Distinction of some induced representations

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Abstract

Let K/F be a quadratic extension of p-adic fields, σ the nontrivial element of the Galois group of K over F, and π a generic representation of $GL(n,K)$. Denoting by $\pi^\vee$ the smooth contragredient of π, and by π^σ the representation $\pi\circ\sigma$, we show that representation of $GL(2n,K)$ obtained by normalized parabolic induction of the representation $\pi^\vee\otimes\pi^\sigma$ is distinguished with respect to $GL(2n,F)$. This is a step towards the classification of distinguished generic representations of general linear groups over p-adic fields.

Introduction

Let K/F be a quadratic extension of p-adic fields, σ the nontrivial element of the Galois group of K over F, and π a generic representation of $GL(n,K)$. We denote by σ again the automorphism of $M_{2n}(K)$ induced by σ.
A smooth representation ρ of $GL(2n,K)$ is said to be distinguished if there is a nonzero $GL(2n,F)$-invariant linear form on its space. If ρ is irreducible, the space of such forms is of dimension at most 1.
Calling $\pi^\vee$ the smooth contragredient of π and π^σ the representation $\pi\circ\sigma$, we denote by $\pi^\sigma\times\pi^\vee$ the representation of $GL(2n,K)$, obtained by normalized induction of the representation $\pi^\sigma\otimes\pi^\vee$ of the standard parabolic subgroup of type (n,n).
The case $n=1$ is treated in [H] for unitary $\pi^\sigma\times\pi^\vee$, using a criterion characterizing distinction in terms of gamma factors. In [F3], Flicker defines a linear form on the space of $\pi^\sigma\times\pi^\vee$ by a formal integral which would define the invariant linear form once the convergence is insured. Finally in [F-H], for $n=1$, the convergence of this linear form is obtained for $\pi^\sigma|\ |_K^s\times\pi^\vee|\ |_K^{-s}$ and s of real part large enough, the conclusion follows from an analytic continuation argument.
We generalize this method here, the first part, which consists of showing that the representation $\pi^\sigma|\ |K^s\times\pi^\vee|\ |_K^{-s}$ when $Re(s)$ is large enough, is quite straightforward. The main difference is the analytic continuation argument, it is more general here, because the precise computations of [F-H] doesn't seem to generalize easily.
The first section is about notations and basic concepts used in the rest of the work.
In the second section, we state a very useful theorem of Bernstein (Theorem 2.1) about rationality of solutions of polynomial systems, and use it as in [C-P] or [Ba], in order to show, in Proposition 2.2, the holomorphy of integrals of Whittaker functions depending on several complex variables.
The third section is devoted to the proof of our main theorem 3.1, which asserts that the representation $\pi^\sigma\times\pi^\vee$ is distinguished.
We end this introduction by recalling a conjecture about classification of distinguished generic representations:

Conjecture. *Let m be a positive integer, and ρ a generic representation of the group $GL(m,K)$, obtained by normalised parabolic induction of quasi-square-integrable representations $\Delta_1,\dots,\Delta_t$. It is distinguished if and only if there exists a reordering of the Δ_i's, and an integer r between 1 and $t/2$, such that we have $\Delta_{i+1}^\sigma=\Delta_i^\vee$ for $i=1,3,..,2r-1$, and Δ_i is distinguished for $i>2r$.*

We denote by η the nontrivial character of F^* trivial on the norms of K^*. According to Proposition 26 in [F1], Proposition 12 of [F2], Theorem 6 of [K], and Corollary 1.6 [A-K-T], our

result reduces the proof of the conjecture to showing that representations of the form $\Delta_1 \times \cdots \times \Delta_t$ with $\Delta_{i+1}^{\sigma} = \Delta_i^{\vee}$ for $i = 1, 3, .., 2r-1$ for some r between 1 and $t/2$, and non isomorphic distinguished or η-distinguished Δ_i's for $i > 2r$ are not distinguished whenever one of the Δ_i's is η-distinguished for $i > 2r$. According to [M3], the preceeding conjecture implies the equality of the analitically defined Asai L-function and the Galois Asai L-function of a generic representation.

