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► **To cite this version:**

Salim Bouzebda, Tarek Zari. Strong Approximation of Empirical Copula Processes by Gaussian Processes. Statistics, 2013. hal-00340276v3

HAL Id: hal-00340276

<https://hal.science/hal-00340276v3>

Submitted on 27 Aug 2010

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Strong Approximation of Empirical Copula Processes by Gaussian Processes

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August 27, 2010

Abstract

We provide the strong approximation of empirical copula processes by a Gaussian process. In addition we establish a strong approximation of the smoothed empirical copula processes and a law of iterated logarithm.

AMS Subject Classifications: Primary 60F17 ; secondary 62G20 ; 62H10 ; 60F15.

Keywords: Empirical Copula processes ; Strong invariance principles ; Kernel-type-estimator ; Kiefer processes ; Gaussian processes.

1 Introduction

The aim of the present paper is to provide the strong (almost sure) approximations of the empirical copula process $\{\mathbb{A}_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$, defined in (1.8) below, by a single Gaussian process $\{\mathcal{H}_C(\mathbf{u}, n) : \mathbf{u} \in [0, 1]^d, n \geq 1\}$. Precisely, we obtain the strong approximations of $\mathbb{A}_n(\mathbf{u})$ in terms of Gaussian process in both \mathbf{u} and n . We will be mainly concerned with the case where $\mathbb{A}_n(\cdot)$ is generated by a sample of random vectors with dependent marginals. Suppose $\mathbf{X} = (X_1, \dots, X_d)$ is

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a random vector with distribution function $\mathbf{F}(\cdot)$ and continuous marginals $F_j(\cdot)$, for $j = 1, \dots, d$. Then the unique copula of \mathbf{X} is defined as

$$\mathbf{C}(u_1, \dots, u_d) = \mathbf{F}(F_1^-(u_1), \dots, F_d^-(u_d)) \text{ for } (u_1, \dots, u_d) \in [0, 1]^d \quad (1.1)$$

where, for $j = 1, \dots, d$, $F_j^-(u) = \inf\{x : F_j(x) \geq u\}$, with $u \in [0, 1]$, is the quantile function of $F_j(\cdot)$. In the monographs by Nelsen (2006) and Joe (1997) the reader may find detailed ingredients of the modelling theory as well as surveys of the commonly used copulas. Copulas have proved to be a very useful tool in the analysis of dependency structures. To be specific, copula $\mathbf{C}(\cdot)$ ‘‘couples’’ the joint distribution function $\mathbf{F}(\cdot)$ to its univariate marginals, capturing as such the dependence structure between the components of $\mathbf{X} = (X_1, \dots, X_d)$. This feature has motivated successful applications in actuarial science and survival analysis (see, e.g., Frees and Valdez (1998), Cui and Sun (2004)). In the literature on risk management and, more generally, in mathematical economics and mathematical finance modelling, a number of illustrations are provided (refer to books of Cherubini *et al.* (2004) and McNeil *et al.* (2005)), in particular, in the context of asset pricing and credit risk management. First, we introduce some definitions and notations.

Let $\mathbf{X}_i = (X_{1i}, \dots, X_{di})$, $i = 1, 2, \dots$, be a sequence an independent replicae of a d -dimensional random vector $\mathbf{X} \in \mathbb{R}^d$. Setting $\mathbb{1}_A(\cdot)$ for the indicator function of A , we define, for each $n \geq 1$, the empirical counterparts of $\mathbf{F}(\cdot)$, $F_1(\cdot), \dots, F_d(\cdot)$ and $F_1^-(\cdot), \dots, F_d^-(\cdot)$, respectively, by setting, for $j = 1, \dots, d$,

$$\mathbf{F}_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\mathbf{x}_i \leq \mathbf{x}\}} = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{1}_{\{X_{ji} \leq x_j\}}, \text{ for } \mathbf{x} \in \overline{\mathbb{R}}^d, \quad (1.2)$$

$$F_{jn}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_{ji} \leq x\}} = \mathbf{F}_n(1, \dots, 1, x, 1, \dots, 1), \text{ for } x \in \overline{\mathbb{R}}, \quad (1.3)$$

$$F_{jn}^-(t) = \begin{cases} \inf\{x : F_{jn}(x) \geq t\} & \text{for } t \in (0, 1), \\ \lim_{t \downarrow 0} F_{jn}^-(t) & \text{for } t = 0, \\ \lim_{t \uparrow 1} F_{jn}^-(t) & \text{for } t = 1. \end{cases} \quad (1.4)$$

