



HAL
open science

Pseudo-euclidean Jordan algebras

Amir Baklouti, Said Benayadi

► **To cite this version:**

| Amir Baklouti, Said Benayadi. Pseudo-euclidean Jordan algebras. 2008. hal-00340666

HAL Id: hal-00340666

<https://hal.science/hal-00340666>

Preprint submitted on 21 Nov 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pseudo-euclidean Jordan algebras

Amir Baklouti and Saïd Benayadi

Université Paul Verlaine-Metz, LMAM, CNRS UMR 7122,
Ile du Saulcy, F-57045 Metz cedex, France

Abstract

A pseudo-euclidean Jordan algebra is a Jordan algebra \mathfrak{J} with an associative non-degenerate symmetric bilinear form B . B is said an associative scalar product on \mathfrak{J} . We study the structure of the pseudo-euclidean Jordan \mathbb{K} -algebras (where \mathbb{K} is a field of null characteristic) and we obtain an inductive description of these algebras in terms of double extensions and generalized double extensions. Next, we study the symplectic pseudo-euclidean Jordan \mathbb{K} -algebras and we give some informations on a particular class of these algebras namely the class of symplectic Jordan-Manin Algebras. Finally, we obtain some characterizations of semi-simple Jordan algebras among the pseudo-euclidean Jordan algebras.

Keywords: Jordan Algebras, Tits-Kantor-Koecher construction, Pseudo-euclidean symplectic Jordan algebras, symplectic quadratic Lie algebras, representations of Jordan algebras, Jordan bialgebras, Jordan-Manin algebras, Jordan Yang Baxter equation, r-matrices, T^* -extension, double extensions.

MSC: 17C10, 17C20, 17C50, 17C55, 17B05, 17B62.

Corresponding Author: Saïd Benayadi, *e-mail:* benayadi@univ-metz.fr

1 Introduction

In this paper, we consider finite dimensional algebras over a commutative field K with characteristic zero. Let \mathfrak{J} be a Jordan algebra. A bilinear form B on \mathfrak{J} is said associative if B satisfies $B(xy, z) = B(x, yz)$, $\forall x, y, z \in \mathfrak{J}$. Moreover, if B is nondegenerate and symmetric, we say that B is an associative scalar product of \mathfrak{J} . In this case, (\mathfrak{J}, B) is called a pseudo-euclidean Jordan algebra. It is well known that, the semi-simple Jordan algebras are pseudo-euclidean (see [7], [8], [14]). But there are many nilpotent Jordan algebras which are pseudo-euclidean (see the second section of this paper and [4]).

It is well known that, to any pseudo-euclidean unital Jordan algebra, one can apply the Tits Kantor Koecher construction (TKK construction) to obtain a quadratic Lie algebra ([9], [10], [15]). In section 2, we slightly modify this construction in order to obtain a 3-graded quadratic Lie algebras $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$, starting from the pseudo-euclidean Jordan algebras (\mathcal{G}, B) which are not necessarily unital. We also call this modification the TKK construction. Recall that the Lie algebra \mathcal{G} of Lie group \mathcal{G} which admits a bi-invariant pseudo-Riemannian structure is quadratic (i.e. \mathcal{G} is endowed with a symmetric non degenerate invariant (or associative) bilinear form B). Conversely, any connected Lie group whose Lie algebra \mathcal{G} is quadratic is endowed with bi-invariant pseudo-Riemannian structure [13]. In [12], A. Medina and Ph. Reyov have introduced the notion of double extension and used this notion to give an inductive classification of quadratic Lie algebras.

The principal objective of this paper is to study structures of pseudo-euclidean Jordan algebras and to give inductive descriptions of these algebras. In [4], it is shown that any pseudo-euclidean Jordan algebra is the orthogonal direct sum of irreducible ideals. So the study of pseudo-euclidean Jordan algebras is reduced to study the irreducible ones. In order to obtain informations and inductive descriptions of the irreducible pseudo-euclidean Jordan algebras, we introduce in section 3 some notions of double extensions namely the double extension of pseudo-euclidean Jordan algebras and the generalized double extension of pseudo-euclidean Jordan algebras by the one dimensional Jordan algebra with zero product. Next, in the section 4, we show that any pseudo-euclidean Jordan algebra can be obtained from a finite number of elements of \mathcal{U} (where \mathcal{U} is the set constituted by the algebra $\{0\}$, the one dimensional Jordan algebra with zero product and the simple Jordan algebras) by a finite sequence of orthogonal direct sums of pseudo-euclidean Jordan algebras and / or generalized double extensions by one dimensional Jordan algebra with zero product and / or double extensions by simple Jordan algebras. recall that the classification of simple Jordan algebra is well known (see. [3], [7], [8], [11]). In particular, we can construct all nilpotent pseudo-euclidean Jordan algebras from the null algebra $\{0\}$ and the one dimensional Jordan algebra with zero product using the generalized double extension by the one dimensional Jordan algebra with zero product. We shall construct, in section 5, all nilpotent pseudo-euclidean Jordan algebras of dimension less then or equal to five. The list of these algebras shows that all nilpotent pseudo-euclidean Jordan algebras of dimension less then or equal to four are associative and that the smallest dimension of non associative nilpotent pseudo-euclidean Jordan algebra is five.

Sections 6 and 7 are devoted to study symplectic pseudo-euclidean Jordan algebras and their connections with the symplectic quadratic Lie algebras. A pseudo-euclidean Jordan algebra (\mathfrak{J}, B) is said symplectic if it is endowed with a skew symmetric nondegenerate bilinear form $\omega : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$ which satisfies

$$\omega(xy, z) + \omega(yz, x) + \omega(zx, y) = 0, \quad \forall x, y, z \in \mathfrak{J}. \quad (1)$$

Such form, is said a symplectic form of \mathfrak{J} . Recall that in [16] and [17], a Jordan algebra \mathfrak{J} is symplectic if it is endowed with an antisymmetric bilinear form not necessarily nondegenerate $\omega : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$ which satisfies (1). In the case where ω is nondegenerate, V.N Zhelyabin ([16], [17]) has obtained some results which similar to known results in the symplectic Lie algebras. If $(\mathfrak{J}, B, \omega)$ is a symplectic pseudo-euclidean Jordan algebra, we show that there exist an invertible derivation D of \mathfrak{J} satisfying $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathfrak{J}$. Next we prove that \mathfrak{J} is nilpotent and $\mathcal{U} = D^{-1}$ is a solution of the Jordan Yang Baxter equation. We show in section 6, that the symplectic form ω may be extended to the quadratic Lie algebra obtained by the TKK construction starting from (\mathfrak{J}, B) , if and only if the derivation D checks the following condition:

$$[D, [R(\mathfrak{J}), R(\mathfrak{J})]] = [R(\mathfrak{J}), R(\mathfrak{J})], \quad (2)$$

where $R(\mathfrak{J}) = \{R_x, x \in \mathfrak{J}\}$ and $R_x : \mathfrak{J} \longrightarrow \mathfrak{J}, y \longmapsto yx$. If D satisfies the condition (2) above, we obtain a symplectic quadratic Lie algebra from a symplectic pseudo-euclidean Jordan algebra. Moreover this symplectic quadratic Lie algebra is a \mathbb{Z}_2 -graded symplectic quadratic Lie algebra (see Remark 8). In section 6, we give examples of non associative symplectic pseudo-euclidean Jordan algebras such that the quadratic Lie algebra obtained from these examples by the TKK construction are symplectic and pseudo-euclidian. Let us recall that the Lie algebra of Lie group which admits a bi-invariant pseudo-Riemannian metric and also a left-invariant symplectic form is a symplectic quadratic Lie algebra. These Lie groups are nilpotent and their geometry (and, consequently, that of their associated homogeneous spaces) is very rich. In particular, they carry two left-invariant affine structures: one defined by the symplectic form and

another which is compatible with a left-invariant pseudo-Riemannian metric. Moreover, if the symplectic form is viewed as a solution r of the classical Yang Baxter equation of Lie algebras (i.e. r is an r-matrix), then the Poisson-Lie tensor $\pi = r^+ - r^-$ and the geometry of double Lie groups $D(r)$ can be nicely described in [6]. In addition, the symplectic quadratic Lie algebras were described by the methods of double extensions in [1] and [2]. Further, in [2], it is proved that every symplectic quadratic Lie algebra (\mathcal{G}, B, ω) , over an algebraically closed fields \mathbb{K} , may be constructed by T^* -extension of nilpotent Lie algebra which admits an invertible derivation.

In order to give an inductive description of the symplectic pseudo-euclidean Jordan algebras, we introduce in section 7 the notion of symplectic pseudo-euclidean double extension. More precisely, we prove that every symplectic pseudo-euclidean Jordan algebra may be constructed from the algebra $\{0\}$ by a finite number of symplectic pseudo-euclidean double extension of symplectic pseudo-euclidean Jordan algebra

In the section 8, we introduce Jordan-Manin algebra (\mathfrak{J}, B) (i.e. $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ where \mathcal{U} and \mathcal{V} are two completely isotropic subalgebras of \mathfrak{J}) and the notion of double extension of Jordan-Manin algebras in order to describe the nilpotent Jordan-Manin algebras. We will show, in Proposition 8.7, that every symplectic pseudo-euclidean Jordan algebra over an algebraically closed fields \mathbb{K} is a symplectic Jordan-Manin algebra (i.e. $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ where \mathcal{U} and \mathcal{V} two subalgebras of \mathfrak{J} completely isotropic with respect to B and ω). Finally, we give an inductive description of symplectic Jordan-Manin algebras over an algebraically closed field by using the notion of symplectic double extension of symplectic Jordan-Manin algebras.

By using some results obtained in the previous sections, we give in the section 9 some new characterizations of semisimple Jordan algebras among the pseudo-euclidean Jordan algebras. These characterizations are based on representations of Jordan algebras, operators of Casimir type and the index of a pseudo-euclidean Jordan algebra.

2 Definitions and preliminary results

Jordan algebra \mathfrak{J} is a commutative non necessary associative algebra which satisfy:

$$x(yx^2) = (xy)x^2, \quad \forall x, y \in \mathfrak{J}, \quad (3)$$

where R_x is the endomorphism of \mathfrak{J} defined by: $R_x(y) := xy = yx, \forall y \in \mathfrak{J}$.

This equality is equivalent to $[R_x, R_{x^2}] = 0, \forall x, y \in \mathfrak{J}$. In Jordan algebra, the following equality are satisfied:

$$2(x, y, zx) + (z, y, x^2) = 0, \quad \forall x, y, z \in \mathfrak{J} \text{ where } (x, y, z) = (xy)z - x(yz), \quad \forall x, y, z \in \mathfrak{J}. \quad (4)$$

$$[R_{wz}, R_x] + [R_{zx}, R_w] + [R_{xw}, R_z] = 0 \quad \forall x, z, w \in \mathfrak{J}. \quad (5)$$

$$R_{xy}R_z - R_xR_yR_z + R_{zx}R_y - R_{y(zx)} + R_{zy}R_x - R_zR_yR_x = 0 \quad x, y, z \in \mathfrak{J}. \quad (6)$$

$$[R_x, [R_y, R_z]] = R_{(y,x,z)} = R_{[R_z, R_y](x)}, \quad \forall x, y, z \in \mathfrak{J}. \quad (7)$$

For more informations about Jordan algebra see [7], [14], [8] and [11].

Definition 2.1 *Let \mathfrak{J} be a Jordan algebra.*

(i) *A bilinear form B on \mathfrak{J} is called associative if $B(xy, z) = B(x, yz), \quad \forall x, y, z \in \mathfrak{J}$.*

(ii) If B is a nondegenerate symmetric and associative bilinear form on \mathfrak{J} , we say that (\mathfrak{J}, B) is a pseudo-euclidean Jordan algebra and B is an associatif scalar product on \mathfrak{J} .

In the following proposition, we are going to give a characterization of pseudo-euclidean Jordan algebra.

Proposition 2.1 *Let \mathfrak{J} be a Jordan algebra. Then \mathfrak{J} is pseudo-euclidean if and only if its adjoint representation and co-adjoint representation are equivalent.*

Proof. Let \mathfrak{J} be a Jordan algebra, R (resp. ρ) be the adjoint (resp. coadjoint) representation of \mathfrak{J} (see the definition or a representation of Jordan algebra in Proposition 3.1 of section 3). Recall that:

$$R : \mathfrak{J} \longrightarrow \text{End}(\mathfrak{J}) \quad \text{and} \quad \rho : \mathfrak{J} \longrightarrow \text{End}(\mathfrak{J}^*)$$

are defined by $R(x)y := R_x(y) = xy$ et $(\rho(x)f) := f \circ R_x$, $\forall x, y \in \mathfrak{J}, \forall f \in \mathfrak{J}^*$.

Assume that \mathfrak{J} is pseudo-euclidean Jordan algebra, then there exists a bilinear form $B : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ which is symmetric, non-degenerate and associative. Consequently, the map $\varphi : \mathfrak{J} \rightarrow \mathfrak{J}^*$ defined by: $\varphi(x) := B(x, \cdot)$, $\forall x \in \mathfrak{J}$, is an isomorphism of vector spaces which verifies:

$$\varphi\left(R_x(y)\right)(z) = B(xy, z) = B(y, xz) = \left(\rho(x)\varphi(y)\right)(z), \quad \forall x, y, z \in \mathfrak{J}.$$

Which proves that $\varphi \circ R_x = \rho(x) \circ \varphi$, $\forall x \in \mathfrak{J}$. It follows that the representations R and ρ are equivalent.

Conversely, suppose that the representations R and ρ are equivalent, then there exists an isomorphism of vector spaces $\phi : \mathfrak{J} \rightarrow \mathfrak{J}^*$ such that $\phi \circ R_x = \rho(x) \circ \phi$, $\forall x \in \mathfrak{J}$. Now, we consider the map $T : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by: $T(x, y) := \phi(x)(y)$, $\forall x, y \in \mathfrak{J}$. Since ϕ is invertible, then T is non-degenerate. Moreover, T is associative, in fact $T(xy, z) = \phi(xy)(z) = \phi(R(y)x)(z) = (\rho(y)(\phi(x)))(z) = \phi(x)(zy) = T(x, zy) = T(x, yz)$, $\forall x, y, z \in \mathfrak{J}$. It is clear that T is not necessarily symmetric. We consider the symmetric (resp. anti-symmetric) part T_s (resp. T_a) of T defined by: $T_s(x, y) = \frac{1}{2}(T(x, y) + T(y, x))$, (resp. $T_a(x, y) = \frac{1}{2}(T(x, y) - T(y, x))$), $\forall x, y \in \mathfrak{J}$. It is clear that T is associative if and only if T_s and T_a are associative.

Let us consider $\mathfrak{J}_s = \{x \in \mathfrak{J} / T_s(x, y) = 0, \forall y \in \mathfrak{J}\}$ and $\mathfrak{J}_a = \{x \in \mathfrak{J} / T_a(x, y) = 0, \forall y \in \mathfrak{J}\}$. The fact that T est non-degenerate implies that $\mathfrak{J}_s \cap \mathfrak{J}_a = \{0\}$. Let $x, y, z \in \mathfrak{J}$, since $T_a(xy, z) = T_a(x, yz) = -T_a(z, xy) = -T_a(x, zy)$ then $T_a(xy, z) = 0$. It follows that $\mathfrak{J}^2 = \mathfrak{J}\mathfrak{J}$ is contained in \mathfrak{J}_a . Moreover \mathfrak{J}_s and \mathfrak{J}_a are ideals of \mathfrak{J} because T_s and T_a are associative. Consequently, $\mathfrak{J}_s^2 \subset \mathfrak{J}_s \cap \mathfrak{J}_a = \{0\}$. Now, let \mathcal{W} be a sub-vector space of \mathfrak{J} such that $\mathfrak{J}_a \subset \mathcal{W}$ and $\mathfrak{J} = \mathcal{W} \oplus \mathfrak{J}_s$. Consider $F : \mathfrak{J}_s \times \mathfrak{J}_s \rightarrow \mathbb{K}$ be a non-degenerate symmetric bilinear on \mathfrak{J}_s . Since $\mathfrak{J}_s^2 = \{0\}$, then F est associative. Therefore the bilinear form $L : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by: $L|_{\mathcal{W} \times \mathcal{W}} = T_s|_{\mathcal{W} \times \mathcal{W}}$, $L|_{\mathfrak{J}_s \times \mathfrak{J}_s} = F$, $L(\mathcal{W}, \mathfrak{J}_s) = L(\mathfrak{J}_s, \mathcal{W}) = \{0\}$, is symmetric non-degenerate and associative. Then (\mathfrak{J}, L) is a pseudo-euclidean Jordan algebra. \square

Definition 2.2 1. *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. An ideal \mathcal{I} is called nondegenerate (resp. degenerate), if the restriction of B on $\mathcal{I} \times \mathcal{I}$ is a nondegenerate (resp. degenerate) bilinear form.*

2. *A pseudo-euclidean Jordan algebra is called B -irreducible, if \mathfrak{J} contains no non-trivial nondegenerate ideals.*

One has the following lemma whose proof is straightforward.

Lemma 2.2 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and \mathcal{I} be an ideal of \mathfrak{J} . Then,*

- (i) \mathcal{I}^\perp is an ideal of \mathfrak{J} and $\mathcal{I}\mathcal{I}^\perp = \{0\}$;
- (ii) If \mathcal{I} is nondegenerate, then $\mathfrak{J} = \mathcal{I} \oplus \mathcal{I}^\perp$ and \mathcal{I}^\perp is a nondegenerate ideal of \mathfrak{J} ;
- (iii) If \mathcal{I} is semisimple, then \mathcal{I} is nondegenerate.

Proposition 2.3 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. Then, $\mathfrak{J} = \bigoplus_{i=1}^r \mathfrak{J}_i$, where for all $1 \leq i \leq r$,

- (i) \mathfrak{J}_i is a non degenerate ideal;
- (ii) \mathfrak{J}_i contains no nondegenerate ideal of \mathfrak{J}_i .
- (iii) For all $i \neq j$, \mathfrak{J}_i and \mathfrak{J}_j are orthogonal.

For more details about this decomposition, see [4].

Definition 2.3 Let \mathfrak{J} be a algebra.

- (i) $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) := \text{Vect}\{(x, y, z) := (xy)z - x(yz); x, y, z \in \mathfrak{J}\}$ is a vector sub-space of \mathfrak{J} called the associator of \mathfrak{J} .
- (ii) The vector subspace $\text{Ann}(\mathfrak{J}) := \{x \in \mathfrak{J}; xy = yx = 0, \forall y \in \mathfrak{J}\}$ of \mathfrak{J} is called the annihilator of \mathfrak{J} .
- (iii) The vector subspace $N(\mathfrak{J}) := \{x \in \mathfrak{J}; (x, y, z) = (y, x, z) = (y, z, x) = 0, \forall y, z \in \mathfrak{J}\}$ is called the nucleus of \mathfrak{J} .

Remark 1 1. In the case of Jordan algebra \mathfrak{J} , the nucleus $N(\mathfrak{J})$ of \mathfrak{J} coincide with the center

$$Z(\mathfrak{J}) := \{x \in N(\mathfrak{J}); xy = yx, \forall y \in \mathfrak{J}\} \text{ of } \mathfrak{J}.$$

- 2. Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. Since $B((x, y, z), t) = B((y, x, t), z) = B((z, t, x), y)$, $\forall x, y, z \in \mathfrak{J}$, then $N(\mathfrak{J}) = Z(\mathfrak{J}) := \{x \in \mathfrak{J}; (x, y, z) = 0, \forall y, z \in \mathfrak{J}\}$.

Proposition 2.4 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. Then,

$$1. \left(\text{Ann}(\mathfrak{J})\right)^\perp = \mathfrak{J}^2. \quad 2. \left(Z(\mathfrak{J})\right)^\perp = (\mathfrak{J}, \mathfrak{J}, \mathfrak{J})$$

Proof. 1. The fact that B is associative implies that $\mathfrak{J}^2 \subset \left(\text{Ann}(\mathfrak{J})\right)^\perp$. Conversely, let $x \in \left(\mathfrak{J}^2\right)^\perp$ and $y \in \mathfrak{J}$. Then $B(xy, z) = B(x, yz) = 0, \forall z \in \mathfrak{J}$. Since B is non-degenerate, then $xy = 0$. Thus, $x \in \text{Ann}(\mathfrak{J})$. Hence, $\left(\mathfrak{J}^2\right)^\perp \subset \text{Ann}(\mathfrak{J})$. We conclude that $\left(\text{Ann}(\mathfrak{J})\right)^\perp = \mathfrak{J}^2$.

2. Let $x, y, z \in \mathfrak{J}$ and let $u \in Z(\mathfrak{J})$, $B(X, u) = B((xy)z - x(yz), u) = B(x, y(zu)) - B(x, (yz)u) = B(x, y(zu)) - (yz)u = 0$. Which proves that $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) \subset \left(Z(\mathfrak{J})\right)^\perp$. Conversely, let $x \in \left(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}\right)^\perp$. $B((x, y, z), u) = B(x, (u, z, y)) = 0$ and $B((y, x, z), u) = B(x, (z, u, y)) = 0, \forall y, z, u \in \mathfrak{J}$. Since B is nondegenerate, then $(x, y, z) = (y, x, z) = 0$. Hence, $x \in Z(\mathfrak{J})$. Therefore, $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J})^\perp \subset \left(Z(\mathfrak{J})\right)$. Consequently, $\left(Z(\mathfrak{J})\right)^\perp = (\mathfrak{J}, \mathfrak{J}, \mathfrak{J})$. \square

Corollary 2.5 *Let (\mathfrak{J}, B) be a Jordan algebra. If $\mathfrak{J} \neq \{0\}$, then $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) \neq \mathfrak{J}$.*

Proof. Suppose that $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) = \mathfrak{J}$. Let $Rad(\mathfrak{J})$ be the radical of \mathfrak{J} , S be a semi-simple subalgebra of \mathfrak{J} such that $\mathfrak{J} = S \oplus Rad(\mathfrak{J})$. The fact that $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) = \mathfrak{J}$ implies that $(S, S, S) = S$. Since, by Proposition 2.4, (S, \mathfrak{A}) is a pseudo-euclidean Jordan algebra (where \mathfrak{A} is the Albert form of S), then $Z(S) = (S, S, S)^\perp = \{0\}$. Therefore, By Theorem 4.7 of [14], $S = \{0\}$. It follows that $\mathfrak{J} = Rad(\mathfrak{J})$, consequently $\mathfrak{J} = \{0\}$ because \mathfrak{J} is nilpotent and $(\mathfrak{J}, \mathfrak{J}, \mathfrak{J}) = \mathfrak{J}$. \square

Corollary 2.6 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. If $\mathfrak{J} \neq \{0\}$, then $Z(\mathfrak{J}) \neq \{0\}$.*

Now, we shall recall the notion of the T^* -extension of the Jordan algebras. This notion was introduced by M.Bordemann in [4] in order to study algebras endowed with associative non-degenerate symmetric bilinear forms. Let \mathfrak{J} be a Jordan algebra, \mathfrak{J}^* be the dual space of \mathfrak{J} and $\theta : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}^*$ be a bilinear map. On the vector space $\tilde{\mathfrak{J}} = \mathfrak{J} \oplus \mathfrak{J}^*$, we define the following product:

$$(x + f)(y + h) = xy + h \circ R_x + f \circ R_y + \theta(x, y), \quad \forall x, y \in \mathfrak{J}, f, g \in \mathfrak{J}^*. \quad (8)$$

Proposition 2.7 *$\tilde{\mathfrak{J}}$ endowed with the product above, is a Jordan algebra if and only if θ is symmetric and satisfies the following identity:*

$$\theta(xy, x^2) + \theta(x, x) \circ R_{xy} + \theta(x, y) \circ R_{x^2} = \theta(x, yx^2) + \theta(y, x^2) \circ R_x + \theta(x, x) \circ R_y R_x, \quad \forall x, y \in \mathfrak{J}. \quad (9)$$

In this case, the bilinear form $B : \tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}} \rightarrow \mathbb{K}$ defined by:

$$B(x + f, y + h) = f(y) + h(x), \quad \forall x, y \in \mathfrak{J}, f, g \in \mathfrak{J}^*,$$

is an associatif scalar product on $\tilde{\mathfrak{J}}$ if and only if θ satisfies

$$\theta(x, y)(z) = \theta(z, x)(y), \quad \forall x, y, z \in \mathfrak{J}. \quad (10)$$

Definition 2.4 *If \mathfrak{J} is a Jordan algebra and $\theta : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}^*$ is a bilinear map which satisfies (9) and (10), then the pseudo-euclidean Jordan algebra $\tilde{\mathfrak{J}}$ is called the T^* -extension of \mathfrak{J} by mean of θ and denoted $T_\theta^* \mathfrak{J}$.*

Remark 2 *If \mathfrak{J} is a nilpotent Jordan algebra, then $T_0^* \mathfrak{J}$ is a nilpotent pseudo-euclidean Jordan algebra. But if \mathfrak{J} is a semi-simple Jordan algebra then $T_0^* \mathfrak{J}$ is a pseudo-euclidean Jordan algebra which is not nilpotent nor semi-simple, because \mathfrak{J}^* is a nilpotent ideal of $T_0^* \mathfrak{J}$ and $(T_0^* \mathfrak{J})^2 = T_0^* \mathfrak{J}$. Which show that the class of pseudo-euclidean Jordan algebra is very rich.*

Now, we will give an inductive description of pseudo-euclidean Jordan algebras by using the notion of double extension. Let us start by introducing some Jordan algebras extensions.

