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On the asymptotic normality of extreme-value estimators in the φ -tail distributions model

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Abstract

We introduce the new model of φ -tail distributions which depends on a function φ and on a parameter θ , called the φ -tail coefficient. Based on an intermediate sequence, estimators of θ and extreme quantiles are proposed. Their asymptotic normality is established under mild assumptions on φ . Depending on the function φ , this model includes very different distribution tail behaviours from the three classical maximum domains of attraction. In the particular cases of Pareto type tails or Weibull tails, our estimators coincide with classical ones proposed in the literature, thus permitting to retrieve their asymptotic normality in an unified way.

AMS Subject Classifications: 62G05, 62G20, 62G30.

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1 Definition of φ -tail distributions

Let F be a cumulative distribution function such that

$$1 - F(x) = \exp(-\varphi^\leftarrow(\log H(x))), \text{ for } x \geq x_*, \quad (1)$$

and verifying the following three conditions

(H.1) H^\leftarrow is an increasing function such that $H^\leftarrow(t) = \inf\{x, H(x) \geq t\} = t^\theta \ell(t)$,

where $\theta > 0$ and ℓ is a slowly varying function *i.e.* $\ell(\lambda x)/\ell(x) \rightarrow 1$ as $x \rightarrow \infty$ for all $\lambda \geq 1$. The function H^\leftarrow is said to be regularly varying at infinity with index θ and this property is denoted by $H^\leftarrow \in \mathcal{R}_\theta$, see [6] for more details on this topic.

(φ .1) φ is increasing and $\varphi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

(φ .2) φ is continuously differentiable, $\varphi' \in \mathcal{R}_\tau$, with $\tau \in [-1, 0]$, and there exists $M > 0$ such that $\varphi'(\cdot) \leq M$.

Remark 1 *If $\tau \neq 0$ in (φ .2), then the assumption $\varphi'(\cdot) \leq M$ is automatically fulfilled.*

In the following, such distributions are referred to as φ -tail distributions. The parameter θ is called the φ -tail coefficient and $x_* = H^\leftarrow(e^{\varphi(0)})$ is the starting point of the φ -tail distribution. We shall see in Section 4 that this model includes very different distribution tails behaviour, depending on the

function φ . For instance, it encompasses Pareto type distributions in the Fréchet Maximum Domain of Attraction as well as Weibull tail-distributions in the Gumbel Maximum Domain of Attraction. We refer to [4] for a general account on Weibull tail-distributions. All these distributions depend on the φ -tail coefficient θ which has to be estimated. In the first mentioned case of Pareto type distributions, θ reads as the tail index, while in the case of Weibull tail-distributions θ is the Weibull tail-coefficient. In the next section, an estimator $\widehat{\theta}_n$ of θ is proposed in the general context of φ -tail distributions and an estimator of extreme quantiles is derived. The asymptotic normality of these estimators is established in Section 3. Some illustrations are provided in Section 4 and concluding remarks are given in Section 5. Proofs are postponed to Section 6.

2 Inference

Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed random variables from a φ -tail distribution. The associated order statistics are denoted by $X_{1,n} \leq \dots \leq X_{n,n}$. In the sequel, (k_n) is an intermediate sequence of integers *i.e.* such that $1 \leq k_n \leq n$, $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. We propose the following estimator of θ :

$$\widehat{\theta}_n = \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})), \quad (2)$$

with, for all $t > 0$ and $q \in \mathbb{N}^*$,

$$\mu_q(t) = \int_0^\infty (\varphi(x+t) - \varphi(t))^q e^{-x} dx.$$

An important point is that the estimator (2) essentially consists in averaging the log-spacings between the upper-order statistics. It only depends on φ through the multiplicative normalizing constant. This characteristics can be intuitively understood by studying the log-spacing between two quantiles x_u and x_v of $1 - F$, with $0 < u < v \leq 1$. The definition of φ -distributions entails

$$\log x_u - \log x_v = \theta (\varphi(-\log u) - \varphi(-\log v)) + \log \left(\frac{\ell(\exp \varphi(-\log u))}{\ell(\exp \varphi(-\log v))} \right). \quad (3)$$

Now, since ℓ is a slowly-varying function, if the orders u and v of the quantiles are small enough, the second term can be neglected in the right-hand side of (3) to obtain

$$\log x_u - \log x_v \simeq \theta (\varphi(-\log u) - \varphi(-\log v)), \quad (4)$$

which shows that log-spacings are approximatively proportional to θ . Note that this property can be checked graphically on a sample by drawing a quantile-quantile plot adapted to φ -tail distributions. It consists in plotting the pairs $(\varphi(\log(n/i)), \log X_{n-i+1,n})$ for $i = 1, \dots, k_n$. From (4), the graph should be approximatively linear. Following the same ideas, an estimator of the extreme quantile x_{p_n} can be deduced from (2) by:

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \exp \left(\widehat{\theta}_n (\varphi(\log(1/p_n)) - \varphi(\log(n/k_n))) \right). \quad (5)$$

Recall that an extreme quantile x_{p_n} of order p_n is defined by the equation

$$1 - F(x_{p_n}) = p_n, \text{ with } 0 < p_n < 1/n.$$

The condition $p_n < 1/n$ is very important in this context. It usually implies that x_{p_n} is larger than the maximum observation of the sample. This necessity to extrapolate sample results to areas where no data are observed occurs in reliability [11], hydrology [27], finance [13],...