Set $U_{ji} := F_j(X_{ji})$ for $i = 1, \dots, n$, $j = 1, \dots, d$ and $\mathbf{U}_i := (U_{1i}, \dots, U_{di})$. It is well known that distribution function of random vectors \mathbf{U}_i coincides with the copula $\mathbf{C}(\cdot)$, we refer to Sklar (1959, 1973), Deheuvels (2009), Philipp and Pinzur (1980), Wichura (1973), Moore and Spruill (1975).

For each $n \geq 1$, $0 \leq u_j \leq 1$ and $1 \leq j \leq d$, set

$$\tilde{\mathbf{C}}_n(u_1, \dots, u_d) := \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{1}\{U_{ji} \leq u_j\} = \mathbf{F}_n(F_1^-(u_1), \dots, F_d^-(u_d)), \quad (1.5)$$

$$G_{jn}(u_j) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{U_{ji} \leq u_j\} = F_{jn}\{F_j^-(u_j)\}. \quad (1.6)$$

$$G_{jn}^-(u_j) := \inf\{t_j \geq 0 : G_{jn}(t_j) \geq u_j\} = F_j\{F_{jn}^-(u_j)\}. \quad (1.7)$$

Then, with $X \stackrel{\mathcal{D}}{=} Y$ which means that X and Y have the same distribution, we have

$$\mathbf{F}_n(x_1, \dots, x_d) \stackrel{\mathcal{D}}{=} \tilde{\mathbf{C}}_n(F_1(x_1), \dots, F_d(x_d))$$

and

$$(F_{1n}(x_1), \dots, F_{dn}(x_d)) \stackrel{\mathcal{D}}{=} (G_{1n}(F_1(x_1)), \dots, G_{dn}(F_d(x_d))).$$

From the definition of $F_{jn}^-(\cdot)$, $j = 1, \dots, d$, it follows that

$$F_{jn}^-(u_j) \stackrel{\mathcal{D}}{=} F_j^-\{G_{jn}^-(u_j)\},$$

and it therefore follows that

$$\mathbf{C}_n(u_1, \dots, u_d) \stackrel{\mathcal{D}}{=} \tilde{\mathbf{C}}_n(G_{1n}^-(u_1), \dots, G_{dn}^-(u_d)).$$

For the preceding distributional equality we may refer to Deheuvels (2009), Rüschendorf (2009) and Tsukahara (2005) including the references therein. Since the copula function associated with $\mathbf{F}_n(\cdot)$ is not unique, it is convenient to investigate the *modified empirical copula process*, which is, in turn, uniquely defined by

$$\mathbb{A}_n(\mathbf{u}) = n^{1/2}(\tilde{\mathbf{C}}_n(G_{1n}^-(u_1), \dots, G_{dn}^-(u_d)) - \mathbf{C}(u_1, \dots, u_d)), \text{ for } \mathbf{u} \in [0, 1]^d. \quad (1.8)$$

The function $\mathbf{C}_n(\cdot)$ was briefly discussed by Ruymgaart (1973), pp. 6–13, in the introduction of his doctoral thesis. Deheuvels (1979) investigated the consistency of $\mathbf{C}_n(\cdot)$ and Deheuvels (1980, 1981) obtained the exact law and the limiting process of $\mathbb{A}_n(\cdot)$ when margins are independent. The empirical copula process $\mathbb{A}_n(\cdot)$ has been studied in full generality in Stute (1984), Gaenssler and Stute (1987) and Rüschendorf (1974, 1976). In the latter references, the normalized empirical copula process was introduced under the name *multivariate rank order process* on discrete grid. In fact, in that paper the sequential version was more generally introduced and analyzed for nonstationary and mixing random variables. van der Vaart and Wellner (1996) utilize the functional delta method

