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Distinction of some induced representations

13th December 2008

Abstract

Let K/F be a quadratic extension of p-adic fields, σ the nontrivial element of the Galois group of K over F, and π a smooth representation of $GL(n,K)$. Denoting by $\pi^\vee$ the smooth contragredient of π, and by π^σ the representation $\pi\circ\sigma$, we show that representation of $GL(2n,K)$ obtained by normalized parabolic induction of the representation $\pi^\vee\otimes\pi^\sigma$ is distinguished with respect to $GL(2n,F)$. This is a step towards the classification of distinguished generic representations of general linear groups over p-adic fields.

Introduction

Let K/F be a quadratic extension of p-adic fields, σ the nontrivial element of the Galois group of K over F, and π a smooth representation of $GL(n,K)$. We denote by σ again the automorphism of $M_{2n}(K)$ induced by σ.
A smooth representation ρ of $GL(2n,K)$ is said to be distinguished if there is a nonzero $GL(2n,F)$-invariant linear form on its space. If ρ is irreducible, the space of such forms is of dimension at most 1.
Calling $\pi^\vee$ the smooth contragredient of π and π^σ the representation $\pi\circ\sigma$, we denote by $\pi^\sigma\times\pi^\vee$ the representation of $GL(2n,K)$, obtained by normalized induction of the representation $\pi^\sigma\otimes\pi^\vee$ of the standard parabolic subgroup of type (n,n). The aim of this work is to show that such a representation is distinguished.
The case $n=1$ is treated in [H] for unitary $\pi^\sigma\times\pi^\vee$, using a criterion characterizing distinction in terms of gamma factors. In [F3], Flicker defines a linear form on the space of $\pi^\sigma\times\pi^\vee$ by a formal integral which would define the invariant linear form once the convergence is insured. Finally in [F-H], for $n=1$, the convergence of this linear form is obtained for $\pi^\sigma|\ |_K^s\times\pi^\vee|\ |_K^{-s}$ and s of real part large enough, the conclusion follows from an analytic continuation argument.
We generalize this method here. The first part, which consists of showing that the representation $\pi^\sigma|\ |_K^s\times\pi^\vee|\ |_K^{-s}$ when $Re(s)$ is large enough. The analytic continuation argument is a consequence of the theory of Igusa local zeta functions.
The first half of the first section is about notations and basic concepts used in the rest of the work. In the second half, we recall properties of the space of flat sections of an parabolically induced representation, and a theorem about rationality of Igusa zeta functions.
The second section is devoted to the proof of our main theorem 2.1, which asserts that the representation $\pi^\sigma\times\pi^\vee$ is distinguished.
We end this introduction by recalling a conjecture about classification of distinguished generic representations:

Conjecture. *Let m be a positive integer, and ρ a generic representation of the group $GL(m,K)$, obtained by normalised parabolic induction of quasi-square-integrable representations $\Delta_1,\dots,\Delta_t$. It is distinguished if and only if there exists a reordering of the Δ_i's, and an integer r between 1 and $t/2$, such that we have $\Delta_{i+1}^\sigma=\Delta_i^\vee$ for $i=1,3,..,2r-1$, and Δ_i is distinguished for $i>2r$.*

We denote by η the nontrivial character of F^* trivial on the norms of K^*. According to Proposition 26 in [F1], Proposition 12 of [F2], Theorem 6 of [K], and Corollary 1.6 [A-K-T], our result reduces the proof of the conjecture to showing that representations of the form $\Delta_1 \times \cdots \times \Delta_t$ with $\Delta_{i+1}^{\sigma} = \Delta_i^{\vee}$ for $i = 1, 3, .., 2r-1$ for some r between 1 and $t/2$, and non isomorphic distinguished or η-distinguished Δ_i's for $i > 2r$ are not distinguished whenever one of the Δ_i's is η-distinguished for $i > 2r$. According to [M3], the preceeding conjecture implies the equality of the analitically defined Asai L-function and the Galois Asai L-function of a generic representation.

1 Notations and preliminary results

1.1 Notations

We denote by $| \ |_K$ and $| \ |_F$ the respective absolute values on K^*, by q_K and q_F the respective cardinalities of their residual field, and by R_K the valuation ring of K.
We call partition of a positive integer n, a family $\bar{n} = (n_1, \ldots, n_t)$ of positive integers (for a certain t in $\mathbb{N} - \{0\}$), such that the sum $n_1 + \cdots + n_t$ is equal to n. To such a partition, we associate a subgroup of $GL(n, K)$ denoted by $P_{\bar{n}}(K)$, given by matrices of the form

$$\begin{pmatrix} g_1 & \star & \star & \star & \star \\ & g_2 & \star & \star & \star \\ & & \ddots & \star & \star \\ & & & g_{t-1} & \star \\ & & & & g_t \end{pmatrix},$$

with g_i in $GL(n_i, K)$ for i between 1 and t. We call it the standard parabolic subgroup associated with the partition $\bar{n}$. We denote by $N_{\bar{n}}(K)$ its unipotent radical subgroup, given by the matrices

$$\begin{pmatrix} I_{n_1} & \star & \star \\ & \ddots & \star \\ & & I_{n_t} \end{pmatrix}.$$

Finally we denote by $P_n(K)$ the affine subgroup of $GL(n, K)$ given by the matrices $\begin{pmatrix} g & \star \\ & 1 \end{pmatrix}$, with g in $GL(n-1, K)$.
Let X be a locally closed space of an l-group G, and H closed subgroup of G, with $H.X \subset X$. If V is a complex vector space, we denote by $C^\infty(X, V)$ the space of smooth functions from X to V, and by $C_c^\infty(X, V)$ the space of smooth functions with compact support from X to V (if one has $V = \mathbb{C}$, we simply denote it by $C_c^\infty(X)$).
If ρ is a complex representation of H in V_ρ, we denote by $C^\infty(H\backslash X, \rho, V_\rho)$ the space of functions f from X to V_ρ, fixed under the action by right translation of some compact open subgroup U_f of G, and which verify $f(hx) = \rho(h)f(x)$ for $h \in H$, and $x \in X$ (if ρ is a character, we denote this space by $C^\infty(H\backslash X, \rho)$. We denote by $C_c^\infty(H\backslash X, \rho, V_\rho)$ subspace of functions with support compact modulo H of $C^\infty(H\backslash X, \rho, V_\rho)$.
We denote by $Ind_H^G(\rho)$ the representation by right translation of G in $C^\infty(H\backslash G, \rho, V_\rho)$ and by $ind_H^G(\rho)$ the representation by right translation of G in $C_c^\infty(H\backslash G, \rho, V_\rho)$. We denote by $Ind'^G_H(\rho)$ the normalized induced representation $Ind_H^G((\Delta_G/\Delta_H)^{1/2}\rho)$ and by $ind'^G_H(\rho)$ the normalized induced representation $ind_H^G((\Delta_G/\Delta_H)^{1/2}\rho)$.
Let n be a positive integer, and $\bar{n} = (n_1, \ldots, n_t)$ be a partition of n, and suppose that we have a representation (ρ_i, V_i) of $GL(n_i, K)$ for each i between 1 and t. Let ρ be the extension to $P_{\bar{n}}(K)$ of the natural representation $\rho_1 \otimes \cdots \otimes \rho_t$ of $GL(n_1, K) \times \cdots \times GL(n_t, K)$, by taking it trivial on $N_{\bar{n}}(K)$. We denote by $\rho_1 \times \cdots \times \rho_t$ the representation $Ind'^{GL(nK)}_{P_{\bar{n}}(K)}(\rho)$.