1 Notations

We denote by $|\ |_K$ and $|\ |_F$ the respective absolute values on K^*, by q_K and q_F the respective cardinalities of their residual field, and by R_K the valuation ring of K.
We call partition of a positive integer n, a family $\bar{n} = (n_1, \ldots, n_t)$ of positive integers (for a certain t in $\mathbb{N} - \{0\}$), such that the sum $n_1 + \cdots + n_t$ is equal to n. To such a partition, we associate a subgroup of $GL(n, K)$ denoted by $P_{\bar{n}}(K)$, given by matrices of the form

$$\begin{pmatrix} g_1 & \star & \star & \star & \star \\ & g_2 & \star & \star & \star \\ & & \ddots & \star & \star \\ & & & g_{t-1} & \star \\ & & & & g_t \end{pmatrix},$$

with g_i in $GL(n_i, K)$ for i between 1 and t. We call it the standard parabolic subgroup associated with the partition $\bar{n}$. We denote by $N_{\bar{n}}(K)$ its unipotent radical subgroup, given by the matrices

$$\begin{pmatrix} I_{n_1} & \star & \star \\ & \ddots & \star \\ & & I_{n_t} \end{pmatrix}.$$

Finally we denote by $P_n(K)$ the affine subgroup of $GL(n, K)$ given by the matrices $\begin{pmatrix} g & \star \\ & 1 \end{pmatrix}$, with g in $GL(n-1, K)$.
Let X be a locally closed space of an l-group G, and H closed subgroup of G, with $H.X \subset X$. If V is a complex vector space, we denote by $C^{\infty}(X, V)$ the space of smooth functions from X to V, and by $C_c^{\infty}(X, V)$ the space of smooth functions with compact support from X to V (if one has $V = \mathbb{C}$, we simply denote it by $C_c^{\infty}(X)$).
If ρ is a complex representation of H in V_ρ, we denote by $C^{\infty}(H\backslash X, \rho, V_\rho)$ the space of functions f from X to V_ρ, fixed under the action by right translation of some compact open subgroup U_f of G, and which verify $f(hx) = \rho(h)f(x)$ for $h \in H$, and $x \in X$ (if ρ is a character, we denote this space by $C^{\infty}(H\backslash X, \rho)$. We denote by $C_c^{\infty}(H\backslash X, \rho, V_\rho)$ subspace of functions with support compact modulo H of $C^{\infty}(H\backslash X, \rho, V_\rho)$.
We denote by $Ind_H^G(\rho)$ the representation by right translation of G in $C^{\infty}(H\backslash G, \rho, V_\rho)$ and by $ind_H^G(\rho)$ the representation by right translation of G in $C_c^{\infty}(H\backslash G, \rho, V_\rho)$. We denote by $Ind'^G_H(\rho)$ the normalized induced representation $Ind_H^G((\Delta_G/\Delta_H)^{1/2}\rho)$ and by $ind'^G_H(\rho)$ the normalized induced representation $ind_H^G((\Delta_G/\Delta_H)^{1/2}\rho)$.
Let n be a positive integer, and $\bar{n} = (n_1, \ldots, n_t)$ be a partition of n, and suppose that we have a representation (ρ_i, V_i) of $GL(n_i, K)$ for each i between 1 and t. Let ρ be the extension to $P_{\bar{n}}(K)$ of the natural representation $\rho_1 \otimes \cdots \otimes \rho_t$ of $GL(n_1, K) \times \cdots \times GL(n_t, K)$, by taking it trivial on $N_{\bar{n}}(K)$. We denote by $\rho_1 \times \cdots \times \rho_t$ the representation $Ind'^{GL(nK)}_{P_{\bar{n}}(K)}(\rho)$.

2 Analytic continuation of Whittaker forms

If ρ is a generic representation of $GL(n, K)$, and ψ is a nontrivial character of K, trivial on F, then for every W in the Whittaker model $W(\rho, \psi)$ of ρ, by standard arguments, the following

integral is convergent for $Re(s)$ large, and defines a rational function in q^{-s}, which has a Laurent series development in q^{-s}:

$$I_{(0)}(W,s) = \int_{N_n(F)\backslash P_n(F)} W(p)|det(p)|_F{}^{s-1} dp.$$

By standard arguments again, the vector space generated by the functions $I_{(0)}(W,s)$, for W in $W(\pi,\psi)$, is a fractional ideal $I_{(0)}(\pi)$ of $\mathbb{C}[q_F^{-s}, q_F^s]$, which has a unique generator which is an Euler factor, independant of ψ, that we denote by $L_{(0)}(\rho,s)$.

