_j(\theta)}{m_1(\theta)} \frac{m_j(\eta)}{m_1(\eta)} = \begin{cases} \frac{1}{2^{\alpha_1} \Gamma(\alpha_1 + 1)} & \text{si } j = 1 ; \\ 0 & \text{si } j \geq 2 ; \end{cases}$$

we get

$$\lim_{M \rightarrow 0} \sum_{j=1}^{\infty} \frac{I_{\alpha_j}(M)}{M^{\alpha_1}} \frac{m_j(\theta)}{m_1(\theta)} \frac{m_j(\eta)}{m_1(\eta)} = \frac{1}{2^{\alpha_1} \Gamma(\alpha_1 + 1)} ,$$

where the convergence is uniform for  $(\theta, \eta) \in C_\Sigma \times C_\Sigma$ .  $\square$

**Lemma 5.4.** *The function of  $y$*

$$\sup_{\|x\| \leq \frac{1}{2}} \left| \frac{p^C(1, x, y)}{g(x)} \right|$$

is integrable on  $\mathbb{R}^d$ .

*Proof.* Using relations (18) and (19), we get

$$\begin{aligned} \left| \frac{p^C(1, x, y)}{g(x)} \right| &\leq \frac{e^{-\frac{(\rho^2 + r^2)}{2}}}{r^{\frac{d}{2} - 1} \rho^{\alpha_1}} \sum_{j=1}^{\infty} I_{\alpha_j}(\rho r) \left| \frac{m_j(\theta)}{m_1(\theta)} m_j(\eta) \right| \\ &\leq \frac{e^{-\frac{r^2}{2}}}{r^{\frac{d}{2} - 1} \rho^{\alpha_1}} \sum_{j=1}^{\infty} \frac{I_{\alpha_j}(\rho r)}{I_{\alpha_j}(1)^{\frac{1}{2}}} \alpha_j^{\frac{d-1}{2}} . \end{aligned}$$

From the integral expression for  $I_{\alpha_j}$ , we find that

$$I_{\alpha_j}(\rho r) \leq \frac{2\left(\frac{\rho r}{2}\right)^{\alpha_j}}{\sqrt{\pi} \Gamma(\alpha_j + \frac{1}{2})} \cosh(\rho r) \int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha_j} dt$$

and

$$I_{\alpha_j}(1) \geq \frac{2\left(\frac{1}{2}\right)^{\alpha_j}}{\sqrt{\pi} \Gamma(\alpha_j + \frac{1}{2})} \int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha_j} dt .$$

So, if we set  $\omega_{\alpha_j} = \int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha_j} dt$ , we get

$$\frac{I_{\alpha_j}(\rho r)}{I_{\alpha_j}(1)^{\frac{1}{2}}} \leq \kappa \cosh(\rho r) \left( \frac{\rho r}{\sqrt{2}} \right)^{\alpha_j} \frac{\sqrt{\omega_{\alpha_j}}}{\Gamma(\alpha_j + \frac{1}{2})^{\frac{1}{2}}}.$$

The Wallis integral  $\int_0^{\frac{\pi}{2}} (\sin t)^{2n} dt$  is equivalent to  $cn^{-\frac{1}{2}}$ , thus  $\omega_{\alpha_j}$  is equivalent to  $c\alpha_j^{-\frac{1}{2}}$ . From Stirling's Formula we also get  $\Gamma(\alpha_j + \frac{1}{2}) \geq c\alpha_j^{\alpha_j} e^{-\alpha_j}$ . Thus,

$$\frac{I_{\alpha_j}(\rho r)}{I_{\alpha_j}(1)^{\frac{1}{2}}} \leq \kappa \cosh(\rho r) \left( \frac{\sqrt{e}\rho r}{\sqrt{2}} \right)^{\alpha_j} \frac{\alpha_j^{-\frac{1}{4}}}{\alpha_j^{\alpha_j/2}}.$$

Therefore,

$$(21) \quad \left| \frac{p^C(1, x, y)}{g(x)} \right| \leq \kappa \frac{e^{-\frac{r^2}{2}}}{r^{\frac{d}{2}-1} \rho^{\alpha_1}} \cosh(\rho r) \sum_{j=1}^{\infty} \left( \frac{\sqrt{e}\rho r}{\sqrt{2}} \right)^{\alpha_j} \frac{\alpha_j^{\frac{2d-3}{4}}}{\alpha_j^{\alpha_j/2}}.$$