to show convergence of $\mathbb{A}_n(\cdot)$ in $\ell^\infty([a, b]^2)$ for some $0 < a < b < 1$, under restrictions on the distribution functions. Fermanian *et al.* (2004) extend their results by proving the weak convergence of this process in $\ell^\infty([0, 1]^2)$ under minimal conditions on the copula function, which coincides with the result obtained by Gaenssler and Stute (1987). We can refer to Deheuvels (2009) and the references therein concerning the strong approximations for the process $\mathbb{A}_n(\cdot)$. In the last reference, a full characterization of empirical copula in general framework is provided. In the present paper, we are concerned with strong approximations of the empirical copula processes $\{\mathbb{A}_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$, based upon $\mathbf{X}_1, \dots, \mathbf{X}_n$, by a single Gaussian process. We first need to introduce some Gaussian processes, which play a central role in strong approximations theory. The d -variate Wiener process $\mathbb{W}(\mathbf{y})$ on the unit cube of \mathbb{R}^d ($\mathbf{y} \in [0, 1]^d$) associated with the copula function $\mathbf{C}(\cdot)$, i.e., $\mathbb{W}_{\mathbf{C}}(\cdot)$ is a d -variate Gaussian process on $[0, 1]^d$ with

$$\mathbb{E}(\mathbb{W}_{\mathbf{C}}(\mathbf{u})) = 0, \quad \mathbb{E}(\mathbb{W}_{\mathbf{C}}(\mathbf{u})\mathbb{W}_{\mathbf{C}}(\mathbf{v})) = \mathbf{C}(\mathbf{u} \wedge \mathbf{v}), \quad (1.9)$$

where $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ and $\mathbf{v} = (v_1, \dots, v_d) \in [0, 1]^d$ and $\mathbb{W}_{\mathbf{C}}(u_1, \dots, u_d) = 0$ whenever $u_j = 0, j = 1, \dots, d$.

A d -variate Brownian bridge process on $[0, 1]^d$ associated with the copula function $\mathbf{C}(\cdot)$ is defined, in terms of $\mathbb{W}_{\mathbf{C}}(\cdot)$, by setting

$$\mathbf{B}_{\mathbf{C}}(\mathbf{u}) := \mathbb{W}_{\mathbf{C}}(\mathbf{u}) - \mathbf{C}(\mathbf{u})\mathbb{W}_{\mathbf{C}}(\mathbf{1}), \quad (1.10)$$

for $\mathbf{u} \in [0, 1]^d$ and $\mathbf{1} := (1, \dots, 1)$. This process has continuous sample paths and fulfills

$$\mathbb{E}(\mathbf{B}_{\mathbf{C}}(\mathbf{u})) = 0, \quad \mathbb{E}(\mathbf{B}_{\mathbf{C}}(\mathbf{u})\mathbf{B}_{\mathbf{C}}(\mathbf{v})) = \mathbf{C}(\mathbf{u} \wedge \mathbf{v}) - \mathbf{C}(\mathbf{u})\mathbf{C}(\mathbf{v}), \quad \text{for } \mathbf{u}, \mathbf{v} \in [0, 1]^d. \quad (1.11)$$

The interested reader may refer to Piterbarg (1996) and Adler (1990) for details on the gaussian processes mentioned above. To state our result we need to formulate the notion of Kiefer random field. Consider a $(d + 1)$ -variate Gaussian process $\mathbb{W}_{\mathbf{C}}(\mathbf{u}, z)$ on $[0, 1]^d \times [0, \infty)$ such that $\mathbb{W}_{\mathbf{C}}(\mathbf{u}, z) = 0$ whenever any of u_1, \dots, u_d or z is zero, $\mathbb{E}\mathbb{W}_{\mathbf{C}}(\mathbf{u}, z) = 0$, and $\mathbb{E}\mathbb{W}_{\mathbf{C}}(\mathbf{u}, z)\mathbb{W}_{\mathbf{C}}(\mathbf{v}, t) = \min(z, t)\mathbf{C}(\mathbf{u}, \mathbf{v})$. Then, a $(d + 1)$ -variate Kiefer process $\mathcal{K}_{\mathbf{C}}(\cdot)$ on $[0, 1]^d \times [0, \infty)$ associated with the copula function $\mathbf{C}(\cdot)$, can be represented as