3 Some extensions of Jordan algebras

3.1 Central extension of Jordan algebras

Let \mathfrak{J}_1 be a Jordan algebra, \mathcal{V} be a vector space and $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \longrightarrow \mathcal{V}$ be a bilinear map. On the vector space $\tilde{\mathfrak{J}} = \mathfrak{J}_1 \oplus \mathcal{V}$ we define the following product:

$$(x + v)(y + w) = xy + \varphi(x, y), \quad \forall x, y \in \mathfrak{J}_1, v, w \in \mathcal{V}. \quad (11)$$

It is easy to check that \mathfrak{J} endowed with the product (11) is a Jordan algebra if and only if φ is symmetric and satisfies:

$$\varphi(xy, x^2) = \varphi(x, yx^2) \quad \forall x, y \in \mathfrak{J}_1.$$

We say that \mathfrak{J} is the central extension of \mathfrak{J}_1 by \mathcal{V} (by means of φ). In this case, \mathcal{V} is contained in the annihilator of \mathfrak{J} .

3.2 Representations and semi-direct products of Jordan algebras

Let \mathfrak{J}_1 be a Jordan algebra, \mathcal{V} be a vector space and $\pi : \mathfrak{J}_1 \longrightarrow \text{End}(\mathcal{V})$ be a linear map. On $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathcal{V}$ we define the following product:

$$(x + v)(y + w) = xy + \pi(x)w + \pi(y)v, \quad \forall x, y \in \mathfrak{J}_1, v, w \in \mathcal{V}. \quad (12)$$

Proposition 3.1 *\mathfrak{J} endowed with the product (12) is a Jordan algebra if and only if π satisfies the following conditions:*

$$\begin{aligned} (i) \pi(x^2)\pi(x) - \pi(x)\pi(x^2) &= 0 \\ (ii) 2\pi(xy)\pi(x) + \pi(x^2)\pi(y) - 2\pi(x)\pi(y)\pi(x) - \pi(x^2y) &= 0, \quad \forall x, y \in \mathfrak{J}_1. \end{aligned} \quad (13)$$

In this case π is called a representation of the Jordan algebra \mathfrak{J}_1 in \mathcal{V} .

Remark 3 Recall that $T^*_0\mathfrak{J} = \mathfrak{J} \oplus \mathfrak{J}^*$ endowed with the product

$$(x + f)(y + h) = xy + h \circ R_x + f \circ R_y, \quad \forall x, y \in \mathfrak{J}, f, g \in \mathfrak{J}^*,$$

is a Jordan algebra. Thus $\rho : \mathfrak{J} \longrightarrow \text{End}(\mathfrak{J}^*)$ defined by $\rho(x)f = f \circ R_x$, $\forall x \in \mathfrak{J}$, is a representation of \mathfrak{J} . This representation is called the co-adjoint representation of \mathfrak{J} .

Corollary 3.2 $\pi : \mathfrak{J}_1 \longrightarrow \text{End}(\mathcal{V})$ is a representation of \mathfrak{J}_1 if and only if

$$\begin{aligned} &\pi(xy)\pi(z) + \pi(yz)\pi(x) + \pi(xz)\pi(y) \\ &= \pi(y)\pi(xz) + \pi(x)\pi(yz) + \pi(z)\pi(xy) \quad \forall x, y, z \in \mathfrak{J}_1. \\ &= \pi((xy)z) + \pi(x)\pi(z)\pi(y) + \pi(y)\pi(z)\pi(x), \end{aligned}$$

Proof. We will proceed by linearization to show this Corollary. Let $x, y, z \in \mathfrak{J}_1$ and $\lambda \in \mathbb{K}$. Replace x by $x + \lambda y$ in (i) of (13). The term of λ is 0. Thus

$$2\pi(xy)\pi(x) + \pi(x^2)\pi(y) - \pi(y)\pi(x^2) - 2\pi(x)\pi(xy) = 0 \quad (14)$$

Replace x in (14) by $x + \lambda z$. The fact that the term of λ is 0 implies that

$$\begin{aligned} &\pi(xy)\pi(z) + \pi(yz)\pi(x) + \pi(xz)\pi(y) \\ &= \pi(y)\pi(xz) + \pi(x)\pi(yz) + \pi(z)\pi(xy). \end{aligned} \quad (15)$$

Conversely, if we replace y and z in (15) by x we obtain (i) of (13).

By the same argument, we show that (ii) of (13) is equivalent to

$$\begin{aligned} &\pi(xy)\pi(z) + \pi(yz)\pi(x) + \pi(xz)\pi(y) \\ &= \pi((xy)z) + \pi(x)\pi(z)\pi(y) + \pi(y)\pi(z)\pi(x). \quad \square \end{aligned} \quad (16)$$

Corollary 3.3 Let \mathfrak{J} be a Jordan algebra, \mathcal{V} be a vector space and $\pi : \mathfrak{J} \longrightarrow \text{End}(\mathcal{V})$ be a representation of \mathfrak{J} . Then,

$$\pi(x, y, z) = [\pi(y), [\pi(x), \pi(z)]], \quad \forall x, y, z \in \mathfrak{J}.$$

Proof. Let $x, y, z \in \mathfrak{J}$. By Proposition 3.2, we have the following equalities:

$$\begin{aligned} \pi((xy)z) &= \pi(xy)\pi(z) + \pi(yz)\pi(x) + \pi(xz)\pi(y) - \pi(x)\pi(z)\pi(y) - \pi(y)\pi(z)\pi(x) \\ \pi(x(yz)) &= \pi((zy)x) = \pi(zy)\pi(x) + \pi(xy)\pi(z) + \pi(xz)\pi(y) - \pi(z)\pi(x)\pi(y) - \pi(y)\pi(x)\pi(z). \end{aligned}$$

Consequently, $\pi(x, y, z) = [\pi(y), [\pi(x), \pi(z)]]$. \square

Remark 4 (i) Let \mathfrak{J} be a Jordan algebra and \mathcal{V}, \mathcal{W} be two vector spaces. If $\pi : \mathfrak{J} \longrightarrow \text{End}(\mathcal{V})$ and $\rho : \mathfrak{J} \longrightarrow \text{End}(\mathcal{W})$ are two representations of Jordan algebras, then $\tilde{\pi} := \pi \oplus \rho : \mathfrak{J} \longrightarrow \text{End}(\mathcal{V} \oplus \mathcal{W})$ defined by

$$(\pi \oplus \rho)(x)(v + w) = \pi(x)v + \rho(x)w, \quad \forall x \in \mathfrak{J}, v \in \mathcal{V}, w \in \mathcal{W},$$

is a representation of \mathfrak{J} called the direct sum of the representations π and ρ .

(ii) Let \mathfrak{J} be a Jordan algebra. Replace z by x in the identity (6), we obtain:

$$2R_{xy}R_x + R_{x^2}R_y - 2R_xR_yR_x - R_{x^2y} = 0, \quad \forall x, y \in \mathfrak{J}.$$

Further, The fact that \mathfrak{J} is a Jordan algebra implies that $R_xR_{x^2} = R_{x^2}R_x, \forall x \in \mathfrak{J}$. Which proves that the linear map $R : \mathfrak{J} \longrightarrow \text{End}(\mathfrak{J})$ defined by $R(x) = R_x, \forall x \in \mathfrak{J}$ is a representation of \mathfrak{J} . This representation is called the adjoint or the regular representation of \mathfrak{J} .

Proposition 3.4 Let $\mathfrak{J}_1, \mathfrak{J}_2$ be two Jordan algebras and $\pi : \mathfrak{J}_1 \longrightarrow \text{End}(\mathfrak{J}_2)$ be a linear map. We define on the vector space $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ the following product:

$$(x + y) \star (x' + y') = xx' + \pi(x)y' + \pi(x')y + yy', \quad \forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Then, (\mathfrak{J}, \star) is a Jordan algebra if and only if π satisfies the following conditions:

$$\begin{aligned} (i) \quad & \pi(x^2)\pi(x)y' + \pi(x^2)(yy') + 2\left(\pi(x)y'\right)\left(\pi(x)y\right) + \left(\pi(x)y'\right)y^2 + 2\left(yy'\right)\left(\pi(x)y\right) \\ & = \pi(x)\pi(x^2)y' + 2\pi(x)\left(y'\left(\pi(x)y\right)\right) + \pi(x)(y'y^2) + \left(\pi(x^2)y'\right)y + 2\left(y'\left(\pi(x)y\right)\right)y, \end{aligned}$$

$$\begin{aligned} (ii) \quad & 2\pi(xx')\pi(x)y + \pi(xx')y^2 + \pi(x^2)\pi(x')y + \left(\pi(x')y\right)y^2 + 2\left(\pi(x')y\right)\left(\pi(x)y\right) \\ & = 2\pi(x)\pi(x')\pi(x)y + \pi(x)\pi(x')y^2 + \pi(x'x^2)y + 2\left(\pi(x')\pi(x)y\right)y + \left(\pi(x')y^2\right)y, \end{aligned}$$

$$\forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Proof. Suppose that (\mathfrak{J}, \star) is a Jordan algebra. Let $x, x' \in \mathfrak{J}_1$ and $y, y' \in \mathfrak{J}_2$.

$$\begin{aligned} 0 &= \left((x + y) \star y'\right) \star (x + y)^2 - (x + y) \star \left(y' \star (x + y)^2\right) \\ &= \pi(x^2)\pi(x)y' + \pi(x^2)(yy') + 2\left(\pi(x)y'\right)\left(\pi(x)y\right) + \left(\pi(x)y'\right)y^2 + 2\left(yy'\right)\left(\pi(x)y\right) \\ &= \pi(x)\pi(x^2)y' + 2\pi(x)\left(y'\left(\pi(x)y\right)\right) + \pi(x)(y'y^2) + \left(\pi(x^2)y'\right)y + 2\left(y'\left(\pi(x)y\right)\right)y, \end{aligned}$$

$$\begin{aligned}
0 &= \left((x+y) \star x' \right) \star (x+y)^2 - (x+y) \star \left(x' \star (x+y)^2 \right) \\
&= 2\pi(xx')\pi(x)y + \pi(xx')y^2 + \pi(x^2)\pi(x')y + \left(\pi(x')y \right) y^2 + 2\left(\pi(x')y \right) \left(\pi(x)y \right) \\
&= 2\pi(x)\pi(x')\pi(x)y + \pi(x)\pi(x')y^2 + \pi(x'x^2)y + 2\left(\pi(x')\pi(x)y \right) y + \left(\pi(x')y^2 \right) y.
\end{aligned}$$

Which give the identities (i) and (ii).

Conversely, suppose that (i) and (ii) are satisfied, then for all $x, x' \in \mathfrak{J}_1$ and $y, y' \in \mathfrak{J}_2$, we have:

$$\begin{aligned}
&\left((x+y) \star y' \right) \star (x+y)^2 - (x+y) \star \left(y' \star (x+y)^2 \right) = 0, \\
&\left((x+y) \star x' \right) \star (x+y)^2 - (x+y) \star \left(x' \star (x+y)^2 \right) = 0.
\end{aligned}$$

Consequently,

$$\left((x+y) \star (x'+y') \right) \star (x+y)^2 - (x+y) \star \left((x'+y') \star (x+y)^2 \right) = 0, \forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Since the product \star is commutatif, then \mathfrak{J} endowed with \star is a Jordan algebra. \square

Corollary 3.5 *Let $\mathfrak{J}_1, \mathfrak{J}_2$ be two Jordan algebras and $\pi : \mathfrak{J}_1 \longrightarrow \text{End}(\mathfrak{J}_2)$ be a representation of Jordan algebra. We define on the vector space $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ the following product:*

$$(x+y) \star (x'+y') = xx' + \pi(x)y' + \pi(x')y + yy', \forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2.$$

Then, (\mathfrak{J}, \star) is a Jordan algebra if and only if π satisfies the following conditions:

- (1) $\pi(x^2)(yy') + 2\left(\pi(x)y' \right) \left(\pi(x)y \right) + \left(\pi(x)y' \right) y^2 + 2\left(yy' \right) \left(\pi(x)y \right)$
 $= 2\pi(x) \left(y' \left(\pi(x)y \right) \right) + \pi(x)(y'y^2) + \left(\pi(x^2)y' \right) y + 2\left(y' \left(\pi(x)y \right) \right) y$
- (2) $\left(\pi(x)y \right) y^2 = \left(\pi(x)y^2 \right) y$
- (3) $\pi(xx')y^2 + 2\left(\pi(x')y \right) \left(\pi(x)y \right) = \pi(x)\pi(x')y^2 + 2\left(\pi(x')\pi(x)y \right) y,$

Proof. Let $x, x' \in \mathfrak{J}_1$, $y, y' \in \mathfrak{J}_2$. First, since π is a representation, then $\pi(x^2)\pi(x)y' = \pi(x)\pi(x^2)y'$. consequently, the identity (i) of the Proposition 3.4 is equivalent to the identity (1). Second, we obtain the identity (2) by replacing x by 0 in (ii) of the Proposition 3.4. Finally, since π is a representation, then

$$2\pi(xx')\pi(x)y + \pi(x^2)\pi(x')y = 2\pi(x)\pi(x')\pi(x)y + \pi(x^2x')y. \quad (17)$$

Hence, the identity (2) implies that (3) is satisfies. Finally, remark that the sum of the identities (2), (3) and (17), give the identity (ii) of the Proposition 3.4. \square

Definition 3.1 Let $\mathfrak{J}_1, \mathfrak{J}_2$ be two Jordan algebras and $\pi : \mathfrak{J}_1 \longrightarrow \text{End}(\mathfrak{J}_2)$ be a representation of Jordan algebras. π is called *admissible*, if π satisfies the identities (1), (2) and (3) of Corollary 3.5. In this case, the Jordan algebra $(\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2, \star)$ where the product \star is defined by :

$$(x + y) \star (x' + y') = xx' + \pi(x)y' + \pi(x')y + yy', \quad \forall x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2,$$

is called the *semi-direct product* of \mathfrak{J}_2 by \mathfrak{J}_1 by means of π .

Corollary 3.6 The adjoint representation of a Jordan algebra \mathfrak{J} is an admissible representation of \mathfrak{J} .

Proof. The identities (1) and (3) of Corollary 3.5 are a consequences of the equality (4). Further, the fact that \mathfrak{J} is a Jordan algebra implies that (2) is satisfies. \square

3.3 Generalized semi-direct product

Let \mathfrak{J} be a Jordan algebra, $(D, x_0) \in \text{End}(\mathfrak{J}) \times \mathfrak{J}$, and $\mathbb{K}a$ be one-dimensional algebra with zero product. On the vector space $\tilde{\mathfrak{J}} = \mathbb{K}a \oplus \mathfrak{J}$, we define the following product:

$$x \star y := xy, \quad x \star a = a \star x = D(x), \quad a \star a = x_0, \quad \forall x, y \in \mathfrak{J}.$$

$\tilde{\mathfrak{J}}$ endowed with the product above is a Jordan algebra, if and only if, for all x, y in \mathfrak{J} , the pair (D, x_0) satisfies the following conditions

$$\begin{aligned} (C_1) D(x^2y) &= x^2D(y) + 2D(x)(xy) - 2x(D(x)y), & (C_5) D^2(x^2) &= 2(D(x))^2 - 2xD^2(x) + x_0x^2, \\ (C_2) : D(x)D(y) - D(D(x)y) &= \frac{1}{2}(x_0, y, x), & (C_6) : D^3(x) &= \frac{3}{2}x_0D(x) - \frac{1}{2}xD(x_0), \\ (C_3) : D(x_0x) &= x_0D(x), & (C_4) : xD(x^2) &= x^2D(x), & (C_7) : D^2(x_0) &= x_0^2. \end{aligned}$$

In this case, (D, x_0) is called an *admissible pair* of \mathfrak{J} and the Jordan algebra $\tilde{\mathfrak{J}}$, is called the *generalized semi-direct product* of \mathfrak{J} by the one-dimensional algebra with zero product by means of the pair (D, x_0) . It is easy to see that \mathfrak{J} is an ideal of $\tilde{\mathfrak{J}}$ and $\mathbb{K}a$ is not in general a subalgebra of $\tilde{\mathfrak{J}}$.

3.4 Double extension of Jordan algebras

Definition 3.2 Let \mathfrak{J} be a pseudo-euclidean Jordan algebra and B be an associatif scalar product on \mathfrak{J} . An endomorphism f of \mathfrak{J} is called *B-symmetric*, (resp. *B-antisymmetric*) if $B(f(x), y) = B(x, f(y))$, (resp. $B(f(x), y) = -B(x, f(y))$), $\forall x, y \in \mathfrak{J}$. Denoted by $\text{End}_s(\mathfrak{J})$ (resp. $\text{End}_a(\mathfrak{J})$) the subspace of *B-symmetric* (resp. *B-antisymmetric*) endomorphism of \mathfrak{J} .

Let \mathfrak{J}_1 be a pseudo-euclidean Jordan algebra, B_1 be an associatif scalar product on \mathfrak{J}_1 , \mathfrak{J}_2 be a Jordan algebra not necessarily pseudo-euclidean and $\pi : \mathfrak{J}_2 \longrightarrow \text{End}_s(\mathfrak{J}_1)$ be an admissible representation of \mathfrak{J}_2 in (\mathfrak{J}_1) . Now, Let $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \longrightarrow \text{End}\left((\mathfrak{J}_2)^*\right)$ be the bilinear map defined by:

$$\varphi(x, y)(z) = B_1(\pi(z)x, y), \quad \forall x, y \in \mathfrak{J}_1, z \in \mathfrak{J}_2.$$

Since, by Corollary 3.5, for all $(x, z) \in \mathfrak{J}_1 \times \mathfrak{J}_2$, $\pi(z)$ is B_1 -symmetric and satisfies $(\pi(z)x)x^2 = (\pi(z)x^2)x$, then φ is a symmetric bilinear mapping which satisfies: $\varphi(xy, x^2) = \varphi(x, yx^2)$, $\forall x, y \in \mathfrak{J}_1$.

Consequently, we can consider the central extension $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2^*$ of \mathfrak{J}_1 by \mathfrak{J}_2^* by means of φ . Recall that the product on \mathfrak{J} is defined by:

$$(x + f)(x' + f') = xx' + \varphi(x, x'), \forall x, x' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*$$

By the Remark 4, the linear map $\tilde{\pi} : \mathfrak{J}_2 \longrightarrow \text{End}(\mathfrak{J})$ defined by

$$\tilde{\pi}(y)(x + f) = \pi(y)x + \rho(y)f = \pi(y)x + f \circ R_y, \forall y \in \mathfrak{J}_2, x \in \mathfrak{J}_1, f \in \mathfrak{J}_2^*,$$

where ρ is the coadjoint representation of \mathfrak{J}_2 , is a representation of \mathfrak{J}_2 in \mathfrak{J} . Now, we shall prove that $\tilde{\pi}$ is admissible.

Since π is an admissible representation, then $\tilde{\pi}$ satisfied the condition (1) of Corollary 3.5 if and only if $\Omega(x, x', y) = 0, \forall x, x' \in \mathfrak{J}_1, \forall y \in \mathfrak{J}_2$, where

$$\begin{aligned} \Omega(x, x', y) := & \rho(y^2)\varphi(x, x') + 2\varphi(\pi(y)x', \pi(y)x) + \varphi(\pi(y)x', x^2) + 2\varphi(xx', \pi(y)x) \\ & - 2\rho(y)\varphi(x', \pi(y)x) - \rho(y)\varphi(x', x^2) - \varphi(\pi(y^2)x', x) - 2\varphi(x'\pi(y)x, x). \end{aligned}$$

It is clear that for all $x, x' \in \mathfrak{J}_1, y, y' \in \mathfrak{J}_2$, we have $\Omega(x, x', y)(y') = B(x', \Gamma(x, y, y'))$, where

$$\begin{aligned} \Gamma(x, y, y') = & \left(\pi(y'y^2) + 2\pi(y)\pi(y')\pi(y) - 2\pi(yy')\pi(y) - \pi(y^2)\pi(y') \right) x \\ & - \left(\pi(yy')x^2 + 2\left(\pi(y)x \right) \left(\pi(y')x \right) - \pi(y)\pi(y')x^2 - 2\left(\pi(y')\pi(y)x \right) x \right) = 0. \end{aligned}$$

By (ii) of (13) and the identity (3) of Corollary 3.5 we have $\Gamma(x, y, y') = 0$. Consequently $\Omega(x, x', y) = 0$. Hence $\tilde{\pi}$ satisfies the condition (1) of corollary 3.5.

we need the following lemma to show that $\tilde{\pi}$ satisfies the conditions (2) et (3) of Corollary 3.5.

Lemma 3.7 *Let $x \in \mathfrak{J}_1, y, z, u \in \mathfrak{J}_2$, then:*

$$(i) \quad B_1([\pi(y), \pi(z)]x^2, x) = 0;$$

$$(ii) \quad B_1\left(\pi((y, z, u)) + 2[\pi(z), \pi(u)\pi(y)]\right)x, x = 0.$$

Proof. Let $x \in \mathfrak{J}_1, y, z, u \in \mathfrak{J}_2$. (i) The identity $B_1([\pi(y), \pi(z)]x^2, x) = 0$ is equivalent to

$$2B_1([\pi(y), \pi(z)]x, xx') + B_1([\pi(y), \pi(z)]x', x^2) = 0.$$

In fact we proceed by linearization, we replace x by $x + \lambda x'$ in the identity $B_1([\pi(y), \pi(z)]x^2, x) = 0$, where $\lambda \in \mathbb{K}$.

Since $[\pi(y), \pi(z)]$ is a B_1 -antisymmetric endomorphism and B_1 is associative, then

$$2B_1([\pi(y), \pi(z)]x, xx') + B_1([\pi(y), \pi(z)]x', x^2) = B_1\left(2\left([\pi(y), \pi(z)]x\right)x - [\pi(y), \pi(z)]x^2, x'\right) = 0.$$

Which equivalent to

$$2\left([\pi(y), \pi(z)]x\right)x - [\pi(y), \pi(z)]x^2 = 0. \tag{18}$$

By (3) of Corollary 3.5 we have:

$$\pi(yz)x^2 + 2\left(\pi(z)x\right)\left(\pi(y)x\right) = \pi(y)\pi(z)x^2 + 2\left(\pi(z)\pi(y)x\right)x \quad (19)$$

and

$$\pi(z)y x^2 + 2\left(\pi(y)x\right)\left(\pi(z)x\right) = \pi(z)\pi(y)x^2 + 2\left(\pi(y)\pi(z)x\right)x. \quad (20)$$

\mathfrak{J}_1 and \mathfrak{J}_2 are commutative, then

$$\pi(y)\pi(z)x^2 + 2\left(\pi(z)\pi(y)x\right)x = \pi(z)\pi(y)x^2 + 2\left(\pi(y)\pi(z)x\right)x.$$

Which give the identity (18) or the identity (i) of Lemma.

(ii) One pose $F(y, z, u) = \pi((y, z, u)) + 2[\pi(z), \pi(u)\pi(y)]$, $\forall y, z, u \in \mathfrak{J}_1$. Then $F(y, z, u)$ is a B_1 -antisymmetric endomorphism. In fact,

$$\begin{aligned} & B_1(F(y, z, u)x, x') + B_1(x, F(y, z, u)x') \\ &= B_1\left(x, \pi((y, z, u))(x')\right) + 2B_1\left(x, [\pi(y)\pi(u), \pi(z)]x'\right) + B_1\left(x, F(y, z, u)x'\right) \\ &= 2B_1\left(x, (\pi((y, z, u)) - [\pi(z), [\pi(y), \pi(u)]])x'\right) = 0, \quad (\text{by Corollary 3.3}). \end{aligned}$$

Which prove that, $F(y, z, u)$ is a B_1 -antisymmetric endomorphism. Consequently, $B_1(F(y, z, u)x, x) = -B_1(x, F(y, z, u)x)$. The fact that B_1 is symmetric implies that $B_1(F(y, z, u)x, x) = 0$. \square

2. Since π satisfies the equality (2) of Corollary 3.3 and φ is symmetric, then $\tilde{\pi}$ satisfies the identity (2) of Corollary 3.5 if and only if

$$\varphi(\pi(y)x, x^2) - \varphi(\pi(y)x^2, x) = 0, \quad \forall x \in \mathfrak{J}_1, \quad y \in \mathfrak{J}_2.$$

By (i) of Lemma 3.7, we have $\left(\varphi(\pi(y)x, x^2) - \varphi(\pi(y)x^2, x)\right)(z) = B_1([\pi(y), \pi(z)]x^2, x) = 0$. Hence,

$$\left(\tilde{\pi}(y)(x + f)\right)(x + f)^2 = \left(\tilde{\pi}(y)(x + f)^2\right)(x + f).$$

3. Since π is admissible, then $\tilde{\pi}$ satisfies the identity (3) of Corollary 3.5 if and only if

$$\rho(y y')\varphi(x, x) + 2\varphi(\pi(y')x, \pi(y)x) - \rho(y)\rho(y')\varphi(x, x) - 2\varphi(\pi(y')\pi(y)x, x) = 0.$$

By (i) and (ii) of lemma 3.7, we have:

$$\begin{aligned} & \left(\rho(y y')\varphi(x, x) + 2\varphi(\pi(y')x, \pi(y)x) - \rho(y)\rho(y')\varphi(x, x) - 2\varphi(\pi(y')\pi(y)x, x)\right)(z) \\ &= B_1\left(\pi((y', y, z) + 2[\pi(y), \pi(z)\pi(y')])x, x\right) - B_1\left([\pi(y'), \pi(z)]x^2, x\right) = 0. \end{aligned}$$

Consequently, $\tilde{\pi}$ satisfies the identity (3) of Corollary 3.5. We conclude that $\tilde{\pi}$ is an admissible representation of \mathfrak{J}_2 in $\mathfrak{J}_1 \oplus \mathfrak{J}_2^*$.

Hence, we can consider, the semi-direct product, $\tilde{\mathfrak{J}} = \mathfrak{J}_2 \oplus \mathfrak{J}$ of \mathfrak{J} by \mathfrak{J}_2 by means of $\tilde{\pi}$. The product in $\tilde{\mathfrak{J}}$ is given by:

$$(x + y + f)(x' + y' + f') = xx' + yy' + \pi(x)y' + \pi(x')y + \rho(x)f' + \rho(x')f + \varphi(y, y'),$$

$\forall x, x' \in \mathfrak{J}_2, y, y' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*$. Further, let γ be an associative symmetric bilinear form not necessarily nondegenerate on $\mathfrak{J}_2 \times \mathfrak{J}_2$. Then, the bilinear form B defined on $\tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}}$ by:

$$\begin{aligned} B : (\mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*) \times (\mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*) &\longrightarrow \mathbb{K} \\ (x + y + f, x' + y' + f') &\longmapsto \gamma(x, x') + B_1(y, y') + f(x') + f'(x) \end{aligned}$$

is an associatif scalar product on $\tilde{\mathfrak{J}}$.

Then we have proved the following Theorem.

Theorem 3.8 *Let (\mathfrak{J}_1, B_1) be a pseudo-euclidean Jordan algebra, \mathfrak{J}_2 be a Jordan algebra and $\pi : \mathfrak{J}_2 \longrightarrow \text{End}_s(\mathfrak{J}_1)$ be an admissible representation of Jordan algebras. Let us consider the symmetric bilinear map $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \longrightarrow \mathfrak{J}_2^*$ defined by: $\varphi(y, y')(x) = B_1(\pi(x)y, y')$, $\forall x \in \mathfrak{J}_2, y, y' \in \mathfrak{J}_1$. Then, the vector space $\tilde{\mathfrak{J}} = \mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*$ endowed with the product*

$$\begin{aligned} (x + y + f)(x' + y' + f') &= xx' + yy' + \pi(x)y' + \pi(x')y + f' \circ R_x + f \circ R_{x'} + \varphi(y, y'), \\ \forall x, x' \in \mathfrak{J}_2, y, y' \in \mathfrak{J}_1, f, f' \in \mathfrak{J}_2^*, \end{aligned}$$

is a Jordan algebra. Moreover, if γ is an associative bilinear form on $\mathfrak{J}_2 \times \mathfrak{J}_2$, then the bilinear form B_γ defined on $\tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}}$ by:

$$\begin{aligned} B_\gamma : (\mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*) \times (\mathfrak{J}_2 \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2^*) &\longrightarrow \mathbb{K} \\ (x + y + f, x' + y' + f') &\longmapsto \gamma(x, x') + B_1(y, y') + f(x') + f'(x) \end{aligned}$$

is an associatif scalar product on $\tilde{\mathfrak{J}}$.

The Jordan algebra $(\tilde{\mathfrak{J}}, B_0)$ (or $\tilde{\mathfrak{J}}$) is called the double extension of (\mathfrak{J}_1, B_1) by \mathfrak{J}_2 by means of π .

Remark 5 *If \mathfrak{J}_2 admits an associative bilinear form $\gamma_1 \neq 0$, then $\tilde{\mathfrak{J}}$ has at least two linearly independent associatif scalar products.*

3.5 Generalized double extension of pseudo-euclidean Jordan algebras by the one-dimensional algebra with zero product

Let (\mathfrak{J}_1, B_1) be a pseudo-euclidean Jordan algebra, $\mathbb{K}b$ be the one-dimensional algebra with zero product and $(D, x_0) \in \text{End}_s(\mathfrak{J}_1, B_1) \times \mathfrak{J}_1$ be an admissible pair. Let $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \longrightarrow \mathbb{K}$ be the symmetric bilinear form defined by:

$$\varphi(x, y) = B_1(Dx, y), \quad \forall x, y \in \mathfrak{J}_1.$$

Then, by the identity (C_4) , $\varphi(x^2, xy) = \varphi(x^2y, x)$, $\forall x, y \in \mathfrak{J}_1$. Hence the vector space $\tilde{\mathfrak{J}} = \mathfrak{J}_1 \oplus \mathbb{K}b$ endowed with the product: $(x + \lambda b)(y + \lambda' b) := xy + \varphi(x, y)b$, $\forall x, y \in \mathfrak{J}_1, \lambda, \lambda' \in \mathbb{K}$ is the central extension of \mathfrak{J}_1 by $\mathbb{K}b$ by means of φ .

Let $(\tilde{D}, w_0) \in \text{End}(\mathfrak{J}) \times \mathfrak{J}$ defined by:

$$\tilde{D}(\alpha b + x) := D(x) + B_1(x_0, x)b, \quad \forall x \in \mathfrak{J}_1, \alpha \in \mathbb{K}, \quad w_0 = x_0 + kb, \quad \text{où } k \in \mathbb{K}.$$

Then the pair (\tilde{D}, w_0) is admissible. Hence, we can consider the generalized semi-direct product $\tilde{\mathfrak{J}} = \mathbb{K}a \oplus \mathfrak{J}$ of \mathfrak{J} by the one-dimensional algebra with zero product $\mathbb{K}a$ by means of (\tilde{D}, w_0) .

Moreover, the symmetric bilinear form $B : \tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}} \rightarrow \mathbb{K}$ defined by:

$$B_{|\mathfrak{J}_1 \times \mathfrak{J}_1} = B_1, \quad B_1(a, b) = 1, \quad B_1(a, \mathfrak{J}_1) = B_1(b, \mathfrak{J}_1) = \{0\} \quad \text{et} \quad B_1(a, a) = B_1(b, b) = 0$$

is an associatif scalar product on $\tilde{\mathfrak{J}}$. Thus, we have the following Theorem:

Theorem 3.9 *Let (\mathfrak{J}_1, B_1) be a pseudo-euclidean Jordan algebra, $\mathbb{K}b$ be the one-dimensional algebra with zero product and $(D, x_0) \in \text{End}_s(\mathfrak{J}_1, B_1) \times \mathfrak{J}_1$ be an admissible pair. Let $\varphi : \mathfrak{J}_1 \times \mathfrak{J}_1 \rightarrow \mathbb{K}$ be the symmetric bilinear form defined by:*

$$\varphi(x, y) = B_1(Dx, y), \quad \forall x, y \in \mathfrak{J}_1.$$

Then, the vector space $\tilde{\mathfrak{J}} = \mathbb{K}a \oplus \mathfrak{J}_1 \oplus \mathbb{K}b$, (where $\mathbb{K}a$ is a one-dimensional vector space), endowed with the following product:

$$\tilde{\mathfrak{J}} \star b = b \star \tilde{\mathfrak{J}} = \{0\}, \quad a \star a = w_0 = x_0 + kb, \quad x \star y = xy + B_1(D(x), y)b, \quad a \star x = x \star a = D(x) + B_1(x_0, x)b,$$

$\forall x, y \in \mathfrak{J}_1$ and with the symmetric bilinear form B defined by:

$$B_{|\mathfrak{J}_1 \times \mathfrak{J}_1} = B_1, \quad B_1(a, b) = 1, \quad B_1(a, \mathfrak{J}_1) = B_1(b, \mathfrak{J}_1) = \{0\} \quad \text{et} \quad B_1(a, a) = B_1(b, b) = 0$$

is a pseudo-euclidean Jordan algebra.

Definition 3.3 *The pseudo-euclidean Jordan algebra $(\tilde{\mathfrak{J}}, B)$ is called the generalized double extension of the pseudo-euclidean Jordan algebra (\mathfrak{J}_1, B_1) by the one-dimensional Jordan algebra with zero product $\mathbb{K}b$ by means of $(D, x_0, k) \in \text{End}_s(\mathfrak{J}_1, B_1) \times \mathfrak{J}_1 \times \mathbb{K}$ (or by means of $(D, x_0) \in \text{End}_s(\mathfrak{J}_1, B_1) \times \mathfrak{J}_1$).*

4 Inductive description of pseudo-euclidean Jordan algebras

Theorem 4.1 *Let (\mathfrak{J}, B) be an irreducible pseudo-euclidean Jordan algebra. If $\mathfrak{J} = \mathcal{I} \oplus \mathcal{V}$, where \mathcal{I} is a maximal ideal of \mathfrak{J} and \mathcal{V} is a subalgebra of \mathfrak{J} , then \mathfrak{J} is the double extension of the pseudo-euclidean Jordan algebra $(\mathcal{W} = \mathcal{I}/\mathcal{I}^\perp, \tilde{B})$ by \mathcal{V} by means of the representation $\pi : \mathcal{V} \rightarrow \text{End}_s(\mathcal{W}, \tilde{\mathfrak{B}})$ defined by $\pi(v)(s(i)) := s(R_v(i)) = s(vi), \forall v \in \mathcal{V}, i \in \mathcal{I}$, where \mathcal{I}^\perp is the orthogonal of \mathcal{I} , s is the canonical surjection of \mathcal{I} onto $\mathcal{I}/\mathcal{I}^\perp$ and \tilde{B} is defined by: $\tilde{B}(s(x), s(y)) := B(x, y), \forall x, y \in \mathcal{I}$.*

Proof. Since \mathcal{I} is a maximal ideal of \mathfrak{J} , Then, \mathcal{I}^\perp is a minimal one. Further, \mathfrak{J} is irreducible, then $\mathcal{I} \cap \mathcal{I}^\perp \neq \{0\}$. Consequently, $\mathcal{I}^\perp \subset \mathcal{I}$. Consider $\mathcal{A} = \mathcal{I}^\perp \oplus \mathcal{V}$ and \mathcal{A}^\perp the orthogonal of \mathcal{A} . It is easy to check that $B_{|\mathcal{A} \times \mathcal{A}}$ is nondegenerate. Thus, $\mathcal{I} = \mathcal{I}^\perp \oplus \mathcal{A}^\perp$. It follows that, $\mathfrak{J} = \mathcal{A}^\perp \oplus \mathcal{I}^\perp \oplus \mathcal{V}$. Now, let $a, b \in \mathcal{A}^\perp$, then $ab = \alpha(a, b) + \beta(a, b)$ where $\alpha(a, b) \in \mathcal{I}^\perp$ and $\beta(a, b) \in \mathcal{A}^\perp$. It is clear that \mathcal{A}^\perp endowed with β is a Jordan algebra. Moreover, the bilinear form $Q := B_{|\mathcal{A}^\perp \times \mathcal{A}^\perp}$ is an associatif scalar product on \mathcal{A}^\perp . Which implies that (\mathcal{A}^\perp, Q) is a pseudo-euclidean Jordan algebra. consider $\theta := s_{|\mathcal{A}^\perp} : \mathcal{A}^\perp \rightarrow \mathcal{I}/\mathcal{I}^\perp$,

θ is an isomorphism of Jordan algebras. Let $v, w \in \mathcal{V}$ and $x \in \mathcal{A}^\perp$. Then, $vx = i + a \in \mathcal{I} = \mathcal{I}^\perp \oplus \mathcal{A}^\perp$ where $i \in \mathcal{I}^\perp$ and $a \in \mathcal{A}^\perp$. Thus,

$$B(i, w) = B(vx - a, w) = B(vx, w) = B(x, vw) = 0,$$

Thus $i = 0$. Which implies that $\mathcal{V}\mathcal{A}^\perp \subset \mathcal{A}^\perp$. Consequently the map $\pi : \mathcal{V} \longrightarrow \text{End}(\mathcal{A}^\perp)$ defined by: $\pi(v)a = R_v(a)$, $\forall v \in \mathcal{V}$, $a \in \mathcal{A}^\perp$ is well defined and it is an admissible representation of \mathcal{V} because R is an admissible representation of \mathfrak{J} . Hence, we can consider the double extension $\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*$ of \mathcal{A}^\perp by \mathcal{V} by means of π . Now, one consider $\nu : \mathcal{I}^\perp \longrightarrow \mathcal{V}^*$ (resp. $\delta : \mathcal{V} \longrightarrow (\mathcal{I}^\perp)^*$) defined by $\nu(i) := B(i, \cdot)$, $\forall i \in \mathcal{I}^\perp$ (resp. $\delta(v) = B(v, \cdot)$, $\forall v \in \mathcal{V}$). Since B is nondegenerate, then ν (resp. δ) is injective. Hence $\dim \mathcal{I}^\perp = \dim \mathcal{V}^*$ and ν is an isomorphism of vector spaces. By the Corollary 3.6, $R : \mathfrak{J} \longrightarrow \text{End}(\mathfrak{J})$; $x \longmapsto R(x) := R_x$ is an admissible representation of \mathfrak{J} . Further, \mathcal{V} is a subalgebra of \mathfrak{J} and \mathcal{I}^\perp is an ideal of \mathfrak{J} , then $\tilde{R} : \mathcal{V} \longrightarrow \text{End}(\mathcal{I}^\perp)$; $v \longmapsto \tilde{R}(v) := R(v)/_{\mathcal{I}^\perp}$, is a representation of \mathcal{V} on \mathcal{I}^\perp . Recall that, $\rho : \mathcal{V} \longrightarrow \text{End}(\mathcal{V}^*)$; $x \longmapsto \rho(x)$ where $\rho(x)(f)(y) = f(xy)$, $\forall x, y \in \mathcal{V}$ is the coadjoint representation of \mathcal{V} . ν is a representations isomorphism (i.e. $\nu \circ \tilde{R}(v) = \rho(v) \circ \nu$, $\forall v \in \mathcal{V}$).

It is clear that the map $\nabla : \mathcal{I}^\perp \oplus \mathcal{A}^\perp \oplus \mathcal{V} \longrightarrow \mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*$; $(i + a + v) \mapsto v + a + \nu(i)$, is an isomorphism of Jordan algebras. In fact, recall that $\mathcal{I} \cap \mathcal{I}^\perp = \{0\}$. Let $X = i + a + v, Y = j + b + w \in \mathfrak{J} = \mathcal{I}^\perp \oplus \mathcal{A}^\perp \oplus \mathcal{V}$ where $i, j \in \mathcal{I}^\perp$, $a, b \in \mathcal{A}^\perp$, $v, w \in \mathcal{V}$.

$$XY = iw + jv + \alpha(a, b) + \beta(a, b) + \pi(w)a + \pi(v)b + vw.$$

$$\nabla(XY) = vw + \beta(a, b) + \pi(w)a + \pi(v)b + \nu(iw) + \nu(jv) + \nu(\alpha(a, b)).$$

Moreover,

$$\nu(iw)(w') = B(iw, w') = B(i, ww') = \nu(i)(ww') = (\nu(i) \circ \pi(w))(w').$$

and,

$$\nu(\alpha(a, b))(w') = B(\alpha(a, b), w') = B(\alpha(a, b) + \beta(a, b), w') = B(ab, w') = \varphi(a, b),$$

where $\varphi(a, b) \in \mathcal{V}^*$ defined by $\varphi(a, b)(v) := Q(va, b)$. It follows that,

$$\nabla(XY) = vw + \beta(a, b) + \pi(w)a + \pi(v)b + \nu(i) \circ \pi(w) + \nu(j) \circ \pi(v) + \varphi(a, b) = \nabla(X)\nabla(Y).$$

Hence ∇ is an algebras isomorphism. Further, if we consider on \mathfrak{J} the symmetric bilinear forms $B' : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$, and $B'' : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$ defined by:

$$B'_{|\mathcal{I} \times \mathfrak{J}} = B/_{|\mathcal{I} \times \mathfrak{J}}, \quad B'_{|\mathcal{V} \times \mathcal{V}} = 0 \quad \text{and} \quad B''_{|\mathcal{V} \times \mathfrak{J}} = 0, \quad B''_{|\mathcal{I} \times \mathcal{I}} = 0, \quad B''_{|\mathcal{V} \times \mathcal{V}} = B_{|\mathcal{V} \times \mathcal{V}},$$

then B' is nondegenerate and associative, B'' is associative and $B = B' + B''$.

Now, let us consider on $\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*$ the associatif scalar products T and T' defined by:

$$\begin{aligned} T'(v + a + f, w + b + h) &= Q(a, b) + f(b) + h(a), \quad \forall v, w \in \mathcal{V}, a, b \in \mathcal{A}^\perp, f, h \in \mathcal{V}^*, \\ T = T' + \gamma \quad \text{where} \quad \gamma : (\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*) \times (\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*) &\longrightarrow \mathbb{K} \\ (v + a + f, w + b + h) &\longmapsto B(v, w). \end{aligned}$$

It follows that ∇ is an isometry of pseudo-euclidean Jordan algebras from (\mathfrak{J}, B) (resp. (\mathfrak{J}, B')) to $(\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*, T)$ (resp. $(\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*, T')$).

It is clear that $\phi : \mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^* \longrightarrow \mathcal{V} \oplus (\mathcal{I}/\mathcal{I}^\perp) \oplus \mathcal{V}^*$; $(v+a+f) \mapsto v+\theta(a)+f$ is an isomorphism of Jordan algebras, where $\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*$ is the double extension of \mathcal{A}^\perp by \mathcal{V} by means of π and $\mathcal{V} \oplus \mathcal{I}/\mathcal{I}^\perp \oplus \mathcal{V}^*$ is the double extension of $\mathcal{I}/\mathcal{I}^\perp$ by \mathcal{V} by means of $\tilde{\pi} : \mathcal{I}/\mathcal{I}^\perp \longrightarrow \text{End}_s(V)$ defined by $\tilde{\pi}(s(i))(v) := \pi(vi), \forall i \in \mathcal{I}, v \in \mathcal{V}$. Now, if one considers the associatif scalar products, Γ and Γ' on $\mathcal{V} \oplus \mathcal{I}/\mathcal{I}^\perp \oplus \mathcal{V}^*$ defined by:

$$\begin{aligned} \Gamma'(v+s(i)+f, w+s(j)+h) &:= \tilde{B}(s(i), s(j)) + f(w) + h(v) = B(i, j) + f(w) + h(v) \\ \forall v, w \in \mathcal{V}, i, j \in \mathcal{I}, f, h \in \mathcal{V}^*, \end{aligned}$$

$$\begin{aligned} \Gamma = \Gamma' + \gamma' \text{ where } \gamma' : (\mathcal{V} \oplus \mathcal{I}/\mathcal{I}^\perp \oplus \mathcal{V}^*) \times (\mathcal{V} \oplus \mathcal{I}/\mathcal{I}^\perp \oplus \mathcal{V}^*) &\longrightarrow \mathbb{K} \\ (v+s(i)+f, w+s(j)+h) &\longmapsto B(v, w), \end{aligned}$$

then ϕ is an isometry of pseudo-euclidean Jordan algebras from $(\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*, B)$ (resp. $(\mathcal{V} \oplus \mathcal{A}^\perp \oplus \mathcal{V}^*, B')$) to $(\mathcal{V} \oplus (\mathcal{I}/\mathcal{I}^\perp) \oplus \mathcal{V}^*, \Gamma)$ (resp. $(\mathcal{V} \oplus (\mathcal{I}/\mathcal{I}^\perp) \oplus \mathcal{V}^*, \Gamma')$). \square

Corollary 4.2 *Let (\mathfrak{J}, B) be an irreducible pseudo-euclidean Jordan algebra which is not simple. If \mathfrak{J} is not nilpotent, then \mathfrak{J} is a double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by a simple Jordan algebra.*

Proof. \mathfrak{J} is not nilpotent, then $\mathfrak{J} = \mathfrak{S} \oplus \text{Ra}(\mathfrak{J})$ where \mathfrak{S} is a semi-simple subalgebra of \mathfrak{J} and $\text{Ra}(\mathfrak{J})$ is the radical of \mathfrak{J} . \mathfrak{J} is irreducible and not simple, then $\text{Ra}(\mathfrak{J}) \neq \{0\}$. Since \mathfrak{S} is semi-simple, then $\mathfrak{S} = \bigoplus_{i=1}^m \mathfrak{S}_i$, where \mathfrak{S}_i is a simple ideal of \mathfrak{S} , for all $i \in \{1, \dots, m\}$. The fact that \mathfrak{S}_1 is a simple ideal of \mathfrak{J} , implies that $I = \mathfrak{S}_2 \oplus \dots \oplus \mathfrak{S}_n \oplus \text{Ra}(\mathfrak{J})$ is a maximal ideal of \mathfrak{J} . By the last theorem, \mathfrak{J} is a double extension of $(W = \mathcal{I}/\mathcal{I}^\perp, T = \tilde{B})$ by \mathfrak{S}_1 by means of the representation $\pi : \mathfrak{S}_1 \longrightarrow \text{End}_s(\mathcal{W}, T)$ defined by: $\pi(x)(s(i)) := s(xi), \forall x \in \mathfrak{S}_1, i \in \mathcal{I}$, where $s : \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}^\perp$ is the canonical surjection. Recall that \tilde{B} is defined by: $\tilde{B}(s(i), s(j)) := B(i, j), \forall i, j \in \mathcal{I}$. \square

Corollary 4.3 *Let (\mathfrak{J}, B) be an irreducible pseudo-euclidean Jordan algebra which is not simple and such that $\mathfrak{J} \neq \{0\}$. If $\text{Ann}(\mathfrak{J}) = \{0\}$, then \mathfrak{J} is a double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by a simple Jordan algebra.*

Proof. If $\text{Ann}(\mathfrak{J}) = \{0\}$, then $\text{Ann}(\mathfrak{J})^\perp = \mathfrak{J}$ (ie. $\mathfrak{J}^2 = \mathfrak{J}$). Consequently, \mathfrak{J} is not nilpotent. Hence, by the Corollary above, \mathfrak{J} is a double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by a simple Jordan algebra. \square

Theorem 4.4 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra which not the one dimensional Jordan algebra with zero product. If $\text{Ann}(\mathfrak{J}) \neq \{0\}$ and if there exist $b \in \text{Ann}(\mathfrak{J}) \setminus \{0\}$ such that $B(b, b) = \{0\}$, then \mathfrak{J} is a generalized double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by one dimensional Jordan algebra with zero product.*

Proof. Let $b \in \text{Ann}(\mathfrak{J}) \setminus \{0\}$ such that $B(b, b) = 0$. Consider the ideal $\mathcal{I} := \mathbb{K}b$ of \mathfrak{J} . There exists $a \in \mathfrak{J}$ such that $B(a, b) = 1, B(a, a) = 0$ and $\mathfrak{J} = \mathcal{I}^\perp \oplus \mathbb{K}a$. Denote $\mathcal{W} := (\mathbb{K}a \oplus \mathbb{K}b)^\perp$, then $\mathcal{I}^\perp = \mathbb{K}b \oplus \mathcal{W}$. It follows that, $\mathfrak{J} = \mathbb{K}a \oplus \mathcal{W} \oplus \mathbb{K}b$.

Let $x, y \in \mathcal{W}$, $xy = \beta(x, y) + \alpha(x, y)b$, where $\beta(x, y) \in \mathcal{W}$ and $\alpha(x, y) \in \mathbb{K}$. It is easy to see that \mathcal{W} endowed with the bilinear form $\beta : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{W}; (x, y) \longmapsto \beta(x, y)$, is a Jordan algebra. Moreover $B_{\mathcal{W}} = B|_{\mathcal{W} \times \mathcal{W}}$ is an associatif scalar product on \mathcal{W} and $\alpha : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{K}$ is a symmetric bilinear form such that

$$\alpha(\beta(x, y), \beta(x, y)) = \alpha(x, \beta(y, \beta(x, x))) \quad \forall x, y \in \mathcal{W}.$$

Now, if $x \in \mathcal{W}$, then $ax = D(x) + \varphi(x)b$ where $D(x) \in \mathcal{W}$ and $\varphi(x) \in \mathbb{K}$ or $D : \mathcal{W} \rightarrow \mathcal{W}; x \mapsto D(x)$ is an endomorphism of \mathcal{W} and $\varphi : \mathcal{W} \rightarrow \mathbb{K}$ is an element of the dual \mathcal{W}^* of \mathcal{W} . Since $B_{\mathcal{W}}$ is nondegenerate, then there exist $w_0 \in \mathcal{W}$ such that $\varphi = B_{\mathcal{W}}(w_0, \cdot)$. Moreover, there exist $x_0 \in \mathcal{W}, k, h \in \mathbb{K}$ such that $a^2 = kb + x_0 + ha$.