Since  $\alpha_j \geq \alpha_1$ , the right-hand side of (21) is increasing with  $\rho$ , so

$$\sup_{\rho \leq \frac{1}{2}} \left| \frac{p^C(1, x, y)}{g(x)} \right| \leq \kappa \frac{e^{-\frac{r^2}{2}}}{r^{\frac{d}{2}-1}} \cosh\left(\frac{r}{2}\right) \sum_{j=1}^{\infty} \left( \frac{\sqrt{e}r}{2\sqrt{2}} \right)^{\alpha_j} \frac{\alpha_j^{\frac{2d-3}{4}}}{\alpha_j^{\alpha_j/2}} =: f(r).$$

Because  $\alpha_j > (\frac{d}{2} - 1)$ , the function  $f$  is integrable on any compact subset of  $[0, +\infty)$ . We shall now find an upper bound for the sum that appears in the definition of  $f$  for large values of  $r$ . Let  $M \geq 1$ . For  $2n \leq \alpha_j \leq 2n + 1$ , we have

$$M^{\alpha_j} \frac{\alpha_j^{\frac{2d-3}{4}}}{\alpha_j^{\alpha_j/2}} \leq M^{2n+1} \frac{(2n+1)^{\frac{2d-3}{4}}}{(2n)^n} = M(M^2/2)^n \frac{(2n+1)^{\frac{2d-3}{4}}}{n^n}.$$

Since  $\alpha_j > c_1 j^{\frac{1}{d-1}}$ , the number of indices  $j$  for which  $\alpha_j \leq 2n + 1$  is bounded by  $\left(\frac{2n+1}{c_1}\right)^{d-1}$ . Thus, there exist  $K = K(d) > 0$  such that

$$\sum_{j=1}^{\infty} M^{\alpha_j} \frac{\alpha_j^{\frac{2d-3}{4}}}{\alpha_j^{\alpha_j/2}} \leq M \sum_{n=1}^{\infty} (M^2/2)^n \frac{n^K}{n^n} \leq P(M) e^{M^2/2},$$

where  $P$  is a polynomial. Applying this result with  $M = \frac{\sqrt{e}r}{2\sqrt{2}}$  and  $r \geq 2\sqrt{2/e}$  gives

$$f(r) \leq \kappa \frac{e^{-(1-e/8)\frac{r^2}{2}}}{r^{\frac{d}{2}-1}} \tilde{P}(r) \cosh\left(\frac{r}{2}\right),$$

where  $\tilde{P}$  is a polynomial whose coefficients depend only on  $d$ . This is sufficient to conclude the proof of Lemma 5.4.  $\square$

**Proposition 5.5.** *For any  $t \in [0, 1]$  and any  $R > 0$ , we have*

$$\lim_{\|x\| \rightarrow 0} e_x(t, y) = \frac{h(t, y)}{\int h(1, z) dz} W_y(\tau_C > 1 - t),$$

uniformly on  $\{y \in C : \|y\| \leq R\}$ .

*Proof.* First recall that  $W_x(\tau_C > 1) = \int p^C(1, x, z) dz$ . It follows from Lemmas 5.3, 5.4 and the dominated convergence theorem that

$$(22) \quad \lim_{\|x\| \rightarrow 0} \frac{W_x(\tau_C > 1)}{g(x)} = \int h(1, z) dz.$$

Since

$$e_x(t, y) = \frac{p^C(t, x, y)}{g(x)} \frac{g(x)}{W_x(\tau_C > 1)} W_y(\tau_C > 1 - t),$$

the desired result is easily deduced from proposition 5.3 and relation (22).  $\square$

**Remark 5.6.** By scaling, it is easily deduced from (22) that

$$\lim_{t \rightarrow +\infty} \frac{W_x(\tau_C > t)}{g(t^{-1/2}x)} = \int h(1, z) dz,$$

uniformly on  $\{x \in C : \|x\| \leq R\}$ , for any positive constant  $R$ . So, for a Lipschitz cone, the convergence is stronger than what was announced by R. Bañuelos and R. Smits ([1], Corollary 1).

Note that a straightforward computation gives

$$\int h(1, z) dz = 2^{-\frac{\alpha_1}{2} + \frac{d-2}{4}} \frac{\Gamma(\frac{\alpha_1}{2} + \frac{d+2}{4})}{\Gamma(\alpha_1 + 1)} \int_{C_\Sigma} m_1(\eta) \sigma(d\eta),$$

where  $\sigma$  is Lebesgue measure on the unit sphere  $\mathbb{S}^{d-1}$ .