According to theorem 9.7 of [Z], there is a partition of n and quasi-square-integrable representations $\Delta_1,\dots,\Delta_t$ associated to it such that ρ is isomorphic to $\Delta_1\times\cdots\times\Delta_t$. The map $u=(u_1,\dots,u_t)\mapsto q_K^u=(q_K^{u_1},\dots,q_K^{u_t})$ defines an isomorphism of varieties between $(\mathcal{D}_K)^t = (\mathbb{C}/\frac{2i\pi}{ln(q_K)\mathbb{Z}})^t$ and $(\mathbb{C}^*)^t$. We also denote by $\mathcal{D}_F$ the variety $(\mathbb{C}/\frac{2i\pi}{ln(q_F)\mathbb{Z}})$ which the isomorphism $s\mapsto q_F^{-s}$ identifies to $(\mathbb{C}^*)^t$, and we denote by $\mathcal{D}$ the product $(\mathcal{D}_K)^t\times\mathcal{D}_F$.
Associate to u and ρ is the representation $\rho_u = \Delta_1|\ |^{u_1}\times\cdots\times\Delta_t|\ |^{u_t}$. In their classical model, for every representation ρ_u, the restrictions of the functions of the space of ρ_u to the maximal compact subgroup $GL(n,R_K)$ of $GL(n,K)$ define the same space $\mathcal{F}_\rho$, which is called the space of flat sections of the series ρ_u. To each f in $\mathcal{F}_\rho$, corresponds a unique function f_u in ρ_u. It is known that for fixed g in $GL(n,K)$ and f in $\mathcal{F}_\rho$, the function $(u,s)\mapsto|det(g)|_K^s\rho_u(g)f$ belongs to $\mathbb{C}[\mathcal{D}]\otimes_{\mathbb{C}}\mathcal{F}_\rho$. For every f in $\mathcal{F}_\rho$ and u in $(\mathcal{D}_K)^t$, there is a function $W_{f,u}=W_{f_u}$ defined in Section 3.1 of [C-P] in the Whittaker model $W(\rho_u,\psi)$, such that $W_{f,u}$ describes $W(\rho_u,\psi)$ when f describes $\mathcal{F}_\rho$. The space $W^{(0)}$ is defined in [C-P] as the complex vector space generated by the functions $(g,u)\mapsto W_{f,u}(gg')$ for g' in $GL(n,K)$.

We will need a theorem of Bernstein insuring rationality of solutions of polynomial systems. The setting is the following.
Let V be a complex vector space of countable dimension. Let R be an index set, and let Ξ be a collection $\{(x_r,c_r)|r\in R\}$ with $x_r\in V$ and $c_r\in\mathbb{C}$. A linear form λ in $V^*=Hom_{\mathbb{C}}(V,\mathbb{C})$ is said to be a solution of the system Ξ if $\lambda(x_r)=c_r$ for all r in R.
Let $\mathcal{D}$ be an irreducible algebraic variety over $\mathbb{C}$, and suppose that to each d, a system $\Xi_d = \{(x_r(d),c_r(d))|r\in R\}$ with the index set R independant of d in $\mathcal{D}$. We say that the family of systems $\{\Xi_d, d\in\mathcal{D}\}$ is polynomial if $x_r(d)$ and $c_r(d)$ belong respectively to $\mathbb{C}[\mathcal{D}]\otimes_{\mathbb{C}}V$ and $\mathbb{C}[\mathcal{D}]$.
Let $\mathcal{M}=\mathbb{C}(\mathcal{D})$ be the field of fractions of $\mathbb{C}[\mathcal{D}]$, we denote by $V_{\mathcal{M}}$ the space $\mathcal{M}\otimes_{\mathbb{C}}V$ and by $V_{\mathcal{M}}^*$ the space $Hom_{\mathcal{M}}(V_{\mathcal{M}},\mathcal{M})$.
The following statement is a consequence of Bernstein's theorem, the discussion preceeding it, and its corollary in Section 1 of [Ba].