$$\mathcal{K}_{\mathbf{C}}(\mathbf{u}, t) = \mathbb{W}_{\mathbf{C}}(\mathbf{u}, t) - \mathbf{C}(\mathbf{u})\mathbb{W}_{\mathbf{C}}(\mathbf{1}, t)$$

and fulfills

$$\mathbb{E}(\mathcal{K}_{\mathbf{C}}(\mathbf{u}, z)) = 0 \quad \text{and} \quad \mathbb{E}(\mathcal{K}_{\mathbf{C}}(\mathbf{u}, z)\mathcal{K}_{\mathbf{C}}(\mathbf{v}, t)) = (z \wedge t) \{\mathbf{C}(\mathbf{u} \wedge \mathbf{v}) - \mathbf{C}(\mathbf{u})\mathbf{C}(\mathbf{v})\},$$

for $\mathbf{u}, \mathbf{v} \in [0, 1]^d$ and $s, t \geq 0$. Clearly, for all fixed $z \geq 0$, we have

$$z^{-1/2} \mathcal{H}_{\mathbf{C}}(\mathbf{u}, z) \stackrel{\mathcal{D}}{=} \mathbf{B}_{\mathbf{C}}(\mathbf{u}).$$

For each $n > 0$, $u_j \in [0, 1]$ and $j = 1, \dots, d$, the *copula Gaussian process* is defined by

$$\begin{aligned} \mathcal{H}_{\mathbf{C}}^*(\mathbf{u}, n) &= \mathcal{H}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathcal{H}_{\mathbf{C}}(1, \dots, 1, u_j, 1, \dots, 1, n) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j} \\ &=: \mathcal{H}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathcal{H}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j}. \end{aligned} \quad (1.12)$$

We say that the strong approximation hold for the process $\mathbb{A}_n(\cdot)$ with rate (b_n) , this means that

$$\sup_{\mathbf{u} \in [0, 1]^d} |\mathbb{A}_n(\mathbf{u}) - \mathbb{Z}_n(\mathbf{u})| = O(b_n), \text{ a.s.} \quad (1.13)$$

on probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathbb{Z}_n(\cdot)$ is a sequence of gaussian processes and a deterministic rate $b_n \rightarrow 0$. The strong approximations are quite useful and have received considerable attention in probability theory. Indeed, many well-known probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences. We mention that the rates of convergence for the distributions of *smooth* functionals of $\mathbb{A}_n(\cdot)$ can be also deduced from the approximation in (1.13). The approximation by Kiefer process is of particular interest, since any kind of law of the iterated logarithm which holds for the partial sums of Gaussian processes may then transferred to the empirical processes $\mathbb{A}_n(\cdot)$. We refer to Komlós *et al.* (1975), (DasGupta, 2008, Chapter 12), (Csörgő and Horváth, 1993, Chapter 3), (Csörgő and Révész, 1981, Chapters 4-5) and (Shorack and Wellner, 1986, Chapter 12) for expositions and references on this problem. We refer to Csörgő and Hall (1984) for a survey of some applications of the strong approximation and many references. There is a huge literature on the strong approximations and their applications. It is not the purpose of this paper to survey this extensive literature.

The remaining of our paper is organized as follows. In the next section we will give our main result concerning the strong approximation of empirical copula processes by a single Gaussian process, which is stated in Theorem 2.1 below. In section 3 we will give some applications of Theorem 2.1, more precisely we will give the limit law of smoothed empirical copula processes and the law of iterated logarithm for the empirical copula processes. The proof of these results will be sketched in section 4.

2 Results

In the sequel, the precise meaning of “suitable probability space” is that an independent sequence of Wiener processes, which is independent of the originally given sequence of i.i.d. r.v., can be constructed on the assumed probability space. This is a technical requirement which allows for the construction of the Gaussian processes in our theorems, and is not restrictive since one can expand the probability space to make it rich enough (cf. Appendix 2 in Csörgő and Horváth (1993)).