Let  $e(t, y)$  be the limit of the densities  $e_x(t, y)$  as  $x \in C \rightarrow 0$ . By Proposition 5.5, we have

$$(23) \quad e(t, y) = ct^{-\alpha_1-1} \|y\|^{\alpha_1 - (\frac{d}{2}-1)} e^{-\frac{\|y\|^2}{2t}} m_1(\vec{y}) W_y(\tau_C > 1 - t),$$

where  $c^{-1} = 2^{\frac{\alpha_1}{2} + \frac{d-2}{4}} \Gamma(\frac{\alpha_1}{2} + \frac{d+2}{4}) \int m_1(\eta) \sigma(d\eta)$ .

Proposition 5.5 is not sufficient to prove weak convergence of the law

$$\widetilde{W}_{x,1}^C(X_t \in dy) = e_x(t, y) dy$$

as  $x \in C$  goes to 0. Indeed, except for  $t = 1$ , we don't know *a priori* if  $e(t, y)$  is a probability density (except for the case  $t = 1$ ), and this seems hard to check by a direct computation. However, when  $C$  is a nice cone, we may handle this by proving the equi-integrability of the family  $\{e_x(t, y) : \|x\| \leq 1\}$ .

**Lemma 5.7.** *For all  $t \in [0, 1)$ , we have*

$$\lim_{R \rightarrow \infty} \sup_{\|x\| \leq 1} \widetilde{W}_{x,1}^C(\|X_t\| > R) = 0 .$$

*In other words, the family of probability  $\{\widetilde{W}_{x,1}^C(X_t \in dy) : x \in C, \|x\| \leq 1\}$  is tight.*

*Proof.* Let  $x \in C$  with  $\|x\| \leq 1$  be given, and let  $R > 2$ . We denote by  $\rho = \tau_{B(0,2)}$  the first exit time from the ball  $B(0, 2)$ . A continuous path started at  $x$  that is outside  $B(0, R)$  at time  $t$  must have left  $B(0, 2)$  before that time, so

$$\begin{aligned} & \widetilde{W}_{x,1}^C(\|X_t\| > R) \\ &= \widetilde{W}_{x,1}^C(\rho < t; \|X_t\| > R) \\ &= \widetilde{W}_{x,1}^C(\rho < t; \widetilde{W}_{X_{\rho}, 1-\rho}^C(\|X_{t-s}\| > R)_{|s=\rho}) \\ &\leq \sup \left\{ \widetilde{W}_{y,1-s}^C(\|X_{t-s}\| > R) : y \in C, \|y\| = 2 \text{ and } s \in [0, t] \right\} . \end{aligned}$$

Suppose that the last expression does not tend to 0 as  $R \rightarrow \infty$ ; then there exist a sequence  $(y_n) \in C$  with  $\|y_n\| = 2$  and a sequence  $(s_n) \in [0, t]$  such that

$$(24) \quad \liminf_{n \rightarrow \infty} \widetilde{W}_{y_n, 1-s_n}^C(\|X_{t-s_n}\| > n) > 0 .$$

Without loss of generality, we can suppose that  $(y_n)$  converges to a point  $y \in \overline{C}$  with  $\|y\| = 2$ , and that  $(s_n)$  converges to  $s \in [0, t]$ . But Lemma 5.1 (or Proposition 2.3 if  $y \in C$ ) then implies that  $(\widetilde{W}_{y_n, 1-s_n}^C(X_{t-s_n} \in dy))$  is a convergent sequence of probability measures : this contradicts (24).  $\square$

**Proposition 5.8.** *The finite-dimensional distributions of  $\widetilde{W}_{x,1}^C$  converge as  $x \in C$  tends to 0.*

*Moreover, the limit distribution of the first transition law  $\widetilde{W}_{x,1}^C(X_t \in dy)$ ,  $t \in (0, 1]$ , has density  $e(t, y)$  w.r.t. Lebesgue measure.*

*Proof.* By Proposition 5.5,  $e_x(t, y)$  converges to  $e(t, y)$  uniformly on every compact set as  $x \in C$  tends to 0. Thus, for all  $R > 0$ , we have

$$\begin{aligned} & \limsup_{\|x\| \rightarrow 0} \int |e_x(t, y) - e(t, y)| dy \\ &\leq \sup_{\|x\| \leq 1} \int_{B(0, R)^c} e_x(t, y) dy + \int_{B(0, R)^c} e(t, y) dy . \end{aligned}$$