Theorem 2.1. *(Bernstein) Suppose that in the above situation, the variety $\mathcal{D}$ is nonsingular and that there exists a non-empty subset $\Omega\subset\mathcal{D}$ open in the usual complex topology of $\mathcal{D}$, such that for each d in Ω, the system Ξ_d has a unique solution λ_d. Then the system $\Xi=\{(x_r(d),c_r(d))|r\in R\}$ over the field $\mathcal{M}=\mathbb{C}(\mathcal{D})$ has a unique solution $\lambda(d)$ in $V_{\mathcal{M}}^*$, and $\lambda(d)=\lambda_d$ is the unique solution of Ξ_d on Ω.*

In order to apply this theorem, we will need the following proposition.

Proposition 2.1. *Let ρ be a generic representation of $GL(n,K)$, there are t affine linear forms L_i, for i between 1 and t, with L_i depending on the variable u_i, such that if the $L_i(u_i)$'s and s have positive real parts, the integral $I_{(0)}(W,s)=\int_{N_n(F)\backslash P_n(F)}W(p)|det(p)|_F^{s-1}dp$ is convergent for any W in $W(\rho_u,\psi)$.*

Proof. We recall the following claim, which is proved in the lemma of Section 4 of [F1].

Claim. *Let τ be a sub-$P_n(K)$-module of $C^\infty(N_n(K)\backslash P_n(K),\psi)$, such that for every k between 0 and n, the central exponents of $\tau^{(k)}$ are positive (i.e. the central characters of all the irreducible subquotients of $\tau^{(k)}$ have positive real parts), then whenever W belongs to τ, the integral $\int_{N_n(F)\backslash P_n(F)}W(p)dp$ is absolutely convergent.*

Applying this to our situation, and noting e_ρ the maximal element of the set of central exponents of ρ (see Section 7.2 of [Ber]), we deduce that as soon as u is such that $L_i(u) = u_i - e_\rho - 1$ has positive real part for i between 1 and t, and as soon as s has positive real part, the integral $\int_{N_n(F)\backslash P_n(F)} W(p)|det(p)|_F^{s-1}dp$ converges for all W in $W(\rho_u, \psi)$. □

We now can prove the following:

Proposition 2.2. *Let ρ be a generic representation of $GL(n, K)$, for every f in $\mathcal{F}_\rho$, the function $I_{(0)}(W_{f,u}, s)$ belongs to $\mathbb{C}(q_F^{-u}, q_F^{-s})$.*

Proof. In our situation, the underlying vector space is $V = \mathcal{F}_\rho$ and is of countable dimension because ρ is admissible. The invariance property satisfied by the functional I, is

$$I_{(0)}(\rho_u(p)W_{f,u}, s) = |det(p)|_F^{1-s} I_{(0)}(W_{f,u}, s) \tag{1}$$

for f in $\mathcal{F}_\rho$, and p in $P_n(F)$.
From the proofs of Lemma 8 and of the unique proposition of [F4], it follows that out of the hyperplanes in (u, s) defined by $c_{\rho_u}|z|^{j(s-1)} = 1$, for ρ_u in the irreducible components of $\rho_u^{(n-j)}$, and for $j > 0$, the space of solutions of equation 1 is of dimension at most one. If we take a basis of $(f_\alpha)_{\alpha\in A}$ of $\mathcal{F}_\rho$, the polynomial family over the irreducible complex variety $\mathcal{D} = (\mathcal{D}_K)^t \times \mathcal{D}_F$ of systems Ξ'_d, for $d = (u, s) \in \mathcal{D}$ expressing the invariance of I is given by:

$$\Xi'_d = \left\{ \begin{array}{l} (\rho_u(p)\rho_u(g_i)f_\alpha - |det(p)|_F^{1-s}\rho_u(g_i)f_\alpha, 0), \\ \alpha \in A, p \in P_n(F), g_i \in GL(n, K) \end{array} \right\}$$

Now we define Ω to be the intersection of the three following subets of $\mathcal{D}$:

- the intersection of the complements of the hyperplanes on which uniqueness up to scalar fails,
- the intersection of the domains $\{Re(L_i(u)) > 0\}$ and $\{Re(s) > 0\}$, on which $I(W_{f,u}, \phi, s)$ is given by an absolutely convergent integral.

The functional I is the unique solution up to scalars of the system Ξ', in order to apply Theorem 2.1, we add for each $d \in \mathcal{D}$ a normalization equation E_d depending polynomially on d. This is done as follows.
From Proposition 3.4 of [M3], if F is a positive function in $C_c^\infty(N_n(K)\backslash P_n(K), \psi)$, we choose a W in $W_\rho^{(0)}$ such that its restriction to $P_n(K)$ is of the form $W(u, p) = F(p)P(q_K^{\pm u})$ for some nonzero P in $\mathcal{P}_0$. We thus have the equality $I_{(0)}(W, u, s) = \int_{N_n(F)\backslash P_n(F)} F(p)|det(p)|_F^{s-1}dpP(q^{\pm u})$. Calling c the constant $r\int_{N_n(F)\backslash P_n(F)} F(p)|det(p)|_F^{s-1}dp$, this latter equality becomes $I(W, u, s) = cP(q_K^{\pm u})$.
Now as W is in $W_\rho^{(0)}$, it can be expressed as a finite linear combination $W(g, u) = \sum_k \rho_u(g_\alpha)W_{f_\alpha,u}(g)$ for appropriate $g_\alpha \in GL(n, K)$. Hence our polynomial family of normalization equations (which is actually independant of s) can be written

$$E_{(u,s)} = \left\{ (\sum_\alpha \rho_u(g_\alpha)f_\alpha, cP(q_K^{\pm u}) \right\}.$$

We now call Ξ the system given by Ξ' and E, it satisfies the hypotheses of Theorem 2.1. We thus conclude that there is a functional I' which is a solution of Ξ such that $(u, s) \mapsto I'(W_{f,u}, s)$ is a rational function of $q_F^{\pm u}$ and $q_F^{\pm s}$ for $f \in \mathcal{F}_\rho$. We also know that $I'(W_{f,u}, s)$ is the unique solution of $\Xi'_{(u,s)}$ on Ω. This implies that for fixed $u \in (\mathcal{D}_K)^t$ and $f \in \mathcal{F}_\rho$, for $Re(s)$ large enough (let's say $Re(s) \geq r$ for some real number r, for which $c_{\tau_u}|z|^{j(s-1)} \neq 1$, for τ_u in the irreducible components of $\rho_u^{(n-j)}$, and for $j > 0$), both functions $I(W_{f,u}, s)$ and $I'(W_{f,u}, s)$ are equal. As they are rational functions of q_F^{-s}, they are equal for all s, and we conclude that $(u, s) \mapsto I(W_{f,u}, s)$ belongs to $\mathbb{C}(q_F^{-u}, q_F^{-s})$. □

We now recall the following theorem of Youngbin Ok:

Proposition 2.3. *([Ok], Theorem 3.1.2 or Proposition 1.1 of [M2]) Let ρ be an irreducible distinguished representation of $GL(n,K)$, if L is a $P_n(F)$-invariant linear form on the space of ρ, then it is actually $GL(n,F)$-invariant.*

In order to apply this theorem later, we also recall the proposition 2.3 of [M2].

Proposition 2.4. *Let ρ be a generic representation of $GL(n,K)$, the functional $\Lambda_{\rho,s} : W \mapsto I_{(0)}(W,s)/L_{(0)}(\rho,s)$ defines a nonzero linear form on $W(\rho,\psi)$ which transforms by $|det(\ )|_F^{1-s}$ under the affine subgroup $P_n(F)$.*
For fixed W in $W(\rho,\psi)$, then $s \mapsto \Lambda_{\rho,s}(W)$ is a polynomial of q_F^{-s}.