1.2 Flat sections

Let π be a representation of $GL(n, K)$, let $(n_1, \dots, n_t)$ be a partition of n, and let $\rho_1, \dots, \rho_t$ be smooth representations of $GL(n_i, K)$, such that $\pi = \rho_1 \times \dots \times \rho_t$. The map $s = (s_1, \dots, s_t) \mapsto q_K^s = (q_K^{s_1}, \dots, q_K^{s_t})$ defines an isomorphism of varieties between $\mathcal{D}_K^t = (\mathbb{C}/\frac{2i\pi}{ln(q_K)\mathbb{Z}})^t$ and $\mathbb{C}^{*t}$. On can twist the representation π by an element s of $\mathcal{D}_K^t$ into the representation $\pi_s = \rho_1|\ |_K^{s_1} \times \dots \times \rho_t|\ |_K^{s_t}$.
Denoting by (ρ, V_ρ) the tensor prodoct of the ρ_i's, the restrictions of functions of the space $C_c^\infty(P_{\bar{n}}(K)\backslash GL(n, K), \Delta\rho_s, V_\rho)$ of π_s (where Δ is $(\Delta_{GL(n,K)}/\Delta_{P_{\bar{n}}(K)})^{1/2}$) to the maximal compact subgroup $K_n = GL(n, R_K)$ of $GL(n, K)$, define the same space $\mathcal{F}_\pi$, which is called the space of flat sections of the series π_s. This restriction being injective, it identifies for each s in $\mathcal{D}_K^t$, the space of π_s with $\mathcal{F}_\pi$. Hence to every f in $\mathcal{F}_\pi$ corresponds a unique function f_s in the space of π_s, and $s \mapsto f_s$ can be seen as a section of the vector bundle over $\mathcal{D}_K^t$ with fibers π_s, which is actually trivial, isomorphic to $\mathcal{D}_K^t \times \mathcal{F}_\pi$.
We still denote by π_s the representation of $GL(n, K)$ on $\mathcal{F}_\pi$, defined by $\pi_s(g)f = (\pi_s(g)f_s)_{|K_n}$ for f in $\mathcal{F}_\pi$ and g in $GL(n, K)$. For fixed g in $GL(n, K)$, the map $u \mapsto \pi_s(g)f$ from $\mathcal{D}_K^t$ to $\mathcal{F}_\pi$ is regular (i.e. belongs to the space $\mathbb{C}(\mathcal{D}_K^t) \otimes \mathcal{F}_\pi$).

Lemma 1.1. *(see [CS], Lemma 2.2) Let the representation π be as above, and let f belong to the space $\mathcal{F}_\pi$. If we denote by N the unipotent radical of $P_{\bar{n}}(K)$, by N^- its image under transposition, and by w a Weyl element such that wNw^{-1} is equal to N^-.*
• *There exists a compact open subgroup U_f of $GL(n, K)$, such that for every s in $\mathcal{D}_K^t$, one has $\pi_s(u)f_s = f_s$.*
• *There exists a compact open subgroup N_f of N, such that for every s in $\mathcal{D}_K^t$, the function $n \mapsto f_s(wn)$ for n in N, has its support in N_f.*

Proof. For the first point, we choose U_f, such that $\pi_s(u)f = f$ for u in U_f.
For the second point, we take a countable exhaustive sequence of N of compact subgroups N_i. Denoting by ϕ_{f_s} the function $n \mapsto f_s(wn)$, as $wNw^{-1} = N^-$ has trivial intersection with $P_{\bar{n}}(K)$, we deduce that for every s, the function ϕ_{f_s} has compact support in N. Hence denoting by $\mathcal{D}_i$ the subset of $\mathcal{D} = \mathcal{D}_K^t$, consisting of s such that ϕ_{f_s} has its support in N_i, we have $\mathcal{D} = \cup_i \mathcal{D}_i$. Baire's theorem then implies that one of the $\mathcal{D}_i$'s contains an open subset of $\mathcal{D}$. Now the condition "ϕ_{f_s} has its support in N_i", is equivalent to the fact that for every n in N, outside N_i, the vector $\phi_{f_s}(n) = (\pi_s(n)f)(w)$ is equal to 0. Calling $\mathcal{D}_{i_0}$ one of the $\mathcal{D}_i$'s containing an open subset $\mathcal{D}$, then if n in N is outside of N_{i_0}, the regular function $s \mapsto (\pi_s(n)f)(w)$ zero for s in $\mathcal{D}_{i_0}$, hence zero everywhere. From this we deduce that $\mathcal{D}$ is equal to $\mathcal{D}_{i_0}$, and this concludes the proof. □

1.3 Igusa local zeta functions

Let m be a positive integer, and P be a non constant polynomial in $K[X_1, \dots, X_m]$. Then to such a polynomial P, one can associate the follwing functional on $C_c^\infty(K^m)$, depending on the complex variable s, formally defined for ϕ in $C_c^\infty(K^m)$ by the formula:

$$I_P(\phi, s) = \int_{K^m} |P(x)|_K^s \phi(x) dx,$$

for dx the Haar measure on K^m. It is clear that such an integral is absolutely convergent, hence well defined, for $Re(s)$ positive. For fixed ϕ, this defines a function of the complex parameter s in the variety $\mathcal{D}_K = \mathbb{C}/\frac{2i\pi}{Ln(q_K)}\mathbb{Z} \simeq \mathbb{C}^*$. Actually, we have the following fact, which is a consequence of a theorem of Igusa:

Theorem 1.1. *([I], theorem 8.2.1) For every ϕ in $C_c^\infty(K^m)$, the function $s \mapsto I_P(\phi, s)$ is a rational function of $\mathcal{D}_K$.*
Moreover, there is a polynomial Q in $\mathbb{C}[X]$, such that for every ϕ in $C_c^\infty(K^m)$, the function $s \mapsto Q(q_K^{-s})I_P(\phi, s)$ is regular (i.e. a polynomial in q_K^{-s}) on $\mathcal{D}_K$.

2 Distinction of representations $\pi^\sigma \times \pi^\vee$

We denote by G the group $GL(2n,K)$, by H its subgroup $GL(2n,F)$, by G' the group $GL(n,K)$ and by M the group $M_n(K)$. We denote by P the group $P_{(n,n)}(K)$, and by N the group $N_{(n,n)}(K)$. We denote by $\bar{H}$ subroup of G given by matrices of the form $\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}$, and by $\bar{T}$ the subgroup of $\bar{H}$ of matrices $\begin{pmatrix} A & 0 \\ 0 & A^\sigma \end{pmatrix}$, with A in G'.

We call U be the matrix $\begin{pmatrix} I_n & -\delta I_n \\ I_n & \delta I_n \end{pmatrix}$ of G, and W the matrix $\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. One has $U^\sigma U^{-1} = W$ and the group H is equal to $U^{-1}\bar{H}U$.

Lemma 2.1. *The double class PUH is opened in G.*

Proof. Call S the space of matrices g in G verifying $g^\sigma = g^{-1}$, which is, from Proposition 3. of chapter 10 of [S], homeomorphic to the quotient space G/H by the map $Q : g \mapsto g^\sigma g^{-1}$. As the map Q sends U on W, the double class PUH corresponds to the open subset of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in S such that $det(C) \neq 0$, the conclusion follows.

□

We prove the following integration formula.

Lemma 2.2. *There is a Haar measure $d\dot{h}$ on the quotient space $\bar{T}\backslash\bar{H}$, and a Haar measure dB on M, such that for any measurable positive function ϕ on the quotient space $\bar{T}\backslash\bar{H}$, then the integrals*

$$\int_{\bar{T}\backslash\bar{H}} \phi(\dot{h})d\dot{h}$$

and

$$\int_M \phi\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix} \frac{dB}{|det(I_n - BB^\sigma)|_K^n}$$

are equal.