Thanks to Lemma 5.7 and the integrability of  $e(t, y)$ , letting  $R \rightarrow +\infty$  gives

$$\limsup_{\|x\| \rightarrow 0} \int |e_x(t, y) - e(t, y)| dy = 0 .$$

This proves that the function  $y \mapsto e(t, y)$  is a probability density and that the probability measures  $\widetilde{W}_{x,1}^C(X_t \in dy)$  converge weakly to  $e(t, y) dy$  as  $x \in C$

tends to 0. The weak convergence of the finite-dimensional distributions then follows from Proposition 2.4 since  $\partial C$  has Lebesgue measure 0.  $\square$

5.1.2. *Tightness.* For any  $T > 0$ , the space  $\mathcal{C}_T$  of continuous paths  $w : [0, T] \rightarrow \mathbb{R}^d$  is endowed with the topology generated by the supremum metric and the corresponding Borel  $\sigma$ -algebra.

**Proposition 5.9.** *For any sequence  $(x_n)$  of points of  $\overline{C}$  converging to 0 and for any  $T > 0$ , the sequence of probability measures  $(\widetilde{W}_{x_n,1}^C)$  is tight in  $\mathcal{C}_T$ .*

*Proof.* Since the arguments do not depend on the value of  $T$ , we will only consider the case  $T = 1$ . It suffices to prove that, for all  $\epsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \widetilde{W}_{x_n,1}^C(\chi(\delta, 0, 1) > \epsilon) = 0 ,$$

where

$$\chi(\delta, a, b)(w) = \sup\{\|w(s) - w(t)\| : |s - t| \leq \delta, s, t \in [a, b]\}$$

is the modulus of continuity of order  $\delta$  of  $w$  on  $[a, b]$  (see Billingsley [2]).

Our proof is a modification of Shimura's one for the two-dimensional case ([8], Theorem 2).

Fix  $\epsilon > 0$  and set  $s = 1/2$ . Since  $\chi(\delta, \cdot, \cdot)$  is sub-additive when considered as a function on the set of intervals, we have

$$\widetilde{W}_{x_n,1}^C(\chi(\delta, 0, 1) > 4\epsilon) \leq \underbrace{\widetilde{W}_{x_n,1}^C(\chi(\delta, 0, s) > 3\epsilon)}_{A_n(\delta)} + \underbrace{\widetilde{W}_{x_n,1}^C(\chi(\delta, s, 1) > \epsilon)}_{B_n(\delta)} .$$

Let us start with  $B_n(\delta)$ . It follows from Proposition 5.8 that

$$\lim_{r \rightarrow 0, R \rightarrow \infty} \liminf_{n \rightarrow \infty} \widetilde{W}_{x_n,1}^C(r \leq \|X_s\| \leq R) = 1 .$$

Hence we can fix  $\alpha > 0$  and choose  $0 < r < R$  such that

$$\inf_n \widetilde{W}_{x_n,1}^C(r \leq \|X_s\| \leq R) \geq 1 - \alpha .$$

We then have

$$B_n(\delta) \leq \widetilde{W}_{x_n,1}^C(r \leq \|X_s\| \leq R; \chi(\delta, s, 1) > \epsilon) + \alpha ,$$

So, by the Markov property,

$$\begin{aligned} B_n(\delta) &\leq \widetilde{W}_{x_n,1}^C\left(r \leq \|X_s\| \leq R; \widetilde{W}_{X_s,1-s}^C(\chi(\delta, 0, 1-s) > \epsilon)\right) + \alpha \\ &\leq \underbrace{\sup\left\{\widetilde{W}_{y,s}^C(\chi(\delta, 0, s) > \epsilon) : y \in C \text{ and } r \leq \|y\| \leq R\right\}}_{D(\delta)} + \alpha . \end{aligned}$$

Now, if  $D(\delta)$  did not tend to 0 as  $\delta$  goes to 0, then we could find a sequence  $(\delta_n)$  converging to 0 and a sequence  $(y_n)$  of points of  $C$  converging to a point  $y \in \overline{C} \setminus \{0\}$  such that

$$\liminf_{n \rightarrow \infty} \widetilde{W}_{y_n,s}^C(\chi(\delta_n, 0, s) > \epsilon) > 0 ,$$

which would contradict the weak convergence of the sequence of probability measures  $(\widetilde{W}_{y_n, s}^C)$  (Lemma 5.1 or Proposition 2.3 if  $y \in C$ ). This proves that  $\lim_{\delta \rightarrow 0} \limsup_n B_n(\delta) \leq \alpha$ , and letting  $\alpha \rightarrow 0$  then gives

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} B_n(\delta) = 0 .$$

We now turn to  $A_n(\delta)$ . Let  $\rho = \tau_{B(0, \epsilon)}$  be the exit time from the ball  $B(0, \epsilon)$  with center at 0 and radius  $\epsilon$ . Since the modulus of continuity of a path  $w$  is less than  $2\epsilon$  as long as it has not left the ball  $B(0, \epsilon)$ , we have

$$\begin{aligned} A_n(\delta) &\leq \widetilde{W}_{x_n, 1}^C(\rho < s; \chi(\delta, \rho, s) > \epsilon) \\ &\leq \widetilde{W}_{x_n, 1}^C(\rho < s; \widetilde{W}_{X(\rho), 1-\rho}^C(\chi(\delta, 0, 1) > \epsilon)) . \end{aligned}$$

Hence

$$\begin{aligned} &\limsup_{n \rightarrow \infty} A_n(\delta) \\ &\leq \sup\{\widetilde{W}_{y, t}^C(\chi(\delta, 0, 1) > \epsilon) : y \in C, \|y\| = \epsilon \text{ and } t \in [s, 1]\} . \end{aligned}$$

In the same way as above, we then get

$$\limsup_{n \rightarrow \infty} A_n(\delta) = 0 ,$$

which is sufficient to prove Proposition 5.9.  $\square$

Together with Proposition 5.8, Proposition 5.9 proves that  $\widetilde{W}_{x, 1}^C$  converges weakly on every  $\mathcal{C}_T$ ,  $T > 0$ , as  $x \in C$  tends to 0. This is equivalent to weak convergence on  $\mathcal{C}_\infty$ ; thus Theorem 1.1 is proven.

The limit law will be denoted by  $\widetilde{W}_{0, 1}^C$  and called the law of  $C$ -Brownian meander. Thanks to Theorem 1.1 we shall say that the  $C$ -Brownian meander is Brownian motion conditioned to stay in  $C$  for a unit of time.

**5.2. Properties of the  $C$ -Brownian meander.** The  $C$ -Brownian meander starts from the vertex of the cone  $C$  and stays in it for a unit of time. From the Markov property and the expression of the densities of the entrance laws, the law of the exit time of  $C$  after time 1 is easily derived :

**Proposition 5.10.** *For any  $t > 1$ , we have*

$$\widetilde{W}_{0, 1}^C(\tau_C > t) = t^{-\frac{\alpha_1}{2} + \frac{d-2}{4}} .$$

*Proof.* By the Markov property, we have

$$\begin{aligned} \widetilde{W}_{0, 1}^C(\tau_C > t) &= \widetilde{W}_{0, 1}^C(W_{X_1}(\tau_C > t - 1)) \\ &= \int_C e(1, y) W_y(\tau_C > t - 1) dy . \end{aligned}$$

With the change of variables  $y = \sqrt{t}u$ , the integral becomes

$$\int_C e(1, \sqrt{t}u) W_{\sqrt{t}u}(\tau_C > t - 1) t^{\frac{d}{2}} du .$$

The scaling property of Brownian motion gives

$$W_{\sqrt{tu}}(\tau_C > t - 1) = W_u(\tau_C > 1 - 1/t),$$

and from relation (23) p. 26 it is easily seen that

$$e(1, \sqrt{tu})W_u(\tau_C > 1 - 1/t)t^{\frac{d}{2}} = t^{-\frac{\alpha_1}{2} + \frac{d-2}{4}} e(1/t, u).$$

The desired result follows from the fact that  $e(1/t, u)$  is a probability density.  $\square$

**5.3. Application to random walks.** Let  $(\xi_n)_{n \geq 1}$  be a sequence of i.i.d. random vectors of  $\mathbb{R}^d$  with  $\mathbb{E}(\xi_n) = 0$  and a covariance matrix equal to the identity. Form the random walk  $S_n = \xi_1 + \xi_2 + \dots + \xi_n$  and let  $\{\mathcal{S}_n(t), t \geq 0\}$  be the process with continuous paths for which  $\mathcal{S}_n(k/n) = S_k/\sqrt{n}$  and which is linearly interpolated elsewhere.

Consider a nice cone  $C$  for which  $\mathbb{P}(\xi_1 \in C) > 0$ , and let

$$T_C = \inf\{n > 0 : S_n \notin C\}$$

be the first exit of the random walk from the cone  $C$ .