3 Distinction of representations $\pi^\sigma \times \pi^\vee$

We denote by G the group $GL(2n,K)$, by H its subgroup $GL(2n,F)$, by G' the group $GL(n,K)$ and by M the group $M_n(K)$. We denote by P the group $P_{(n,n)}(K)$, and by N the group $N_{(n,n)}(K)$. We denote by $\bar{H}$ subroup of G given by matrices of the form $\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}$, and by $\bar{T}$ the subgroup of $\bar{H}$ of matrices $\begin{pmatrix} A & 0 \\ 0 & A^\sigma \end{pmatrix}$, with A in G'.
We call U be the matrix $\begin{pmatrix} I_n & -\delta I_n \\ I_n & \delta I_n \end{pmatrix}$ of G, and W the matrix $\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. One has $U^\sigma U^{-1} = W$ and the group H is equal to $U^{-1}\bar{H}U$.

Lemma 3.1. *The double class PUH is opened in G.*

Proof. Call S the space of matrices g in G verifying $g^\sigma = g^{-1}$, which is, from Proposition 3. of chapter 10 of [S], homeomorphic to the quotient space G/H by the map $Q : g \mapsto g^\sigma g^{-1}$. As the map Q sends U on W, the double class PUH corresponds to the open subset of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in S such that $det(C) \neq 0$, the conclusion follows.

□

We prove the following integration formula.

Lemma 3.2. *There is a Haar measure $d\dot{h}$ on the quotient space $\bar{T}\backslash\bar{H}$, and a Haar measure dB on M, such that for any measurable positive function ϕ on the quotient space $\bar{T}\backslash\bar{H}$, then the integrals*

$$\int_{\bar{T}\backslash\bar{H}} \phi(\dot{h}) d\dot{h}$$

and

$$\int_M \phi\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix} \frac{dB}{|det(I_n - BB^\sigma)|_K^n}$$

are equal.

Proof. It suffices to show this equality when ϕ is positive, continuous with compact support in $\bar{T}\backslash\bar{H}$. We fix Haar measures dt on $\bar{T}$ and dg on $\bar{H}$, such that $d\dot{h}dt = dg$. It is known that there exists some positive function $\tilde{\phi}$ with compact support in $\bar{H}$, such that $\phi = \tilde{\phi}^{\bar{T}}$, which means that for any $\dot{h}$ in $\bar{H}$, one has $\phi(\dot{h}) = \int_{\bar{T}} \tilde{\phi}(tg)dt$. One then has the relation

$$\int_{\bar{T}\backslash\bar{H}} \phi(\dot{h}) d\dot{h} = \int_{\bar{H}} \tilde{\phi}(g) dg.$$

Now as $\bar{H}$ is conjugate to H, there are Haar measures dA and dB on M such that dt is equal to $d^*A = \frac{dA}{|det(A)|_K^n}$, and the Haar measure on $\bar{H}$ is described by the relation

$$d\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix} = \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_F^{2n}} = \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_K^{n}}.$$

Hence we have

$$\begin{aligned} \int_{\bar{T}\backslash\bar{H}} \phi(\dot{h})d\dot{h} &= \int_{M\times M} \tilde{\phi}\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix} \frac{dAdB}{\left|det\begin{pmatrix} A & B \\ B^\sigma & A^\sigma \end{pmatrix}\right|_K^{n}} \\ &= \int_{M\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & A^{-1}B \\ (A^{-1}B)^\sigma & I_n \end{pmatrix}\right] \frac{dAdB}{|det(A)|_K^{2n}|det(I_n - A^{-1}B(A^{-1}B)^\sigma)|_K^n} \end{aligned}$$

as the complementeray of G' is a negligible set of M.

This becomes after the change of variable $B := A^{-1}B$ equal to

$$\int_{M\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix}\right] \frac{dA}{|det(A)|_K^n}\frac{dB}{|det(I_n - BB^\sigma|_K^n}$$

which is itself equal to

$$\int_{G'\times M} \tilde{\phi}\left[\begin{pmatrix} A & \\ & A^\sigma \end{pmatrix}\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix}\right] d^*A\frac{dB}{|det(I_n - BB^\sigma)|_K^n}.$$

The conclusion follows from the fact that $\tilde{\phi}^{\bar{T}}$ is equal to ϕ.

□

Theorem 3.1. *Let n be a positive integer, and π a generic representation of G'. Then the representation $\pi^\sigma \times \pi^\vee$ is a distinguished representation of G.*

Proof. We will first show that the representation $\pi^\sigma|\ |^s \times \pi^\vee|\ |^{-s}$ of G is distinguished for a complex number s of real part large enough. Then we show that this property still holds for s equal to 0.

Step 1.
The representation $\pi^\sigma|\ |^s \otimes \pi^\vee|\ |^{-s}$ of $G'\times G'$ is distinguished by $\bar{T}$. Call L a nonzero $\bar{T}$-invariant linear form on the space of $\pi^\sigma|\ |_K^s \otimes \pi^\vee|\ |_K^{-s}$.
We denote by ρ_s the representation P, which is the extension of $\pi^\sigma|\ |_K^s \otimes \pi^\vee|\ |_K^{-s}$ to P by the trivial representation of $N_{(n,n)}(K)$. Let f belong to the space $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$ of $\pi^\sigma|\ |_K^s \times \pi^\vee|\ |_K^{-s}$, and A, B and g belong respectively to G', G' and G, one has the following relation:

$$f\left[\begin{pmatrix} A & \star \\ 0 & B \end{pmatrix} g\right] = \frac{|det(A)|_K^{n/2}}{|det(B)|_K^{n/2}}\pi^\sigma(A)\otimes\pi^\vee(B)f(g).$$

In particular for f in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho)$, the restriction to $\bar{H}$ of the function $\phi_f : g \mapsto L(f(g))$ belongs to the space $C^\infty(\bar{T}\backslash\bar{H})$, but its support modulo $\bar{T}$ is generally not compact, we will show later that the space of functions obtained this way contains $C_c^\infty(\bar{T}\backslash\bar{H})$ as a proper subspace. We must show that for s of real part large enough, the integral $\int_{\bar{T}\backslash\bar{H}} |\phi_f(\dot{h})|d\dot{h}$ converges.
According to lemma 3.2, this integral is equal to

$$\int_M |\phi_f|\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix}\frac{dB}{|det(I_n - BB^\sigma)|_K^n}.$$

As before, we cane suppose that B belongs to G', hence the followin decomposition holds

$$\begin{pmatrix} I_n & B \\ B^\sigma & I_n \end{pmatrix} = \begin{pmatrix} (I_n - BB^\sigma)B^{-\sigma} & I_n \\ & B^\sigma \end{pmatrix} \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} \begin{pmatrix} I_n & B^{-\sigma} \\ & I_n \end{pmatrix}.$$

Denoting by ϕ the function

$$B \mapsto |\phi_f| \left[\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} \right],$$

the following equalities follow:

$$\begin{aligned} \int_{\bar{T}\backslash\bar{H}} |\phi_f(\dot{h})| d\dot{h} &= \int_M (\tfrac{|det(I_n - BB^\sigma)|_K}{|det(B)|_K^2})^{s+n/2} \phi(B^{-\sigma}) \tfrac{dB}{|det(I_n - BB^\sigma)|_K^n} \\ &= \int_{G'} (\tfrac{|det(I_n - BB^\sigma)|_K}{|det(B)|_K^2})^{s+n/2} \phi(B^{-\sigma}) \tfrac{|det(B)|_K^n d^*B}{|det(I_n - BB^\sigma)|_K^n} \\ &= \int_{G'} (|det(I_n - C^{-\sigma}C^{-1})|_K |det(C)|_K^2)^{s+n/2} \phi(C) \tfrac{d^*C}{|det(C)|_K^n |det(I_n - C^{-\sigma}C^{-1})|_K^n} \\ &= \int_M |det(CC^\sigma - I_n)|_K^{s-n/2} \phi(C) dC \end{aligned}$$

Recalling that the map f belongs to $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, and as the map $(P, N') \in P \times N \mapsto PWN' \in G$ is a homeomorphism, we deduce that

$$C \mapsto f \left[\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} \begin{pmatrix} I_n & C \\ & I_n \end{pmatrix} \right]$$

has compact support in M, hence so does ϕ.
This eventually implies that $\int_{\bar{T}\backslash\bar{H}} L(f(\dot{h}))d\dot{h}$ is convergent as soon as s has real part greater than $n/2$.

Step 2.
Suppose that the complex number s has real part greater than $n/2$. We are going to show that the linear form $\Lambda : f \mapsto \int_{\bar{T}\backslash\bar{H}} \phi_f(\dot{h})d\dot{h}$ is nonzero. More precisely we are going to show that the space of functions $L(f)$ on $\bar{T}\backslash\bar{H}$ for f in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, contain $C_c^\infty(\bar{T}\backslash\bar{H})$.
According to Lemma 3.1 , the double class PUH is opened in G, hence the extension by zero outside PUH gives an injection of the space $C_c^\infty(P\backslash PUH, \Delta_P^{-1/2}\rho_s)$ into the space $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$.
But the right translation by U, which is a vector space automorphism of $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, sends $C_c^\infty(P\backslash PUH, \Delta_P^{-1/2}\rho_s)$ onto $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$, hence $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$ is a subspace of $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$.
Now restriction to $\bar{H}$ defines an isomorphism between $C_c^\infty(P\backslash P\bar{H}, \Delta_P^{-1/2}\rho_s)$ and $C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s)$ because Δ_P has trivial restriction to the unimodular group $\bar{T}$. But then the map $f \mapsto L(f)$ defines a morphism of $\bar{H}$-modules from $C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s)$ to $C_c^\infty(\bar{T}\backslash\bar{H})$, which is surjective because of the the commutativity of the following diagram,

$$\begin{array}{ccc} C_c^\infty(\bar{H}) \otimes V_{\rho_s} & \stackrel{Id\otimes L}{\longrightarrow} & C_c^\infty(\bar{H}) \\ \downarrow & & \downarrow \\ C_c^\infty(\bar{T}\backslash\bar{H}, \rho_s) & \longrightarrow & C_c^\infty(\bar{T}\backslash\bar{H}) \end{array},$$

where the vertical arrows defined in Lemma 2.9 of [M1] and the upper arrow are surjective.
We thus proved that space of restrictions to $\bar{H}$ of functions of $L(f)$, for f in $C_c^\infty(P\backslash G, \Delta_P^{-1/2}\rho_s)$, contain $C_c^\infty(\bar{T}\backslash\bar{H})$, hence Λ is nonzero and the reresentation $\pi^\sigma| \ |_K^s \times \pi^\vee| \ |_K^{-s}$ is distinguished for $Re(s) \geq n/2$.

Step 3.
We are going to show that $\pi^\sigma \times \pi^\vee$ is distinguished. According to Proposition 2.3 and Proposition

2.4, an irreducible representation of G is distinguished if and only if for any function W of its Whittaker model, and any h in H, the rational function $I_{(0)}(W,s)/I_{(0)}(\rho(h)W,s)$ of $\mathbb{C}(q_F^s)$ is equal to 1 when s is zero.
We recall using notaions of Proposition 2.2, that for any f in $\mathcal{F}_{\pi^\sigma\times\pi^\vee}$, the function $I_{(0)}(W_{f,u},s)$ belongs to $\mathbb{C}(q_F^s,q_F^u)$. Hence for any element h of H, the function $I_{(0)}(W_{f,u},s)/I_{(0)}(\rho(h)W_{f,u},s)$ belongs to $\mathbb{C}(q_F^s,q_F^u)$, which implies that the function $I_{(0)}(W_{f,u},0)/I_{(0)}(\rho(h)W_{f,u},0)$ belongs to $\mathbb{C}(q_F^t)$, when we take u of the form $(t,-t)$. But this latter function is equal to one when $Re(t)$ is large enough, because for such values of t, the representation $\pi^\sigma|\ |_K^t\times\pi^\vee|\ |_K^{-t}$ is distinguished and irreducible, hence it is one. Taking $t=0$, we have $I_{(0)}(W_f,0)/I_{(0)}(\rho(h)W_f,0)=1$ for any f in $\mathcal{F}_{\pi^\sigma\times\pi^\vee}$. The conclusion follows from the fact that W_f describes $W(\pi,\psi)$ when f describes $\mathcal{F}_{\pi^\sigma\times\pi^\vee}$. □

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