Let $x, y \in \mathcal{W}$, $B(xy, a) = B(x, ya)$ because B is associative. Consequently, $\alpha(x, y) = B(x, D(y))$. On the other hand, $B(x, ya) = B(xa, y)$. Which proves that $B(x, D(y)) = B(D(x), y)$. Then $D \in \text{End}_s(\mathcal{W}, B_{\mathcal{W}})$ and $xy = \beta(x, y) + B(D(x), y)$, $\forall x, y \in \mathcal{W}$. Now, $B(a^2, b) = B(a, ab) = 0$ which implies that, $hB(a, b) = 0$ (i.e. $h = 0$). Hence, $a^2 = w_0 + kb$.

Let $x \in \mathcal{W}, B(ax, a) = B(x, a^2)$. Thus $\varphi(x) = B(x, x_0)$. We conclude that, $w_0 = x_0$. Consequently, $ax = D(x) + B(x, x_0)b$.

It is easy to show that $(D, x_0) \in \text{End}_s(\mathcal{W}, B_{\mathcal{W}}) \times \mathcal{W}$ is an admissible pair of the pseudo-euclidean Jordan algebra $(\mathcal{W}, B_{\mathcal{W}})$. Consequently, (\mathfrak{J}, B) is a generalized double extension of $(\mathcal{W}, T := B_{\mathcal{W}})$ by the one dimensional Jordan algebra with zero product $\mathbb{K}a$ by means of the pair $(D, x_0) \in \text{End}_s(\mathcal{W}, T) \times \mathcal{W}$. \square

Corollary 4.5 *Let (\mathfrak{J}, B) be an irreducible nilpotent pseudo-euclidean Jordan algebra. If \mathfrak{J} is not the one dimensional Jordan algebra with zero product, then \mathfrak{J} is the generalized double extension of a nilpotent pseudo-euclidean Jordan algebra (\mathcal{W}, T) by the one dimensional Jordan algebra with zero product.*

Proof. If \mathfrak{J} is nilpotent, then $\mathfrak{J}^2 \neq \mathfrak{J}$. Consequently $\text{Ann}(\mathfrak{J}) \neq \{0\}$. By the last Theorem \mathfrak{J} is a generalized double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by the one dimensional Jordan algebra to null product. In the proof of the same Theorem, $\mathcal{W} = \mathcal{I}^\perp/\mathcal{I}$ where $\mathcal{I} = \mathbb{K}b \subset \text{Ann}(\mathfrak{J})$. Since \mathfrak{J} is nilpotent, then \mathcal{I}^\perp is nilpotent. Consequently, \mathcal{W} is nilpotent. \square

Let \mathcal{U} be the set constituted with $\{0\}$, the one dimensional Jordan algebra with zero product and all simple Jordan algebras.

Theorem 4.6 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. If $\mathfrak{J} \notin \mathcal{U}$, then \mathfrak{J} is obtained from elements $\mathfrak{J}_1, \dots, \mathfrak{J}_n$ of \mathcal{U} , by a finite number of orthogonal direct sums of pseudo-euclidean Jordan algebras or/and double extensions by a simple Jordan algebra or/and generalized double extension by the one dimensional Jordan algebra with zero product.*

Proof. We proceed by induction on $\dim \mathfrak{J}$. If $\dim \mathfrak{J} = 0$ or 1 , then $\mathfrak{J} \in \mathcal{U}$.

Assume that $\dim \mathfrak{J} = 2$. If \mathfrak{J} is not irreducible, then $\mathfrak{J} = \mathcal{I}_1 \oplus \mathcal{I}_2$ where $\mathcal{I}_1, \mathcal{I}_2$ are two nondegenerate ideals of \mathfrak{J} which satisfies $B(\mathcal{I}_1, \mathcal{I}_2) = \{0\}$ and $\dim \mathcal{I}_1 = \dim \mathcal{I}_2 = 1$. Therefore \mathcal{I}_1 (resp. \mathcal{I}_2) is either a one dimensional simple Jordan algebra or the one dimensional algebra with zero product. Now, suppose that \mathfrak{J} is irreducible. If \mathfrak{J} is neither simple nor nilpotent, then $\mathfrak{J} = \mathcal{S} \oplus \text{Ra}(\mathfrak{J})$ where \mathcal{S} is a one dimensional semi-simple Jordan subalgebra and $\dim \text{Ra}(\mathfrak{J}) = 1$. In this case, $\mathfrak{J} = \mathcal{S} \oplus \mathcal{S}^*$ is the double extension of $\{0\}$ by \mathcal{S} by means of the representation 0 . Now if \mathfrak{J} nilpotent, then \mathfrak{J} is the generalized double extension of $\{0\}$ by the one dimensional Jordan algebra with zero product. We conclude that if $\dim \mathfrak{J} = 2$, the theorem is satisfied.

Now suppose that the theorem is satisfied for $\dim(\mathfrak{J}) < n \in \mathbb{N}$. We shall prove it in the case where $\dim \mathfrak{J} = n$. If $\mathfrak{J} \notin \mathcal{U}$ and \mathfrak{J} is irreducible, then \mathfrak{J} is either a generalized double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by the one dimensional Jordan algebra with zero product or a double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by a simple Jordan algebra. Since $\dim \mathcal{W} < \dim \mathfrak{J}$, then (\mathcal{W}, T) satisfies the theorem, so \mathfrak{J} satisfies the theorem. Now, if $\mathfrak{J} \notin \mathcal{U}$ and \mathfrak{J} is not irreducible, then $\mathfrak{J} = \mathcal{I}_1 \oplus \dots \oplus \mathcal{I}_m$ where $\mathcal{I}_i \neq \{0\}, 1 \leq i \leq m$ are nondegenerate irreducible ideals of \mathfrak{J} such that $B(\mathcal{I}_i, \mathcal{I}_j) = \{0\} \forall i \neq j \in \{1, \dots, m\}$. Let $i \in \{1, \dots, m\}$, the fact that $\dim \mathcal{I}_i < \dim \mathfrak{J}$ implies that $(\mathcal{I}_i, B/\mathcal{I}_i \times \mathcal{I}_i)$ satisfies the theorem. Hence, (\mathfrak{J}, B) satisfies the theorem. \square

Let \mathcal{E} be the set constituted by $\{0\}$ and the one dimensional Jordan algebra with zero product.

Theorem 4.7 *Let (\mathfrak{J}, B) be a nilpotent pseudo-euclidean Jordan algebra. If $\mathfrak{J} \notin \mathcal{E}$, then \mathfrak{J} is obtained from elements $\mathfrak{J}_1, \dots, \mathfrak{J}_n$ of \mathcal{E} by a finite number of orthogonal direct sums of pseudo-euclidean Jordan algebras or/and generalized double extension by the one dimensional Jordan algebra with zero product.*

Proof. We proceed by induction on $\dim \mathfrak{J}$. If $\dim \mathfrak{J} = 0$ or 1 , then $\mathfrak{J} \in \mathcal{E}$.

If $\dim \mathfrak{J} = 2$, then \mathfrak{J} is either the orthogonal direct sum of two one dimensional Jordan algebras with zero product or \mathfrak{J} is the generalized double extension of $\{0\}$ by the one dimensional Jordan algebra with zero product.

Now, suppose that the theorem is satisfied for $\dim \mathfrak{J} < n$. We shall prove it in the case where $\dim \mathfrak{J} = n$. Suppose that $\mathfrak{J} \notin \mathcal{E}$ and \mathfrak{J} is irreducible. since \mathfrak{J} is nilpotent, then \mathfrak{J} is the generalized double extension of a nilpotent pseudo-euclidean Jordan algebra (\mathcal{W}, T) by the one dimensional algebra with zero product. Since $\dim(\mathcal{W}) < \dim \mathfrak{J}$, then \mathcal{W} satisfies the theorem. Then \mathfrak{J} satisfies the theorem. If $\mathfrak{J} \notin \mathcal{E}$ and \mathfrak{J} is not irreducible, then $\mathfrak{J} = \mathcal{I}_1 \oplus \dots \oplus \mathcal{I}_m$ where $\mathcal{I}_i, 1 \leq i \leq m$, are nilpotent nondegenerate irreducible ideals of \mathfrak{J} which satisfy $B(\mathcal{I}_i, \mathcal{I}_j) = \{0\}, \forall i \neq j \in \{1, \dots, m\}$. Since $\dim \mathcal{I}_i < \dim \mathfrak{J}, \forall i \in \{1, \dots, m\}$, then, any $(\mathcal{I}_i, B|_{\mathcal{I}_i \times \mathcal{I}_i})$ satisfies the theorem. Hence, (\mathfrak{J}, B) satisfies the theorem. \square

5 Nilpotent pseudo-euclidean Jordan algebras with dimension less than or equal to 5

In this section, we shall use inductive description obtained in section 4 to construct the nilpotent pseudo-euclidean Jordan algebras with dimensions $n \leq 5$. In particular, we shall prove that all nilpotent pseudo-euclidean Jordan algebras with dimensions $n \leq 4$ are associative, but there are nilpotent pseudo-euclidean Jordan algebras with dimension equal to 5 which are nonassociative.

First case: $n = 1$. The product of nilpotent one dimensional Jordan algebra vanish (see [14]), (i.e. $\mathfrak{J}_{1,1} = \mathbb{K}a$ where $a^2 = 0$). The associatif scalar product of $\mathfrak{J}_{1,1}$ is given by $B_1(a, a) = 1$.

Second case: $n = 2$. The generalized double extension of null algebra is the nilpotent pseudo-euclidean Jordan algebra $\mathfrak{J}_2 = Vect\{a_1, b_1\}$ where $b_1 \in Ann(\mathfrak{J}_2)$ and $a_1 a_1 = \lambda b_1$ where $\lambda \in \mathbb{K}$.

If $\lambda = 0$, we obtain $\mathfrak{J}_{2,0} = Vect\{a_1, b_1\}$ where $\mathfrak{J}_{2,0}^2 = \{0\}$. If $\lambda \neq 0$, we obtain $\mathfrak{J}_{2,\lambda}$ such that $b_1 \in Ann(\mathfrak{J}_2)$ and $a_1 a_1 = \lambda b_1$. Now, let $\lambda \in \mathbb{K}^*$, then $\varphi_2 : \mathfrak{J}_{2,\lambda} \longrightarrow \mathfrak{J}_{2,1}$ defined by

$$\varphi_2(a_1) = a_1 \text{ and } \varphi_2(b_1) = \frac{1}{\lambda_1} b_1$$

is an algebras isomorphism. Thus, for all $\lambda \neq 0$, $\mathfrak{J}_{2,\lambda}$ is isomorph to $\mathfrak{J}_{2,1}$. The associatif scalar product B_2 on $\mathfrak{J}_{2,0}$ and on $\mathfrak{J}_{2,1}$ is given by $B_2(a_1, b_1) = 1, B_2(a_1, a_1) = B_2(b_1, b_1) = 0$.

Third case: $n = 3$. Let us consider the algebra $\mathfrak{J}_{1,1} = \mathbb{K}a$, where $aa = 0$. The set of admissible pairs of this algebra is given by:

$$\{(D, x_0) \in \text{End}_s(\mathfrak{J}_{1,1}) \times \mathfrak{J}_{1,1}; D = 0, x_0 = \alpha a, \text{ where } \alpha \in \mathbb{K}\}.$$

If $\alpha = 0$, the nilpotent pseudo-euclidean Jordan algebra with dimension equal to 3 obtained by generalized double extension of $\mathfrak{J}_{1,1}$ by means of the pair $(D, x_0) = (0, 0)$ is $\mathfrak{J}_{3,0,k} = \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2$ where $\{a, b_2\} \in Ann(\mathfrak{J}_{3,0,k})$ and $a_2 a_2 = k b_2, k \in \mathbb{K}$. If $k = 0$, then $\mathfrak{J}_{3,0,0} = \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2$ where $\mathfrak{J}_{3,0,0}^2 = \{0\}$. Let $k \in \mathbb{K}^*$, then the map $\varphi_3 : \mathfrak{J}_{3,0,k} \longrightarrow \mathfrak{J}_{3,0,1}$ defined by

$$\varphi_3(a_2) = a_2, \varphi_3(a) = a \text{ et } \varphi_3(b_2) = \frac{1}{k} b_2,$$

is a isomorphism of Jordan algebras. Thus for all $k \in \mathbb{K}^*$, $\mathfrak{J}_{3,0,k}$ is isomorph to $\mathfrak{J}_{3,0,1}$. Now if $\alpha \neq 0$, then the nilpotent pseudo-euclidean Jordan algebra with dimension equal to 3 obtained by generalized double extension of $\mathfrak{J}_{1,1}$ by means of $(D, x_0) = (0, \alpha a)$ is given by $\mathfrak{J}_{3,\alpha,k} = \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2$ where $b_2 \in \text{Ann}(\mathfrak{J}_{3,\alpha,k})$, $aa = 0$, $a_2a = \alpha b_2$ and $a_2a_2 = \alpha a + kb_2$, $k \in \mathbb{K}$. Let $k \in \mathbb{K}$, $\alpha \in \mathbb{K}^*$. Let us consider the map $\varphi : \mathfrak{J}_{3,\alpha,k} \longrightarrow \mathfrak{J}_{3,1,0}$, defined by,

$$\varphi(a_2) = \alpha a_2 + \frac{k}{2}a, \quad \varphi(b_2) = \alpha b_2, \quad \varphi(a) = \alpha a.$$

φ is an isomorphism of Jordan algebras. Thus, $\forall \alpha \in \mathbb{K}^*, \forall k \in \mathbb{K}$, $\mathfrak{J}_{3,\alpha,k}$ is isomorph to $\mathfrak{J}_{3,1,0}$. The associatif scalar product on $\mathfrak{J}_{3,0,0}$, $\mathfrak{J}_{3,0,1}$ and on $\mathfrak{J}_{3,1,0}$ is given by

$$\begin{aligned} B(a_2, b_2) &= 1 & B(a_2, a_2) &= B(b_2, b_2) = 0 \\ B(a, a) &= 1 & B(a_2, a) &= B(b_2, a) = 0 \end{aligned}$$

Fourth case: $n = 4$. Recall that in the second case, we have proved that the two dimension nilpotent pseudo-euclidean Jordan algebras are $\mathfrak{J}_{2,0} = \text{Vect}\{a_1, b_1\}$ where $\mathfrak{J}_{2,0}^2 = \{0\}$ and $\mathfrak{J}_{2,1} = \text{Vect}\{a_1, b_1\}$ where $b_1 \in A(\mathfrak{J}_2)$ and $a_1a_1 = b_1$. Moreover, the bilinear form defined on $\mathfrak{J}_{2,0}$ (resp. $\mathfrak{J}_{2,1}$) by $B_2(a_1, b_1) = 1$, $B_2(a_1, a_1) = B_2(b_1, b_1) = 0$ is an associatif scalar product on $\mathfrak{J}_{2,0}$ (resp. $\mathfrak{J}_{2,1}$).

Let us consider $\mathfrak{J}_{2,0} = \mathbb{K}a_1 \oplus \mathbb{K}b_1$, where $\mathfrak{J}_{2,0}^2 = \{0\}$. Denote by, A and B the subsets of $\text{End}(\mathfrak{J}_{2,0})$ defined by:

$$\begin{aligned} A &= \{D \in \text{End}(\mathfrak{J}_{2,0}); D(a) = \alpha b \text{ and } D(b) = 0, \text{ where } \alpha \in \mathbb{K}\}, \\ B &= \{D \in \text{End}(\mathfrak{J}_{2,0}); D(a) = 0 \text{ and } D(b) = \alpha a, \text{ where } \alpha \in \mathbb{K}\}. \end{aligned}$$

The pair $(D, x_0) \in \text{End}(\mathfrak{J}_{2,0}) \times \mathfrak{J}_{2,0}$ is an admissible pair of $\mathfrak{J}_{2,0}$, if and only if $(D, x_0) \in (A \cup B) \times \mathfrak{J}_{2,0}$. Let $D \in A$ (resp. $\in B$) defined by $D(a_1) = \alpha b_1$ and $D(b_1) = 0$ (resp. $D(a_1) = 0$ and $D(b_1) = \alpha a_1$) and $x_0 = \eta a_1 + \varepsilon b_1 \in \mathfrak{J}_{2,0}$, where $\alpha, \eta, \varepsilon \in \mathbb{K}$. Let $k \in \mathbb{K}$. Then, the product in the nilpotent pseudo-euclidean Jordan algebra $\mathfrak{J}_{4,0} = \mathbb{K}a_3 \oplus \mathbb{K}a_1 \oplus \mathbb{K}b_1 \oplus \mathbb{K}b_3$ obtained by the generalized double extension of $\mathfrak{J}_{2,0}$ by the one dimensional Jordan algebra with zero product by means of $(D, x_0) \in A \times \mathfrak{J}_{2,0}$ (resp. $\in B \times \mathfrak{J}_{2,0}$) is given by:

$$\begin{aligned} b_3 \in \text{Ann}(\mathfrak{J}_{4,0}), \quad a_3a_3 &= \eta a_1 + \varepsilon b_1 + kb_3, \quad a_3b_1 = \eta b_3, \\ a_3a_1 &= \alpha b_1 + \varepsilon b_3, \quad a_1a_1 = b_1 + \alpha b_3, \quad a_1b_1 = b_1b_1 = 0. \end{aligned}$$

resp.

$$\begin{aligned} b_3 \in \text{Ann}(\mathfrak{J}_{4,0}), \quad a_3a_3 &= \eta a_1 + \varepsilon b_1 + kb_3, \quad a_3b_1 = \alpha a_1 + \eta b_3, \\ a_3a_1 &= \varepsilon b_3, \quad a_1a_1 = b_1, \quad a_1b_1 = 0, \quad b_1b_1 = \alpha b_3. \end{aligned}$$

Now let us consider the Jordan algebra $\mathfrak{J}_{2,1}$ obtained in the second case. $(D, x_0) \in \text{End}(\mathfrak{J}_{2,1}) \times \mathfrak{J}_{2,1}$ is admissible, if and only if, there exists $\beta, \varepsilon \in \mathbb{K}$, such that

$$D(a_1) = \beta b_1, \quad D(b_1) = 0, \text{ and } x_0 = \varepsilon b_1.$$

Let (D, x_0) be an admissible pair of $\mathfrak{J}_{2,1}$. Then, the product on the nilpotent pseudo-euclidean Jordan algebra $\mathfrak{J}_{4,1} = \mathbb{K}a_3 \oplus \mathbb{K}a_1 \oplus \mathbb{K}b_1 \oplus \mathbb{K}b_3$ obtained by the generalized double extension of $\mathfrak{J}_{2,1}$ by the one dimensional Jordan algebra with zero product by means of (D, x_0) is given by:

$$b_1, b_3 \in \text{Ann}(\mathfrak{J}_3), \quad a_3a_3 = \varepsilon b_1 + kb_3, \quad a_3a_1 = \beta b_1 + \varepsilon b_3, \quad a_1a_1 = b_1 + \beta b_3$$

An associatif scalar product on $\mathfrak{J}_{4,0}$ and $\mathfrak{J}_{4,1}$ is given by:

$$\begin{aligned} B_4(a_1, a_1) = B_4(b_1, b_1) = 0, & \quad B_4(a_1, b_1) = B_4(a_3, b_3) = 1, & \quad B_4(a_3, a_3) = B_4(b_3, b_3) = 0, \\ B_4(a_3, a_1) = B_4(a_3, b_1) = 0, & \quad B_4(b_3, a_1) = B_4(b_3, b_1) = 0 \end{aligned}$$

Fifth case: $n = 5$. We shall construct with the same process all nilpotent pseudo-euclidean Jordan algebras with dimension equal to 5. In the first time, let us consider the nilpotent pseudo-euclidean Jordan algebra $\mathfrak{J}_{3,0,0}$ and let $(D, x_0) \in \text{End}(\mathfrak{J}_{3,0,0}) \times \mathfrak{J}_{3,0,1}$, which satisfies the conditions $(C_1), \dots, (C_7)$. These conditions are equivalent to $D^3(x) = 0, \forall x \in \mathfrak{J}_{3,0,0}$ and $D^2(x_0) = 0$.

In the second time, let us consider $\mathfrak{J}_{3,0,1} = \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2$. The product on $\mathfrak{J}_{3,0,1}$ is defined by $a_2a_2 = b_2$ and $a, b_2 \in A(\mathfrak{J}_{3,0,1})$. $(D, x_0) \in \text{End}(\mathfrak{J}_{3,1,0}) \times \mathfrak{J}_{3,0,1}$ is admissible, if and only if there exists, $\alpha, \beta, \gamma, \eta_1, \eta_2, \eta_3$ in \mathbb{K} , such that

$$D(a_2) = \alpha a + \beta b_2, \quad D(a) = \gamma b_2, \quad D(b_2) = 0 \text{ and } x_0 = \eta_1 a_2 + \eta_2 a + \eta_3 b_2. \text{ where } \eta_1 \in \{0, \alpha^2\}.$$

Let (D, x_0) an admissible pair of $\mathfrak{J}_{3,0,1}$. The product on the Jordan algebra $\mathfrak{J}_{5,0,1} = \mathbb{K}a_4 \oplus \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2 \oplus \mathbb{K}b_4$, obtained by the generalized double extension of $\mathfrak{J}_{3,0,1}$ by the one dimensional Jordan algebra with zero product is given by: $b_4 \in A(\mathfrak{J}_3)$,

$$\begin{aligned} a_4a_4 = \eta_1 a_2 + \eta_2 a + \eta_3 b_2 + kb_4, & \quad aa = b_2a_2 = b_2a = b_2b_2 = 0, & \quad b_2a_4 = \eta_1 b_4 \\ a_2a = \alpha_2 b_4 & \quad aa_4 = \alpha_2 b_2, & \quad a_2a_4 = \alpha_2 a + \alpha_3 b_2 + \eta_3 b_4, & \quad a_2a_2 = b_2 + \alpha_3 b_4. \end{aligned}$$

Now let us consider $\mathfrak{J}_{3,1,0} = \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2$. The product on $\mathfrak{J}_{3,0,1}$ is defined by:

$$b_2 \in A(\mathfrak{J}_{3,1,0}), \quad a_2a_2 = a, \quad a_2a = b_2.$$

$(D, x_0) \in \text{End}(\mathfrak{J}_{3,1,0}) \times \mathfrak{J}_{3,1,0}$ is admissible, if and only if, there exists $\alpha, \beta, \gamma, \varepsilon, \eta$ in \mathbb{K} , such that

$$D(a_2) = \alpha a + \beta b_2, \quad D(a) = \gamma b_2, \quad D(b_2) = 0 \text{ and } x_0 = \varepsilon a + \eta b_2.$$

Let (D, x_0) be an admissible pair of $\mathfrak{J}_{3,1,0}$. Then, the product on the Jordan algebra $\mathfrak{J}_{5,1,0} = \mathbb{K}a_4 \oplus \mathbb{K}a_2 \oplus \mathbb{K}a \oplus \mathbb{K}b_2 \oplus \mathbb{K}b_4$, obtained by the generalized double extension of $\mathfrak{J}_{3,1,0}$ by the one dimensional Jordan algebra with zero product is given by:

$$\begin{aligned} b_2, b_4 \in \text{Ann}(\mathfrak{J}_{5,0,1}), & \quad a_4a_4 = \varepsilon a + \eta b_2 + kb_4, & \quad a_2a_2 = b_2 + \beta b_4, \\ a_2a = \gamma b_4, & \quad aa = 0, & \quad a_4a_2 = \alpha a + \beta b_2 + \eta b_4, & \quad a_4a = \gamma b_2, \end{aligned}$$

Now let us consider the pseudo-euclidean Jordan algebra $\mathfrak{J} = \mathfrak{J}_{5,1,0}$ where $k = \alpha_3 = \eta_2 = \eta_3 = 0$ and $\alpha_2 = 0$. The product on \mathfrak{J} is given by:

$$b_2, b_4 \in \text{Ann}(\mathfrak{J}), \quad a_4a_4 = aa = 0, \quad a_2a_2 = a, \quad a_2a = b_2 + b_4, \quad a_4a_2 = a, \quad a_4a = b_2.$$

The fact that $a_2(a_2a_4) - (a_2a_2)a_4 = b_4 \neq 0$ implies that \mathfrak{J} is not associative.

6 Symplectic Forms and Jordan bialgebras

In this section we study the relation between solutions of Yang Baxter equation (EYB) in the case of Jordan algebras (see [17]) and symplectic structures on pseudo euclidean Jordan algebras. We study also the connection between symplectic pseudo euclidean Jordan algebras and some symplectic quadratic Lie algebras.

Definition 6.1 Let ω be a skew-symmetric non-degenerate bilinear form on \mathfrak{J} which satisfies $\omega(xy, z) + \omega(yz, x) + \omega(zx, y) = 0$, $\forall x, y, z \in \mathfrak{J}$. ω is called a symplectic form and the pair (\mathfrak{J}, ω) is called symplectic Jordan algebra and if moreover B is an associatif scalar product on \mathfrak{J} , then $(\mathfrak{J}, B, \omega)$ is called a symplectic pseudo-euclidean Jordan algebra.

Definition 6.2 Let \mathcal{V} be a vector space and $\Delta : \mathcal{V} \longrightarrow \mathcal{V} \otimes \mathcal{V}$ be a map. Then the pair (\mathcal{V}, Δ) is called a coalgebra and Δ is called a comultiplication.

For $v \in \mathcal{V}$, one writes: $\Delta(v) = \sum_v v_{(1)} \otimes v_{(2)}$. The dual space \mathcal{V}^* of \mathcal{V} endowed by the following product:

$$\langle fg, v \rangle = \sum_v \langle f, v_{(1)} \rangle \langle g, v_{(2)} \rangle, \quad \forall f, g \in \mathcal{V}^*, v \in \mathcal{V}.$$

is an algebra called the dual of the coalgebra (\mathcal{V}, Δ) . Recall that if $f, g \in \mathcal{V}^*$, we can consider $f \otimes g \in (\mathcal{V} \otimes \mathcal{V})^*$ defined by:

$$\langle f \otimes g, \sum_i a_i \otimes b_i \rangle = \sum_i \langle f, a_i \rangle \langle g, b_i \rangle, \quad a_i, b_i \in \mathcal{V},$$

which proves that $\mathcal{V}^* \otimes \mathcal{V}^* \subseteq (\mathcal{V} \otimes \mathcal{V})^*$.

The space \mathcal{V} endowed with the following actions is a \mathcal{V}^* -bimodule:

$$f.v = \sum_v v_{(1)} \langle f, v_{(2)} \rangle \quad \text{and} \quad v.f = \sum_v \langle f, v_{(1)} \rangle v_{(2)},$$

$$\text{where } f \in \mathcal{V}^*, v \in \mathcal{V} \text{ and } \Delta(v) = \sum_v v_{(1)} \otimes v_{(2)}.$$

It is clear that for all $f, g \in \mathcal{V}^*$ and $v \in \mathcal{V}$ the equalities $\langle fg, v \rangle = \langle f, g.v \rangle = \langle g, v.f \rangle$, hold for all $f, g \in \mathcal{V}^*$ and $v \in \mathcal{V}$. Now, let \mathcal{V} be an algebra, Δ be a comultiplication of \mathcal{V} and \mathcal{V}^* the dual algebra of the coalgebra (\mathcal{V}, Δ) . \mathcal{V}^* with the following actions is a \mathcal{V} -bimodule:

$$\langle f * v, w \rangle = \langle f, vw \rangle = \langle w * f, v \rangle, \quad \forall f \in \mathcal{V}^*, v, w \in \mathcal{V}.$$

Consider the space $D(\mathcal{V}) = \mathcal{V} \oplus \mathcal{V}^*$ on which we define the following product:

$$(v + f)(w + g) = (vw + f.w + v.g) + (fg + f * w + v * g), \quad \forall f, g \in \mathcal{V}^*, v, w \in \mathcal{V}.$$

Then, $D(\mathcal{V})$ endowed with the product above is an algebra in which \mathcal{V} and \mathcal{V}^* are two subalgebras. The algebra $D(\mathcal{V})$ is called the (Drinfeld-)double of the bialgebra (\mathcal{V}, m, Δ) where $m : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ defined by $m(v, w) = vw$, $\forall v, w \in \mathcal{V}$ (i.e. m is the multiplication in \mathcal{V}).

Definition 6.3 Let \mathfrak{J} be a Jordan algebra not necessarily unital and Δ be a comultiplication of \mathfrak{J} . (\mathfrak{J}, Δ) is a Jordan bialgebra if $D(\mathfrak{J})$ is a Jordan algebra.

Let \mathfrak{J} be a Jordan algebra and consider the linear map

$$\begin{aligned} \tau : \mathfrak{J} \otimes \mathfrak{J} &\longrightarrow \mathfrak{J} \otimes \mathfrak{J} \\ \sum_i x_i \otimes y_i &\longmapsto \sum_i y_i \otimes x_i. \end{aligned}$$

Let $r = \sum_i a_i \otimes b_i \in \mathfrak{J} \otimes \mathfrak{J}$ such that $\tau(r) = -r$ (i.e. r is skew-symmetric). We define on \mathfrak{J} the comultiplication Δ_r by $\Delta_r(x) = \sum_{i=1}^n a_i x \otimes b_i - a_i \otimes x b_i$, $\forall x \in \mathfrak{J}$. Let

$$C_{\mathfrak{J}}(r) = r_{12}r_{13} - r_{12}r_{23} + r_{13}r_{23}, \quad \text{where}$$

$$\begin{aligned} r_{12}r_{13} &= \sum_{1 \leq i, j \leq n} a_i a_j \otimes b_i \otimes b_j, \\ r_{12}r_{23} &= \sum_{1 \leq i, j \leq n} a_i \otimes b_i a_j \otimes b_j = - \sum_{1 \leq i, j \leq n} b_j \otimes a_i a_j \otimes b_i \\ \text{and } r_{13}r_{23} &= \sum_{1 \leq i, j \leq n} a_i \otimes a_j \otimes b_i b_j = \sum_{1 \leq i, j \leq n} b_i \otimes b_j \otimes a_i a_j \end{aligned}$$

because r is antisymmetric. Thus,

$$C_{\mathfrak{J}}(r) = \sum_{1 \leq i, j \leq n} a_i a_j \otimes b_i \otimes b_j + b_i \otimes b_j \otimes a_i a_j + b_j \otimes a_i a_j \otimes b_i.$$

$C_{\mathfrak{J}}(r)$ is an element of $\mathfrak{J} \otimes \mathfrak{J} \otimes \mathfrak{J}$ and it is well defined even if \mathfrak{J} is not unital. The Jordan Yang Baxter equation of \mathfrak{J} is $C_{\mathfrak{J}}(r) = 0$ (see [16]). In this case, r is said an antisymmetric solution of The Jordan Yang Baxter equation or r is an antisymmetric r -matrix. In [16], it is proven that if $r \in \mathfrak{J} \otimes \mathfrak{J}$ is an antisymmetric r -matrix, then the pair (\mathfrak{J}, Δ_r) is a Jordan bialgebra.

Let r be an antisymmetric r -matrix. One poses the linear map $R : \mathfrak{J}^* \rightarrow \mathfrak{J}$ defined by $R(f) = \sum_i f(a_i) b_i$, $\forall f \in \mathfrak{J}^*$. The equality $C_{\mathfrak{J}}(r) = 0$ is equivalent to the following identity:

$$\langle f, R(h)R(l) \rangle + \langle h, R(l)R(f) \rangle + \langle l, R(f)R(h) \rangle = 0.$$

In fact,

$$\begin{aligned} (f \otimes h \otimes l)(r_{12}r_{13}) &= \sum_{i,j} f(a_i a_j) h(b_i) l(b_j) = f(h(b_i) a_i l(b_j) a_j) = \langle f, R(h)R(l) \rangle, \\ (f \otimes h \otimes l)(r_{12}r_{23}) &= \sum_{i,j} f(a_i) h(b_i a_j) l(b_j) = - \langle h, R(l)R(f) \rangle, \\ (f \otimes h \otimes l)(r_{13}r_{23}) &= \sum_{i,j} f(a_i) h(a_j) l(b_i b_j) = \langle l, R(f)R(h) \rangle. \end{aligned}$$

Moreover, since r is antisymmetric, then R is antisymmetric. In fact, let $f, h \in \mathfrak{J}^*$.

$$\langle f, R(h) \rangle = -f\left(\sum_{i=1}^n h(b_i) a_i\right) = -h\left(\sum_{i=1}^n f(a_i) b_i\right) = - \langle h, R(f) \rangle = - \langle R(f), h \rangle.$$

In this case we say that R satisfies the Yang Baxter equation. If R is bijective, we say that r is a nondegenerate r -matrix.

Proposition 6.1 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and $r = \sum_{i=1}^n a_i \otimes b_i$ be an antisymmetric r -matrix of \mathfrak{J} . Let $\phi : \mathfrak{J} \rightarrow \mathfrak{J}^*$ be a linear isomorphism defined by: $\phi(x) = B(x, \cdot)$, $\forall x \in \mathfrak{J}$ and $R : \mathfrak{J}^* \rightarrow \mathfrak{J}$ the linear map defined by: $R(f) = \sum_{i=1}^n f(a_i) b_i$, $\forall f \in \mathfrak{J}^*$. Then, $\mathcal{U} = R \circ \phi$ is B -antisymmetric. Further, \mathfrak{J} endowed with the product \star defined by $x \star y = \mathcal{U}(x) y + x \mathcal{U}(y)$, $\forall x, y \in \mathfrak{J}$ is a Jordan algebra for which $\mathcal{U} : (\mathfrak{J}, \star) \rightarrow \mathfrak{J}$ is a Jordan algebra isomorphism.*

Proof. Let $x, y, z \in \mathfrak{J}$. Consider $f := \phi(x) = B(x, \cdot)$, $h := \phi(y) = B(y, \cdot)$ and $l := \phi(z) = B(z, \cdot)$.

$$1. \quad x \star y = x \mathcal{U}(y) + y \mathcal{U}(x) = x(R \circ \phi)(y) + y(R \circ \phi)(x) = xR(h) + yR(f)$$

$$= \phi^{-1}(f \circ R_{R(h)}) + \phi^{-1}(h \circ R_{R(f)}) = \phi^{-1}(\phi(x)\phi(y)).$$

Hence, $\phi(x \star y) = \phi(x)\phi(y)$. Since \mathfrak{J}^* is a Jordan algebra and ϕ is bijective, then (\mathfrak{J}, \star) is isomorph to \mathfrak{J}^* .

$$2. \quad \text{Since } R \text{ is antisymmetric, then } \langle \phi(x), R(\phi(y)) \rangle = - \langle \phi(y), R(\phi(x)) \rangle. \text{ Hence, } B(x, \mathcal{U}(y)) = -B(y, \mathcal{U}(x)). \text{ Thus } \mathcal{U} \text{ is } B\text{-antisymmetric.}$$

$$3. \quad C_{\mathfrak{J}}(r) = 0. \text{ Thus, } \langle f, R(h)R(l) \rangle + \langle h, R(l)R(f) \rangle + \langle l, R(f)R(h) \rangle = 0. \text{ Thus, } B(x, \mathcal{U}(y)\mathcal{U}(z)) + B(y, \mathcal{U}(z)\mathcal{U}(x)) + B(z, \mathcal{U}(x)\mathcal{U}(y)) = 0. \text{ It follows that, } B(-\mathcal{U}(x)\mathcal{U}(y) - \mathcal{U}(y)\mathcal{U}(x) + \mathcal{U}(x)\mathcal{U}(y), z) = 0. \text{ Hence, } \mathcal{U}(x\mathcal{U}(y)) + \mathcal{U}(y\mathcal{U}(x)) = \mathcal{U}(x)\mathcal{U}(y). \text{ Consequently, } \mathcal{U}(x \star y) = \mathcal{U}(x)\mathcal{U}(y). \text{ Thus } \mathcal{U} \text{ is a Jordan algebra morphism. } \square$$

Remark 6 *The converse of the last proposition is true. If \mathcal{U} is a B -antisymmetric endomorphism of \mathfrak{J} such that the product \star given by: $x \star y = \mathcal{U}(x)y + x\mathcal{U}(y)$, $\forall x, y \in \mathfrak{J}$, define a Jordan structure on \mathfrak{J} , then the linear map $R : \mathfrak{J}^* \rightarrow \mathfrak{J}$ defined by $R = \mathcal{U} \circ \phi^{-1}$, where $\phi(x) = B(x, \cdot)$, $\forall x \in \mathfrak{J}$, is a solution of the Yang Baxter equation.*

Proposition 6.2 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and $r = \sum_{i=1}^n a_i \otimes b_i$ be an antisymmetric r -matrix. Let $\mathcal{U} : \mathfrak{J} \rightarrow \mathfrak{J}$ be the linear map defined by $\mathcal{U} = R \circ \phi$. Then, $Im(\mathcal{U}) = \{\mathcal{U}(x); x \in \mathfrak{J}\}$ is a Jordan subalgebra of \mathfrak{J} . Further, the bilinear form $\omega : Im(\mathcal{U}) \times Im(\mathcal{U}) \rightarrow \mathbb{K}$ defined by:*

$$\omega(\mathcal{U}(x), \mathcal{U}(y)) = B(\mathcal{U}(x), y), \quad \forall x, y \in \mathfrak{J},$$

is a symplectic form on $Im(\mathcal{U})$.

Proof. Let $x \in \ker \mathcal{U}$ and $y \in Im(\mathcal{U})$. There exists z in \mathfrak{J} such that $y = \mathcal{U}(z)$, then $B(x, y) = B(x, \mathcal{U}(z)) = -B(\mathcal{U}(x), z) = 0$. Thus, $Im(\mathcal{U}) \subset (\ker \mathcal{U})^\perp$. Since $dim(Im(\mathcal{U})) = dim((\ker \mathcal{U})^\perp)$, then $Im(\mathcal{U}) \subset (\ker \mathcal{U})^\perp$.

Let $x = \mathcal{U}(x_0)$, $y = \mathcal{U}(y_0) \in Im(\mathcal{U})$ where $x_0, y_0 \in \mathfrak{J}$ and $z \in \ker \mathcal{U}$.

$$B(xy, z) = B(\mathcal{U}(x_0)\mathcal{U}(y_0), z) = -B(\mathcal{U}(z)\mathcal{U}(x_0), y_0) - B(\mathcal{U}(y_0)\mathcal{U}(z), x_0) = 0.$$

Thus, $Im(\mathcal{U})$ is a subalgebra of \mathfrak{J} . Now, let $x_0, y_0, z_0 \in \mathfrak{J}$ and $x = \mathcal{U}(x_0)$, $y = \mathcal{U}(y_0)$ et $z = \mathcal{U}(z_0) \in Im(\mathcal{U})$.

$$\begin{aligned} \omega(x, y) &= \omega(\mathcal{U}(x_0), \mathcal{U}(y_0)) = B(\mathcal{U}(x_0), y_0) = -B(x_0, \mathcal{U}(y_0)) \\ &= -\omega(\mathcal{U}(y_0), \mathcal{U}(x_0)) = -\omega(y, x). \end{aligned}$$

Thus ω is antisymmetric. Further,

$$\sum_{cyc} \omega(xy, z) = \sum_{cyc} \omega(\mathcal{U}(x \star y), \mathcal{U}(z_0)) = \sum_{cyc} B(\mathcal{U}(x_0)\mathcal{U}(y_0), z_0) = 0,$$

because $C_{\mathfrak{J}}(r) = 0$. Now, let $x \in \mathfrak{J}$. If $\omega(\mathcal{U}(x), \mathcal{U}(y)) = 0$ for all y in \mathfrak{J} , then $B(\mathcal{U}(x), y) = 0$, so $\mathcal{U}(x) = 0$. Consequently, $\omega_{Im(\mathcal{U}) \times Im(\mathcal{U})}$ is nondegenerate. It follows that, $(Im(\mathcal{U}), \omega_{Im(\mathcal{U}) \times Im(\mathcal{U})})$ is a symplectic Jordan algebra. \square

Corollary 6.3 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and $r = \sum_{i=1}^n a_i \otimes b_i \in \mathfrak{J} \otimes \mathfrak{J}$ such that $\tau(r) = -r$, $C_{\mathfrak{J}}(r) = 0$ and r nondegenerate. Then \mathfrak{J} endowed with the bilinear form $\omega_{\mathcal{U}} : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$ defined by

$$\omega_{\mathcal{U}}(x, y) := B(\mathcal{U}^{-1}(x), y), \quad \forall x, y \in \mathfrak{J},$$

is a symplectic Jordan algebra.

Proof. The fact that r is nondegenerate implies that \mathcal{U} is invertible. It follows that, $Im(\mathcal{U}) = \mathfrak{J}$. The last proposition gives the result. \square

Remark 7 Let \mathcal{U} be an invertible endomorphism of \mathfrak{J} . Then, \mathcal{U} satisfies:

$$\mathcal{U}(\mathcal{U}(x)y + x\mathcal{U}(y)) = \mathcal{U}(x)\mathcal{U}(y), \quad \forall x, y \in \mathfrak{J}.$$

if and only if \mathcal{U}^{-1} is a derivation of \mathfrak{J} .

Proposition 6.4 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. \mathfrak{J} have a symplectic form ω , if and only if there exists a B -antisymmetric invertible derivation D of \mathfrak{J} such that $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathfrak{J}$.

Proof. Since B and ω are two nondegenerate bilinear form of \mathfrak{J} , then there exists an invertible endomorphism δ of \mathfrak{J} such that $\omega(x, y) = B(\delta(x), y)$, $\forall x, y \in \mathfrak{J}$. Further, since ω is symplectic, then

$$B(\delta(xy), z) = B(\delta(y), xz) + B_1(\delta(x), yz) = B(\delta(y)x, z) + B(\delta(x)y, z), \quad \forall x, y, z \in \mathfrak{J}.$$

The fact that B is nondegenerate implies that $\delta(xy) = \delta(x)y + x\delta(y)$, $\forall x, y, z \in \mathfrak{J}$. Hence δ is an invertible derivation of \mathfrak{J} . Conversely, if δ is a B -antisymmetric invertible derivation of \mathfrak{J} , then it is clear that the bilinear form $\omega : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{K}$ defined by: $\omega(x, y) = B(\delta(x), y)$, $\forall x, y \in \mathfrak{J}$ is a symplectic form of \mathfrak{J} . \square

Corollary 6.5 Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra. Then, there exists a B -antisymmetric endomorphism $\mathcal{U} : \mathfrak{J} \longrightarrow \mathfrak{J}$ satisfying $\mathcal{U}(\mathcal{U}(x)y + x\mathcal{U}(y)) = \mathcal{U}(x)\mathcal{U}(y)$, $\forall x, y \in \mathfrak{J}$ and such that $\omega = \omega_{\mathcal{U}}$.

Proof. Consider the B -antisymmetric invertible derivation D of \mathfrak{J} defined by $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathfrak{J}$. Then, $\mathcal{U} = D^{-1}$ is B -antisymmetric and satisfies

$$\mathcal{U}(\mathcal{U}(x)y + x\mathcal{U}(y)) = \mathcal{U}(x)\mathcal{U}(y), \quad \forall x, y \in \mathfrak{J}. \square$$

The following Theorem gives a generalization of the Corollary 6.5.

Theorem 6.6 Let (\mathfrak{J}, ω) be a symplectic Jordan algebra. Then, there exists a nondegenerate r -matrix which satisfies $\tau(r) = -r$, $C_{\mathfrak{J}}(r) = 0$ and such that $\omega = \omega_{\mathcal{U}}$ where \mathcal{U} is the endomorphism associate to r .

Proof. Let $\{e_1, \dots, e_n\}$ be a base of \mathfrak{J} . One poses $\alpha_{ij} = \omega(a_i, a_j)$. Since the form ω is nondegenerate, then the matrix $(\alpha_{ij})_{1 \leq i, j \leq n}$ is invertible. Let $(\beta_{ij})_{1 \leq i, j \leq n}$ be the inverse of the matrix $(\alpha_{ij})_{1 \leq i, j \leq n}$ and $r = \sum_{ij} \beta_{ij}(a_i \otimes a_j)$. It is clear that $(\beta_{ij})_{1 \leq i, j \leq n}$ is an antisymmetric matrix. Thus $\tau(r) = -r$. Now we shall prove that $C_{\mathfrak{J}}(r) = 0$. Let $\phi : \mathfrak{J} \longrightarrow \mathfrak{J}^*$ be the linear isomorphism given by $\phi(x) = \omega(x, \cdot)$ and $R = \phi^{-1}$. Then, R satisfies the Yang Baxter equation. In fact, Let $f, h, l \in \mathfrak{J}^*$. Then, there exists

$x, y, z \in \mathfrak{J}$ such that $f = \phi(x)$, $h = \phi(y)$ and $l = \phi(z)$. Thus $x = R(f)$, $y = R(h)$ and $z = R(l)$. It is easy to see that

$$\langle f, R(h)R(l) \rangle + \langle h, R(l)R(f) \rangle + \langle l, R(f)R(h) \rangle = \omega(x, yz) + \omega(y, xz) + \omega(z, xy) = 0.$$

Hence, R satisfies the YBE . It follows that, $C_{\mathfrak{J}}(r) = 0$. \square

Recall that to any unital pseudo-euclidean Jordan algebra one can apply the Tits Kantor Koecher construction (TKK construction) to obtain a quadratic Lie algebra ([9], [10]). In the following, we are going to make a slight modification to this construction in order to obtain a quadratic Lie algebras $(\mathfrak{L}ie(\mathfrak{J}), B_{\mathcal{L}})$ starting from the pseudo-euclidean Jordan algebras (\mathfrak{J}, B) which are not necessarily unital.

Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra which is not necessarily unital. $\bar{\mathfrak{J}}$ is a copy of \mathfrak{J} . One pose $L(\mathfrak{J}) := \text{vect}\{L_x, x \in \mathfrak{J}\}$ and $\mathcal{H}(\mathfrak{J}) := L(\mathfrak{J}^2) \oplus [L(\mathfrak{J}), L(\mathfrak{J})]$. On $\mathcal{H}(\mathfrak{J})$ we define the bilinear form Γ by:

$$\Gamma(R_a + D_1, R_b + D_2) = B(a, b) + \Omega(D_1, D_2), \quad \forall a, b \in \mathfrak{J}^2, D_1, D_2 \in [L(\mathfrak{J}), L(\mathfrak{J})].$$

where Ω is given by: $\Omega : [L(\mathfrak{J}), L(\mathfrak{J})] \times [L(\mathfrak{J}), L(\mathfrak{J})] \longrightarrow \mathbb{K}$

$$(D_1, D_2 = \sum_{i=1}^n [R_{c_i}, R_{d_i}]) \longmapsto \sum_{i=1}^n B(D_1(c_i), d_i).$$

The bilinear form Γ is well defined because the orthogonal of $(\text{Ann}(\mathfrak{J}))$ with respect B is \mathfrak{J}^2 .

The vector space $\mathfrak{L}ie(\mathfrak{J}) = \mathfrak{J} \oplus \mathcal{H}(\mathfrak{J}) \oplus \bar{\mathfrak{J}}$ with the following bracket:

$$\begin{aligned} [T_1, T_2] &= [T_1, T_2]_{\mathcal{H}}, [T_1, a_2] = T_1(a_2), [T_1, \bar{b}_2] = \overline{-T_1(b_2)}, \\ [a_1, \bar{b}_2] &= R_{a_1 b_2}, [a_1, a_2] = [\bar{b}_1, \bar{b}_2] = 0, \quad \forall T_i \in \mathcal{H}(\mathfrak{J}), a_i, b_i \in \mathfrak{J}, i \in \{1, 2\}, \end{aligned}$$

is a Lie algebra. Moreover $(\mathfrak{L}ie(\mathfrak{J}), B_{\mathcal{L}})$ is a quadratic Lie algebra, where the invariant scalar product $B_{\mathcal{L}}$ is defined by:

$$\begin{aligned} B_{\mathcal{L}} : \mathfrak{L}ie(\mathfrak{J}) &= \mathfrak{J} \oplus \mathcal{H}(\mathfrak{J}) \oplus \bar{\mathfrak{J}} \times \mathfrak{L}ie(\mathfrak{J}) = \mathfrak{J} \oplus \mathcal{H}(\mathfrak{J}) \oplus \bar{\mathfrak{J}} \longrightarrow \mathbb{K} \\ (x_1 = a_1 + T_1 + \bar{b}_1, x_2 = a_2 + T_2 + \bar{b}_2) &\longmapsto \Gamma(T_1, T_2) + 2B(a_1, b_2) + 2B(a_2, b_1). \end{aligned}$$

Before starting the description of symplectic pseudo-euclidean Jordan algebras, it is natural to ask the following question: If (\mathfrak{J}, B) is a pseudo-euclidean Jordan algebra and $(\mathfrak{L}ie(\mathfrak{J}), B_{\mathcal{L}})$ the Lie algebra constructed by the TKK construction from \mathfrak{J} . Suppose that (\mathfrak{J}, B) has a symplectic structure. Then, does $(\mathfrak{L}ie(\mathfrak{J}), B_{\mathcal{L}})$ admit a symplectic structure?

We give the answer of this question in the following proposition.