For  $d = 1$  and  $C = (0, +\infty)$ , Bolthausen proved in [3] that conditionally on  $\{T_C > n\}$  the process  $\mathcal{S}_n$  converges in law to a Brownian meander; in other words, we have

$$\mathbb{P}(\mathcal{S}_n \in * | T_C > n) \Rightarrow \widetilde{W}_{0,1}^+.$$

This means that a random walk conditioned to stay positive converges in law to a Brownian motion conditioned to stay positive. By analogy, we are led to conjecture that a multidimensional random walk conditioned to stay in a cone should converge, in good cases, to a Brownian motion conditioned to stay in the same cone, that is

$$(25) \quad \mathbb{P}(\mathcal{S}_n \in * | T_C > n) \Rightarrow \widetilde{W}_{0,1}^C.$$

Shimura proved it to be true in the two-dimensional case, under extra symmetry assumptions on the random walk and the cone (see [9]). It is clear that such an invariance principle can not hold without extra assumptions. For example, consider the simple random on  $\mathbb{Z}^3$  and the cone  $C = \{(x, y, z) : x \geq 0 \text{ and } x/2 \leq y \leq 2x\}$ . Given  $\{T_C > n\}$ ,  $\{S_k, k = 1 \dots n\}$  is a simple random walk on  $\{(0, 0, z) : z \in \mathbb{Z}\}$ . Hence it converges in law to a one-dimensional Brownian motion that lives on the  $z$ -axis. Note that in this case  $\mathbb{P}(T_C > n) = 1/3^n$  decreases exponentially to 0. The next result states that a necessary condition for the above-mentioned limit theorem to hold is that  $\mathbb{P}(T_C > n)$  decreases polynomially to 0

**Proposition 5.11.** *Suppose  $d \geq 2$  and let  $C$  be a nice cone in  $\mathbb{R}^d$ .*

*If*

$$\mathbb{P}(\mathcal{S}_n \in * | T_C > n) \Rightarrow \widetilde{W}_{0,1}^C,$$

*then*

$$\mathbb{P}(T_C > n) = n^{-\frac{\alpha_1}{2} + \frac{d-2}{4}} L(n),$$

where  $L$  is a slowly varying function.

*Proof.* Set  $\mu_n(*) = \mathbb{P}(\mathcal{S}_n \in * | T_C > n)$ . Let  $t > 1$  and observe that

$$\frac{\mathbb{P}(T_C > [tn])}{\mathbb{P}(T_C > n)} = \mu_n(\tau_C > t_n),$$

where  $t_n = [tn]/n$ . Since  $\mu_n \Rightarrow \widetilde{W}_{0,1}^C$  and  $t_n \rightarrow t$ , an argument similar to the one used in the proof of Lemma 4.2 shows that

$$\lim_{n \rightarrow \infty} \mu_n(\tau_C > t_n) = \widetilde{W}_{0,1}^C(\tau_C > t).$$

By Proposition 5.10 we have therefore

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(T_C > [tn])}{\mathbb{P}(T_C > n)} = t^{-\frac{\alpha_1}{2} + \frac{d-2}{4}}.$$

This proves that  $L(n) = n^{\frac{\alpha_1}{2} - \frac{d-2}{4}} \mathbb{P}(T_C > n)$  is slowly varying.  $\square$

For a two-dimensional random walk with bounded increments, the conditional limit theorem (25) holds under the assumption that  $\mathbb{P}(T_C > n)$  is “slowly” decreasing to 0. We will present this result in a forthcoming paper.

#### APPENDIX A. THE CONTINUOUS MAPPING THEOREM

For the reader convenience we state here the continuous mapping theorem. The proof can be found for example in Billingsley [2].

**Theorem A.1.** *Let  $X$  and  $Y$  be two metric spaces equipped with their Borel  $\sigma$ -algebras. Let  $(\mu_n), \mu$  be probability measures on  $X$  such that  $\mu_n \Rightarrow \mu$ , and let  $(\phi_n), \phi : X \rightarrow Y$  be measurable mappings.*

*If there exists a measurable subset  $X'$  of  $X$  such that  $\mu(X') = 1$  and such that  $\phi_n(x_n) \rightarrow \phi(x)$  for all sequence  $(x_n)$  converging to a point  $x \in X'$ , then*

$$\mu_n \circ \phi_n^{-1} \Rightarrow \mu \circ \phi^{-1}.$$

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