3 Asymptotic normality

To establish the asymptotic normality of $\widehat{\theta}_n$, a second-order condition on ℓ is necessary:

(H.2) There exist $\rho \leq 0$ and $b(x) \rightarrow 0$ such that uniformly locally on $\lambda \geq 1$

$$\log\left(\frac{\ell(\lambda x)}{\ell(x)}\right) \sim b(x)K_\rho(\lambda), \text{ when } x \rightarrow \infty,$$

$$\text{with } K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du.$$

It can be shown that necessarily $|b| \in \mathcal{R}_\rho$ (see [19]). The second order parameter $\rho \leq 0$ tunes the rate of convergence of $\ell(\lambda x)/\ell(x)$ to 1. The closer ρ is to 0, the slower is the convergence. Condition **(H.2)** is the cornerstone in all the proofs of asymptotic normality for extreme value estimators. It is used in [3, 24, 25] to prove the asymptotic normality of several estimators of the extreme value index.

Theorem 1 *Suppose **(H.1)**, **(H.2)**, **(\varphi.1)** and **(\varphi.2)** hold. Let (k_n) be an intermediate sequence such that $k_n^{1/2}b(\exp \varphi(\log(n/k_n))) \rightarrow 0$. Then,*

$$k_n^{1/2}(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Let us emphasize that the asymptotic distribution of $\widehat{\theta}_n$ does not depend on φ . The asymptotic normality of the extreme quantile estimator (5) can be deduced from Theorem 1:

Theorem 2 *Suppose the assumptions of Theorem 1 hold with $\rho < 0$. If, moreover,*

$$k_n^{1/2} \frac{b(\exp \varphi(\log(n/k_n)))}{\log(n/k_n)\varphi'(\log(n/k_n))} \rightarrow 0 \tag{6}$$

and there exists $c > 1$ such that

$$\frac{\log(1/p_n)}{\log(n/k_n)} \rightarrow c \tag{7}$$

then,

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))} \left(\frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2 K_{\tau+1}^2(c)).$$

Unsurprisingly, since $\tau+1 > 0$, the smaller is p_n , the larger is c and the larger is the asymptotic variance. It is also worth noticing that the speed of convergence, defined as $k_n^{1/2}/\{\log(n/k_n)\varphi'(\log(n/k_n))\}$, is larger for light tails ($\varphi'(t) \rightarrow 0$ rapidly) than for heavy tails.

Finally, we shall see in the next section that it can be of interest to slightly extend the class of φ -tail distributions thanks to a change of variable.

Remark 2 *Consider a sequence Y_1, Y_2, \dots, Y_n of independent and identically distributed random variables with distribution tail such that*

$$1 - G(x) = \exp(-\varphi^\leftarrow(\log H(\psi(x))), \text{ for } x \in (x_*, x^*), \tag{8}$$

where $x_* = \psi^\leftarrow(H^\leftarrow(e^{\varphi(0)}))$, $x^* = \lim_{t \rightarrow \infty} \psi^\leftarrow(t)$, H and φ are defined as previously and where ψ is an increasing function. Then, it is readily seen that the random variables defined as $X_i = \psi(Y_i)$, $i = 1, \dots, n$ follow a φ -tail distribution. As a straightforward consequence, Theorem 1 and Theorem 2 also hold for the estimators defined by

$$\begin{aligned} \widehat{\theta}_n &= \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(\psi(Y_{n-i+1,n})) - \log(\psi(Y_{n-k_n+1,n}))) \\ \widehat{y}_{p_n} &= \psi^\leftarrow\left(\psi(Y_{n-k_n+1,n}) \exp\left(\widehat{\theta}_n (\varphi(\log(1/p_n)) - \varphi(\log(n/k_n)))\right)\right). \end{aligned} \tag{9}$$

4 Examples of φ -tail distributions

4.1 Fréchet Maximum Domain of Attraction

Choosing $\varphi(t) = t$ in model (1) yields $1 - F \in \mathcal{R}_{-1/\theta}$, $\theta > 0$, which is verified if and only if F belongs to the Fréchet Maximum Domain of Attraction. In this case, θ is interpreted as the tail index,

$$\hat{\theta}_n = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))$$

is the Hill estimator [25] (since $\mu_q(t) = q!$ for all $t > 0$ and $q \in \mathbb{N}^*$) and

$$\hat{x}_{p_n} = X_{n-k_n+1,n} \left(\frac{k_n}{np_n} \right)^{\hat{\theta}_n}$$

is the Weissman estimator [28]. A straightforward application of the above theorems gives back the classical results:

Corollary 1 *Suppose (H.1), (H.2) hold and let $\varphi(t) = t$. Let (k_n) be an intermediate sequence such that $k_n^{1/2}b(n/k_n) \rightarrow 0$. Then,*

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover $\rho < 0$, and there exists $c > 1$ such that (7) holds then,

$$\frac{k_n^{1/2}}{\log(n/k_n)} \left(\frac{\hat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2(c-1)^2).$$

Note that such estimators exhibit a severe bias, which justifies the recent techniques of bias reduction. For instance, we can mention the exponential regression model for Pareto type distributions (see [3, 15]) or the estimation of the second order parameter ρ (see [21, 22]).

4.2 Weibull Maximum Domain of Attraction

Choosing $\varphi(t) = t$ and $\psi(t) = 1/(x^* - t)$ in (8) of Remark 2 yields the characterization of the Weibull Maximum Domain of Attraction. In this case, $-\theta$ represents the extreme value index and x^* is the endpoint of the distribution. Replacing x^* by its empirical counterpart $Y_{n,n}$ in (9) yields the negative Hill estimator (see for instance [14] or [23], paragraph 3.6.2).

4.3 Gumbel Maximum Domain of Attraction

Weibull tail-distributions. Choosing $\varphi(t) = \log t$ in model (1) yields $-\log(1 - F) \in \mathcal{R}_{1/\theta}$, $\theta > 0$, which is the definition of the so-called Weibull tail-distributions. They represent an important sub-family of the Gumbel Maximum Domain of Attraction and include, for instance, Gaussian, Gamma, and Weibull distributions. Note that, in this considered case, the estimator (2) corresponds to $\hat{\theta}_n^{(1)}$ introduced in [18], although there is no closed form available for μ_q , $q \in \mathbb{N}^*$, and

$$\hat{x}_{p_n} = X_{n-k_n+1,n} \left(\frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\hat{\theta}_n}$$

is the estimator introduced in [17]. As a consequence of Theorem 1 and Theorem 2, we obtain the asymptotic results given in [18], Corollary 3.1.

Corollary 2 Suppose **(H.1)**, **(H.2)** hold and let $\varphi(t) = \log t$. Let (k_n) be an intermediate sequence such that $k_n^{1/2}b(\log(n/k_n)) \rightarrow 0$. Then,

$$k_n^{1/2}(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover $\rho < 0$, and there exists $c > 1$ such that (7) holds then,

$$k_n^{1/2} \left(\frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2 \log^2(c)).$$

Let us also note that estimators $\widehat{\theta}_n^{(2)}$ and $\widehat{\theta}_n^{(3)}$ in [18] can be respectively deduced from $\widehat{\theta}_n$ by approximating μ_1 by a Riemann's sum or using the first order approximation $\mu_1(t) \sim \varphi'(t)$ as $t \rightarrow \infty$ given in Lemma 2. Other approaches based on the mean excess function [10], quantile function [2, 7] or record values [5] have also been investigated. Finally, most recent approaches include a bias correction based on an exponential regression model for Weibull tail-distributions (see [8, 9]).

Log-Weibull tail-distributions. Choosing $\varphi(t) = \psi(t) = \log t$ in model (8) gives rise to a new family of distributions defined by $-\log(1 - G(\exp(\cdot))) \in \mathcal{R}_{1/\theta}$, $\theta > 0$. If $\theta < 1$ and $H'' \in \mathcal{R}_{1/\theta-2}$, one can show that these « log-Weibull tail-distributions » belong to the Gumbel maximum domain of attraction. Indeed, there exists z and a positive constant c such that

$$1 - G(x) = c \exp \left(- \int_z^x 1/a(t) dt \right),$$

where $a(t) = t/H'(\log(t))$. Since

$$a'(t) = \frac{H'(\log(t)) - H''(\log(t))}{[H'(\log(t))]^2} \sim \frac{1}{H'(\log(t))} \rightarrow 0 \text{ as } t \rightarrow \infty,$$

Theorem 3.3.26 in [13] entails that G belongs to the Gumbel maximum domain of attraction. This family of distributions includes for instance the log-normal distribution ($\theta = 1/2$).

Remark 3 If $\theta > 1$ and $H' \in \mathcal{R}_{1/\theta-1}$, note that for every $\lambda > 1$,

$$H(\log(\lambda x)) - H(\log(x)) = \log(\lambda)H'(\eta \log(x)),$$

where $\eta \in [1, 2]$. Since $\theta > 1$, it is clear that $\lim_{x \rightarrow \infty} H'(\eta \log(x)) = 0$ and thus,

$$\lim_{x \rightarrow \infty} \frac{1 - G(\lambda x)}{1 - G(x)} = 1.$$

In other words, if $\theta > 1$, these log-Weibull tail-distributions belong to the class of super-heavy tailed distributions [16] (log-Cauchy, log-Pareto, etc ...). The case $\theta = 1$ is more complicated, the domain of attraction depending on the slowly-varying function ℓ in **(H.1)**. For example the log-gamma distribution belongs to the Fréchet maximum domain of attraction.

5 Concluding remarks and further work

As illustrated in the previous sections, φ -tail distributions provide a general framework for the analysis of extreme-value estimators. They cover several maximum domains of attraction and thus may be considered as an unifying tool for designing and studying estimators. In this paper, we limited ourselves to the estimation of the φ -tail coefficient and extreme quantiles. From the theoretical point of view, this work could be extended to estimators including a bias correction. To this end, an exponential regression model for φ -tail distributions extending [3, 8, 15] would be of interest. From the practical point of view, the φ -tail distribution model can be used to devise a test discriminating between Pareto type and Weibull tail-distributions. Indeed, denoting by

H_0 : There exists $\theta_0 > \theta_{\min} > 0$ such that $1 - F \in \mathcal{R}_{-1/\theta_0}$ (where θ_{\min} is known),
 H_1 : There exists $\theta_1 > 0$ such that $-\log(1 - F) \in \mathcal{R}_{1/\theta_1}$ and

$$\hat{H}_n := \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))$$

the Hill statistics, it appears that, from Corollary 1, \hat{H}_n is asymptotically Gaussian with mean θ_0 under H_0 , whereas, from Corollary 2, \hat{H}_n converges to zero in probability under H_1 . A natural idea is therefore to reject the null hypothesis if \hat{H}_n is small. To be more specific, the null hypothesis is rejected if

$$\left\{ \hat{H}_n < z_{n,\alpha} := \theta_{\min} \left(1 + \frac{\Phi^{-1}(\alpha)}{\sqrt{k_n}} \right) \right\},$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution and α is the nominal size of the test. Note that the introduction of θ_{\min} in H_0 permits to obtain a critical region independent of the unknown θ_0 . From Corollary 1, it is easy to obtain the following bound for the probability that the test statistics fall in the critical region when the null hypothesis is correct:

$$\mathbb{P} \left(\hat{H}_n < z_{n,\alpha} \mid H_0 \right) \leq \alpha(1 + o(1)),$$

whereas from Corollary 2, we deduce that

$$\mathbb{P} \left(\hat{H}_n > z_{n,\alpha} \mid H_1 \right) = o(1).$$

An extensive simulation study will be necessary in order to compare the actual size of the test with the nominal one and to compute the power of the test. A discussion on the selection of θ_{\min} will be also of great interest. This should lead to a forthcoming paper.

6 Proofs

We first give some preliminary lemmas. Their proofs are postponed to the appendix.

6.1 Preliminary lemmas

The first lemma is of general interest. It provides some uniform approximations based on **(H.1)** and **(H.2)**.

Lemma 1 *Suppose **(H.1)** and **(H.2)** hold. Then, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that, for x large enough,*

$$\sup_{\lambda \geq 1} \frac{\min(1, \lambda^{-\rho-\varepsilon})}{\varepsilon |b(x)|} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta} b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| \leq C.$$

If, moreover, $\rho < 0$, then

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)),$$

when $x \rightarrow \infty$.

The following lemma is of analytical nature. It provides first-order expansions which will be useful in the sequel.

Lemma 2 Under $(\varphi.1)$ and $(\varphi.2)$, for all $q \in \mathbb{N}^*$,

$$\begin{aligned} \mu_q(t) &\sim q!(\varphi'(t))^q \\ \mu'_1(t)/\mu_1(t) &\rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$.

The next lemma presents an expansion of $\widehat{\theta}_n$.

Lemma 3 Let (k_n) be an intermediate sequence. Then, under **(H.1)** and $(\varphi.1)$, the following expansions hold:

$$\widehat{\theta}_n = \frac{1}{\mu_1(\log(n/k_n))} \{ \theta_{n,1}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \},$$

with

$$\begin{aligned} \theta_{n,1}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\varphi(F_i + t) - \varphi(t)) \\ \theta_{n,2}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{\ell(\exp \varphi(F_i + t))}{\ell(\exp \varphi(t))} \right), \end{aligned}$$

and where $E_{n-k_n+1,n}$ is the $(n - k_n + 1)$ th order statistics associated to n independent standard exponential variables and $\{F_1, \dots, F_{k_n-1}\}$ are independent standard exponential variables and independent from $E_{n-k_n+1,n}$.

The next two lemmas provide the key results for establishing the asymptotic distribution of $\widehat{\theta}_n$. They describe the asymptotic behavior of the random terms appearing in Lemma 3.

Lemma 4 Let (k_n) be an intermediate sequence. Then, under $(\varphi.1)$ and $(\varphi.2)$,

$$\frac{k_n^{1/2}}{\sigma_1(E_{n-k_n+1,n})} \{ \theta_{n,1}(E_{n-k_n+1,n}) - \mu_1(E_{n-k_n+1,n}) \} \xrightarrow{d} \mathcal{N}(0, 1),$$

where we have defined for all $t > 0$, $\sigma_1^2(t) = \mu_2(t) - \mu_1^2(t)$.

Lemma 5 Suppose **(H.1)**, **(H.2)**, $(\varphi.1)$ and $(\varphi.2)$ hold. Let (k_n) be an intermediate sequence. Then,

$$\theta_{n,2}(E_{n-k_n+1,n}) = \theta_{n,1}(E_{n-k_n+1,n}) O_{\mathbb{P}}(b(\exp \varphi(E_{n-k_n+1,n}))).$$

6.2 Proofs of the main results

Proof of Theorem 1 – Lemma 4 states that

$$\frac{k_n^{1/2}}{\sigma_1(E_{n-k_n+1,n})} \{\theta_{n,1}(E_{n-k_n+1,n}) - \mu_1(E_{n-k_n+1,n})\} = \xi_n$$

where $\xi_n \xrightarrow{d} \mathcal{N}(0, 1)$. Then, by Lemma 3,

$$\begin{aligned} k_n^{1/2}(\widehat{\theta}_n - \theta) &= \theta \frac{\sigma_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} \xi_n + k_n^{1/2} \theta \left(\frac{\mu_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} - 1 \right) + k_n^{1/2} \frac{\theta_{n,2}(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} \\ &=: T_n^{(1)} + T_n^{(2)} + T_n^{(3)}, \end{aligned}$$

and the three terms are studied separately. First,

$$T_n^{(1)} = \frac{\sigma_1(E_{n-k_n+1,n})}{\mu_1(E_{n-k_n+1,n})} \frac{\mu_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} \theta \xi_n$$

and, from Lemma 2,

$$\frac{\sigma_1(E_{n-k_n+1,n})}{\mu_1(E_{n-k_n+1,n})} = \left(\frac{\mu_2(E_{n-k_n+1,n})}{\mu_1^2(E_{n-k_n+1,n})} - 1 \right)^{1/2} \xrightarrow{\mathbb{P}} 1. \quad (10)$$

Besides, since φ' is regularly varying and $E_{n-k_n+1,n}/\log(n/k_n) \xrightarrow{\mathbb{P}} 1$ (see [20], Lemma 1) we have

$$\frac{\mu_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} = \frac{\varphi'(E_{n-k_n+1,n})}{\varphi'(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) \xrightarrow{\mathbb{P}} 1. \quad (11)$$

As a preliminary conclusion, (10) and (11) yield

$$T_n^{(1)} = \theta \xi_n (1 + o_{\mathbb{P}}(1)). \quad (12)$$

Second, using a first-order expansion of μ_1 , one can show that there exists a random variable $\eta_n \in (\min(E_{n-k_n+1,n}, \log(n/k_n)), \max(E_{n-k_n+1,n}, \log(n/k_n)))$ such that

$$\begin{aligned} T_n^{(2)} &= \theta k_n^{1/2} (E_{n-k_n+1,n} - \log(n/k_n)) \frac{\mu_1'(\eta_n)}{\mu_1(\log(n/k_n))} \\ &= O_{\mathbb{P}}(1) \frac{\mu_1'(\eta_n)}{\mu_1(\eta_n)} \frac{\mu_1(\eta_n)}{\mu_1(\log(n/k_n))}, \end{aligned}$$

since $k_n^{1/2} (E_{n-k_n+1,n} - \log(n/k_n)) \xrightarrow{d} \mathcal{N}(0, 1)$ (see [20], Lemma 1). Applying Lemma 2 twice yields

$$T_n^{(2)} = o_{\mathbb{P}}(1) \frac{\varphi'(\eta_n)}{\varphi'(\log(n/k_n))} = o_{\mathbb{P}}(1), \quad (13)$$

since $\eta_n/\log(n/k_n) \xrightarrow{\mathbb{P}} 1$ entails $\varphi'(\eta_n)/\varphi'(\log(n/k_n)) \xrightarrow{\mathbb{P}} 1$. In view of Lemma 4 and Lemma 5, $T_n^{(3)}$ can be expanded as

$$\begin{aligned} T_n^{(3)} &= O_{\mathbb{P}} \left(k_n^{1/2} b(\exp \varphi(E_{n-k_n+1,n})) \right) \left(\frac{\mu_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} + \frac{\sigma_1(E_{n-k_n+1,n})}{\mu_1(\log(n/k_n))} k_n^{-1/2} \xi_n \right) \\ &= O_{\mathbb{P}} \left(k_n^{1/2} b(\exp \varphi(E_{n-k_n+1,n})) \right), \end{aligned}$$

from (10) and (11). Finally, there exists $\eta'_n \in (\min(E_{n-k_n+1,n}, \log(n/k_n)), \max(E_{n-k_n+1,n}, \log(n/k_n)))$ such that

$$\begin{aligned} \frac{\exp \varphi(E_{n-k_n+1,n})}{\exp \varphi(\log(n/k_n))} &= \exp((E_{n-k_n+1,n} - \log(n/k_n))\varphi'(\eta'_n)) \\ &= \exp\left(O_{\mathbb{P}}(k_n^{-1/2})\varphi'(\eta'_n)\right) \xrightarrow{\mathbb{P}} 1, \end{aligned} \quad (14)$$

in view of $(\varphi.2)$. Therefore, b being regularly varying,

$$\frac{b(\exp \varphi(E_{n-k_n+1,n}))}{b(\exp \varphi(\log(n/k_n)))} \xrightarrow{\mathbb{P}} 1$$

as well, and consequently

$$T_n^{(3)} = O_{\mathbb{P}}\left(k_n^{1/2}b(\exp \varphi(\log(n/k_n)))\right) = o_{\mathbb{P}}(1) \quad (15)$$

by assumption. Collecting (12), (13) and (15) concludes the proof. \blacksquare

Proof of Theorem 2 – From (5), one can infer that

$$\begin{aligned} \log \widehat{x}_{p_n} - \log x_{p_n} &= \left(\log X_{n-k_n+1,n} - \log F^{\leftarrow}\left(1 - \frac{k_n}{n}\right) \right) \\ &\quad + \left(\widehat{\theta}_n - \theta \right) (\varphi(\log(1/p_n)) - \varphi(\log(n/k_n))) \\ &\quad + \log \frac{\ell(\exp \varphi(\log(n/k_n)))}{\ell(\exp \varphi(\log(1/p_n)))} \\ &=: Q_n^{(1)} + Q_n^{(2)} + Q_n^{(3)}. \end{aligned}$$

The three terms are studied separately. First, note that in view of **(H.1)** and **(H.2)**, $Q_n^{(1)}$ can be expanded as

$$\begin{aligned} Q_n^{(1)} &= \log H^{\leftarrow}(\exp \varphi(E_{n-k_n+1,n})) - \log H^{\leftarrow}(\exp \varphi(\log(n/k_n))) \\ &= \theta(\varphi(E_{n-k_n+1,n}) - \varphi(\log(n/k_n))) + \log \frac{\ell(\exp \varphi(E_{n-k_n+1,n}))}{\ell(\exp \varphi(\log(n/k_n)))} \\ &=: Q_n^{(4)} + Q_n^{(5)}. \end{aligned}$$

Keeping in mind the notations introduced in the proof of Theorem 1, we obtain

$$\begin{aligned} Q_n^{(4)} &\stackrel{d}{=} \theta(E_{n-k_n+1,n} - \log(n/k_n))\varphi'(\eta'_n) \\ &= \theta(E_{n-k_n+1,n} - \log(n/k_n))\varphi'(\log(n/k_n))(1 + o_{\mathbb{P}}(1)), \end{aligned}$$

since φ' is regularly varying. Now, taking into account that $k_n^{1/2}(E_{n-k_n+1,n} - \log(n/k_n))$ converges in distribution to a standard Gaussian random variable yields

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))}Q_n^{(4)} = o_{\mathbb{P}}(1). \quad (16)$$

From (14) and **(H.2)**,

$$\begin{aligned} Q_n^{(5)} &= \log \frac{\ell(\exp \varphi(\log(n/k_n))(1 + o_{\mathbb{P}}(1)))}{\ell(\exp \varphi(\log(n/k_n)))} \\ &= b(\exp \varphi(\log(n/k_n)))K_{\rho}(1 + o_{\mathbb{P}}(1)) \\ &= b(\exp \varphi(\log(n/k_n)))o_{\mathbb{P}}(1), \end{aligned}$$

since $K_\rho(u) \rightarrow 0$ as $u \rightarrow 1$ for all $\rho \leq 0$. As a preliminary result, we obtain

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))}Q_n^{(1)} = o_{\mathbb{P}}(1), \quad (17)$$

as a simple consequence of (16) and condition (6). The second term $Q_n^{(2)}$ can be rewritten as follows

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))}Q_n^{(2)} = k_n^{1/2}(\hat{\theta}_n - \theta) \int_1^\infty \frac{\varphi'(u \log(n/k_n))}{\varphi'(\log(n/k_n))} \mathbb{1}_{\{u \leq \frac{\log(1/p_n)}{\log(n/k_n)}\}} du.$$

Assumption (7) and Potter bounds entail that, for all $\varepsilon > 0$ and n large enough,

$$\frac{\varphi'(u \log(n/k_n))}{\varphi'(\log(n/k_n))} \mathbb{1}_{\{u \leq \frac{\log(1/p_n)}{\log(n/k_n)}\}} \leq 2u^{\tau+\varepsilon} \mathbb{1}_{\{u \leq c+1\}}$$

and thus Lebesgue theorem yields

$$\int_1^\infty \frac{\varphi'(u \log(n/k_n))}{\varphi'(\log(n/k_n))} \mathbb{1}_{\{u \leq \frac{\log(1/p_n)}{\log(n/k_n)}\}} du \longrightarrow \int_1^c u^\tau du = K_{\tau+1}(c),$$

as $n \rightarrow \infty$ since $\varphi' \in \mathcal{R}_\tau$. Consequently,

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))}Q_n^{(2)} = k_n^{1/2}(\hat{\theta}_n - \theta) K_{\tau+1}(c)(1 + o_{\mathbb{P}}(1)) \stackrel{d}{\rightarrow} \mathcal{N}(0, \theta^2 K_{\tau+1}^2(c)), \quad (18)$$

from Theorem 1. Finally, since $\rho < 0$ and $0 \leq K_\rho(u) \leq -1/\rho$ for all $u \geq 1$, Lemma 1 yields

$$\frac{\ell(\exp \varphi(\log(n/k_n)))}{\ell(\exp \varphi(\log(1/p_n)))} \rightarrow 1$$

as $n \rightarrow \infty$ and thus

$$\begin{aligned} Q_n^{(3)} &= \left(\frac{\ell(\exp \varphi(\log(n/k_n)))}{\ell(\exp \varphi(\log(1/p_n)))} - 1 \right) (1 + o(1)) \\ &= -b(\exp \varphi(\log(n/k_n))) \{K_\rho(\exp[\varphi(\log(1/p_n)) - \varphi(\log(n/k_n))]) + o(1)\} (1 + o(1)) \\ &= O(b(\exp \varphi(\log(n/k_n)))) \end{aligned}$$

since K_ρ is bounded. It follows that

$$\frac{k_n^{1/2}}{\log(n/k_n)\varphi'(\log(n/k_n))}Q_n^{(3)} = o(1), \quad (19)$$

under condition (6). Combining (17), (18) and (19), Theorem 2 follows from a simple application of the delta method. \blacksquare

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Appendix: Proof of lemmas

Proof of Lemma 1 – It is based on an application of Lemma 5.2 in [12]. Indeed, from (H.1) and (H.2), it is easy to infer that, for any constant $\tilde{C} > 0$, we have

$$\begin{aligned} \frac{\frac{H^-(\lambda x) - H^-(x)}{\theta H^-(x)(1 + \frac{1}{\theta}b(x))} - \frac{\lambda^\theta - 1}{\theta}}{\tilde{C}b(x)} &= \frac{\lambda^\theta}{\tilde{C}\theta} K_\rho(\lambda) - \frac{1}{\tilde{C}\theta} \frac{\lambda^\theta - 1}{\theta} + o(1) \\ &= \frac{\theta + \rho}{\tilde{C}\theta} \frac{1}{\rho} [K_{\theta+\rho}(\lambda) - K_\theta(\lambda)] + o(1). \end{aligned}$$

Then, by choosing \tilde{C} such that $\frac{\theta + \rho}{\tilde{C}\theta} = 1$, a direct application of this lemma yields, for any $\varepsilon > 0$ and $\lambda \geq 1$,

$$\begin{aligned} &\min(1, \lambda^{-\rho-\varepsilon}) \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| \\ &\leq \varepsilon \tilde{C}\theta |b(x)| \left| 1 + \frac{1}{\theta}b(x) \right| \min(1, \lambda^{-\rho-\varepsilon}) [\lambda^{-\theta} + 1 + 2\lambda^{\rho+\varepsilon}] \\ &\leq 4\varepsilon \tilde{C}\theta |b(x)| \min(1, \lambda^{-\rho-\varepsilon}) [1 + \lambda^{\rho+\varepsilon}] \\ &\leq 8\varepsilon \tilde{C}\theta |b(x)| \end{aligned}$$

for x large enough. Letting $C = 8\tilde{C}\theta$, the first part of the lemma is proved. Moreover, if $\rho < 0$, choosing $\varepsilon < -\rho$ yields

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| = o(b(x)). \quad (20)$$

Besides, $K_\rho(\lambda) - K_{-\theta}(\lambda)$ is bounded when $\rho < 0$, and therefore (20) can be simplified as

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)),$$

as $x \rightarrow \infty$. ■

Proof of Lemma 2 – Let us consider for $t > 1$ and $q \in \mathbb{N}^*$,

$$Q_q(t) = \int_0^\infty \left(\frac{\varphi(x+t) - \varphi(t)}{\varphi'(t)} \right)^q e^{-x} dx.$$

Under (φ.1) and (φ.2), there exists $\eta \in (0, 1)$ and $\varepsilon_1 > -\tau$ such that

$$\left| \frac{\varphi(x+t) - \varphi(t)}{x\varphi'(t)} \right| = \frac{\varphi'(t+\eta x)}{\varphi'(t)} \leq 2 \left(1 + \frac{\eta x}{t} \right)^{\tau+\varepsilon_1} \leq 2(1+x)^{\tau+\varepsilon_1},$$

for t large enough, from Potter bounds (see [6], Theorem 1.5.6 (iii)). Thus, Lebesgue Theorem implies that

$$\lim_{t \rightarrow \infty} Q_q(t) = \int_0^\infty \lim_{t \rightarrow \infty} \left(\frac{\varphi'(t+\eta x)}{\varphi'(t)} \right)^q x^q e^{-x} dx.$$

Finally, since φ' is regularly varying and $(t+\eta x)/t \rightarrow 1$ as $t \rightarrow \infty$ for all $x \geq 0$, it follows that $\varphi'(t+\eta x)/\varphi'(t) \rightarrow 1$ as well and therefore

$$\lim_{t \rightarrow \infty} Q_q(t) = \int_0^\infty x^q e^{-x} dx = q!$$

which concludes the first part of the proof. Potter bounds imply that there exists $\varepsilon_2 > -\tau$ such that

$$|\varphi'(x+t) - \varphi'(t)| \leq \varphi'(t) \left(\frac{\varphi'(x+t)}{\varphi'(t)} + 1 \right) \leq M \left(2 \left(1 + \frac{x}{t} \right)^{\tau+\varepsilon_2} + 1 \right) \leq 2M(1+x)^{\tau+\varepsilon_2} + M.$$

Therefore, $\mu'_1(t)$ can be written as

$$\begin{aligned} \mu'_1(t) &= \int_0^\infty (\varphi'(x+t) - \varphi'(t))e^{-x} dx \\ &= \int_0^\infty \varphi'(x+t)e^{-x} dx - \varphi'(t) \\ &= \int_0^\infty \varphi(x+t)e^{-x} dx - \varphi(t) - \varphi'(t) \\ &= \mu_1(t) - \varphi'(t), \end{aligned}$$

after integrating by parts and remarking that $\varphi(t)e^{-t} \rightarrow 0$ as $t \rightarrow \infty$. Finally, the first part of the lemma states that $\varphi'(t)/\mu_1(t) \rightarrow 1$ as $t \rightarrow \infty$ which entails $\mu'_1(t)/\mu_1(t) \rightarrow 0$ as $t \rightarrow \infty$. \blacksquare

Proof of Lemma 3 – Recall that

$$\begin{aligned} \widehat{\theta}_n &= \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \\ &\stackrel{d}{=} \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^\leftarrow(\exp \varphi(E_{n-i+1,n}))}{H^\leftarrow(\exp \varphi(E_{n-k_n+1,n}))} \right), \end{aligned}$$

where $E_{1,n}, \dots, E_{n,n}$ are ordered statistics generated by n independent standard exponential random variables. The Rényi representation of the Exp(1) ordered statistics (see [1], p. 72) yields

$$\{E_{n-i+1,n}\}_{i=1, \dots, k_n-1} \stackrel{d}{=} \{F_{k_n-i, k_n-1} + E_{n-k_n+1,n}\}_{i=1, \dots, k_n-1}, \quad (21)$$

where $\{F_{1, k_n-1}, \dots, F_{k_n-1, k_n-1}\}$ are ordered statistics independent from $E_{n-k_n+1,n}$ and generated by $k_n - 1$ independent standard exponential variables $\{F_1, \dots, F_{k_n-1}\}$. We thus have

$$\begin{aligned} \widehat{\theta}_n &\stackrel{d}{=} \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^\leftarrow(\exp \varphi(F_{k_n-i, k_n-1} + E_{n-k_n+1,n}))}{H^\leftarrow(\exp \varphi(E_{n-k_n+1,n}))} \right) \\ &\stackrel{d}{=} \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left(\frac{H^\leftarrow(\exp \varphi(F_i + E_{n-k_n+1,n}))}{H^\leftarrow(\exp \varphi(E_{n-k_n+1,n}))} \right) \\ &\stackrel{d}{=} \frac{1}{\mu_1(\log(n/k_n))} (\theta_{n,1}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n})) \end{aligned}$$

in view of **(H.1)** and the conclusion follows. \blacksquare

Proof of Lemma 4 – Let us introduce for all $t \geq 1$,

$$S_n(t) = \frac{(k_n - 1)^{1/2}}{\sigma_1(t)} (\theta_{n,1}(t) - \mu_1(t)) = \frac{(k_n - 1)^{-1/2}}{\sigma_1(t)} \sum_{i=1}^{k_n-1} Y_i(t),$$

where $Y_i(t) := \varphi(F_i + t) - \varphi(t) - \mu_1(t)$, $i = 1, \dots, k_n - 1$ are centered, independent and identically distributed random variables with variance $\sigma_1^2(t)$. Clearly, in view of the Central Limit Theorem,

for all $t \geq 1$, $S_n(t)$ converges in distribution to a standard Gaussian distribution. Our goal is to prove that for $x \in \mathbb{R}$,

$$\mathbb{P}(S_n(E_{n-k_n+1,n}) \leq x) - \Phi(x) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where Φ is the cumulative distribution function of the standard Gaussian distribution. Lemma 2 implies that for all $\varepsilon \in (0, 1)$, and $q \in \mathbb{N}^*$, there exists $T_\varepsilon \geq 1$ such that for all $t \geq T_\varepsilon$,

$$(1 - \varepsilon)q!(\varphi'(t))^q \leq \mu_q(t) \leq (1 + \varepsilon)q!(\varphi'(t))^q. \quad (22)$$

Furthermore, for $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(S_n(E_{n-k_n+1,n}) \leq x) - \Phi(x) &= \int_0^{T_\varepsilon} (\mathbb{P}(S_n(t) \leq x) - \Phi(x))h_n(t)dt \\ &+ \int_{T_\varepsilon}^\infty (\mathbb{P}(S_n(t) \leq x) - \Phi(x))h_n(t)dt =: A_n + B_n, \end{aligned}$$

where h_n is the density of the random variable $E_{n-k_n+1,n}$. First, let us focus on the term A_n . We have,

$$|A_n| \leq 2\mathbb{P}(E_{n-k_n+1,n} \leq T_\varepsilon).$$

Since $E_{n-k_n+1,n}/\log(n/k) \xrightarrow{\mathbb{P}} 1$ (see [20], Lemma 1), it is easy to show that $A_n \rightarrow 0$. Now, let us consider the term B_n . For all $t \geq T_\varepsilon$,

$$\begin{aligned} \mathbb{E}(|Y_1(t)|^3) &\leq \mathbb{E}(\varphi(F_1 + t) - \varphi(t) + \mu_1(t))^3 \\ &= \mu_3(t) + 3\mu_1(t)\mu_2(t) + 4\mu_1^3(t) \\ &\leq C_1(\varepsilon)(\varphi'(t))^3 < \infty, \end{aligned}$$

from (22) where $C_1(\varepsilon)$ is a constant independent of t . Thus, from Berry-Esséen's inequality (see [26], Theorem 3), we have:

$$\sup_x |\mathbb{P}(S_n(t) \leq x) - \Phi(x)| \leq C_2 L_n,$$

where C_2 is a positive constant and

$$L_n = \frac{(k_n - 1)^{-1/2}}{\sigma_1^3(t)} \mathbb{E}(|Y_1(t)|^3).$$

From (22), since $t \geq T_\varepsilon$,

$$\sigma_1^2(t) = \mu_2(t) - \mu_1^2(t) \geq C_3(\varepsilon)(\varphi'(t))^2,$$

where $C_3(\varepsilon)$ is a constant independent of t . Thus, $L_n \leq C_4(\varepsilon)(k_n - 1)^{-1/2}$ where $C_4(\varepsilon)$ is a constant independent of t , and therefore

$$|B_n| \leq C_2 C_4(\varepsilon)(k_n - 1)^{-1/2} \mathbb{P}(E_{n-k_n+1,n} \geq T_\varepsilon) \leq C_2 C_4(\varepsilon)(k_n - 1)^{-1/2} \rightarrow 0,$$

which concludes the proof. ■

Proof of Lemma 5 – Let us consider the random variables $x_n = \exp \varphi(E_{n-k_n+1,n})$ and $\lambda_{i,n} = \exp\{\varphi(F_i + E_{n-k_n+1,n}) - \varphi(E_{n-k_n+1,n})\}$, $i = 1, \dots, k_n - 1$. It is clear that $x_n \xrightarrow{\mathbb{P}} \infty$, in view of ($\varphi.1$) and Lemma 1 in [20]. Moreover from ($\varphi.2$), there exists $\eta_{i,n} \in (0, 1)$ such that

$$1 \leq \lambda_{i,n} = \exp\{F_i \varphi'(E_{n-k_n+1,n} + \eta_{i,n} F_i)\} \leq \exp(M F_i),$$

and thus **(H.2)** entails

$$\theta_{n,2}(E_{n-k_n+1,n}) \stackrel{d}{=} (1+o_{\mathbb{P}}(1))b(\exp \varphi(E_{n-k_n+1,n}))\frac{1}{k_n-1} \sum_{i=1}^{k_n-1} K_{\rho}(\exp(\varphi(F_i + E_{n-k_n+1,n}) - \varphi(E_{n-k_n+1,n}))).$$

Since $0 \leq K_{\rho}(u) \leq \log u$ for all $u \geq 1$, we have

$$\begin{aligned} \theta_{n,2}(E_{n-k_n+1,n}) &= O_{\mathbb{P}}(b(\exp \varphi(E_{n-k_n+1,n})))\frac{1}{k_n-1} \sum_{i=1}^{k_n-1} (\varphi(F_i + E_{n-k_n+1,n}) - \varphi(E_{n-k_n+1,n})) \\ &= \theta_{n,1}(E_{n-k_n+1,n})O_{\mathbb{P}}(b(\exp \varphi(E_{n-k_n+1,n}))) \end{aligned}$$

and the conclusion follows. ■