Proposition 6.7 *Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra. Let D be the derivation of \mathfrak{J} defined by: $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathfrak{J}$. Define the linear map $D_{\mathcal{L}} : \mathfrak{L}ie(\mathfrak{J}) \longrightarrow \mathfrak{L}ie(\mathfrak{J})$ by*

$$D_{\mathcal{L}}(a) = D(a), \quad D_{\mathcal{L}}(\bar{a}) = \overline{D(a)}, \quad D_{\mathcal{L}}(R_a) = R_{D(a)}, \quad \text{and } D_{\mathcal{L}}([R_a, R_b]) = [R_{D(a)}, b] + [R_a, R_{D(b)}],$$

$\forall a, b \in \mathfrak{J}$. Then, the bilinear form $\omega_{\mathcal{L}}$ defined on $\mathfrak{L}ie(\mathfrak{J})$, by: $\omega_{\mathcal{L}}(x, y) = B_{\mathcal{L}}(D_{\mathcal{L}}(x), y)$, $\forall x, y \in \mathfrak{L}ie(\mathfrak{J})$, is a symplectic form if and only if $D_{\mathcal{L}}$ satisfies the following condition:

$$D_{\mathcal{L}}([R(\mathfrak{J}), R(\mathfrak{J})]) = [R(\mathfrak{J}), R(\mathfrak{J})]. \quad (21)$$

Proof. Since D is a derivation of \mathfrak{J} , then

$$D((x, y, z)) = (D(x), y, z) + (x, D(y), z) + (x, y, D(z)), \quad \forall x, y, z \in \mathfrak{J}.$$

Consequently $D_{\mathcal{L}}$ is a derivation of $\mathfrak{L}ie(\mathfrak{J})$. Moreover the fact that D is B -antisymmetric implies that $D_{\mathcal{L}}$ is $B_{\mathcal{L}}$ -antisymmetric. Next, it is clear that $D_{\mathcal{L}}$ is invertible if and only if $D_{\mathcal{L}}([R(\mathfrak{J}), R(\mathfrak{J})]) = [R(\mathfrak{J}), R(\mathfrak{J})]$. \square

Definition 6.4 Let $(\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}, B)$ be a quadratic \mathbb{Z}_2 -graded Lie algebra. We say that $(\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}, B, \omega)$ is a symplectic quadratic \mathbb{Z}_2 -graded Lie algebra, if $\omega : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{K}$ is a nondegenerate anti-symmetric bilinear form which satisfies:

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y), \quad \forall x, y, z \in \mathcal{G}$$

and such that $\omega(\mathcal{G}_{\bar{0}}, \mathcal{G}_{\bar{1}}) = \{0\}$.

Remark 8 Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra and D be the antisymmetric invertible derivation of \mathfrak{J} such that $\omega(x, y) = B(D(x), y)$, $\forall x, y \in \mathfrak{J}$. If $D_{\mathfrak{L}}$ satisfies the condition (21), then $(\mathfrak{L}ie(\mathfrak{J}) = \mathfrak{L}ie(\mathfrak{J})_{\bar{0}} \oplus \mathfrak{L}ie(\mathfrak{J})_{\bar{1}}, B_{\mathfrak{L}}, \omega_{\mathfrak{L}})$, where $\omega_{\mathfrak{L}}(x, y) = B_{\mathfrak{L}}(D_{\mathfrak{L}}(x), y)$, $\forall x, y \in \mathfrak{L}ie(\mathfrak{J})$, is a symplectic quadratic \mathbb{Z}_2 -graded Lie algebra.

We are going to give an example of nonassociative Jordan algebra which admits a derivation satisfying the condition (21).

Example. Let $A = \mathbb{K}[X]$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) and $\mathcal{O} = XA$ the ideal of A generated by $\{X\}$. Let us consider $\tilde{\mathcal{O}} := \mathcal{O}/X^n\mathcal{O}$, $n \in \mathbb{N}^*$. $\tilde{\mathcal{O}}$ is an associative algebra generated by $\{\overline{X}, \overline{X^2}, \dots, \overline{X^n}\}$. Let \mathfrak{J} be an r dimension Jordan algebra ($r \in \mathbb{N}^*$). The vector space $\tilde{\mathfrak{J}} = \mathfrak{J} \otimes \tilde{\mathcal{O}}$, endowed with the following commutatif product:

$$(x \otimes \overline{P})(y \otimes \overline{Q}) = (xy) \otimes \overline{PQ}, \quad \forall x, y \in \mathfrak{J}, \forall P, Q \in \mathcal{O},$$

is a Jordan algebra. Now, let $\tilde{\mathfrak{J}} \oplus (\tilde{\mathfrak{J}})^*$ be the trivial T^* -extension of $\tilde{\mathfrak{J}}$ and $I := \{1, \dots, r\} \times \{1, \dots, n\}$. Let $(e_i)_{1 \leq i \leq r}$ be a base of \mathfrak{J} . Then, $(e_{ij} := e_i \otimes \overline{X^j})_{(i,j) \in I}$ is a base of $\tilde{\mathfrak{J}}$. Let $(e_{ij}^*)_{(i,j) \in I}$ be the dual base of $\tilde{\mathfrak{J}}$ associate to $(e_{ij})_{(i,j) \in I}$. For all $j \in \{1, \dots, n\}$, denote by $\mathfrak{J}_j = \mathfrak{J} \otimes \overline{X^j}$. It is easy to see that if $a \in \mathfrak{J}_i$ and $b \in \mathfrak{J}_j$, then $ab \in \mathfrak{J}_{i+j}$ if $i+j \leq n$ and $ab = 0$ if $i+j > n$. Let us consider the endomorphism D of $\tilde{\mathfrak{J}}$ defined by: $D(x \otimes \overline{X^i}) = ix \otimes \overline{X^i}$, $\forall x \in \mathfrak{J}, \forall i \in \{1, \dots, n\}$. Then, D is a derivation of $\tilde{\mathfrak{J}}$. It is easy to verify that the map $\bar{D} : \tilde{\mathfrak{J}} \oplus (\tilde{\mathfrak{J}})^* \rightarrow \tilde{\mathfrak{J}} \oplus (\tilde{\mathfrak{J}})^*$ defined by:

$$\bar{D}(a + f) = D(a) - f \circ D, \quad \forall a \in \tilde{\mathfrak{J}}, f \in (\tilde{\mathfrak{J}})^*$$

is a derivation of $\tilde{\mathfrak{J}} \oplus (\tilde{\mathfrak{J}})^*$. On the other hand, let $i, k, t \in \{1, \dots, r\}$ and $j, l \in \{1, \dots, n\}$. We have:

$$\left[R_{e_{ij}}, R_{e_{kj}^*} \right] (e_{tl}) = e_{kj}^* \circ (R_{e_{tl}} R_{e_{ij}} - R_{e_{ij} e_{tl}}).$$

Since

$$e_{kj}^* \circ (R_{e_{tl}} R_{e_{ij}} - R_{e_{ij} e_{tl}}) (e_{pq}) = e_{kj}^* (e_{tl} (e_{ij} e_{pq})) - e_{kj}^* ((e_{tl} e_{ij}) e_{pq}) = 0$$

because

$$\begin{aligned} e_{tl}(e_{ij}e_{pq}), (e_{tl}e_{ij})e_{pq} &\in \mathfrak{J}_{j+l+q} \text{ if } j+l+q \leq n \\ \text{and } (e_{tl}(e_{ij}e_{pq}) = (e_{tl}e_{ij})e_{pq} &= 0 \text{ if } j+l+q > n. \end{aligned}$$

Hence, if we consider $I_1 := \{(i, j, k, l), (i, j), (k, l) \in I, \text{ where } j \neq l\}$, then

$$[R(\bar{\mathfrak{J}}), R(\bar{\mathfrak{J}})] = Vect \left(\bigcup_{(i,j),(k,l) \in I} \left(\{ [R_{e_{ij}}, R_{e_{kl}}], [R_{e_{ij}^*}, R_{e_{kl}^*}] \} \right) \bigcup \left(\bigcup_{(i,j,k,l) \in I_1} \{ [R_{e_{ij}}, R_{e_{kl}^*}] \} \right) \right).$$

By the definition of \bar{D} , we have: $\bar{D}(e_{ij}) = ie_{ij}$ and $\bar{D}(e_{ij}^*)(e_{kl}) = -e_{ij}^*(D(e_{kl})) = -le_{ij}^*(e_{kl})$. Hence $\bar{D}(e_{ij}^*)(e_{kl}) = 0$ if $(i, j) \neq (k, l)$ and $\bar{D}(e_{ij}^*)(e_{ij}) = -j$. It follows that, $\bar{D}(e_{ij}^*) = -je_{ij}^*$.

$$[\bar{D}, [R_{e_{ij}}, R_{e_{kl}}]] = (j+l) [R_{e_{ij}}, R_{e_{kl}}].$$

$$[\bar{D}, [R_{e_{ij}^*}, R_{e_{kl}^*}]] = -(j+l) [R_{e_{ij}^*}, R_{e_{kl}^*}].$$

It follows that,

$$[\bar{D}, [R(\bar{\mathfrak{J}}), R(\bar{\mathfrak{J}})]] = [R(\bar{\mathfrak{J}}), R(\bar{\mathfrak{J}})].$$

Hence, \bar{D} satisfies the condition (21).

7 Descriptions of symplectic pseudo-euclidean Jordan algebras

In this section, \mathbb{K} is algebraically closed.

Theorem 7.1 *Let $(\mathfrak{J}_1, B_1, \omega_1)$ be a symplectic pseudo-euclidean Jordan algebras, $a_0 \in \mathfrak{J}_1$, $\lambda \in \mathbb{K}$ and δ the invertible derivation of \mathfrak{J}_1 such that*

$$\omega_1(x, y) = B_1(\delta(x), y), \quad \forall x, y \in \mathfrak{J}_1.$$

Let $(D, x_0) \in \text{End}_s(\mathfrak{J}_1) \times \mathfrak{J}_1$ be an admissible pair such that

$$D(a_0) = \lambda x_0 + \frac{1}{2}\delta(x_0) \quad R_{a_0} = \delta D - D\delta + \lambda D, \quad \forall x \in \mathfrak{J}_1.$$

Let $(\mathfrak{J} = \mathbb{K}a \oplus \mathfrak{J}_1 \oplus \mathbb{K}b, B)$ be the generalized double extension of (\mathfrak{J}_1, B_1) by the one dimensional Jordan algebra with zero product by means of (D, x_0) . Then, the endomorphism Δ of \mathfrak{J} defined by:

$$\Delta(b) = \lambda b, \quad \Delta(x) = \delta(x) - B_1(a_0, x)b, \quad \Delta(a) = a_0 - \lambda a, \quad \forall x \in \mathfrak{J}_1,$$

is a B -antisymmetric invertible derivation of \mathfrak{J} . Thus the antisymmetric bilinear form ω of \mathfrak{J} defined by: $\omega(x, y) = B(\Delta(x), y)$, $\forall x, y \in \mathfrak{J}$, is a symplectic form on \mathfrak{J} .

The algebra $(\mathfrak{J}, B, \omega)$ obtained above is called the symplectic pseudo-euclidean double extension of $(\mathfrak{J}_1, B_1, \omega_1)$ (by means of $(D, x_0, a_0, \lambda) \in \text{End}(\mathfrak{J}_1) \times \mathfrak{J}_1 \times \mathfrak{J}_1 \times \mathbb{K}$).

Proof. The product on \mathfrak{J} is denoted by \star . Let $x, y \in \mathfrak{J}_1$, then, we have

$$\Delta(x \star a) - \Delta(x) \star a - x \star \Delta(a) = \delta D(x) - D\delta(x) - a_0x + \lambda D(x) + B_1(2\lambda x_0 - a_0 + \delta(x_0) - D(a_0), x)b = 0.$$

$$\Delta(x \star y) - \Delta(x) \star y - x \star \Delta(y) = \delta(xy) - \delta(x)y - x\delta(y) + B_1((\lambda D - R_{a_0} - D\delta + \delta D)(x), y)b = 0.$$

$$\Delta(a \star a) - 2a \star D(a) = \delta(x_0) - 2D(a_0) + 2\lambda x_0 + 3(k\lambda - B_1(a_0, x_0))b = 0.$$

Hence, Δ is a derivation of \mathfrak{J} . Moreover, since δ is invertible, then Δ is invertible too. Thus, the bilinear form ω defined on \mathfrak{J} by: $\omega(x, y) = B(\Delta(x), y)$, $\forall x, y \in \mathfrak{J}$, is a symplectic form on \mathfrak{J} . \square

Lemma 7.2 *Let \mathfrak{J} be a Jordan algebra. If \mathfrak{J} admits an invertible derivation, then \mathfrak{J} is nilpotent.*

Proof. Let $Ra(\mathfrak{J})$ be the nilpotent radical of \mathfrak{J} and let \mathcal{S} be a semi-simple subalgebra of \mathfrak{J} such that $\mathfrak{J} = Ra(\mathfrak{J}) \oplus \mathcal{S}$. The map

$$\begin{aligned} \delta & : \mathcal{S} \longrightarrow \mathcal{S} \\ x & \longmapsto \delta(x) := p_s \circ D(x), \end{aligned}$$

where $p_s : \mathfrak{J} \longrightarrow \mathcal{S}$ is defined by $p_s(s + r) = s$, $\forall (s, r) \in \mathcal{S} \times Ra(\mathfrak{J})$, is a derivation of \mathcal{S} . In fact, Let $x, y \in \mathcal{S}$. One poses $D(x) = x_1 + x_2$, $D(y) = y_1 + y_2$ where $x_1, y_1 \in \mathcal{S}$ and $x_2, y_2 \in Ra(\mathfrak{J})$.

$$\delta(xy) = p_s(D(xy)) = p_s(xD(y) + D(x)y) = xy_2 + x_1y = x\delta(y) + \delta(x)y.$$

Thus δ is a derivation of \mathcal{S} . Now, let $x \in \mathcal{S}$ such that $\delta(x) = 0$. Thus, $D(x) = r \in Ra(\mathfrak{J})$. Hence, $D^{-1}(r) = x \in \mathcal{S}$, which is impossible because $D^{-1}(Ra(\mathfrak{J})) = Ra(\mathfrak{J})$. It follows that, δ is injective. On the other hand, since \mathcal{S} is semi-simple, then it admits a unital element e . Thus $\delta(e) = 0$. It follows that $e = 0$. Which show that $\mathcal{S} = \{0\}$. Hence $\mathfrak{J} = Ra(\mathfrak{J})$. Thus \mathfrak{J} is nilpotent. \square

Theorem 7.3 *Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra such that $\mathfrak{J} \neq \{0\}$. Then $(\mathfrak{J}, B, \omega)$ is a symplectic pseudo-euclidean double extension of a symplectic pseudo-euclidean Jordan algebra (\mathcal{W}, T, Ω) .*

Proof. Let Δ be the invertible B -antisymmetric derivation of \mathfrak{J} defined by, $\omega(x, y) = B(\Delta(x), y)$ for all $x, y \in \mathfrak{J}$. Since \mathfrak{J} is nilpotent (see the Lemma 7.2), then $Ann(\mathfrak{J}) \neq \{0\}$. Let $x \in Ann(\mathfrak{J}) \setminus \{0\}$, then

$$\Delta(x)y = \Delta(xy) - x\Delta(y) = 0, \quad \forall y \in \mathfrak{J}.$$

Hence $\Delta(x) \in Ann(\mathfrak{J})$. It follows that, $\Delta(Ann(\mathfrak{J})) = Ann(\mathfrak{J})$. Let λ be an eigenvalue of $\Delta|_{Ann(\mathfrak{J})}$. Let $b \in Ann(\mathfrak{J})$ such that $\Delta(b) = \lambda b$. Then $0 = \omega(b, b) = \lambda B(b, b)$. Which implies that $B(b, b) = 0$. One pose $\mathcal{I} = \mathbb{K}b$ and \mathcal{I}^\perp the B -orthogonal of \mathcal{I} . Let $a \in \mathfrak{J}$ such that $B(a, a) = 0$, $B(a, b) = 1$ and $\mathfrak{J} = \mathbb{K}a \oplus \mathcal{I}^\perp$. By the Theorem 4.4, the pseudo-euclidean Jordan algebra (\mathfrak{J}, B) is a generalized double extension of the pseudo-euclidean Jordan algebra $(\mathcal{W} = (\mathbb{K}a \oplus \mathbb{K}b)^\perp, T := B|_{\mathcal{W} \times \mathcal{W}})$ by the one dimensional Jordan algebra with zero product by means of the pair $(D, x_0) \in End_s(\mathcal{W}) \times \mathcal{W}$ defined by: $D = p \circ R_a|_{\mathcal{W}}$ and $x_0 = p \circ R_a(a)$, where $p : \mathcal{W} \oplus \mathbb{K}b \longrightarrow \mathcal{W}$ is the projection defined by $p(x + \alpha b) = x$, $\forall x \in \mathcal{W}$, $\alpha \in \mathbb{K}$. The product in $\mathfrak{J} = \mathbb{K}a \oplus \mathcal{W} \oplus \mathbb{K}b$ is denoted by \star and given by:

$$\begin{cases} b \star \mathfrak{J} = \{0\}, \\ a \star a = w_0 = x_0 + kb, \\ x \star y := xy + B(D(x), y)b, \\ a \star x = D(x) + B(x_0, x)b \end{cases} \quad \forall x, y \in \mathcal{W}.$$

where xy is the multiplication of x by y in \mathcal{W} . Let $x \in \mathcal{W}$. We have $B(\Delta(x), b) = -B(x, \Delta(b)) = -\lambda B(x, b) = 0$. Thus $\Delta(\mathcal{W}) \subset \mathcal{I}^\perp = \mathcal{W} \oplus \mathbb{K}b$. We obtain:

$$\Delta(b) = \lambda b, \quad \Delta(x) = \delta(x) + \psi(x)b, \quad \forall x \in \mathcal{W}, \quad \Delta(a) = \alpha a + a_0 + \beta b,$$

where δ is an endomorphism of \mathcal{W} , ψ is a linear form of \mathcal{W} , $\alpha, \beta \in \mathbb{K}$ and $a_0 \in \mathcal{W}$. The fact that Δ is B -antisymmetric and $B(a, a) = 0$ implies, that δ is T -antisymmetric, $\beta = 0$, $\alpha = -\lambda$ and $\psi = -B(a_0, \cdot)$. Thus,

$$\Delta(x) = \delta(x) - B(a_0, x), \quad \forall x \in \mathcal{W} \quad \Delta(a) = \alpha a + a_0.$$

Further, the fact that Δ is an invertible derivation of \mathfrak{J} , implies that δ is an invertible derivation of \mathcal{W} and

$$\delta(x_0) = 2D(a_0) - 2\lambda x_0, \quad k\lambda = B(a_0, x_0), \quad R_{a_0|_{\mathcal{W}}} = \delta D - D\delta + \lambda D.$$

Since δ is an invertible T -antisymmetric derivation of \mathcal{W} , then the bilinear form $\Omega : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{K}$ defined by: $\Omega(x, y) := T(\delta(x), y)$, $\forall x, y \in \mathcal{W}$ is a symplectic form on \mathcal{W} . \square

Corollary 7.4 *Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra. Then $(\mathfrak{J}, B, \omega)$ is obtained from the algebras $\{0\}$ by a finite sequence of symplectic pseudo-euclidean double extensions of symplectic pseudo-euclidean Jordan algebras.*

8 Jordan-Manin Algebras

8.1 Double extensions of Jordan-Manin algebras

Definition 8.1 *A pseudo-euclidean Jordan algebra (\mathfrak{J}, B) is said Jordan-Manin algebra if there exists two totally isotropic subalgebras \mathcal{U}, \mathcal{V} of \mathfrak{J} such that $\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}$. In this case the triple $(\mathfrak{J}, \mathcal{U}, \mathcal{V})$ is said a Jordan-Manin triple.*

Theorem 8.1 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ be a Jordan-Manin algebra. Let $(D, x_0) \in \text{End}_s(\mathfrak{J}) \times \mathfrak{J}$ be an admissible pair of \mathfrak{J} such that $D(\mathcal{V}) \subset \mathcal{V}$ and $x_0 \in \mathcal{V}$. The Jordan algebra obtained by generalized double extension of \mathfrak{J} by means of $(D, x_0, 0)$ is a Jordan-Manin algebra $(\tilde{\mathfrak{J}} = \mathcal{U}' \oplus \mathcal{V}', \tilde{B})$, where $\mathcal{U}' = \mathcal{U} \oplus \mathbb{K}b$ is the direct product of \mathcal{U} and $\mathbb{K}b$ and $\mathcal{V}' = \mathcal{V} \oplus \mathbb{K}a$ is the generalized semi-direct product of \mathcal{V} by $\mathbb{K}a$ by means of $(D|_{\mathcal{V}}, x_0)$.*

Proof. The associatif scalar product on $\tilde{\mathfrak{J}}$ is given by

$$\tilde{B}|_{\tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}}} = B, \quad \tilde{B}(a, b) = 1, \quad \tilde{B}(a, \mathfrak{J}) = \tilde{B}(b, \mathfrak{J}) = \{0\} \text{ et } \tilde{B}(a, a) = \tilde{B}(b, b) = 0.$$

Let $\mathcal{U}' = \mathcal{U} \oplus \mathbb{K}b$ and $\mathcal{V}' = \mathcal{V} \oplus \mathbb{K}a$. It is easy to see that \mathcal{U}' and \mathcal{V}' are two totally isotropic subalgebra of $\tilde{\mathfrak{J}}$. Further, it is clear that $\tilde{\mathfrak{J}} = \mathcal{U}' \oplus \mathcal{V}'$. Thus $\tilde{\mathfrak{J}}$ is a Jordan-Manin algebra. \square

Definition 8.2 *The Jordan-Manin algebra $(\tilde{\mathfrak{J}} = \mathcal{U}' \oplus \mathcal{V}', \tilde{B})$ is said the double extension of the Jordan-Manin algebra $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ by the one dimensional algebra with zero product by means of (D, x_0) .*

Theorem 8.2 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ be a Jordan-Manin algebra. If $\mathcal{U} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$ or $\mathcal{V} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$, then $\tilde{\mathfrak{J}}$ is the double extension of the Jordan-Manin algebra $(\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}', T)$ by the one dimensional Jordan algebra to null product.*

Proof. Suppose that $\mathcal{U} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$. Let $b \in \mathcal{U} \cap \text{Ann}(\mathfrak{J}) \setminus \{0\}$. There exists $a \in \mathcal{V}$ such that $B(a, a) = 0$, $B(a, b) = 1$ and $\mathfrak{J} = (\mathbb{K}b)^\perp \oplus \mathbb{K}a$. By the proof of the Theorem 4.4, \mathfrak{J} is the generalized double extension of $\mathcal{W} = (\mathbb{K}a \oplus \mathbb{K}b)^\perp$ by the one dimensional Jordan algebra with zero product by means of the pair $(D, x_0) \in \text{End}_s(\mathcal{W}) \times \mathcal{W}$ defined by: $D = p \circ R_{a|_{\mathcal{W}}}$ and $x_0 = p \circ R_a(a)$, where $p : \mathcal{W} \oplus \mathbb{K}b \rightarrow \mathcal{W}$ is the projection defined by $p(x + \alpha b) = x$, $\forall x \in \mathcal{W}$, $\alpha \in \mathbb{K}$. The product in $\mathfrak{J} = \mathbb{K}a \oplus \mathcal{W} \oplus \mathbb{K}b$ is given by:

$$\begin{cases} b \star \mathfrak{J} = \{0\}, \\ a \star a = x_0 + kb, \\ x \star y = \beta(x, y) + B(D(x), y)b, \\ a \star x = D(x) + B(x_0, x)b, \end{cases} \quad \forall x, y \in \mathcal{W}.$$

where β is a Jordan product on \mathcal{W} and $k \in \mathbb{K}$. We have $\mathcal{I} = \mathbb{K}b \subset \mathcal{U}$, thus $\mathcal{U}^\perp \subset \mathcal{I}^\perp$. Since $\mathcal{U} = \mathcal{U}^\perp$, then $\mathcal{U} \subset \mathcal{I}^\perp$. It follows that, $\mathcal{I}^\perp = \mathcal{U} \oplus (\mathcal{V} \cap \mathcal{I}^\perp)$. Further, $\mathcal{V} \cap \mathcal{I}^\perp \subset \mathcal{W}$ because if $x \in \mathcal{V} \cap \mathcal{I}^\perp$, then $B(a, x) = B(b, x) = 0$. Hence, $\mathcal{W} = (\mathcal{U} \cap \mathcal{W}) \oplus (\mathcal{V} \cap \mathcal{I}^\perp)$. One poses $\mathcal{U}' = \mathcal{U} \cap \mathcal{W}$ and $\mathcal{V}' = \mathcal{V} \cap \mathcal{I}^\perp$. It is clear that \mathcal{V}' is an ideal of \mathcal{V} . Thus, for all $v \in \mathcal{V}'$, $a \star v \in \mathcal{V}'$. Hence, $D(v) + B(v, x_0)b \in \mathcal{V}' \forall v \in \mathcal{V}'$. It follows that, $B(v, x_0) = 0$ and $D(v) \in \mathcal{V}'$, $\forall v \in \mathcal{V}'$. Hence, $x_0 \in (\mathcal{V}')^\perp \cap \mathcal{W} = \mathcal{V}'$ and $D(\mathcal{V}') \subset \mathcal{V}'$. Further, \mathcal{U}' and \mathcal{V}' are two subalgebra of \mathcal{W} . In fact, let $x, y \in \mathcal{U}'$, then $\alpha(x, y) = x \star y - B(D(x), y)b \in \mathcal{U}'$ because $b \in \mathcal{U}$. Since $\alpha(x, y) \in \mathcal{W}$, then $\alpha(x, y) \in \mathcal{U}'$. The fact that \mathcal{V}' is stable under D and that $\mathcal{V}' \subset \mathcal{V}$ is totally isotropic implies that \mathcal{V}' is a subalgebra of \mathcal{W} . Further, it is clear that \mathcal{U}' and \mathcal{V}' are totally isotropic. It follows that, (D, x_0) is an admissible pair of $\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}'$ such that $x_0 \in \mathcal{V}'$ and $D(\mathcal{V}') \subset \mathcal{V}'$. Hence $\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}$ is a double extension of the Jordan-Manin algebra $\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}'$ by means of (D, x_0) . For the other case $\mathcal{V} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$, the proof is similar. \square

Lemma 8.3 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ be a Jordan-Manin algebra. If \mathfrak{J} is nilpotent, then $\mathcal{U} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$ and $\mathcal{V} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$.*

Proof. Suppose that $\mathcal{U} \cap \text{Ann}(\mathfrak{J}) = \{0\}$. One poses $L_0 = \mathcal{V} \text{Ann}(\mathcal{U}) = \{xy; x \in \mathcal{V} \text{ and } y \in \text{Ann}(\mathcal{U})\}$. It is easy to check that $L_0 \neq \{0\}$ and include in \mathcal{U} . Consider \mathcal{I}_0 the ideal of \mathcal{U} generated by L_0 . Then \mathcal{I}_0 is a non-zero ideal of the nilpotent algebra \mathcal{U} . Thus, $\mathcal{I}_0 \cap \text{Ann}(\mathcal{U}) \neq \{0\}$. Now, let $L_1 = \mathcal{V}(\mathcal{I}_0 \cap \text{Ann}(\mathcal{U}))$. It is clear that $L_1 \neq \{0\}$ and include in \mathcal{U} . Let \mathcal{I}_1 be the ideal of \mathcal{U} generated by L_1 . By the same argument, we show that $\mathcal{I}_1 \cap \text{Ann}(\mathcal{U}) \neq \{0\}$. We repeat this process and we obtain the subsets sequence $(L_n)_{n \in \mathbb{N}}$ of \mathcal{U} and the ideals sequence $(\mathcal{I}_n)_{n \in \mathbb{N}}$ of \mathcal{U} defined by $L_0 = \mathcal{V} \text{Ann}(\mathcal{U})$, $L_n = \mathcal{V}(\mathcal{I}_{n-1} \cap \text{Ann}(\mathcal{U}))$, and \mathcal{I}_n is an ideal generated by $L_n \forall n \in \mathbb{N}$. It is easy to see that for all $n \in \mathbb{N}$, \mathcal{I}_n is a non-zero ideal of \mathcal{U} and that for all $n \in \mathbb{N}$ $\mathcal{I}_n \subset C^{n+1}(\mathfrak{J})$, which contradict the fact that \mathfrak{J} is nilpotent. For the other case $\mathcal{V} \cap \text{Ann}(\mathfrak{J}) \neq \{0\}$ the proof is similar. \square

The following Corollary is a consequence of the Theorem 8.2 and the Lemma 8.3.

Corollary 8.4 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ be a Jordan-Manin algebra. If \mathfrak{J} is nilpotent, then \mathfrak{J} is the double extension of a Jordan-Manin algebra (\mathcal{W}, T) by the one dimensional algebra with zero product.*

Corollary 8.5 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ be a nilpotent Jordan-Manin algebra. Then, \mathfrak{J} is obtained from the two dimension Jordan-Manin algebra by a finite sequence of Manin double extensions of nilpotent Jordan-Manin algebras.*

8.2 Symplectic Jordan-Manin algebras

Definition 8.3 *Jordan-Manin $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ is said symplectic Jordan-Manin algebra if there exists a symplectic structure ω on \mathfrak{J} satisfying $\omega(\mathcal{U}, \mathcal{U}) = \omega(\mathcal{V}, \mathcal{V}) = \{0\}$.*

The study of symplectic Jordan-Manin algebra is interesting because every pseudo-euclidean symplectic Jordan algebras is a symplectic Jordan-Manin algebra (see. Proposition 8.7). This result, will be proved in the Proposition 8.7. The proof of the Proposition 8.7 is based on the following Lemma (Proposition 2. (ii). p. 8 of [5]).

Lemma 8.6 [5]) *Let \mathcal{E} , \mathcal{E}' , \mathcal{F} be a vector spaces and S be a set. Let $r : S \longrightarrow \text{End}(\mathcal{E})$, $r' : S \longrightarrow \text{End}(\mathcal{E}')$ and $q : S \longrightarrow \text{End}(\mathcal{F})$. For all $\lambda : S \longrightarrow \mathbb{K}$, one pose:*

$$\mathcal{E}(\lambda) = \{x \in \mathcal{E}; \forall s \in S, (r(s) - \lambda(s))^n(x) = 0, \text{ for } n \in \mathbb{N}^*\}.$$

Let $\sigma : \mathcal{E} \times \mathcal{E}' \longrightarrow \mathcal{F}$ be a bilinear map such that:

$$q(s)\sigma(x, x') = \sigma(r(s)x, x') + \sigma(x, r'(s)x'), \quad \forall s \in S, x \in \mathcal{E}, x' \in \mathcal{E}'.$$

Then, For all maps $\lambda, \mu : S \longrightarrow \mathbb{K}$, we have $\sigma(\mathcal{E}(\lambda), \mathcal{E}'(\mu)) \subset \mathcal{F}(\lambda + \mu)$.

Proposition 8.7 *Let $(\mathfrak{J}, B, \omega)$ be a symplectic pseudo-euclidean Jordan algebra over an algebraically closed fields \mathbb{K} . Then, there exists two subalgebras \mathcal{U} and \mathcal{V} of \mathfrak{J} such that $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ is a symplectic Jordan-Manin algebra.*

Proof. Let D be the derivation defined by $\omega(x, y) = B(D(x), y)$ for all $x, y \in \mathfrak{J}$. Let $Sp(D)$ the set of the eigenvalues of D . One pose

$$\begin{aligned} Sp^+ &= \{\lambda \in Sp(D); \text{Re}(\lambda) > 0\} \cup \{\lambda \in Sp(D); \text{Re}(\lambda) = 0 \text{ and } \text{Im}(\lambda) > 0\} \\ &= (Sp(D) \cup i\mathbb{R}_+) \quad \text{and} \quad Sp^- = \{\lambda \in Sp(D); -\lambda \in Sp^+\}. \end{aligned}$$

It is clear that $Sp(D) = Sp^+ \cup Sp^-$ and $Sp^+ \cap Sp^- = \emptyset$. Now, one poses

$$\mathcal{U} = \sum_{\lambda \in Sp^+} \mathfrak{J}(\lambda), \quad \mathcal{V} = \sum_{\lambda \in Sp^-} \mathfrak{J}(\lambda),$$

where $\mathfrak{J}(\lambda) = \{x \in \mathfrak{J}; (D - \lambda \text{id}_{\mathfrak{J}})^{\dim(\mathfrak{J})}(x) = 0\}$. \mathcal{U} and \mathcal{V} are two subalgebras of \mathfrak{J} . In fact, if we consider in the Proposition 8.6 $\mathcal{E} = \mathcal{E}' = \mathcal{F} = S = \mathfrak{J}$ and we define

$$\begin{aligned} \sigma : \mathfrak{J} \times \mathfrak{J} &\longrightarrow \mathfrak{J}, & r : \mathfrak{J} &\longrightarrow \text{End}(\mathfrak{J}) \quad \text{and} \quad r' = q = r. \\ (x, y) &\longmapsto xy, & x &\longmapsto D \end{aligned}$$

Since D is a derivation, then

$$q(x)\sigma(y, z) = \sigma(r(x)y, z) + \sigma(y, r'(x)z), \quad \forall x, y, z \in \mathfrak{J}.$$

Thus, by the Proposition 8.6, we have $\sigma(\mathfrak{J}(\lambda), \mathfrak{J}(\mu)) \subset \mathfrak{J}(\lambda + \mu)$, $\forall \lambda, \mu \in \mathbb{K}$. Thus $\mathfrak{J}(\lambda)\mathfrak{J}(\mu) \subset \mathfrak{J}(\lambda + \mu)$, $\forall \lambda, \mu \in \mathbb{K}$. Thus, \mathcal{U} and \mathcal{V} are two subalgebras of \mathfrak{J} . Now, we use the same proposition, to prove that \mathcal{U} and \mathcal{V} are totally isotropic. One pose $\mathcal{E} = \mathcal{E}' = S = \mathfrak{J}$, $\mathcal{F} = \mathbb{K}$ and define

$$\begin{aligned} r = r' : \mathfrak{J} &\longrightarrow \text{End}(\mathfrak{J}) \quad \text{et} \quad q : \mathfrak{J} \longrightarrow \text{End}(\mathbb{K}) \\ x &\longmapsto D, & x &\longmapsto 0, \end{aligned}$$

Since D is B -antisymmetric, then

$$q(x)B(y, z) = B(r(x)y, z) + B(y, r(x)z) = 0, \quad \forall x, y, z \in \mathfrak{J}.$$

It follows that, by the Proposition 8.6, we have $B(\mathfrak{J}(\lambda), \mathfrak{J}(\mu)) \subset \mathbb{K}(\lambda + \mu)$.

However, $\mathbb{K}(\lambda) := \{\alpha \in \mathbb{K}; (q(x) - \lambda)^n(\alpha) = 0; \text{ where } n \in \mathbb{N}^*\} = \{\alpha \in \mathbb{K}; \alpha\lambda = 0\}$.

It follows that, for all $\lambda \neq 0$, we have $\mathbb{K}(\lambda) = \{0\}$. Consequently, since the sum of two elements of Sp^+ (resp. Sp^-) is not zero, then \mathcal{U} and \mathcal{V} are totally isotropic (i.e. $B(\mathcal{U}, \mathcal{U}) = \mathcal{U}(\mathcal{V}, \mathcal{V}) = 0$). It follows that, \mathcal{U} and \mathcal{V} are two D -stable totally isotropic subalgebras of \mathfrak{J} which satisfies $\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}$. Hence, $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ is a symplectic Jordan-Manin algebra. \square

Theorem 8.8 *Let $(\mathfrak{J}_1 = \mathcal{U}_1 \oplus \mathcal{V}_1, B_1, \omega_1)$ be a symplectic Jordan-Manin algebra. Let $a_0 \in \mathcal{V}_1$, $\lambda \in \mathbb{K}$ and δ be the invertible derivation of \mathfrak{J}_1 defined by*

$$\omega_1(x, y) = B_1(\delta(x), y), \quad \forall x, y \in \mathfrak{J}_1.$$

Let $(D, x_0) \in \text{End}_s(\mathfrak{J}_1) \times \mathfrak{J}_1$ be an admissible pair such that:

$$D(\mathcal{V}) \subset \mathcal{V}, \quad x_0 \in \mathcal{V} \quad D(a_0) = \lambda x_0 + \frac{1}{2}\delta(x_0) \quad R_{a_0} = \delta D - D\delta + \lambda D.$$

Then, the symplectic pseudo-euclidean double extension $(\mathfrak{J}, B, \omega)$ of $(\mathfrak{J}_1 = \mathcal{U}_1 \oplus \mathcal{V}_1, B_1, \omega_1)$ by means of (D, x_0, a_0, λ) is a symplectic Jordan-Manin algebra.

Proof. By the Theorem 7.1, the Jordan algebra $(\mathfrak{J}, B, \omega)$ is symplectic and pseudo-euclidean, where $\omega(x, y) = B(\Delta(x), y)$, $\forall x, y \in \mathfrak{J}$, and Δ is the B -antisymmetric invertible derivation of (\mathfrak{J}, B) defined by:

$$\Delta(b) = \lambda b, \quad \Delta(x) = \delta(x) - B_1(a_0, x)b, \quad \Delta(a) = a_0 - \lambda a, \quad \forall x \in \mathfrak{J}_1.$$

Further, by the Theorem 8.1, The algebra $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ where $\mathcal{U} = \mathcal{U}_1 \oplus \mathbb{K}b$, $\mathcal{V} = \mathcal{V}_1 \oplus \mathbb{K}a$ est is a Jordan-Manin algebra. It remains to be shown that $\Delta(\mathcal{U}) = \mathcal{U}$ and $\Delta(\mathcal{V}) = \mathcal{V}$. In fact, let $u = u_1 + \alpha b \in \mathcal{U}$.

$$\Delta(u) = \Delta(u_1) + \alpha\Delta(b) = \delta(u_1) + (\alpha\lambda b - B_1(a_0, u_1))b \in \mathcal{U}.$$

Let $v = v_1 + \beta a \in \mathcal{V}$.

$$\Delta(v) = \Delta(v_1) + \beta\Delta(a) = \delta(v_1) - \beta a_0 - \beta\lambda a \in \mathcal{V}. \square$$

Definition 8.4 *The algebra $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ constructed above is said the symplectic-Manin double extension of the symplectic Jordan-Manin algebra $(\mathfrak{J}_1 = \mathcal{U}_1 \oplus \mathcal{V}_1, B_1, \omega_1)$ by means of (D, x_0, a_0, λ) .*

Theorem 8.9 *Let $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ be a symplectic Jordan-Manin algebra. Let Δ be the B -antisymmetric invertible derivation defined by $\omega(x, y) = B(\Delta(x), y)$, $\forall x, y \in \mathfrak{J}$. If Δ admits an eigenvector $c \in \text{Ann}(\mathfrak{J}) \cap \mathcal{U} + \text{Ann}(\mathfrak{J}) \cap \mathcal{V}$, then $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ is the symplectic-Manin double extension of the symplectic Jordan-Manin algebra $(\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}', T, \Omega)$ by means of $(D, x_0, a_0, \lambda) \in \text{End}(\mathcal{W}) \times \mathcal{V}' \times \mathcal{V}' \times \mathbb{K}$ satisfying $D(\mathcal{V}') \subset \mathcal{V}'$.*

Proof. Let $c \in \text{Ann}(\mathfrak{J}) \cap \mathcal{U} + \text{Ann}(\mathfrak{J}) \cap \mathcal{V}$ such that $c \neq 0$ and $\Delta(c) = \lambda c$ where $\lambda \in \mathbb{K}$. One poses $c = b + b'$ where $b \in \text{Ann}(\mathfrak{J}) \cap \mathcal{U}$ and $b' \in \text{Ann}(\mathfrak{J}) \cap \mathcal{V}$. Since \mathcal{U} and \mathcal{V} are stable by Δ , then $\Delta(b) = \lambda b$ and $\Delta(b') = \lambda b'$. Since $c \neq 0$, then $b \neq 0$ or $b' \neq 0$. Suppose that $b \neq 0$ (if $b' \neq 0$ the same demonstration is remade). Since $b \in \text{Ann}(\mathfrak{J}) \cap \mathcal{U} \setminus \{0\}$, then there exists $a \in \mathcal{V}$ such that $B(a, a) = 0$, $B(a, b) = 1$ and $\mathfrak{J} = \mathbb{K}a \oplus (\mathbb{K}b)^\perp$. By the Theorem 7.3 $(\mathfrak{J}, B, \omega)$ is the symplectic pseudo-euclidean double extension of $(\mathcal{W} = (\mathbb{K}a \oplus \mathbb{K}b)^\perp, T, \delta)$ by means of $(D, x_0, a_0, \lambda) \in \text{End}(\mathcal{W}) \times \mathcal{W} \times \mathcal{W} \times \mathbb{K}$, where $a_0 = p' \circ \Delta(a)$ and (D, x_0) an admissible pair defined by: $D = p \circ R_{a|_{\mathcal{W}}}$ and $x_0 = p \circ R_a(a)$, where p (resp. p') is the projection of $\mathcal{W} \oplus \mathbb{K}b$ (resp. $\mathcal{W} \oplus \mathbb{K}a$) in \mathcal{W} defined by $p(x + ab) = x$ (resp. $p'(x + \alpha a) = x$), $\forall x \in \mathcal{W}, \alpha \in \mathbb{K}$.

Recall that, $\delta = p \circ \Delta|_{\mathcal{W}}$. On the other hand, by the Theorem 8.2, $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B)$ is the Manin double extension of the Jordan-Manin algebra $(\mathcal{W} = (\mathbb{K}u \oplus \mathbb{K}w)^\perp, T)$ by the one dimensional algebra with zero product by means of the admissible pair (D, x_0) where $D(\mathcal{V}') \subset \mathcal{V}'$ and $x_0 \in \mathcal{V}'$. Recall that, $\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}'$ where $\mathcal{U}' = \mathcal{U} \cap \mathcal{W}$ and $\mathcal{V}' = \mathcal{V} \cap (\mathbb{K}u)^\perp$. Since \mathcal{U} and \mathcal{V} are stable by Δ , then \mathcal{U}' et \mathcal{V}' are stable by δ too. Moreover, $\Delta(a) = a_0 - \lambda a$. Thus, $a_0 - \lambda a \in \mathcal{V}$ because $a \in \mathcal{V}$. Hence, $a_0 \in \mathcal{V}$. On the other hand,

$$B(\Delta(a), b) = -B(a, \Delta(b)) = -\lambda B(a, b) = B(-\lambda a, b).$$

Thus, $B(a_0, b) = 0$. It follows that $a_0 \in (\mathbb{K}b)^\perp$. Hence, $a_0 \in \mathcal{V}'$. We conclude by the Theorem 8.8 that, $(\mathfrak{J} = \mathcal{U} \oplus \mathcal{V}, B, \omega)$ is the symplectic-Manin double extension of a symplectic Jordan-Manin algebra $(\mathcal{W} = \mathcal{U}' \oplus \mathcal{V}', T, \Omega)$ by means of (D, x_0, a_0, λ) . \square

9 Characterizations of semisimple Jordan algebras

In this section we are going to give some new characterizations of semisimple Jordan algebras among the pseudo-euclidean Jordan algebras.

9.1 Characterization via the representations of Jordan algebra

Recall that, by Theorem of Albert ([14], Theorem 4.5), a Jordan algebra \mathfrak{J} is semisimple if and only if its Albert form $\mathfrak{A} : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by: $\mathfrak{A}(x, y) = \text{tr}(R_{xy})$, $\forall x, y \in \mathfrak{J}$ is non-degenerate. Now, we are going to give an answer to the following question: Let \mathfrak{J} be a Jordan algebra and $\pi : \mathfrak{J} \rightarrow \text{End}(\mathcal{V})$ be a representation of finite dimension of \mathfrak{J} . Consider the bilinear form $B_\pi : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by: $B_\pi(x, y) = \text{tr}(\pi(xy))$, $\forall x, y \in \mathfrak{J}$ (we say that B_π is the bilinear form of \mathfrak{J} associate to representation π). What about the structure of Jordan algebra \mathfrak{J} such that B_π is non-degenerate?

Proposition 9.1 *Let \mathfrak{J} be a Jordan algebra and $\pi : \mathfrak{J} \rightarrow \text{End}(\mathcal{V})$ be a finite-dimensional representation of \mathfrak{J} . Then, the bilinear form $B : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by:*

$$B_\pi(x, y) = \text{tr}(\pi(xy)), \forall x, y \in \mathfrak{J},$$

is symmetric and associative.

Proof. Let $x, y, z \in \mathfrak{J}$. $B_\pi(x, y) = \text{tr}(\pi(xy)) = \text{tr}(\pi(yx)) = B_\pi(y, x)$, then B_π is symmetric. Now,

$$B_\pi(xy, z) - B_\pi(x, yz) = \text{tr}(\pi((xy)z - x(yz))) = \text{tr}(\pi((x, y, z))).$$

By Corollary 3.3, we have $\pi((x, y, z)) = [\pi(y), [\pi(x), \pi(z)]]$. It follows that $\text{tr}(\pi((x, y, z))) = \text{tr}([\pi(y), [\pi(x), \pi(z)]]) = 0$, so $B_\pi(xy, z) - B_\pi(x, yz) = 0$. Consequently, B is associative. \square

The following Lemma is the first step to study the non-degeneracy of B_π .

Lemma 9.2 *Let \mathfrak{J} be a Jordan Algebra. Then, \mathfrak{J} is nilpotent if and only if $\pi(x)$ is nilpotent, for all finite-dimensional representation π of \mathfrak{J} and for all $x \in \mathfrak{J}$.*

Proof. Let \mathfrak{J} be a Jordan Algebra and let $\pi : \mathfrak{J} \rightarrow \text{End}(\mathcal{V})$ be a finite-dimensional representation of \mathfrak{J} . Then the vector space $\tilde{\mathfrak{J}} = \mathfrak{J} \oplus \mathcal{V}$ with the following product

$$(x + v)(y + w) = xy + \pi(x)w + \pi(y)v, \forall x, y \in \mathfrak{J}, v, w \in \mathcal{V},$$

is a Jordan Algebra. It is clear that \mathcal{V} is an ideal of $\tilde{\mathfrak{J}}$ such that $vw = 0, \forall v, w \in \mathcal{V}$, then \mathcal{V} is an ideal nilpotent of $\tilde{\mathfrak{J}}$. Consequently, \mathcal{V} is contained in the radical $Ra(\tilde{\mathfrak{J}})$ of $\tilde{\mathfrak{J}}$.

Suppose that $\tilde{\mathfrak{J}}$ is not nilpotent, then, $\tilde{\mathfrak{J}} = S \oplus Ra(\tilde{\mathfrak{J}})$ where $S \neq \{0\}$ is a semi-simple subalgebra of $\tilde{\mathfrak{J}}$. Consequently, the Jordan algebras $\tilde{\mathfrak{J}}/\mathcal{V}$ and $S \oplus Ra(\tilde{\mathfrak{J}}/\mathcal{V})$ are isomorphic. Let $\phi : \tilde{\mathfrak{J}} \rightarrow \tilde{\mathfrak{J}}/\mathcal{V}$ be the canonical surjection. Since $S \cap \mathcal{V} = \{0\}$, then $\phi : S \rightarrow \phi(S)$ is an isomorphism of Jordan Algebras. It follows that $\phi(S)$ is semi-simple. Moreover $\tilde{\mathfrak{J}}/\mathcal{V}$ is nilpotent because \mathfrak{J} is nilpotent, then $\phi(S)$ is also nilpotent, so $\phi(S) = \{0\}$. Therefore, $S \subset \mathcal{V}$, which contradicts $S \neq \{0\}$. We conclude that $\tilde{\mathfrak{J}}$ is nilpotent. consequently, $\pi(x)$ is nilpotent for all $x \in \mathfrak{J}$.

Conversely, we have, in particular, R_x is nilpotent for all $x \in \mathfrak{J}$ where R is the adjoint representation of \mathfrak{J} , so \mathfrak{J} est nilpotente ([14], Theorem 4.3). \square

Theorem 9.3 *Let \mathfrak{J} be a Jordan algebra. Then the following assertions are equivalent:*

- (i) \mathfrak{J} is semi-simple;
- (ii) There exists a finite-dimensional representation $\pi : \mathfrak{J} \times \mathfrak{J} \rightarrow \text{End}(\mathcal{V})$ of \mathfrak{J} such that the bilinear form $B : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by $B(x, y) = \text{tr}(\pi(xy)), \forall x, y \in \mathfrak{J}$, is non-degenerate.

Proof.

If \mathfrak{J} is semi-simple, then by Theorem of Albert ([14], Theorem 4.5), the bilinear form $B : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{K}$ defined by $B(x, y) = \text{tr}(R_{xy}), \forall x, y \in \mathfrak{J}$, is non-degenerate.

Conversely, suppose that there exists a representation $\pi : \mathfrak{J} \times \mathfrak{J} \rightarrow \text{End}(\mathcal{V})$ such that the bilinear form B de \mathfrak{J} defined by: $B(x, y) = \text{tr}(\pi(xy)), \forall x, y \in \mathfrak{J}$, is non-degenerate. Let $x \in \mathfrak{J}$ and $r \in Ra(\mathfrak{J})$, where $Ra(\mathfrak{J})$ is the radical de \mathfrak{J} . The previous Lemma implie that $\pi(rx)$ is nilpotent, hence $B(r, x) = \text{tr}(\pi(rx)) = 0$. The fact that B is non-degenerate implies that $r = 0$. Which proves that $Ra(\mathfrak{J}) = \{0\}$. We conclude that \mathfrak{J} is semi-simple. \square

9.2 Opertors of Casimir type of pseudo-euclidean Jordan algebra

Now, we are going to give a characterization of semi-simple Jordan algebra by using an operator called opertor of Casimir type. Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra of dimension n . We consider $\beta = \{e_1, \dots, e_n\}, \beta' = \{e'_1, \dots, e'_n\}$, two basis of \mathfrak{J} such that $B(e_i, e'_j) = \delta_{ij}, \forall i, j \in \{1, \dots, n\}$ where δ_{ij} is Kronecker's symbol. Let us consider $c = \sum_{i=1}^n e_i e'_i$, then the operator $R_c : \mathfrak{J} \rightarrow \mathfrak{J}$ is defined by: $R_c(x) = \sum_{i=1}^n x(e_i e'_i), \forall x \in \mathfrak{J}$. If $\Gamma = \{f_1, \dots, f_n\}, \Gamma' = \{f'_1, \dots, f'_n\}$ two other basis of \mathfrak{J} such that $B(f_i, f'_j) = \delta_{ij}, \forall i, j \in \{1, \dots, n\}$, in the following Lemma, we prove in particular that $R_c = R_{c'}$, where $c' = \sum_{i=1}^n f_i f'_i$.

Lemma 9.4 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra.

(1) For all $x, y \in \mathfrak{J}$, we have $\mathfrak{A}(x, y) = B(R_c(x), y)$ (where \mathfrak{A} is the Albert form of \mathfrak{J});

(2) $R_c = R_{c'}$;

(3) For all $x \in \mathfrak{J}$, on a $R_c \circ R_x = R_x \circ R_c$.

Proof. (1) Let $x, y \in \mathfrak{J}$, $B(R_c(x), y) = \sum_{i=1}^n B(R_{e_i e'_i}(x), y) = \sum_{i=1}^n B(e_i e'_i, xy) = \sum_{i=1}^n B((xy)e_i, e'_i) = \text{tr} R_{xy} = \mathfrak{A}(x, y)$.

(2) By (1), it is clear that $B(R_{c'}(x), y) = \mathfrak{A}(x, y) = B(R_c(x), y)$, $\forall x, y \in \mathfrak{J}$. The fact that B is non-degenerate implies that $R_c = R_{c'}$.

(3) Let $x, y, z \in \mathfrak{J}$, $B(R_c R_x(y), z) - B(R_x R_c(y), z) = B(R_c(yx), z) - B(R_c(y), xz) = \mathfrak{A}(yx, z) - \mathfrak{A}(y, xz) = 0$. The fact that B is non-degenerate implies that $R_c R_x(y) = R_x R_c(y)$. \square

Definition 9.1 R_c is called the operator of Casimir type of the pseudo-euclidean Jordan algebra (\mathfrak{J}, B) .

Proposition 9.5 \mathfrak{J} is semi-simple if and only if R_c is invertible.

Proof. Let $x, y \in \mathfrak{J}$, by Lemma 9.4 we have $\mathfrak{A}(x, y) = B(R_c(x), y)$. Consequently, the Albert form \mathfrak{A} is non-degenerate if and only if the operator R_c est invertible. Then we conclude, by Theorem 4.5 of [14], that \mathfrak{J} is semi-simple if and only if R_c is invertible. \square

Proposition 9.6 Let (\mathfrak{J}, B) be an B -irreducible pseudo-euclidean Jordan algebra. Then the operator of Casimir Type R_c of (\mathfrak{J}, B) is either nilpotent or invertible.

Proof. It is known that there exists $n \in \mathbb{N}$ such that $\mathfrak{J} = \ker(R_c)^n \oplus \text{Im}(R_c)^n$. It is easy to see that $B(\ker(R_c)^n, \text{Im}(R_c)^n) = \{0\}$, so $(\ker(R_c)^n)^\perp = \text{Im}(R_c)^n$. Moreover, by Lemma 9.4, $\ker(R_c)^n$ is an ideal of \mathfrak{J} . Consequently, $\ker(R_c)^n = \{0\}$ or $\ker(R_c)^n = \mathfrak{J}$. We conclude that R_c is either nilpotent or invertible. \square

Proposition 9.7 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and R_c be the operator of Casimir Type of (\mathfrak{J}, B) . Then R_c is nilpotent if and only if \mathfrak{J} is a Jordan algebra without nonzero semisimple ideal.

Proof. Suppose that \mathfrak{J} contain a semisimple \mathcal{I} . By Lemma 2.2, \mathcal{I} and \mathcal{I}^\perp are non-degenerate ideals of \mathfrak{J} and $\mathfrak{J} = \mathcal{I} \oplus \mathcal{I}^\perp$. It follows that $R_{c|_{\mathcal{I}}}$ is the operator of Casimir type of $(\mathcal{I}, B|_{\mathcal{I} \times \mathcal{I}})$ where R_c is the operator of Casimir type of (\mathfrak{J}, B) . If $\mathcal{I} \neq \{0\}$, then $R_{c|_{\mathcal{I}}}$ is invertible, so R_c is not nilpotent.

Conversely, suppose that \mathfrak{J} is without nonzero semisimple ideal then \mathfrak{J} is an orthogonal direct sum of irreducible non-degenerate ideals \mathfrak{J}_k , $k \in \{1, \dots, p\}$. Let us set R_k the operator of Casimir type of the pseudo-euclidean Jordan algebra $(\mathfrak{J}_k, B|_{\mathfrak{J}_k \times \mathfrak{J}_k})$, where $k \in \{1, \dots, p\}$. It is clear that $R_{c|_{\mathfrak{J}_k}} = R_k$, for all $k \in \{1, \dots, p\}$. Since, by Propositions 9.5 and 9.6, R_k is nilpotent, for all $k \in \{1, \dots, p\}$, then R_c is nilpotent. \square

Corollary 9.8 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan Algebra and R_c be its operator of Casimir type. Let S be the largest semisimple ideal of \mathfrak{J} . Then $\mathfrak{J} = S \oplus S^\perp$ is the Fitting decomposition of \mathfrak{J} relative to R_c .

Proof. By Lemma 2.2 S and S^\perp are non-degenerate ideals of \mathfrak{J} and $\mathfrak{J} = S \oplus S^\perp$. It follows that the operator of Casimir type of $(S, B|_{S \times S})$ (resp. $(S^\perp, B|_{S^\perp \times S^\perp})$) is $T := (R_c)|_S$ (resp. $L := (R_c)|_{S^\perp}$). Moreover, T is invertible because S is semisimple and L is nilpotent because S^\perp is a Jordan algebra without nonzero semisimple ideal. \square

9.3 Characterization by means of the index of Jordan algebra

In this subsection, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. We denote by $\mathfrak{F}(\mathfrak{J})$ the vector space of all associative symmetric bilinear forms on \mathfrak{J} and by $\mathfrak{B}(\mathfrak{J})$ the subspace of $\mathfrak{F}(\mathfrak{J})$ spanned by the set of the associative scalar products on \mathfrak{J} .

Lemma 9.9 $\mathfrak{B}(\mathfrak{J}) = \mathfrak{F}(\mathfrak{J})$.

Proof. Let B be an associative scalar product on \mathfrak{J} and $\varphi \in \mathfrak{F}(\mathfrak{J})$. Let T be a basis of \mathfrak{J} , $M(B)$ and $M(\varphi)$ the associated matrices of B and φ in T . Since B is non-degenerate, then there exists f , a linear map of \mathfrak{J} into itself, such that $\varphi(x, y) = B(f(x), y)$, $\forall x, y \in \mathfrak{J}$. Let $M(f)$ be the matrix of f in T , so $M(\varphi) = {}^tM(f)M(B)$. Consider the polynomial $P(X) = \det(M(\varphi) - XM(B)) \in \mathbb{C}[X]$, then $P(X) = \det(M(B))\det(M(f) - XI_n)$ where n is the dimension of \mathfrak{J} . Consequently, $P(X)$ is a non-zero polynomial; so there exists $\lambda \in \mathbb{K}$ such that $P(\lambda) \neq 0$. Therefore $\varphi - \lambda B$ is non-degenerate, it follows that $\varphi = (\varphi - \lambda B) + \lambda B \in \mathfrak{B}(\mathfrak{J})$. We conclude that $\mathfrak{F}(\mathfrak{J}) = \mathfrak{B}(\mathfrak{J})$. \square

Definition 9.2 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. The dimension of $\mathfrak{B}(\mathfrak{J})$ is called the index of \mathfrak{J} and will be denoted by $ind(\mathfrak{J})$.

Proposition 9.10 If \mathfrak{J} is a simple Jordan algebra over \mathbb{C} , then $ind(\mathfrak{J}) = 1$.

Proof. Since \mathfrak{J} is simple then, by Theorem of Albert ([14], Theorem 4.5), the Albert form \mathfrak{A} of \mathfrak{J} is an associative scalar product on \mathfrak{J} . Now, let B be an associative scalar product on \mathfrak{J} . Then there exists an endomorphism D of \mathfrak{J} such that $B(x, y) = \mathfrak{A}(D(x), y)$, $\forall x, y \in \mathfrak{J}$. If $\lambda \in \mathbb{K}$, the bilinear form B' of \mathfrak{J} defined by: $B'(x, y) = \mathfrak{A}(D(x) - \lambda x, y)$, $\forall x, y \in \mathfrak{J}$, is associative. If λ is an eigenvalue of D and v is an eigenvector for λ , then $B'(v, y) = 0$, $\forall y \in \mathfrak{J}$ and $rad(B') := \{x \in \mathfrak{J}; B'(x, y) = 0, \forall y \in \mathfrak{J}\} \neq \{0\}$. Moreover, $rad(B')$ is an ideal of \mathfrak{J} , it follows that $rad(B') = \mathfrak{J}$. Therefore $D - \lambda id_{\mathfrak{J}} = 0$, so $B(x, y) = \lambda \mathfrak{A}(x, y)$, $\forall x, y \in \mathfrak{J}$. \square

Proposition 9.11 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra. If $ind(\mathfrak{J}) = 1$, then \mathfrak{J} is either a simple Jordan algebra or \mathfrak{J} is the one-dimensional algebra with zero product.

Proof. If $ind(\mathfrak{J}) = 1$, then \mathfrak{J} is irreducible. Assume that \mathfrak{J} is neither simple nor the one-dimensional algebra with zero product. Then, by Corollary 4.3 and Theorem 4.4, \mathfrak{J} is either a double extension of a pseudo-euclidean Jordan algebra by a simple Jordan algebra or a generalized double extension of a pseudo-euclidean Jordan algebra by the one-dimensional algebra with zero product. If $\mathfrak{J}^2 = \mathfrak{J}$ (i.e. $Ann(\mathfrak{J}) = \{0\}$), then \mathfrak{J} is a double extension of a pseudo-euclidean Jordan algebra (\mathcal{W}, T) by a simple Jordan algebra \mathcal{S} . Moreover, by Theorem 3.8, if γ is an associative symmetric bilinear form on \mathcal{S} , then the bilinear form $\tilde{\gamma}$ on $\mathcal{S} \oplus \mathcal{W} \oplus \mathcal{S}^*$ defined by: $\tilde{\gamma}(x + y + f, x' + y' + f') = \gamma(x, x') + B(y, y') + f(x') + f'(x)$, $\forall (x, y, f), (x', y', f') \in \mathcal{S} \times \mathcal{W} \times \mathcal{S}^*$, is an associative scalar product on \mathfrak{J} . Let us consider $\gamma_1 = \mathfrak{A}$ the Albert form of \mathcal{S} and the bilinear form $\gamma_2 = 0$ on \mathcal{S} . It is clear that $\tilde{\gamma}_1$ et $\tilde{\gamma}_2$ are two linearly independant elements of $\mathfrak{B}(\mathfrak{J})$, so $ind(\mathfrak{J}) \geq 2$ which contradicts the fact that $ind(\mathfrak{J}) = 1$. Now, if $Ann(\mathfrak{J}) \neq \{0\}$, then there exists $b \in Ann(\mathfrak{J})$ such that $b \neq 0$. Since \mathfrak{J} is irreducible and \mathfrak{J} is not the one-dimensional algebra with zero product, then $B(b, b) = 0$. By Theorem 4.4, $\mathfrak{J} = \mathbb{K}a \oplus \mathcal{W} \oplus \mathbb{K}b$, where $a \in \mathfrak{J}$ such that $B(a, a) = 0$, $B(a, b) = 1$ and $\mathcal{W}^\perp = \mathbb{K}a \oplus \mathbb{K}b$. Let us consider the associative symmetric bilinear form T on \mathfrak{J} defined by: $T(a, a) = 1$, and $T(x, y) = 0$, $\forall (x, y) \in \mathfrak{J} \times \mathfrak{J} \setminus \mathbb{K}a \times \mathbb{K}a$. It is easy to see that B et T are two linearly independant elements of $\mathfrak{B}(\mathfrak{J})$. therefore, $ind(\mathfrak{J}) \geq 2$, which contradicts the hypothesis $ind(\mathfrak{J}) = 1$.

We conclude that if $\text{ind}(\mathfrak{J}) = 1$, then \mathfrak{J} is either a simple Jordan algebra or \mathfrak{J} is the one-dimensional algebra with zero product. \square

For any real Jordan algebra \mathfrak{J} , we will denote by $\mathfrak{J}^{\mathbb{C}} := \mathfrak{J} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification which is a Jordan algebra over \mathbb{C} . It is clear if (\mathfrak{J}, B) is a real pseudo-euclidean Jordan algebra, then $(\mathfrak{J}^{\mathbb{C}}, B^{\mathbb{C}})$ is a complex pseudo-euclidean Jordan algebra where $B^{\mathbb{C}}$ is the bilinear form on defined $\mathfrak{J}^{\mathbb{C}}$ defined by: $B^{\mathbb{C}}(x \otimes a, y \otimes b) := abB(x, y), \forall x, y \in \mathfrak{J}, \forall a, b \in \mathbb{C}$.

Corollary 9.12 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra over \mathbb{C} . Then the two following assertions are equivalent:*

1. $\text{ind}(\mathfrak{J}) = 1$
2. \mathfrak{J} is either a simple Jordan algebra or \mathfrak{J} is the one-dimensional algebra with zero product.

Corollary 9.13 *Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra over \mathbb{R} such that $\mathfrak{J}^2 \neq \{0\}$. Then the following assertions are equivalent:*

- (i) $\text{ind}(\mathfrak{J}) = 1$,
- (ii) $\text{ind}(\mathfrak{J}^{\mathbb{C}}) = 1$ (i.e. $\dim_{\mathbb{C}} B(\mathfrak{J}^{\mathbb{C}}) = 1$),
- (iii) $\mathfrak{J}^{\mathbb{C}}$ is simple.

Proof. By Corollary 9.12, the assertions (ii) and (iii) are equivalent. In the following we are going to prove that (i) and (ii) are equivalent. Assume that $\dim_{\mathbb{C}} B(\mathfrak{J}^{\mathbb{C}}) = 1$. Let T_1, T_2 be two associative scalar products on \mathfrak{J} . For $k \in \{1, 2\}$, we consider the symmetric bilinear form \tilde{T}_k on $\mathfrak{J}^{\mathbb{C}}$ defined by:

$$\tilde{T}_k(x, y) = T_k(x, y), \quad \tilde{T}_k(ix, iy) = -T_k(x, y), \quad \tilde{T}_k(ix, y) = iT_k(x, y), \quad \forall x, y \in \mathfrak{J}.$$

It is easy to verify that \tilde{T}_k is an associative scalar product on $\mathfrak{J}^{\mathbb{C}}$. Since $\dim_{\mathbb{C}} B(\mathfrak{J}^{\mathbb{C}}) = 1$, then there exists $\lambda := \alpha + i\beta \in \mathbb{C}$ such that $\tilde{T}_1 = \lambda \tilde{T}_2$. In particular, for all $x, y \in \mathfrak{J}$ we have $\tilde{T}_1(x, y) = \lambda \tilde{T}_2(x, y)$, so $T_1(x, y) = (\alpha + i\beta)T_2(x, y)$. It follows that $\beta = 0$, consequently $T_1 = \alpha T_2$. We conclude that $\dim_{\mathbb{R}} B(\mathfrak{J}) = 1$.

Conversely, assume that $\dim_{\mathbb{R}} B(\mathfrak{J}) = 1$. By Proposition 9.11, \mathfrak{J} is simple. Suppose that $\dim_{\mathbb{C}} B(\mathfrak{J}^{\mathbb{C}}) \neq 1$. Then, by Theorem 9.12, $\mathfrak{J}^{\mathbb{C}}$ is not simple. Therefore \mathfrak{J} admits a complex structure. This means that there exists a simple complex Jordan algebra \mathfrak{S} such that $\mathfrak{J} = \mathfrak{S}_{\mathbb{R}}$ is the underlying real algebra \mathfrak{S} . By Theorem 8.5.2 of [7], \mathfrak{S} admits an euclidean real form \mathcal{E} , so $\mathfrak{J} = \mathcal{E} \oplus i\mathcal{E}$. Let \mathfrak{A} be the Albert form of \mathcal{E} , we consider the two bilinear forms B_1 and B_2 on \mathfrak{J} defined by:

$$\begin{aligned} B_1(x_1 + ix_2, y_1 + iy_2) &= \mathfrak{A}(x_1, y_1) - \mathfrak{A}(x_2, y_2), \\ B_2(x_1 + ix_2, y_1 + iy_2) &= \mathfrak{A}(x_1, y_2) + \mathfrak{A}(x_2, y_1), \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{E}. \end{aligned}$$

An easy calculation proves that B_1 and B_2 are linearly independent associative scalar product on \mathfrak{J} which contradicts that $\text{ind}(\mathfrak{J}) = 1$. We conclude that $\dim_{\mathbb{C}} B(\mathfrak{J}^{\mathbb{C}}) = 1$. \square

Definition 9.3 *Let \mathfrak{J} be a Jordan algebra. \mathfrak{J} is called reductive if $\mathfrak{J} = \mathfrak{S} \oplus \text{Ann}(\mathfrak{J})$ where \mathfrak{S} is a semisimple ideal of \mathfrak{J} .*

Remark 9 If $\mathfrak{J} = \sigma \oplus \text{Ann}(\mathfrak{J})$ where \mathfrak{S} is a semisimple ideal of \mathfrak{J} , then \mathfrak{S} is the greatest semisimple ideal of \mathfrak{J} and $\mathfrak{S} = \bigoplus_{i=1}^n \mathfrak{S}_i$ where $\{\mathfrak{S}_i, 1 \leq i \leq n\}$ is the set of all simple ideals of \mathfrak{J} .

Corollary 9.14 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and $\mathfrak{J}_1, \dots, \mathfrak{J}_r$ ($r \in \mathbb{N}$) be B -irreducible non-degenerate ideals of \mathfrak{J} such that $\mathfrak{J} = \bigoplus_{i=1}^r \mathfrak{J}_i$ and $B(\mathfrak{J}_k, \mathfrak{J}_l) = \{0\}$, for all $k, l \in \{1, \dots, r\}$. Then the following assertions are equivalent:

1. $\text{ind}(\mathfrak{J}) = r$.
2. \mathfrak{J} is reductive and $\dim \text{Ann}(\mathfrak{J}) \leq 1$.

Proof. Suppose that $\text{ind}(\mathfrak{J}) = r$. Let $i \in \{1, \dots, r\}$, we consider $\{T_{i1}, \dots, T_{in_i}\}$ a basis of $\mathfrak{B}(\mathfrak{J}_i)$, where $n_i = \text{ind}(\mathfrak{J}_i)$. Now, for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\}$, we consider the bilinear form $\tilde{T}_{ij} : \mathfrak{J} \times \mathfrak{J} \rightarrow \mathbb{C}$ defined by: $\tilde{T}_{ij}|_{\mathfrak{J}_i \times \mathfrak{J}_i} = T_{ij}$, and $\tilde{T}_{ij}(x, y) = 0, \forall (x, y) \in \mathfrak{J} \times \mathfrak{J} \setminus \mathfrak{J}_i \times \mathfrak{J}_i$. It is clear that the elements of $\bigcup_{1 \leq i \leq r} \{\tilde{T}_{ij}, 1 \leq j \leq n_i\}$ are linearly independent, so $\text{ind}(\mathfrak{J}) \geq \sum_{1 \leq i \leq r} n_i = \sum_{1 \leq i \leq r} \text{ind}(\mathfrak{J}_i)$. Since $\text{ind}(\mathfrak{J}) = r$, then $\text{ind}(\mathfrak{J}_i) = 1, \forall i \in \{1, \dots, r\}$ (i.e. $\mathfrak{B}(\mathfrak{J}_i) = \text{Vect}\{T_{i1}\}$). Consequently, by Corollary 9.12, for all i in $\{1, \dots, r\}$, \mathfrak{J}_i is either simple or the one-dimensional algebra with zero product. Let us suppose that there exist $i \neq j \in \{1, \dots, r\}$ such that $\mathfrak{J}_i = \mathbb{C}a$ and $\mathfrak{J}_j = \mathbb{C}b$ with $a^2 = b^2 = 0$ and consider the symmetric bilinear form defined on \mathfrak{J} by: $T(a, b) = 1$ and $T(x, y) = 0, \forall (x, y) \in \mathfrak{J} \times \mathfrak{J} \setminus (\mathbb{C}a \times \mathbb{C}b)$. It is clear that T is an element of $\mathfrak{F}(\mathfrak{J})$. Therefore, there exist $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ such that $T = \sum_{i=1}^r \tilde{T}_{i1}$. Which contradicts the fact that $T(a, b) \neq 0$, so \mathfrak{J} is reductive with $\dim \text{Ann}(\mathfrak{J}) \leq 1$.

Conversely, Assume that \mathfrak{J} is reductive and $\dim \text{Ann}(\mathfrak{J}) \leq 1$. Then, without loss of generality, we can suppose that for all $i \in \{2, \dots, r\}$ \mathfrak{J}_i is simple and \mathfrak{J}_1 is either simple or the one-dimensional algebra with zero product. It follows that for all $i \in \{1, \dots, r\}$, we have $\text{ind}(\mathfrak{J}_i) = 1$ and $\mathfrak{B}(\mathfrak{J}_i) = \mathbb{C}B_i$, where $B_i := B|_{\mathfrak{J}_i \times \mathfrak{J}_i}$. Now, If T is an associative symmetric bilinear form on \mathfrak{J} , then $T_i := T|_{\mathfrak{J}_i \times \mathfrak{J}_i}$ is an associative symmetric bilinear form on \mathfrak{J}_i . Consequently, there exists $(\alpha_i)_{1 \leq i \leq r}$ in \mathbb{C} such that $T_i = \alpha_i B_i, \forall i \in \{1, \dots, r\}$. Moreover, since $T_i(\mathfrak{J}_i, \mathfrak{J}_k) = 0, \forall i \neq k \in \{1, \dots, r\}$, then $T = \sum_{i=1}^r \alpha_i \tilde{B}_i$ where \tilde{B}_i is the bilinear form on \mathfrak{J} defined by: $\tilde{B}_i|_{\mathfrak{J}_i \times \mathfrak{J}_i} = B_i$ and $\tilde{B}_i(x, y) = 0, \forall x, y \in \mathfrak{J} \times \mathfrak{J} \setminus \mathfrak{J}_i \times \mathfrak{J}_i$. Moreover, it is easy to see that the elements of $\{\tilde{B}_i, 1 \leq i \leq r\}$ are linearly independent. We conclude that $\text{ind}(\mathfrak{J}) = r$. \square

In the following result, we give a characterization of semisimple Jordan algebra by using the notion of index

Corollary 9.15 Let (\mathfrak{J}, B) be a pseudo-euclidean Jordan algebra and $\mathfrak{J}_1, \dots, \mathfrak{J}_r$ ($r \in \mathbb{N}$) be B -irreducible non-degenerate ideals of \mathfrak{J} such that $\mathfrak{J} = \bigoplus_{i=1}^r \mathfrak{J}_i$ and $B(\mathfrak{J}_k, \mathfrak{J}_l) = \{0\}$, for all $k, l \in \{1, \dots, r\}$. Then the two following assertions are equivalent:

1. \mathfrak{J} is semisimple,
2. $\mathfrak{J}^2 = \mathfrak{J}$ and $\text{ind}(\mathfrak{J}) = r$.

Acknowledgments

We thank W. Bertram, M. Bordemann and A. Elduque for very interesting discussions. Moreover, we are very grateful to A. Elduque for his remarks, which improved the readability of this article.

References

- [1] A. Aubert, Structures affines et pseudo-métriques invariantes à gauche sur des groupes de Lie, Thèse, Université Montpellier II, 1996.
- [2] I. Bajo, S. Benayadi, A. Medina. “Symplectic structures on quadratic Lie algebras”, *Journal of Algebra* 316 (1) (2007) 174–188.
- [3] W. Bertram, The geometry of Jordan and Lie structures, *Lecture Notes in Math.*, Vol. 1754, Springer-Verlag, Berlin, 2000.
- [4] M. Bordemann, “Nondegenerate invariant bilinear forms on nonassociative algebras, *Acta Math. Univ. Com LXVI* (2) (1997) 151–201.
- [5] N. Bourbaki, *Groupe et algèbres de Lie*. Ch. 7 et 8, Hermann, Paris, (1975).
- [6] A. Diatta, A. Medina, “Classical Yang-Baxter equation and left invariant affine geometry on Lie groups”, *Manuscr. Math* 114 (4) (2004) 477–486.
- [7] J.Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, (1994).
- [8] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications, Vol.XXXIX. American Mathematical Society, Providence, R.I.(1968).
- [9] M. Koecher, Imbedding of Jordan algebras into Lie algebras. I. *Amer. J. Math* 89 (1967) 787–816.
- [10] M. Koecher, Imbedding of Jordan algebras into Lie algebras. II. *Amer. J. Math* 90 (1968) 476–510.
- [11] K. McCrimmon, *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, (2004).
- [12] A. Medina, Ph. Revoy, Algèbre de Lie et produit scalaire invariant, *Ann. Scient. Ec. Norm. Sup.*, 4ème série, 18 (1985) 553–561.
- [13] O’Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, (1983).
- [14] R.D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York, (1966).
- [15] J. Tits, Une classe d’algèbres de Lie en relation avec les algèbres de Jordan, *Nederl. Akad. Wetensch. Proc. Ser.A* 65 = *Indag. Math* 24 (1962) 530–535.
- [16] V.N Zhelyabin. “Jordan D-Bialgebras And Symplectic Forms On Jordan Algebras”, *Siberian Adv. Math* 10 (2) (2000) 142–150.
- [17] V.N Zhelyabin. “On a class of Jordan D -bialgebras”. *St. Petersburg. Math. J* 11 (4) (2000) 589–609 ; translation from *Algebra Anal* 11 (4) (1999) 64–94.