The main result to be proved here may now be stated precisely as follows.

Theorem 2.1 *Assume that $\mathbf{C}(\cdot)$, associated with $\mathbf{F}(\cdot)$, is twice continuously differentiable on $(0, 1)^d$ and all the partial derivatives of second order are continuous on $[0, 1]^d$. On a suitable probability space, we may define the empirical copula processes $\{\mathbb{A}_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d; n > 0\}$ in combination with a Gaussian process $\{\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d; t \geq 0\}$, in such a way that, almost surely as $n \rightarrow \infty$*

$$\sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n}\mathbb{A}_n(\mathbf{u}) - \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n)| = O(n^{1/2-1/(4d)}(\log n)^{3/2}), \quad (2.1)$$

where $\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, t)$ is defined in (1.12).

The proof of Theorem 2.1 is given in Section 4.

Remark 2.2 *In the particular case of independence, i.e.,*

$$\mathbf{C}(\mathbf{u}) = \prod_{i=1}^d u_i,$$

the process $\{\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n) : \mathbf{u} \in [0, 1]^d; n \geq 0\}$ is equal to

$$\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n) =: \mathcal{K}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathcal{K}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n) \prod_{i \neq j}^d u_i, \quad \mathbf{u} \in [0, 1]^d,$$

with mean zero and covariance functions

$$\mathbb{E}(\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, s)\mathcal{K}_{\mathbf{C}}^*(\mathbf{v}, t)) = (s \wedge t) \left\{ \prod_{i=1}^d (u_i \wedge v_i) + (d-1) \prod_{i=1}^d u_i v_i - \sum_{i=1}^d (u_i \wedge v_i) \prod_{i \neq j} u_i v_j \right\}$$

where $\mathbf{u}, \mathbf{v} \in [0, 1]^d$ and $s, t \geq 0$. For more details the reader may refer to Csörgő (1979). Note that in the case where $\mathbb{A}_n(\cdot)$ is generated by a sample of random vectors with independent marginals then the limit distribution in (2.1) is free.

Remark 2.3 *Theorem 2.1 may be used to derive the limiting laws of some statistics like Kendall's sample rank correlation coefficient and Spearman's sample rank correlation coefficient. More generally, let us define, for any function $J(\cdot)$ on $[0, 1]^3$*

$$\mathbb{S}(\mathbf{C}) = \int_0^1 \int_0^1 J(u, v, \mathbf{C}(u, v)) dudv.$$

The corresponding sample quantity $\mathbb{S}(\mathbf{C}_n)$ may be called Spearman type rank statistic, the interested reader may refer to Gaenssler and Stute (1987) for more details. To see this, suppose that $z \rightarrow J(u, v, z)$ has a continuous derivative $J^3(u, v, z)$ with $\sup_{u,v,z} |J^3(u, v, z)| = \sup_{u,v,z} |\partial J(u, v, z)/\partial z| < \infty$. Then we can write

$$\begin{aligned} \sqrt{n}(\mathbb{S}(\mathbf{C}_n) - \mathbb{S}(\mathbf{C})) &= \sqrt{n} \left(\int_0^1 \int_0^1 J(u, v, \mathbf{C}_n(u, v)) dudv - \int_0^1 \int_0^1 J(u, v, \mathbf{C}(u, v)) dudv \right) \\ &= \int_0^1 \int_0^1 J^3(u, v, \delta_n(u, v)) \mathbb{A}_n(u, v) dudv, \end{aligned}$$

where $\delta_n(u, v)$ is a point between $\mathbf{C}_n(u, v)$ and $\mathbf{C}(u, v)$, so that $\delta_n \rightarrow \mathbf{C}$ uniformly with probability one. Making use of Theorem 2.1 we have

$$\left| \sqrt{n}(\mathbb{S}(\mathbf{C}_n) - \mathbb{S}(\mathbf{C})) - \int_0^1 \int_0^1 J^3(u, v, \mathbf{C}(u, v)) \frac{1}{\sqrt{n}} \mathcal{K}_{\mathbf{C}}^*(u, v, n) dudv \right| = o_P(1).$$

Corresponding to the Spearman type rank statistic, we put, for any function $J(\cdot)$ on $[0, 1]^3$,

$$\mathbb{T}(\mathbf{C}) = \int_0^1 \int_0^1 J(u, v, \mathbf{C}(u, v)) d\mathbf{C}(u, v),$$

the integration being understood as multiple integral based on the bivariate copula. We call $\mathbb{T}(\mathbf{C}_n)$ a Kendall type rank statistic. Similarly, using Theorem 2.1 we can obtain the limiting law of $\sqrt{n}(\mathbb{T}(\mathbf{C}_n) - \mathbb{T}(\mathbf{C}))$.

3 Applications

3.1 Smoothed empirical copula processes

The smoothed empirical copula function $\widehat{\mathbf{C}}_n(\cdot)$ is defined by

$$\widehat{\mathbf{C}}_n(\mathbf{u}) = \frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}}\right) \mathbf{C}_n(\mathbf{v}) d\mathbf{v} \text{ for } \mathbf{u} \in [0, 1]^d, \quad (3.1)$$

where $k(\cdot)$ is a kernel function and $h = h(n)$ is the smoothing parameter. For notational convenience, we have chosen the same bandwidth sequence for each margins. This assumption can be dropped

easily. Similarly to the previous section, we define the smoothed empirical copulas process by

$$\widehat{\mathbb{A}}_n(\mathbf{u}) := \sqrt{n} \left(\widehat{\mathbf{C}}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u}) \right) \quad \text{for } \mathbf{u} \in [0, 1]^d. \quad (3.2)$$

We will describe the asymptotic properties of the smoothed empirical copulas process $\widehat{\mathbb{A}}_n(\cdot)$ under the following assumptions.

(F.1). There exists a constant $0 < \mathfrak{C} < \infty$ such that

$$\sup_{\mathbf{u} \in [0, 1]^d} \left| \frac{\partial^s \mathbf{C}(\mathbf{u})}{\partial^{j_1} u_1 \dots \partial^{j_d} u_d} \right| \leq \mathfrak{C}, \quad j_1 + \dots + j_d = s.$$

Suppose that $\{h(n)\}_{n \geq 1}$ is a sequence of positive constants which satisfies the condition.

(C.1). $h = h(n) \rightarrow 0$, $nh \rightarrow \infty$ and $\sqrt{nh^{s/d}} \rightarrow 0$ as $n \rightarrow \infty$;

and the kernel function $k(\cdot)$ fulfills the following conditions.

(C.2). $k(\cdot)$ is a continuous density function and compactly supported;

(C.3). $k(\cdot)$ is of order s , i.e.,

$$\begin{aligned} \int_{\mathbb{R}^d} k(\mathbf{u}) d\mathbf{u} &= 1, \\ \int_{\mathbb{R}^d} u_1^{j_1} \dots u_d^{j_d} k(\mathbf{u}) d\mathbf{u} &= 0, \quad j_1, \dots, j_d \geq 0, \quad j_1 + \dots + j_d = 1, \dots, s-1, \\ \int_{\mathbb{R}^d} |u_1^{j_1} \dots u_d^{j_d}| k(\mathbf{u}) d\mathbf{u} &< \infty, \quad j_1, \dots, j_d \geq 0, \quad j_1 + \dots + j_d = s. \end{aligned}$$

The following corollary establishes the limiting behaviour of the smoothed empirical copulas process $\widehat{\mathbb{A}}_n(\cdot)$.

Corollary 3.1 *Assume that (F.1) and (C.1)-(C.3) hold. Then, on a suitable probability space, we may define the smoothed empirical copula processes $\{\widehat{\mathbb{A}}_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d; n > 0\}$ in combination with a Gaussian process $\{\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d; t \geq 0\}$, in such a way that, as $n \rightarrow \infty$*

$$\sup_{\mathbf{u} \in [0, 1]^d} \left| \widehat{\mathbb{A}}_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n) \right| = o_P(1). \quad (3.3)$$

The proof of Corollary 3.1 is given in Section 4.

The result of Corollary 3.1 is motivated by the following remark.

Remark 3.2 *The empirical copula provides an universal way for estimation purposes. