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Weibull tail-distributions revisited: a new look at some tail estimators

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Abstract

In this paper, we propose to include Weibull tail-distributions in a more general family of distributions. In particular, the considered model also encompasses the whole Fréchet maximum domain of attraction as well as log-Weibull tail-distributions. The asymptotic normality of some tail estimators based on the log-spacings between the largest order statistics is established in a unified way within the considered family. This result permits to understand the similarity between most estimators of the Weibull tail-coefficient and the Hill estimator. Some different asymptotic properties, in terms of bias, rate of convergence, are also highlighted.

AMS Subject Classifications: 62G05, 62G20, 62G30.

Keywords: Weibull tail-distributions, extreme quantile, maximum domain of attraction, asymptotic normality.

1 Motivations

Weibull tail-distributions encompass a variety of light-tailed distributions, *i.e.* distributions in the Gumbel maximum domain of attraction, see [20] for further details. Weibull tail-distributions include for instance Weibull, Gaussian, gamma and logistic distributions. Let us recall that a cumulative distribution function F has a Weibull tail if its associated survival function $\bar{F} = 1 - F$ satisfies the following property: There exists $\theta > 0$ such that for all $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{\log \bar{F}(\lambda t)}{\log \bar{F}(t)} = \lambda^{1/\theta}. \quad (1)$$

The parameter θ is called the Weibull tail-coefficient. We refer to [7] for a general account on Weibull tail-distributions and to [6] for an application to the modeling of large claims in non-life insurance. Dedicated methods have been proposed to estimate the Weibull tail-coefficient since the relevant information is only contained in the extreme upper part of the sample denoted hereafter by X_1, \dots, X_n . A first direction was investigated in [8] where an estimator based on the record values is proposed. Another family of approaches [3, 4, 10, 13] consists of using the k_n upper order statistics $X_{n-k_n+1,n} \leq \dots \leq X_{n,n}$ where (k_n) is an intermediate sequence of integers *i.e.* such that

$$\lim_{n \rightarrow \infty} k_n = \infty \text{ and } \lim_{n \rightarrow \infty} k_n/n = 0. \quad (2)$$

More specifically, most recent estimators are based on the log-spacings between the k_n upper order statistics [7, 11, 22, 23, 25, 26, 27]. All these estimators are thus similar to the Hill statistics [34] defined as

$$H_n(k_n) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}). \quad (3)$$

As an example, all three estimators proposed in [22] are proportional to $H_n(k_n)$. This similarity may be surprising since $H_n(k_n)$ is dedicated to the estimation of the tail index γ for heavy-tailed distribution *i.e.* such that

$$\lim_{t \rightarrow \infty} \frac{\bar{F}(\lambda t)}{\bar{F}(t)} = \lambda^{-1/\gamma},$$

for all $\lambda > 0$. This property characterizes distributions belonging to the Fréchet maximum domain of attraction and sometimes called Pareto-type distributions.

The main goal of this work is therefore to explain why statistics based on log-spacings could be efficient in estimating tail parameters of both Weibull-tail and Pareto-type distributions. To this end, we introduce a family of distributions, indexed by two parameters $\tau \in [0, 1]$ and $\theta > 0$, which includes these two type of distributions. The first parameter τ allows to represent a large panel of distribution tails ranging from Weibull-type tails ($\tau = 0$) to Pareto-type tails ($\tau = 1$). The second parameter θ is the parameter to be estimated. It coincides with the Weibull tail-coefficient when $\tau = 0$ and with the tail index when $\tau = 1$.

An estimator $\hat{\theta}_n(k_n)$ of θ is then introduced for the new family of distributions and an estimator of extreme quantiles is derived. The asymptotic normality of these estimators is established in Section 3 in a unified way and illustrated on some simulated data in Section 4. Some concluding remarks are given in Section 5. Proofs are postponed to Section 6.

2 Model and estimators

2.1 Definition and first properties

Let us consider the family of survival distribution functions defined as

$(\mathbf{A}_1(\tau, \theta))$ $\bar{F}(x) = \exp(-K_\tau^-(\log H(x)))$ for $x \geq x_*$ with $x_* > 0$ and

- $K_\tau(x) = \int_1^x u^{\tau-1} du$ where $\tau \in [0, 1]$,
- H an increasing function such that $H^-(t) = \inf\{x, H(x) \geq t\} = t^\theta \ell(t)$, where $\theta > 0$ and ℓ is a slowly varying function *i.e.* $\ell(\lambda x)/\ell(x) \rightarrow 1$ as $x \rightarrow \infty$ for all $\lambda \geq 1$.

The function H^- is the so-called generalized inverse of H . Note that K_τ^- coincides with the classical inverse since K_τ is continuous. The expansion $H^-(t) = t^\theta \ell(t)$ is equivalent to supposing that H^- is regularly varying at infinity with index θ . This property is denoted by $H^- \in \mathcal{R}_\theta$, see [9] for more details on regular variations theory. Let us first highlight that the tail heaviness of \bar{F} is mainly driven by $\tau \in [0, 1]$ and secondarily by $\theta > 0$:

Proposition 1 *Let $\bar{F}_{\tau_1, \theta_1}$ and $\bar{F}_{\tau_2, \theta_2}$ be two survival distribution functions satisfying respectively $(\mathbf{A}_1(\tau_1, \theta_1))$ and $(\mathbf{A}_1(\tau_2, \theta_2))$.*

- (i) *If $\tau_1 < \tau_2$ then $\bar{F}_{\tau_1, \theta_1}(x)/\bar{F}_{\tau_2, \theta_2}(x) \rightarrow 0$ as $x \rightarrow \infty$ for all $(\theta_1, \theta_2) \in (0, \infty)^2$.*
- (ii) *If $\tau_1 = \tau_2 = \tau$ and $\theta_1 < \theta_2$ then $\bar{F}_{\tau, \theta_1}(x)/\bar{F}_{\tau, \theta_2}(x) \rightarrow 0$ as $x \rightarrow \infty$.*

Thus, the larger is τ , the heavier is the tail. Let us consider the two extremal cases $\tau = 0$ and $\tau = 1$. Clearly, under $(\mathbf{A}_1(0, \theta))$, $\bar{F}(x) = \exp(-H(x))$ is the survival function of a Weibull-tail distribution, see (1). At the opposite, $(\mathbf{A}_1(1, \theta))$ entails $\bar{F}(x) = e^{1/H(x)} = x^{-1/\theta} \tilde{\ell}(x)$ where $\tilde{\ell}$ is a slowly varying function. As a consequence, F belongs to the Fréchet maximum domain of attraction and θ coincides with the tail index. In view of the above remarks, intermediate values of $\tau \in (0, 1)$ correspond to distribution tails lighter than Pareto tails but heavier than Weibull tails. Indeed, we have $\bar{F}(x) = \exp(-h(x))$ with $h(x) \sim ((\tau/\theta) \log x)^{1/\tau}$ and thus $h(x)/x^\beta \rightarrow 0$ for all $\beta > 0$ while $h(x)/\log(x) \rightarrow \infty$ as $x \rightarrow \infty$, this property characterizing an “exponential type” distribution, see [32]. The next proposition provides a more precise characterization while examples are provided in Paragraph 2.2.

Proposition 2

- (i) F verifies $(\mathbf{A}_1(0, \theta))$ if and only if F is a Weibull-tail distribution function with Weibull tail-coefficient θ .
- (ii) If F verifies $(\mathbf{A}_1(\tau, \theta))$, $\tau \in [0, 1)$ and if H is twice differentiable then F belongs to the Gumbel maximum domain of attraction.
- (iii) F verifies $(\mathbf{A}_1(1, \theta))$ if and only if F is in the Fréchet maximum domain of attraction with tail-index θ .

2.2 Examples

In view of Proposition 2(i), Gaussian, gamma, Weibull, Benktander II, logistic and extreme-value distributions all verify $(\mathbf{A}_1(0, \theta))$ since they are examples of Weibull tail-distributions (see [23], Table 1). Examples of distributions verifying $(\mathbf{A}_1(\tau, \theta))$ with $\tau \in (0, 1)$ include some log-Weibull tail-distributions. Let us recall that a random variable Y is distributed from a log-Weibull tail-distribution if $\log(Y)$ follows a Weibull tail-distribution.

Proposition 3 *Suppose that F verifies $(\mathbf{A}_1(0, \theta))$ with $\theta \in (0, 1]$. If, moreover, the slowly-varying function ℓ is differentiable and $\ell(t) \rightarrow \ell_\infty > 0$ as $t \rightarrow \infty$ then $F(\log \cdot)$ verifies $(\mathbf{A}_1(\theta, \theta \ell_\infty))$.*

As an example, the standard log-normal distribution can be looked at as a log-Weibull tail-distribution and thus verifies $(\mathbf{A}_1(1/2, \sqrt{2}/2))$. Similarly, the gamma distribution verifies $(\mathbf{A}_1(0, 1))$ and the log-gamma distribution belongs to the Fréchet maximum domain of attraction, see for instance [16], Table 3.4.2. Finally, other examples of distributions satisfying $(\mathbf{A}_1(1, \theta))$ can be found in the above mentioned table.

2.3 Definition of the estimators

Denoting by (k_n) an intermediate sequence of integers (see (2)), the following estimator of θ is considered:

$$\hat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})), \tag{4}$$

with, for all $t > 0$ and $q \in \mathbb{N} \setminus \{0\}$,

$$\mu_{q,\tau}(t) = \int_0^\infty (K_\tau(x+t) - K_\tau(t))^q e^{-x} dx.$$

A crucial point is that the estimator (4) essentially consists in averaging the log-spacings between the upper-order statistics. Even more strongly, $\widehat{\theta}_n(k_n)$ only differs from the Hill statistics (3) by a non-random normalizing sequence: $\widehat{\theta}_n(k_n) = H_n(k_n)/\mu_{1,\tau}(\log(n/k_n))$. This similarity can be intuitively understood by studying the log-spacing between two quantiles x_u and x_v of \bar{F} , with $0 < u < v \leq 1$. Under $(\mathbf{A}_1(\tau, \theta))$ we have

$$\log x_u - \log x_v = \theta (K_\tau(-\log u) - K_\tau(-\log v)) + \log \left(\frac{\ell(\exp K_\tau(-\log u))}{\ell(\exp K_\tau(-\log v))} \right). \quad (5)$$

Now, since ℓ is a slowly-varying function, if the orders u and v of the quantiles are small enough, the second term can be neglected in the right-hand side of (5) to obtain

$$\log x_u - \log x_v \simeq \theta (K_\tau(-\log u) - K_\tau(-\log v)), \quad (6)$$

which shows that log-spacings are approximately proportional to θ . Since this key property holds for all $\tau \in [0, 1]$, it is thus shared by Pareto-type, Weibull tail and log-Weibull tail-distributions. Note that this property can be checked graphically on a sample by drawing a quantile-quantile plot. It consists in plotting the pairs $(K_\tau(\log(n/i)), \log(X_{n-i+1,n}))$ for $i = 1, \dots, k_n$. From (6), the graph should be approximately linear. Following the same ideas, an estimator of the extreme quantile x_{p_n} can be deduced from (4) by:

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \exp \left(\widehat{\theta}_n(k_n) (K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \right). \quad (7)$$

Recall that an extreme quantile x_{p_n} of order p_n is defined by $x_{p_n} = \bar{F}^{\leftarrow}(p_n)$ with $p_n \rightarrow 0$ as $n \rightarrow \infty$. For instance, if $np_n \rightarrow 0$ then x_{p_n} is larger than the maximum observation $X_{n,n}$ of the sample (with probability tending to one). This requires to extrapolate sample results to areas where no data are observed and occurs in reliability [14], hydrology [36], finance [16],...

3 Asymptotic properties

We show in the next paragraph that the asymptotic normality of $\widehat{\theta}_n(k_n)$ and \widehat{x}_{p_n} can be established for all $\tau \in [0, 1]$ in a unified way. In this sense, the asymptotic behavior of these estimators is more a consequence of the log-spacings property than of a tail behavior (which can be exponential as well as polynomial). Paragraphs 3.2 and 3.3 illustrate our general result on the two extremal cases $\tau = 0$ and $\tau = 1$.

3.1 Main results

To establish the asymptotic normality of $\widehat{\theta}_n(k_n)$, a second-order condition on ℓ is necessary:

$(\mathbf{A}_2(\rho))$ There exist $\rho < 0$ and $b(x) \rightarrow 0$ such that uniformly locally on $\lambda \geq \lambda_0 > 0$

$$\log \left(\frac{\ell(\lambda x)}{\ell(x)} \right) \sim b(x) K_\rho(\lambda), \text{ when } x \rightarrow \infty.$$

It can be shown that necessarily $|b| \in \mathcal{R}_\rho$ (see [24]). The second order parameter $\rho < 0$ tunes the rate of convergence of $\ell(\lambda x)/\ell(x)$ to 1. The closer is ρ to 0, the slower is the convergence. Condition $(\mathbf{A}_2(\rho))$ is the cornerstone in all the proofs of asymptotic normality for extreme value estimators. It is used in [5, 33, 34] to prove the asymptotic normality of several estimators of the extreme value index.

Theorem 1 Suppose that $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$ hold. Let (k_n) be an intermediate sequence such that

$$\sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \rightarrow \lambda. \quad (8)$$

Then, introducing $a_{\tau,\rho} = 1$ if $\tau \in [0, 1)$ and $a_{1,\rho} = 1/(1 - \rho)$, we have

$$\sqrt{k_n} \left(\widehat{\theta}_n(k_n) - \theta - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n))) \right) \xrightarrow{d} \mathcal{N}(0, \theta^2). \quad (9)$$

It appears that the asymptotic variance of $\widehat{\theta}_n(k_n)$ given by $\mathcal{AV} = \theta^2/k_n$ is independent of τ . In particular, it remains constant whatever the maximum domain of attraction of F . The asymptotic squared bias is given by $\mathcal{ASB}(\tau, \rho) = a_{\tau,\rho}^2 b^2(\exp K_\tau(\log(n/k_n)))$. If b^2 is ultimately decreasing, then \mathcal{ASB} is a decreasing function of $\tau \in [0, 1)$ with a jump at $\tau = 1$. These remarks are illustrated on simulated data in Section 4. The next result allows us to establish the rate of convergence of $\widehat{\theta}_n(k_n)$ to θ in (9).

Proposition 4 Condition (8) with $\lambda \neq 0$ implies $\log(k_n) = -2\rho a_{\tau,2\rho} K_\tau(\log n)(1 + o(1))$.

The rate of convergence is thus of order $\exp(-\rho a_{\tau,2\rho} K_\tau(\log n)(1 + o(1)))$. A geometrical rate of convergence is obtained only in the Fréchet maximum domain of attraction, $\tau = 1$ yields $\sqrt{k_n} = n^{-\rho/(1-2\rho)+o(1)}$ which is consistent with the conclusions of [31]. Weibull tail-distributions give rise to logarithmic rates of convergence, $\tau = 0$ yields $\sqrt{k_n} = (\log n)^{-\rho+o(1)}$ which is consistent with the results of [22]. More generally, the heavier is the tail, the better the rate of convergence is. The next result provides an extension of Statement 1 in [2], which was initially proved only for Weibull tail-distributions ($\tau = 0$).

Proposition 5 Suppose condition (8) holds with $\lambda \neq 0$. If $\tau \in [0, 1/2]$ then

$$\mathcal{ASB}(\tau, \rho) = c_{\tau,\rho} b^2(\exp K_\tau(\log n))(1 + o(1)),$$

where $c_{\tau,\rho} = 1$ if $\tau \in [0, 1/2)$ and $c_{1/2,\rho} = \exp(8\rho^2)$.

It follows that, when $\tau \in [0, 1/2]$, the first order of the asymptotic bias is asymptotically independent of k_n . As a consequence, the asymptotic mean-squared error defined as $\mathcal{ASB}(\tau, \rho) + \mathcal{AV}$ is eventually decreasing with respect to k_n . This remark, already made in [2] in the particular case $\tau = 0$, is only of theoretical interest. Indeed, in finite sample situations, condition (8) does not hold and the empirical mean-squared error is a convex function of k_n , see for instance [22]. Now, the asymptotic normality of the extreme quantile estimator (7) can be deduced from Theorem 1:

Theorem 2 Suppose the assumptions of Theorem 1 hold with $\lambda = 0$. If, moreover,

$$(\log(n/k_n))^{1-\tau} (K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \rightarrow \infty \quad (10)$$

then,

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} \left(\frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Let us now focus on the two particular cases $\tau = 0$ (Weibull tail-distributions) and $\tau = 1$ (Fréchet maximum domain of attraction).

3.2 Application to Weibull tail-distributions

If $\tau = 0$, the estimator (4) coincides with $\widehat{\theta}_n^{(1)}$ introduced in [22], and

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \left(\frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\widehat{\theta}_n(k_n)}$$

is the estimator proposed in [21]. As a consequence of Theorem 1 and Theorem 2, we obtain:

Corollary 1 *Suppose that $(\mathbf{A}_1(0, \theta))$ and $(\mathbf{A}_2(\rho))$ hold. Let (k_n) be an intermediate sequence such that $\sqrt{k_n} b(\log(n/k_n)) \rightarrow 0$. Then,*

$$\sqrt{k_n} (\widehat{\theta}_n(k_n) - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover

$$\log(n/k_n) (\log \log(1/p_n) - \log \log(n/k_n)) \rightarrow \infty \quad (11)$$

then,

$$\frac{\sqrt{k_n}}{\log \log(1/p_n) - \log \log(n/k_n)} \left(\frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

This result is very similar to Corollary 3.1 in [22] except that condition (11) is weaker than the one used in the above mentioned paper. Let us also note that estimators $\widehat{\theta}_n^{(2)}$ and $\widehat{\theta}_n^{(3)}$ in [22] can be respectively deduced from $\widehat{\theta}_n$ by approximating $\mu_{1,0}$ by a Riemann's sum or using the first order approximation $\mu_{1,0}(t) \sim 1/t$ as $t \rightarrow \infty$ given in Lemma 2(i).

3.3 Application to the Fréchet maximum domain of attraction

Letting $\tau = 1$ and remarking that $\mu_{q,1}(t) = q!$ for all $t > 0$ and $q \in \mathbb{N} \setminus \{0\}$, the estimator (4) coincides with (3) which is the Hill estimator [34] of the tail index. Besides,

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \left(\frac{k_n}{np_n} \right)^{\widehat{\theta}_n(k_n)}$$

is the Weissman estimator [37]. A straightforward application of the above theorems gives back the classical results:

Corollary 2 *Suppose that $(\mathbf{A}_1(1, \theta))$ and $(\mathbf{A}_2(\rho))$ hold. Let (k_n) be an intermediate sequence such that $\sqrt{k_n} b(n/k_n) \rightarrow 0$. Then,*

$$\sqrt{k_n} (\widehat{\theta}_n(k_n) - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover $k_n/(np_n) \rightarrow \infty$ then,

$$\frac{\sqrt{k_n}}{\log(k_n/(np_n))} \left(\frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

4 Illustration on simulations

The section is dedicated to the illustration of the conclusions drawn from Theorem 1 on simulated data. To this end, we consider a cumulative distribution function $F_{\theta, \tau, \rho}$ verifying $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$ with $\theta = 1/2$, $\tau \in \{0, 1/2, 1\}$ and $\rho \in \{-1/2, -1/4\}$. More specifically, the slowly-varying function is given by

$$\ell(x) = 1 - \frac{\theta}{\rho}(1+x)^\rho \left(1 + \frac{1}{x}\right)^\theta.$$

Following Proposition 2(i), it appears that the case $\tau = 0$ corresponds to a Weibull tail-distribution (with Weibull tail-coefficient $1/2$) similar to a Gaussian distribution. When $\tau = 1/2$, in view of Paragraph 2.1, $\bar{F}(x) = \exp\{-\log x)^2(1 + o(1))\}$, the distribution has a tail behavior similar to the log-normal distribution. Finally, Proposition 2(iii) shows that, when $\tau = 1$, the distribution belongs to the Fréchet maximum domain of attraction with tail-index $1/2$.

For each considered combination of τ and ρ , $N = 500$ samples $(\mathcal{X}_{n,j})_{j=1, \dots, N}$ of size $n = 500$ were simulated from $F_{1/2, \tau, \rho}$. On each sample $(\mathcal{X}_{n,j})$, the estimate $\hat{\theta}_{n,j}$ is computed for $k = 2, \dots, 250$, the associated empirical squared bias $\mathcal{E}\mathcal{S}\mathcal{B}$ and empirical variance $\mathcal{E}\mathcal{V}$ plots are built by plotting the pairs $(k, (\bar{\theta}_n^{(1)}(k) - \theta)^2)$ and $(k, \bar{\theta}_n^{(2)}(k) - (\bar{\theta}_n^{(1)}(k))^2)$ where for $i \in \{1, 2\}$,

$$\bar{\theta}_n^{(i)}(k) = \frac{1}{N} \sum_{j=1}^N (\hat{\theta}_{n,j}(k))^i.$$

The empirical squared bias and the empirical variance are depicted on Figure 1 and Figure 2 respectively. Both graphs are represented on the same scale for the sake of comparison. As expected, the squared bias, for a fixed value of k , is an increasing function of ρ and a decreasing function of τ . At the opposite, the variance seems to be independent of ρ and is not much dependent of τ .

5 Concluding remarks

As illustrated in the previous sections, the model $(\mathbf{A}_1(\tau, \theta))$ provides a new tool for the analysis of tail estimators based on log-spacings. It allows us to encompass Weibull tail-distributions in a more general framework and thus to explain why their dedicated tail estimators are very similar to Hill or Weissman statistics. The next step would be to estimate the parameter τ . For instance, one can consider the following estimator based on the log-spacing between two Hill statistics

$$\hat{\tau}_n = 1 + \frac{\log H_n(k'_n) - \log H_n(k_n)}{\log \log(n/k'_n) - \log \log(n/k_n)},$$

where (k_n) and (k'_n) are two intermediate sequences such that

$$\liminf_{n \rightarrow \infty} \frac{\log(n/k'_n)}{\log(n/k_n)} > 1.$$

Let us note that

$$\begin{aligned} (\log \log(n/k'_n) - \log \log(n/k_n))(\hat{\tau}_n - \tau) &= \log(\hat{\theta}_n(k'_n)/\theta) - \log(\hat{\theta}_n(k_n)/\theta) \\ &+ \log\left(\frac{\mu_{1,\tau}(\log(n/k'_n))}{\log^{\tau-1}(n/k'_n)}\right) - \log\left(\frac{\mu_{1,\tau}(\log(n/k_n))}{\log^{\tau-1}(n/k_n)}\right). \end{aligned}$$

This implies that the consistency of $\widehat{\tau}_n$ is a simple consequence of Theorem 1 and Lemma 2(i) whereas the asymptotic distribution is much more difficult to handle as it requires the joint distribution of $\widehat{\theta}_n(k'_n)$ and $\widehat{\theta}_n(k_n)$. Also in practice, the choice of the parameters k_n and k'_n is an open question. These two points are currently under investigation.

Other extensions are possible, among others bias correction. To this end, an exponential regression model for these tail distributions extending [5, 11, 12, 18] would be of interest. We also plan to adapt our results to the case $\tau > 1$ and to investigate the possible links with super-heavy tails [19]. Finally, this work could be further extended to random variables $Y = \psi(X)$ where X has a parent distribution satisfying $(\mathbf{A}_1(\tau, \theta))$. For instance, choosing $\psi(x) = x^* - 1/x$ would allow to consider distributions (with finite endpoint x^*) in the Weibull maximum domain of attraction. This may help for including the negative Hill estimator (see for instance [17] or [30], paragraph 3.6.2) in our framework.

6 Proofs

We first give some preliminary lemmas. Their proofs are postponed to the appendix.

6.1 Preliminary lemmas

The first lemma provides some uniform approximations based on $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$.

Lemma 1 *If $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$ hold then*

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)), \text{ when } x \rightarrow \infty.$$

Let us define for all $q \in \mathbb{N} \setminus \{0\}$, $\tau \in [0, 1]$ and $t > 0$, $\sigma_{q,\tau}^2(t) = \mu_{2q,\tau}(t) - \mu_{q,\tau}^2(t)$. The following lemma is of analytical nature. It provides first-order expansions which will be useful in the sequel.

Lemma 2 *For all $q \in \mathbb{N} \setminus \{0\}$ and $\tau \in [0, 1]$, when $t \rightarrow \infty$:*

- (i) $\mu_{q,\tau}(t) \sim q! t^{(\tau-1)q}$,
- (ii) $\sigma_{q,\tau}^2(t)/\mu_{q,\tau}^2(t) \rightarrow (2q)!/(q!)^2 - 1$,
- (iii) $\mu'_{1,\tau}(t)/\mu_{1,\tau}(t) \rightarrow 0$.

The next lemma presents an expansion of $\widehat{\theta}_n(k_n)$.

Lemma 3 *Let (k_n) be an intermediate sequence. Then, under $(\mathbf{A}_1(\tau, \theta))$, the following expansions hold:*

$$\widehat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \left(\theta \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \right),$$

with, for all $q \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned} \theta_{n,1}^{(q)}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (K_\tau(F_i + t) - K_\tau(t))^q, \\ \theta_{n,2}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{\ell(\exp K_\tau(F_i + t))}{\ell(\exp K_\tau(t))} \right), \end{aligned}$$

and where $E_{n-k_n+1,n}$ is the $(n - k_n + 1)$ th order statistic associated to n independent standard exponential variables and $\{F_1, \dots, F_{k_n-1}\}$ are independent standard exponential variables and independent from $E_{n-k_n+1,n}$.

The asymptotic behavior of the $(n - k_n + 1)$ th standard exponential order statistic is described in the following lemma.

Lemma 4 *Let (k_n) be an intermediate sequence. Then, for all differentiable function g , we have*

$$\sqrt{k_n}(g(E_{n-k_n+1,n}) - g(\log(n/k_n))) = O_{\mathbb{P}}(1)g'(\log(n/k_n))(1 + o_{\mathbb{P}}(1)).$$

The next two lemmas provide the key results for establishing the asymptotic distribution of $\widehat{\theta}_n(k_n)$. They describe the asymptotic behavior of the random terms appearing in Lemma 3.

Lemma 5 *Let (k_n) be an intermediate sequence. Then, for all $q \in \mathbb{N} \setminus \{0\}$,*

$$\frac{\sqrt{k_n}}{\sigma_{q,\tau}(E_{n-k_n+1,n})} \left(\theta_{n,1}^{(q)}(E_{n-k_n+1,n}) - \mu_{q,\tau}(E_{n-k_n+1,n}) \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

Lemma 6 *Suppose that $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$ hold. Let (k_n) be an intermediate sequence. Then,*

$$\theta_{n,2}(E_{n-k_n+1,n}) = b(\exp K_{\tau}(E_{n-k_n+1,n}))\theta_{n,3}(E_{n-k_n+1,n})(1 + o_{\mathbb{P}}(1)),$$

where

$$\left| \theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) \right| \leq -\frac{\rho}{2}\theta_{n,1}^{(2)}(E_{n-k_n+1,n}).$$

Moreover, if $\tau = 1$, then $\theta_{n,3}(E_{n-k_n+1,n}) \xrightarrow{P} 1/(1 - \rho)$.

6.2 Proofs of the main results

Proof of Proposition 1 – Assumptions $(\mathbf{A}_1(\tau_1, \theta_1))$ and $(\mathbf{A}_1(\tau_2, \theta_2))$ entail

$$\frac{\bar{F}_{\tau_1, \theta_1}(x)}{\bar{F}_{\tau_2, \theta_2}(x)} = \exp \left[-K_{\tau_1}^{\leftarrow}(\log H_1(x)) \left(1 - \frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_{\tau_1}^{\leftarrow}(\log H_1(x))} \right) \right], \quad (12)$$

where $H_1 \in \mathcal{R}_{1/\theta_1}$ and $H_2 \in \mathcal{R}_{1/\theta_2}$. As a consequence, for all $q \in \{1, 2\}$, $\log H_q(x) \sim \log(x)/\theta_q$ when $x \rightarrow \infty$, see [9], Proposition 1.3.6. Let us first prove (i): $0 < \tau_1 < \tau_2$ implies

$$K_{\tau_q}^{\leftarrow}(\log H_q(x)) \sim (\tau_q/\theta_q)^{1/\tau_q} (\log x)^{1/\tau_q} \rightarrow \infty, \quad (13)$$

and thus

$$\frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_{\tau_1}^{\leftarrow}(\log H_1(x))} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{(\tau_1/\theta_1)^{1/\tau_1}} (\log x)^{1/\tau_2 - 1/\tau_1} \rightarrow 0. \quad (14)$$

Collecting (12), (13) and (14) gives the result: $\bar{F}_{\tau_1, \theta_1}(x)/\bar{F}_{\tau_2, \theta_2}(x) \rightarrow 0$ as $x \rightarrow \infty$. Similarly, if $\tau_1 = 0$, then

$$\frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_0^{\leftarrow}(\log H_1(x))} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{H_1(x)} (\log x)^{1/\tau_2} \rightarrow 0,$$

which concludes the first part of the proof. Let us now focus on (ii) and suppose $\theta_1 < \theta_2$. If $\tau > 0$ then

$$\frac{K_{\tau}^{\leftarrow}(\log H_2(x))}{K_{\tau}^{\leftarrow}(\log H_1(x))} \rightarrow \left(\frac{\theta_1}{\theta_2} \right)^{1/\tau} < 1,$$

as $x \rightarrow \infty$, while, if $\tau = 0$,

$$\frac{K_0^{\leftarrow}(\log H_2(x))}{K_0^{\leftarrow}(\log H_1(x))} = \frac{H_2(x)}{H_1(x)} \rightarrow 0,$$

as $x \rightarrow \infty$. In both cases, for x large enough,

$$1 - \frac{K_{\tau}^{\leftarrow}(\log H_2(x))}{K_{\tau}^{\leftarrow}(\log H_1(x))} > 0, \quad (15)$$

and collecting (12), (13) and (15) concludes the proof: $\bar{F}_{\tau, \theta_1}(x)/\bar{F}_{\tau, \theta_2}(x) \rightarrow 0$ as $x \rightarrow \infty$. \blacksquare

Proof of Proposition 2 – Proofs of (i) and (iii) are straightforward consequences of Paragraph 2.1. Let us focus on (ii). In view of the characterization (3.35) in [16] of the Gumbel maximum domain of attraction, it is sufficient to prove that there exists a positive function a , differentiable with $a'(t) \rightarrow 0$ as $t \rightarrow \infty$, such that

$$\bar{F}(x) = \exp \left\{ - \int_{x_*}^x \frac{dt}{a(t)} \right\}, \quad x \geq x_*. \quad (16)$$

Letting $a = 1/(K_\tau^\leftarrow(\log H))'$, it thus remains to prove that $a'(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $\tau \in [0, 1)$. To this end, let us remark that

$$\begin{aligned} a'(t) &= \frac{1}{K_\tau^\leftarrow(\log H(t))} \left(\tau - 1 + \left(1 - \frac{H''(t)H(t)}{H'(t)^2} \right) (1 + \tau \log H(t)) \right) \\ &= \frac{1}{K_\tau^\leftarrow(\log H(t))} (\tau - 1 + (\theta + o(1))(1 + \tau \log H(t))), \end{aligned}$$

since $H' \in \mathcal{R}_{1/\theta-1}$ implies $H''(t)H(t)/H'(t)^2 \rightarrow 1 - \theta$ as $t \rightarrow \infty$. Two cases arise:

- If $\tau \in (0, 1)$ then $a'(t) \sim \theta(\tau \log H(t))^{1-1/\tau} \rightarrow 0$ as $t \rightarrow \infty$.
- Otherwise, when $\tau = 0$, we have $a'(t) = (\theta - 1 + o(1))/H(t) \rightarrow 0$ as $t \rightarrow \infty$.

In both situations, the conclusion follows. \blacksquare

Proof of Proposition 3 – Let us suppose that F verifies $(\mathbf{A}_1(0, \theta))$ with $\theta \in (0, 1]$. Then, introducing $W(x) = \exp K_\theta(H(\log x))$, we have $\bar{F}(\log x) = \exp(-K_\theta^\leftarrow(\log W(x)))$. It thus remains to prove that $W^\leftarrow \in \mathcal{R}_{\theta\ell_\infty}$. Simple calculations show that

$$\begin{aligned} W^\leftarrow(t) &= \exp \{ H^\leftarrow(K_\theta^\leftarrow(\log t)) \} \\ &= \exp \{ (1 + \theta \log t) \ell(K_\theta^\leftarrow(\log t)) \} \\ &= e^{\ell_\infty} t^{\theta\ell_\infty} \varphi(t), \end{aligned}$$

where we have defined $\varphi(t) = \psi(\log t)$ with $\psi(x) = \exp\{(1 + \theta x)[\ell(K_\theta^\leftarrow(x)) - \ell_\infty]\}$. As a consequence,

$$\begin{aligned} t(\log \varphi(t))' &= (\log \psi)'(\log t) \\ &= \theta(\ell(K_\theta^\leftarrow(\log t)) - \ell_\infty) + K_\theta^\leftarrow(\log t) \ell'(K_\theta^\leftarrow(\log t)) \\ &= o(1), \end{aligned}$$

since, from [9], p. 15, $u\ell'(u)/\ell(u) \rightarrow 0$ as $u \rightarrow \infty$. Using again [9], p. 15, it follows that φ is a slowly varying function. Thus, $W^\leftarrow \in \mathcal{R}_{\theta\ell_\infty}$ and $F(\log \cdot)$ verifies $(\mathbf{A}_1(\theta, \theta\ell_\infty))$. \blacksquare

Proof of Theorem 1 – Lemma 5 states that for $q \in \{1, 2\}$,

$$\frac{\sqrt{k_n}}{\sigma_{q,\tau}(E_{n-k_n+1,n})} \left(\theta_{n,1}^{(q)}(E_{n-k_n+1,n}) - \mu_{q,\tau}(E_{n-k_n+1,n}) \right) = \xi_n^{(q)}$$

where $\xi_n^{(q)} \xrightarrow{d} \mathcal{N}(0, 1)$. Then, by Lemma 3,

$$\begin{aligned} \sqrt{k_n} (\hat{\theta}_n(k_n) - \theta - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n)))) &= \sqrt{k_n} \theta \left(\frac{\mu_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} - 1 \right) + \theta \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} \xi_n^{(1)} \\ &+ \sqrt{k_n} \left(\frac{\theta_{n,2}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n))) \right) \\ &\stackrel{\text{def}}{=} T_n^{(1)} + T_n^{(2)} + T_n^{(3)}, \end{aligned}$$

and the three terms are studied separately. First, applying Lemma 4 to $g = \mu_{1,\tau}$ yields

$$T_n^{(1)} = O_{\mathbb{P}}(1) \frac{\mu'_{1,\tau}(\log(n/k_n)(1 + o_{\mathbb{P}}(1)))}{\mu_{1,\tau}(\log(n/k_n))} = o_{\mathbb{P}}(1), \quad (17)$$

in view of Lemma 2(i, iii). Second,

$$T_n^{(2)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left(1 + \frac{T_n^{(1)}}{\theta\sqrt{k_n}} \right) \theta\xi_n^{(1)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \theta\xi_n^{(1)} (1 + o_{\mathbb{P}}(1))$$

and, from Lemma 2(ii), $\sigma_{1,\tau}(E_{n-k_n+1,n})/\mu_{1,\tau}(E_{n-k_n+1,n}) \xrightarrow{P} 1$. As a preliminary conclusion,

$$T_n^{(2)} = \theta\xi_n^{(1)} (1 + o_{\mathbb{P}}(1)). \quad (18)$$

From Lemma 6, $T_n^{(3)}$ can be expanded as

$$\begin{aligned} T_n^{(3)} &= \sqrt{k_n} b(\exp K_{\tau}(\log(n/k_n))) \left(\frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) \\ &= \lambda \left(\frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)). \end{aligned}$$

Introducing $T_n^{(3,1)} = K_{\tau}(E_{n-k_n+1,n}) - K_{\tau}(\log(n/k_n))$ and applying Lemma 4 with $g = K_{\tau}$ yield

$$\exp T_n^{(3,1)} = \exp \left(O_{\mathbb{P}}(1) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}} \right) \xrightarrow{P} 1, \quad (19)$$

since $\tau \in [0, 1]$. Therefore, b being regularly varying,

$$b(\exp K_{\tau}(E_{n-k_n+1,n})/b(\exp K_{\tau}(\log(n/k_n))) \xrightarrow{P} 1$$

as well, and consequently

$$\begin{aligned} T_n^{(3)} &= \lambda \left(\frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)) \\ &= \lambda \left(\frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left(1 + \frac{T_n^{(1)}}{\theta\sqrt{k_n}} \right) (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)) \\ &= \lambda \left(\frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)), \end{aligned}$$

from (17). Two situations occur. If $\tau = 1$, then, in view of Lemma 6, $\theta_{n,3}(E_{n-k_n+1,n}) \xrightarrow{P} a_{1,\rho} = 1/(1-\rho)$, $\mu_{1,1}(E_{n-k_n+1,n}) = 1$ and thus $T_n^{(3)} \xrightarrow{P} 0$. If $\tau \in [0, 1)$, $T_n^{(3)}$ can be rewritten as

$$T_n^{(3)} = \lambda \left((T_n^{(3,2)} + T_n^{(3,3)})(1 + o_{\mathbb{P}}(1)) - 1 \right) (1 + o(1)),$$

where

$$T_n^{(3,2)} \stackrel{\text{def}}{=} \frac{\theta_{n,1}^{(1)}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} = 1 + \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \frac{\xi_n^{(1)}}{\sqrt{k_n}} = 1 + o_{\mathbb{P}}(1)$$

$$\begin{aligned}
|T_n^{(3,3)}| &\stackrel{\text{def}}{=} \frac{|\theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n})|}{\mu_{1,\tau}(E_{n-k_n+1,n})} \\
&\leq \frac{\rho \theta_{n,1}^{(2)}(E_{n-k_n+1,n}) \mu_{2,\tau}(E_{n-k_n+1,n})}{2 \mu_{2,\tau}(E_{n-k_n+1,n}) \mu_{1,\tau}(E_{n-k_n+1,n})} \\
&\stackrel{d}{=} -\rho(\log(n/k_n))^{\tau-1}(1 + o_{\mathbb{P}}(1)) \left(1 + \frac{\sigma_{2,\tau}(E_{n-k_n+1,n}) \xi_n^{(2)}}{\mu_{2,\tau}(E_{n-k_n+1,n}) \sqrt{k_n}}\right) \\
&= O_{\mathbb{P}}(\log(n/k_n))^{\tau-1},
\end{aligned}$$

in view of Lemma 2, Lemma 5 and Lemma 6. Thus, for all $\tau \in [0, 1)$, $T_n^{(3)} \xrightarrow{P} 0$. Taking (17) and (18) into account concludes the proof. \blacksquare

Proof of Proposition 4 – From (8), we have

$$\frac{1}{2} \log k_n + \log |b(\exp K_\tau(\log(n/k_n)))| \rightarrow \log |\lambda|$$

as $n \rightarrow \infty$, and since $K_\tau(\log(n/k_n)) \rightarrow \infty$ as $n \rightarrow \infty$, it follows that

$$\frac{\log k_n}{2K_\tau(\log(n/k_n))} + \frac{\log |b(\exp K_\tau(\log(n/k_n)))|}{K_\tau(\log(n/k_n))} \rightarrow 0$$

as $n \rightarrow \infty$. Now, $|b|$ is a regularly-varying function with index ρ and thus $\log |b(x)|/\log x \rightarrow \rho$ for all $x \rightarrow \infty$, see [9], Proposition 1.3.6. As a consequence, we obtain

$$\frac{\log k_n}{K_\tau(\log(n/k_n))} \rightarrow -2\rho \tag{20}$$

as $n \rightarrow \infty$. Let us first remark that, if $\tau = 1$ then (20) implies

$$\log k_n = \frac{2\rho}{2\rho - 1}(\log n)(1 + o(1)) = \frac{2\rho}{2\rho - 1}K_1(\log n)(1 + o(1))$$

and the conclusion follows. Otherwise, if $\tau \in [0, 1)$, condition (20) can be rewritten as

$$\frac{\log k_n}{\log n} \frac{\log n}{K_\tau(\log(n/k_n))} \rightarrow -2\rho. \tag{21}$$

Besides, since K_τ is non-decreasing,

$$\frac{\log n}{K_\tau(\log(n/k_n))} \geq \frac{\log n}{K_\tau(\log n)} \rightarrow \infty$$

for all $\tau \in [0, 1)$ and thus, in view of (21), necessarily $\log k_n/\log n \rightarrow 0$ as $n \rightarrow \infty$. As a consequence, $\log(n/k_n)$ is asymptotically equivalent to $\log n$ and thus $K_\tau(\log(n/k_n))$ is asymptotically equivalent to $K_\tau(\log(n))$ as well. Replacing in (20), the conclusion follows. \blacksquare

Proof of Proposition 5 – Let us consider $\tau \in [0, 1/2)$ and suppose that (8) holds with $\lambda \neq 0$. Following Proposition 4, $\log(k_n) = -2\rho K_\tau(\log n)(1 + o(1))$ and thus $\log(k_n)/\log(n) \rightarrow 0$ as $n \rightarrow \infty$. A first order Taylor expansion shows that there exists $\eta_n \in [0, 1]$ such that

$$\begin{aligned}
\Delta_n &\stackrel{\text{def}}{=} \exp\{K_\tau(\log(n/k_n)) - K_\tau(\log n)\} = \exp\{-(\log k_n)K'_\tau(\log(n) - \eta_n \log(k_n))\} \\
&= \exp\{-(\log k_n)K'_\tau(\log n)(1 + o(1))\},
\end{aligned}$$

since K'_τ is regularly-varying. As a consequence,

$$\Delta_n = \exp\{2\rho K_\tau(\log n)K'_\tau(\log n)(1 + o(1))\}$$

and thus $\Delta_n \rightarrow 1$ if $\tau \in [0, 1/2)$ or $\Delta_n \rightarrow \exp(4\rho)$ if $\tau = 1/2$. Since b^2 is regularly varying with index 2ρ it follows that

$$\begin{aligned} ASB(\tau, \rho) &= b^2(\exp K_\tau(\log n)) \frac{b^2(\Delta_n \exp K_\tau(\log n))}{b^2(\exp K_\tau(\log n))} \\ &= c_{\tau, \rho} b^2(\exp K_\tau(\log n))(1 + o(1)), \end{aligned}$$

and the conclusion follows. ■

Proof of Theorem 2 – From (7), one can infer that

$$\begin{aligned} \log \widehat{x}_{p_n} - \log x_{p_n} &= (\log(X_{n-k_n+1, n}) - \log \bar{F}^{\leftarrow}(k_n/n)) \\ &+ (\widehat{\theta}_n(k_n) - \theta)(K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \\ &+ \log \frac{\ell(\exp K_\tau(\log(n/k_n)))}{\ell(\exp K_\tau(\log(1/p_n)))} \\ &\stackrel{\text{def}}{=} Q_n^{(1)} + Q_n^{(2)} + Q_n^{(3)}. \end{aligned}$$

The three terms are studied separately. First, note that in view of $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$, $Q_n^{(1)}$ can be expanded as

$$\begin{aligned} Q_n^{(1)} &= \log H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1, n})) - \log H^{\leftarrow}(\exp K_\tau(\log(n/k_n))) \\ &= \theta(K_\tau(E_{n-k_n+1, n}) - K_\tau(\log(n/k_n))) + \log \frac{\ell(\exp K_\tau(E_{n-k_n+1, n}))}{\ell(\exp K_\tau(\log(n/k_n)))} \\ &\stackrel{\text{def}}{=} \theta T_n^{(3,1)} + Q_n^{(1,2)}, \end{aligned}$$

where $T_n^{(3,1)}$ is defined in the proof of Theorem 1 as

$$T_n^{(3,1)} = K_\tau(E_{n-k_n+1, n}) - K_\tau(\log(n/k_n)) = O_{\mathbb{P}}(1) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}, \quad (22)$$

in view of (19). Moreover, $Q_n^{(1,2)} \stackrel{\text{def}}{=} \log \ell(\lambda_n x_n) - \log \ell(x_n)$, where $x_n = \exp K_\tau(\log(n/k_n)) \rightarrow \infty$ and $\lambda_n = \exp T_n^{(3,1)} \xrightarrow{P} 1$. Thus, from $(\mathbf{A}_2(\rho))$ we have

$$\begin{aligned} Q_n^{(1,2)} &= b(\exp K_\tau(\log(n/k_n))) K_\rho(\lambda_n)(1 + o_{\mathbb{P}}(1)) \\ &= b(\exp K_\tau(\log(n/k_n))) \log(\lambda_n)(1 + o_{\mathbb{P}}(1)) \\ &= O_{\mathbb{P}}(1) b(\exp K_\tau(\log(n/k_n))) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}, \end{aligned}$$

in view of (22). Since $b(x) \rightarrow 0$ as $x \rightarrow \infty$, it follows that

$$Q_n^{(1,2)} = o_{\mathbb{P}}\left(\frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}\right),$$

entailing

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(1)} = O_{\mathbb{P}}\left(\frac{(\log(n/k_n))^{\tau-1}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))}\right) = o_{\mathbb{P}}(1),$$

from (10). Now, concerning the second term, Theorem 1 entails that

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(2)} = \sqrt{k_n} \left(\hat{\theta}_n(k_n) - \theta \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Finally, $Q_n^{(3)} = \log \ell(x_n^*) - \log \ell(\lambda_n^* x_n^*)$ where $\lambda_n^* = \exp[K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))] \geq 1$ in view of (10) and $x_n^* = \exp K_\tau(\log(n/k_n)) \rightarrow \infty$. Thus, Lemma 1 entails

$$\frac{\sqrt{k_n} Q_n^{(3)}}{\log \lambda_n^*} \sim -\sqrt{k_n} b(x_n^*) \frac{K_\rho(\lambda_n^*)}{\log \lambda_n^*} = o\left(\frac{K_\rho(\lambda_n^*)}{\log \lambda_n^*}\right),$$

since $\sqrt{k_n} b(x_n^*) = \sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \rightarrow 0$. Taking account of the inequality $K_\rho(x) \leq \log x$ for all $x \geq 1$ yields

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(3)} = o(1).$$

Combining the above results, Theorem 2 follows. ■

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References

- [1] Arnold, B.C, Balakrishnan, N., Nagaraja H.N., (1992), *A First course in order statistics*, Wiley and sons.
- [2] Asimit, V., Li, D., Peng, L., (2010), Pitfalls in using Weibull tailed distributions, *Journal of Statistical Planning and Inference* **140**, 2018–2024.
- [3] Beirlant, J., Bouquiaux, C., Werker, B., (2006), Semiparametric lower bounds for tail index estimation, *Journal of Statistical Planning and Inference*, **136**, 705–729.
- [4] Beirlant, J., Broniatowski, M., Teugels, J.L., Vynckier, P., (1995), The mean residual life function at great age: Applications to tail estimation, *Journal of Statistical Planning and Inference*, **45**, 21–48.
- [5] Beirlant, J., Dierckx, G., Goegebeur, Y., Matthys, G., (1999), Tail index estimation and an exponential regression model, *Extremes*, **2**, 177–200.
- [6] Beirlant, J., Teugels, J.L., (1992), Modeling large claims in non-life insurance. *Insurance: Mathematics and Economics*, **11**, 17–29.
- [7] Beirlant, J., Teugels, J., Vynckier, P., (1996), *Practical analysis of extreme values*, Leuven university press, Leuven.
- [8] Berred, M., (1991), Record values and the estimation of the Weibull tail-coefficient, *Comptes Rendus de l'Académie des Sciences*, **T. 312**, Série I, 943–946.
- [9] Bingham, N.H., Goldie, C.M., Teugels, J.L., (1987), *Regular Variation*, Cambridge University Press.

- [10] Broniatowski, M., (1993), On the estimation of the Weibull tail coefficient, *Journal of Statistical Planning and Inference*, **35**, 349–366.
- [11] Diebolt, J., Gardes, L., Girard, S., Guillou, A., (2008a), Bias-reduced estimators of the Weibull tail-coefficient, *Test*, **17**, 311–331.
- [12] Diebolt, J., Gardes, L., Girard, S., Guillou, A., (2008b), Bias-reduced extreme quantiles estimators of Weibull tail-distributions, *Journal of Statistical Planning and Inference*, **138**, 1389–1401.
- [13] Dierckx, G., Beirlant, J., De Waal, D., Guillou, A. (2009), A new estimation method for Weibull-type tails based on the mean excess function, *Journal of the Statistical Planning and Inference*, **139**, 1905–1920.
- [14] Ditlevsen, O., (1994), Distribution Arbitrariness in Structural Reliability, *Structural Safety and Reliability*, 1241–1247, Balkema, Rotterdam.
- [15] Draisma, G., de Haan, L., Peng, L., Peireira, T. T., (1999), A Bootstrap-based method to achieve optimality in estimating the extreme-value index, *Extremes*, **2:4**, 367–404.
- [16] Embrechts, P., Klüppelberg, C., Mikosch, T., (1997), *Modelling extremal events*, Springer.
- [17] Falk, M., (1995), Some best parameter estimates for distributions with finite endpoint, *Statistics*, **27**, 115–125.
- [18] Feuerverger, A., Hall, P., (1999), Estimating a tail exponent by modelling departure from a Pareto distribution, *Annals of Statistics*, **27**, 760–781.
- [19] Fraga Alves, M.I, de Haan, L., Neves, C., (2005), Statistical inference for heavy and super-heavy tailed distributions, *4th International Conference on Extreme Value Analysis*, Gothenburg, Sweden.
- [20] Galambos, J., (1987), *The Asymptotic theory of extreme order statistics*, R.E. Krieger publishing company.
- [21] Gardes, L., Girard, S., (2005), Estimating extreme quantiles of Weibull tail-distributions, *Communication in Statistics - Theory and Methods*, **34**, 1065–1080.
- [22] Gardes, L., Girard, S., (2006), Comparison of Weibull tail-coefficient estimators, *REVSTAT - Statistical Journal*, **4**, 163–188.
- [23] Gardes, L., Girard, S., (2008), Estimation of the Weibull tail-coefficient with linear combination of upper order statistics, *Journal of Statistical Planning and Inference*, **138**, 1416–1427.
- [24] Geluk, J.L., de Haan, L., (1987), *Regular Variation, Extensions and Tauberian Theorems*, Math Centre Tracts, **40**, Centre for Mathematics and Computer Science, Amsterdam.
- [25] Girard, S., (2004), A Hill type estimate of the Weibull tail-coefficient, *Communication in Statistics - Theory and Methods*, **33**(2), 205–234.
- [26] Goegebeur, Y., Beirlant, J., de Wet, T. (2010), Generalized kernel estimators for the Weibull-tail coefficient, *Communications in Statistics - Theory and Methods*, to appear, <http://bib.mathematics.dk/preprint.php?id=IMADA-PP-2009-09>
- [27] Goegebeur, Y., Guillou, A. (2010), Goodness-of-fit testing for Weibull-type behavior, *Journal of Statistical Planning and Inference*, **140**, 1417–1436.

- [28] Gomes, I., Caeiro, F., (2002), A class of asymptotically unbiased semi-parametric estimators of the tail index, *Test*, **11**(2), 345–364.
- [29] Gomes, I., Caeiro, F., Figueiredo, F., (2004), Bias reduction of a tail index estimator through an external estimation of the second-order parameter, *Statistics*, **38**(6), 497–510.
- [30] de Haan, L., Ferreira, A., (2006), *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering, Springer.
- [31] de Haan, L., Peng, L., (1998), Comparison of tail index estimators, *Statistica Neerlandica*, **52**(1), 60–70.
- [32] Harris, R.J., (1996), Gumbel re-visited - a new look at extreme value statistics applied to wind speeds, *Journal of Wind Engineering and Industrial Aerodynamics*, **59**, 1–22.
- [33] Häusler, E., Teugels, J.L. (1985), On asymptotic normality of Hill’s estimator for the exponent of regular variation, *The Annals of Statistics*, **13**, 743–756.
- [34] Hill, B.M., (1975), A simple general approach to inference about the tail of a distribution, *The Annals of Statistics*, **3**, 1163–1174.
- [35] Petrov, V.V., (1975), *Sums of independent random variables*, Springer-Verlag, Berlin, Eidelberg, New York.
- [36] Smith, J., (1991), Estimating the upper tail of flood frequency distributions, *Water Resources Research*, **23**(8), 1657–1666.
- [37] Weissman, I., (1978), Estimation of parameters and large quantiles based on the k largest observations, *Journal of the American Statistical Association*, **73**, 812–815.

Appendix: Proof of auxiliary results

Proof of Lemma 1 – From $(\mathbf{A}_1(\tau, \theta))$ and $(\mathbf{A}_2(\rho))$, it is easy to infer that, for any constant $\tilde{C} > 0$, we have

$$\begin{aligned} \frac{1}{\tilde{C}b(x)} \left(\frac{H^\leftarrow(\lambda x) - H^\leftarrow(x)}{\theta H^\leftarrow(x)(1 + b(x)/\theta)} - \frac{\lambda^\theta - 1}{\theta} \right) &= \frac{\lambda^\theta}{\tilde{C}\theta} K_\rho(\lambda) - \frac{1}{\tilde{C}\theta} \frac{\lambda^\theta - 1}{\theta} + o(1) \\ &= \frac{\theta + \rho}{\tilde{C}\theta} \frac{1}{\rho} [K_{\theta+\rho}(\lambda) - K_\theta(\lambda)] + o(1). \end{aligned}$$

Then, choosing \tilde{C} such that $(\theta + \rho)/(\tilde{C}\theta) = 1$, a direct application of Lemma 5.2 in [15] yields, for any $\varepsilon > 0$ and $\lambda \geq 1$,

$$\begin{aligned} &\min(1, \lambda^{-\rho-\varepsilon}) \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| \\ &\leq \varepsilon \tilde{C}\theta |b(x)| |1 + b(x)/\theta| \min(1, \lambda^{-\rho-\varepsilon}) [\lambda^{-\theta} + 1 + 2\lambda^{\rho+\varepsilon}] \\ &\leq 4\varepsilon \tilde{C}\theta |b(x)| \min(1, \lambda^{-\rho-\varepsilon}) [1 + \lambda^{\rho+\varepsilon}] \\ &\leq 8\varepsilon \tilde{C}\theta |b(x)| \end{aligned}$$

for x large enough. Moreover, letting $0 < \varepsilon < -\rho$ yields

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| = o(b(x)). \quad (23)$$

Besides, $K_\rho(\lambda) - K_{-\theta}(\lambda)$ is bounded when $\rho < 0$, and therefore (23) can be simplified as

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)), \text{ as } x \rightarrow \infty,$$

and the conclusion follows. ■

Proof of Lemma 2 – (i) Let us consider for $t > 1$ and $q \in \mathbb{N} \setminus \{0\}$,

$$Q_q(t) = \int_0^\infty \left(\frac{K_\tau(x+t) - K_\tau(t)}{K'_\tau(t)} \right)^q e^{-x} dx.$$

There exists $\eta \in (0, 1)$ such that

$$\left| \frac{K_\tau(x+t) - K_\tau(t)}{xK'_\tau(t)} \right| = \left(1 + \frac{\eta x}{t} \right)^{\tau-1} \leq 1.$$

Thus, Lebesgue Theorem implies that

$$\lim_{t \rightarrow \infty} Q_q(t) = \int_0^\infty \lim_{t \rightarrow \infty} \left(1 + \frac{\eta x}{t} \right)^{q(\tau-1)} x^q e^{-x} dx = \int_0^\infty x^q e^{-x} dx = q!$$

which concludes the first part of the proof.

(ii) is a straightforward consequence of (i).

(iii) We have

$$\begin{aligned} \mu'_{1,\tau}(t) &= \int_0^\infty (K'_\tau(x+t) - K'_\tau(t)) e^{-x} dx \\ &= \int_0^\infty K'_\tau(x+t) e^{-x} dx - K'_\tau(t) \\ &= \int_0^\infty K_\tau(x+t) e^{-x} dx - K_\tau(t) - K'_\tau(t) \\ &= \mu_{1,\tau}(t) - t^{\tau-1}. \end{aligned}$$

Finally, (i) states that $t^{\tau-1}/\mu_{1,\tau}(t) \rightarrow 1$ as $t \rightarrow \infty$ which entails $\mu'_{1,\tau}(t)/\mu_{1,\tau}(t) \rightarrow 0$ as $t \rightarrow \infty$. ■

Proof of Lemma 3 – Recall that

$$\begin{aligned} \hat{\theta}_n &= \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \\ &\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^{\leftarrow}(\exp K_\tau(E_{n-i+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right), \end{aligned}$$

where $E_{1,n}, \dots, E_{n,n}$ are ordered statistics generated by n independent standard exponential random variables. The Rényi representation of the $\text{Exp}(1)$ ordered statistics (see [1], p. 72) yields

$$\{E_{n-i+1,n}\}_{i=1,\dots,k_n-1} \stackrel{d}{=} \{F_{k_n-i,k_n-1} + E_{n-k_n+1,n}\}_{i=1,\dots,k_n-1}, \quad (24)$$

where $\{F_{1,k_n-1}, \dots, F_{k_n-1,k_n-1}\}$ are ordered statistics independent from $E_{n-k_n+1,n}$ and generated by $k_n - 1$ independent standard exponential variables $\{F_1, \dots, F_{k_n-1}\}$. We thus have

$$\hat{\theta}_n(k_n) \stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^{\leftarrow}(\exp K_\tau(F_{k_n-i,k_n-1} + E_{n-k_n+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right)$$

$$\begin{aligned}
&\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^{\leftarrow}(\exp K_\tau(F_i + E_{n-k_n+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right) \\
&\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \left(\theta_{n,1}^{(1)}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \right)
\end{aligned}$$

in view of $(\mathbf{A}_1(\tau, \theta))$ and the conclusion follows. \blacksquare

Proof of Lemma 4 – A first order expansion of the function g leads to,

$$\sqrt{k_n}(g(E_{n-k_n+1,n}) - g(\log n/k_n)) = \sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n))g'(\tilde{\eta}_n),$$

with $\tilde{\eta}_n \in [\min(E_{n-k_n+1,n}, \log(n/k_n)), \max(E_{n-k_n+1,n}, \log(n/k_n))]$. Now, Lemma 1 in [25] shows that $\sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n)) \xrightarrow{d} \mathcal{N}(0, 1)$ which implies that $\tilde{\eta}_n = \log(n/k_n)(1 + o_{\mathbb{P}}(1))$ and the result follows. \blacksquare

Proof of Lemma 5 – Let us introduce for all $t \geq 1$ and $q \in \mathbb{N} \setminus \{0\}$,

$$S_n^{(q)}(t) = \frac{(k_n - 1)^{1/2}}{\sigma_{q,\tau}(t)} (\theta_{n,1}^{(q)}(t) - \mu_{q,\tau}(t)) = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}(t)} \sum_{i=1}^{k_n-1} Y_i^{(q)}(t),$$

where $Y_i^{(q)}(t) \stackrel{\text{def}}{=} (K_\tau(F_i + t) - K_\tau(t))^q - \mu_{q,\tau}(t)$, $i = 1, \dots, k_n - 1$ are centered, independent and identically distributed random variables with variance $\sigma_{q,\tau}^2(t)$. Clearly, in view of the Central Limit Theorem, for all $t \geq 1$ and $q \in \mathbb{N} \setminus \{0\}$, $S_n^{(q)}(t)$ converges in distribution to a standard Gaussian distribution. Our goal is to prove that, for all $x \in \mathbb{R}$ and $q \in \mathbb{N} \setminus \{0\}$,

$$\mathbb{P}(S_n^{(q)}(E_{n-k_n+1,n}) \leq x) \rightarrow \Phi(x) \text{ as } n \rightarrow \infty,$$

where Φ is the cumulative distribution function of the standard Gaussian distribution. Lemma 2(i) implies that for all $\varepsilon \in (0, 1)$, and $r \in \mathbb{N} \setminus \{0\}$, there exists $T_\varepsilon \geq 1$ such that for all $t \geq T_\varepsilon$,

$$(1 - \varepsilon) r! t^{r(\tau-1)} \leq \mu_{r,\tau}(t) \leq (1 + \varepsilon) r! t^{r(\tau-1)}. \quad (25)$$

Furthermore, for $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}(S_n^{(q)}(E_{n-k_n+1,n}) \leq x) - \Phi(x) &= \int_0^{T_\varepsilon} (\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)) h_n(t) dt \\
&+ \int_{T_\varepsilon}^\infty (\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)) h_n(t) dt \stackrel{\text{def}}{=} A_n + B_n,
\end{aligned}$$

where h_n is the density of the random variable $E_{n-k_n+1,n}$. First, let us focus on the term A_n . We have,

$$|A_n| \leq 2\mathbb{P}(E_{n-k_n+1,n} \leq T_\varepsilon).$$

Since $E_{n-k_n+1,n}/\log(n/k) \xrightarrow{P} 1$ (see [25], Lemma 1), it is easy to show that $A_n \rightarrow 0$. Now, let us consider the term B_n . For all $t \geq T_\varepsilon$,

$$\begin{aligned}
\mathbb{E}(|Y_1^{(q)}(t)|^3) &\leq \mathbb{E}((K_\tau(F_1 + t) - K_\tau(t))^q + \mu_{q,\tau}(t))^3 \\
&= \mu_{3q,\tau}(t) + 3\mu_{q,\tau}(t)\mu_{2q,\tau}(t) + 4\mu_{q,\tau}^3(t) \\
&\leq C_1(\varepsilon) t^{3q(\tau-1)} < \infty,
\end{aligned}$$

from (25). Here, and in the following, $C_1(\varepsilon)$, C_2 , $C_3(\varepsilon)$ and $C_4(\varepsilon)$ are positive constants independent of t . Thus, from Berry-Esséen's inequality (see [35], Theorem 3), we have:

$$\sup_x |\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)| \leq C_2 L_n \quad \text{with} \quad L_n = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}^3(t)} \mathbb{E}(|Y_1^{(q)}(t)|^3).$$

From (25), since $t \geq T_\varepsilon$,

$$\sigma_{q,\tau}^2(t) = \mu_{2q,\tau}(t) - \mu_{q,\tau}^2(t) \geq C_3(\varepsilon) t^{2q(\tau-1)}.$$

Thus, $L_n \leq C_4(\varepsilon)(k_n - 1)^{-1/2}$ and therefore

$$|B_n| \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \mathbb{P}(E_{n-k_n+1,n} \geq T_\varepsilon) \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \rightarrow 0,$$

which concludes the proof. \blacksquare

Proof of Lemma 6 – Let us consider the random variables $x_n = \exp[K_\tau(E_{n-k_n+1,n})]$ and $\lambda_{i,n} = \exp[K_\tau(F_i + E_{n-k_n+1,n}) - K_\tau(E_{n-k_n+1,n})]$, $i = 1, \dots, k_n - 1$. It is clear that $x_n \xrightarrow{P} \infty$ in view of Lemma 1 in [25] and $\lambda_{i,n} \geq 1$. Thus, letting

$$\theta_{n,3}(E_{n-k_n+1,n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho[\exp(K_\tau(F_i + E_{n-k_n+1,n}) - K_\tau(E_{n-k_n+1,n}))],$$

Lemma 1 entails

$$\theta_{n,2}(E_{n-k_n+1,n}) \stackrel{d}{=} b(\exp K_\tau(E_{n-k_n+1,n})) \theta_{n,3}(E_{n-k_n+1,n}) (1 + o_{\mathbb{P}}(1)).$$

Since $|K_\rho(\exp u) - u| \leq -\rho u^2/2$ for all $u \geq 0$, we have

$$\left| \theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) \right| \leq -\frac{\rho}{2} \theta_{n,1}^{(2)}(E_{n-k_n+1,n}).$$

Moreover, if $\tau = 1$, then

$$\theta_{n,3}(E_{n-k_n+1,n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho(\exp F_i) \xrightarrow{P} \int_0^{+\infty} K_\rho(\exp u) \exp(-u) du = \frac{1}{1 - \rho},$$

in view of the law of large numbers, and the conclusion follows. \blacksquare

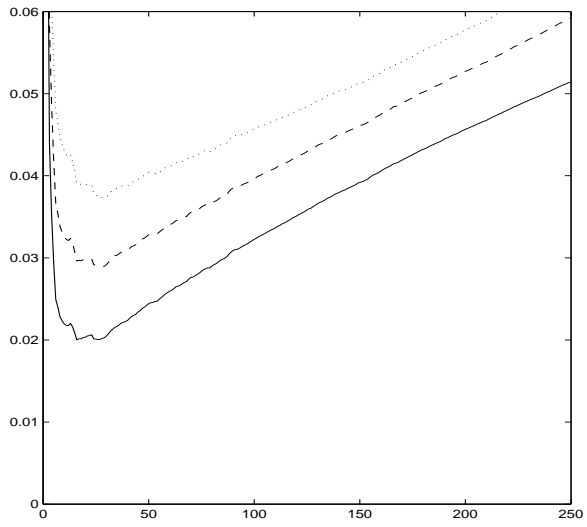
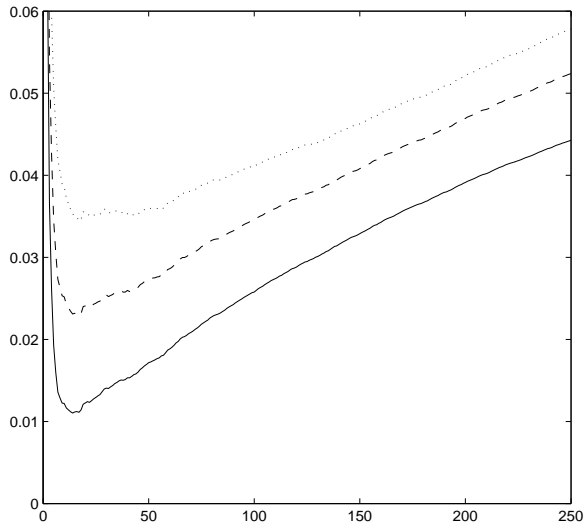


Figure 1: Empirical squared bias as a function of k obtained with $\widehat{\theta}_n(k_n)$ computed on 500 samples of size 500 from $F_{1/2,\tau,\rho}$. Up: $\rho = -1/2$, down: $\rho = -1/4$, solid line: $\tau = 1$, dashed line: $\tau = 1/2$, dotted line: $\tau = 0$.

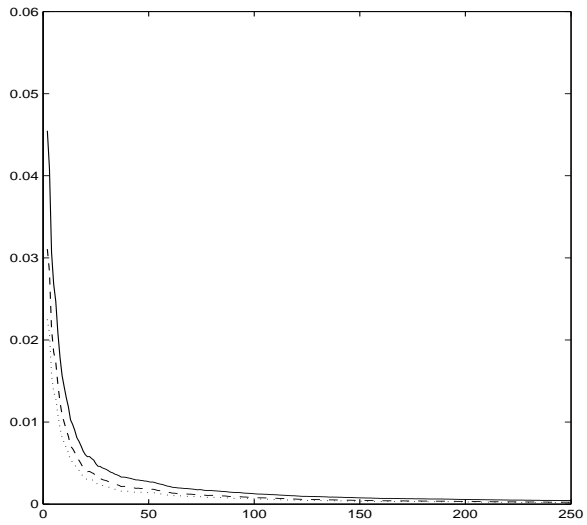
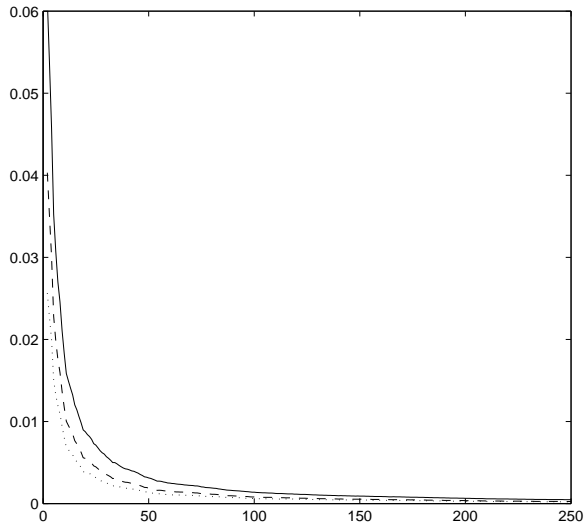


Figure 2: Empirical variance as a function of k obtained with $\widehat{\theta}_n(k_n)$ computed on 500 samples of size 500 from $F_{1/2,\tau,\rho}$. Up: $\rho = -1/2$, down: $\rho = -1/4$, solid line: $\tau = 1$, dashed line: $\tau = 1/2$, dotted line: $\tau = 0$.