Proof. It suffices to show this equality when ϕ is positive, continuous with compact support in $\bar{T}\backslash\bar{H}$. We fix Haar measures dt on $\bar{T}$ and dg on $\bar{H}$, such that $d\dot{h}dt = dg$. It is known that there exists some positive function $\tilde{\phi}$ with compact support in $\bar{H}$, such that $\phi = \tilde{\phi}^{\bar{T}}$, which means that for any $\dot{h}$ in $\bar{H}$, one has $\phi(\dot{h}) = \int_{\bar{T}} \tilde{\phi}(tg)dt$. One then has the relation

$$\int_{\bar{T}\backslash\bar{H}} \phi(\dot{h})d\dot{h} = \int_{\bar{H}} \tilde{\phi}(g)dg.$$

Now as $\bar{H}$ is conjugate to H, there are Haar measures dA and dB on M such that dt is equal to $d^*A = \frac{dA}{|det(A)|_K^n}$, and the Haar measure on $\bar{H}$ is described by the relation

$$d\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix} = \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_F^{2n}} = \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_K^{n}}.$$

Hence we have

$$\begin{aligned} \int_{\bar{T}\backslash\bar{H}} \phi(\dot{h})d\dot{h} &= \int_{M\times M} \tilde{\phi}\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix} \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_K^{n}} \\ &= \int_{M\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & A^{-1}B \\ (A^{-1}B)^\sigma & I_n \end{pmatrix}\right] \frac{dAdB}{|det(A)|_K^{2n}|det(I_n - A^{-1}B(A^{-1}B)^\sigma)|_K^n} \end{aligned}$$

as the complementeray of G' is a negligible set of M.

This becomes after the change of variable $B := A^{-1}B$ equal to

$$\int_{M\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix}\right] \frac{dA}{|det(A)|_K^n} \frac{dB}{|det(I_n - BB^\sigma|_K^n}$$

which is itself equal to

$$\int_{G'\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix}\right] d^*A \frac{dB}{|det(I_n - BB^\sigma)|_K^n}.$$

The conclusion follows from the fact that $\tilde{\phi}^{\bar{T}}$ is equal to ϕ.

□

Theorem 2.1. *Let n be a positive integer, and let π be a smooth irreducible representation of G'. Then the representation $\pi^\sigma \times \pi^\vee$ is a distinguished representation of G.*

Proof. We will first show that the representation $\Pi_s = \pi^\sigma|\ |^s \times \pi^\vee|\ |^{-s}$ of G is distinguished for a complex number s of real part large enough. Then we show that this property still holds for s equal to 0.

Step 1.

The representation $\pi^\sigma|\ |^s \otimes \pi^\vee|\ |^{-s}$ of $G' \times G'$ is distinguished by $\bar{T}$. Call L a nonzero $\bar{T}$-invariant linear form on the space of $\pi^\sigma|\ |_K^s \otimes \pi^\vee|\ |_K^{-s}$.

We denote by ρ_s the representation P, which is the extension of $\pi^\sigma|\ |_K^s \otimes \pi^\vee|\ |_K^{-s}$ to P by the trivial representation of $N_{(n,n)}(K)$. Let f_s belong to the space $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$ of $\pi^\sigma|\ |_K^s \times \pi^\vee|\ |_K^{-s}$, corresponding to f in the space of flat sections $\mathcal{F}_\Pi$, where $\Pi = \Pi_0 = \pi^\sigma \times \pi^\vee$. If A, B and g belong respectively to G', G' and G, one has the following relation:

$$f_s\left[\begin{pmatrix} A & \star \\ 0 & B \end{pmatrix} g\right] = \frac{|det(A)|_K^{n/2+s}}{|det(B)|_K^{n/2+s}} \pi^\sigma(A) \otimes \pi^\vee(B) f(g).$$

In particular for f_s in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, corrsponding to , the restriction to $\bar{H}$ of the function $L_{f_s} : g \mapsto L(f_s(g))$ belongs to the space $C^\infty(\bar{T}\backslash\bar{H})$, but its support modulo $\bar{T}$ is generally not compact, we will show later that the space of functions obtained this way contains $C_c^\infty(\bar{T}\backslash\bar{H})$ as a proper subspace. We must show that for s of real part large enough, the integral $\int_{\bar{T}\backslash\bar{H}} |L_{f_s}(\dot{h})| d\dot{h}$ converges.

According to lemma 2.2, this integral is equal to

$$\int_M |L_{f_s}| \begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix} \frac{dB}{|det(I_n - BB^\sigma)|_K^n}.$$

As before, we cane suppose that B belongs to G', hence the followin decomposition holds

$$\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix} = \begin{pmatrix} (I_n - BB^\sigma)B^{-\sigma} & I_n \\ & B^\sigma \end{pmatrix}\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}\begin{pmatrix} I_n & B^{-\sigma} \\ & I_n \end{pmatrix}.$$

Denoting by Φ_{f_s} the function

$$L(\phi_{f_s}) : B \mapsto L_{f_s}\left[\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}\begin{pmatrix} I_n & B \\ & I_n \end{pmatrix}\right],$$

the following equalities follow:

$$\begin{aligned} \int_{\bar{T}\backslash\bar{H}} |L_{f_s}(\dot{h})| d\dot{h} &= \int_M \left(\tfrac{|det(I_n - BB^\sigma)|_K}{|det(B)|_K^2}\right)^{s+n/2} |\Phi_{f_s}(B^{-\sigma})| \tfrac{dB}{|det(I_n - BB^\sigma)|_K^n} \\ &= \int_{G'} \left(\tfrac{|det(I_n - BB^\sigma)|_K}{|det(B)|_K^2}\right)^{s+n/2} |\Phi_{f_s}(B^{-\sigma})| \tfrac{|det(B)|_K^n d^*B}{|det(I_n - BB^\sigma)|_K^n} \\ &= \int_{G'} (|det(I_n - C^{-\sigma}C^{-1})|_K |det(C)|_K^2)^{s+n/2} |\Phi_{f_s}(C)| \tfrac{d^*C}{|det(C)|_K^n |det(I_n - C^{-\sigma}C^{-1})|_K^n} \\ &= \int_M |det(CC^\sigma - I_n)|_K^{s-n/2} |\Phi_{f_s}(C)| dC \end{aligned}$$

Recalling that the map f_s belongs to $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, and as the map $(P, N') \in P \times N \mapsto PWN' \in G$ is a homeomorphism, we deduce that

$$C \mapsto f_s\left[\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}\begin{pmatrix} I_n & C \\ & I_n \end{pmatrix}\right]$$

has compact support in M, hence so does Φ_{f_s}.
This eventually implies that $\int_{\bar{T}\backslash\bar{H}} L(f_s(\dot{h}))d\dot{h}$ is convergent as soon as s has real part greater than $n/2$.

Step 2.
Suppose that the complex number s has real part greater than $n/2$. We are going to show that the linear form $\Lambda : f_s \mapsto \int_{\bar{T}\backslash\bar{H}} L_{f_s}(\dot{h})d\dot{h}$ is nonzero. More precisely we are going to show that the space of functions $L(f)$ on $\bar{T}\backslash\bar{H}$ for f in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, contain $C_c^\infty(\bar{T}\backslash\bar{H})$.
According to Lemma 2.1 , the double class PUH is opened in G, hence the extension by zero outside PUH gives an injection of the space $C_c^\infty(P\backslash PUH, \Delta_P^{-1/2}\rho_s)$ into the space $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$.
But the right translation by U, which is a vector space automorphism of $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, sends $C_c^\infty(P\backslash PUH, \Delta_P^{-1/2}\rho_s)$ onto $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$, hence $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$ is a subspace of $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$.
Now restriction to $\bar{H}$ defines an isomorphism between $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$ and $C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s)$ because Δ_P has trivial restriction to the unimodular group $\bar{T}$. But then the map $f \mapsto L(f)$ defines a morphism of $\bar{H}$-modules from $C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s)$ to $C_c^\infty(\bar{T}\backslash\bar{H})$, which is surjective because of the the commutativity of the following diagram,

$$\begin{array}{ccc} C_c^\infty(\bar{H}) \otimes V_{\rho_s} & \overset{Id\otimes L}{\longrightarrow} & C_c^\infty(\bar{H}) \\ \downarrow & & \downarrow \\ C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s) & \longrightarrow & C_c^\infty(\bar{T}\backslash\bar{H}) \end{array},$$

where the vertical arrows defined in Lemma 2.9 of [M1] and the upper arrow are surjective.
We thus proved that space of restrictions to $\bar{H}$ of functions of $L(f)$, for f in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, contain $C_c^\infty(\bar{T}\backslash\bar{H})$, hence Λ is nonzero and the reresentation $\pi^\sigma| \, |_K^s \times \pi^\vee| \, |_K^{-s}$ is distinguished for $Re(s) \geq n/2$.

Step 3.
We are going to show that $\Pi = \pi^\sigma \times \pi^\vee$ is distinguished. Let f belong to $\mathcal{F}_\Pi$, it is a consequence of Lemma 1.1 that there exists two compact open subgroups $U_1 \subset U_2$ of $(M_n(K), +)$, such that for every s in $\mathcal{D}_K$, the function Φ_{f_s} is invariant under addition by elements of U_1, and has support in U_2. Denoting by P the polynomial function on $M_n(K)$ given by $C \mapsto det(CC^\sigma - I_n)$, we recall, using notations of subsection 1.3, that for $Re(s) \geq n/2$, the linear form $f \mapsto I_P(\Phi_{f_s}, s - n/2)$ is nonzero and $G_n(F)$-invariant. For such s, one has $I_P(\Phi_{f_s}, s-n/2) = \int_{U_2} |P(C)|_K^{s-n/2}\Phi_{f_s}(C)dC = \sum_{C_i\in U_2/U_1} \Phi_{f_s}(C_i)\int_{C_i+U_1} |P(C)|_K^{s-n/2}dC = \sum_{C_i\in U_2/U_1} \Phi_{f_s}(C_i)I_P(1_{C_i+U_1}, s - n/2)$.
In particular,from subsection 1.2, we know that the functions $s \mapsto \Phi_{f_s}(C_i)$ are polynomial in q_K^{-s}, and according to theorem 1.1, there is Q in $\mathbb{C}[X]$, such that for every $f \in \mathcal{F}_\Pi$, the function $Q(q_K^{-s})I_P(\Phi_{f_s}, s - n/2)$ is regular on $\mathcal{D}_K$. As the function $s \mapsto I_P(\Phi_{f_s}, s - n/2)$ is meromorphic, one has for every h in $GL(n, F)$ the equality $I_P(\Phi_{\pi_s(h)f_s}, s - n/2) = I_P(\Phi_{f_s}, s - n/2)$ (because the equality holds for $Re(s) \geq n/2$). Moreover one can choose an integer d such that the function $(q_K^{-s} - 1)^d I_P(\Phi_{f_s}, s - n/2)$ is regular at zero for every $f \in \mathcal{F}_\Pi$, and doesn't vanish at $s = 0$ for a good choice of f. Hence the linear form $f_0 \mapsto \underset{s\to 0}{lim}(q_K^{-s} - 1)^d I_P(\Phi_{f_s}, s - n/2)$ defines a nonzero $G_n(F)$-invariant linear form on the space of Π. We thus showed that the representation Π is distinguished. $\square$

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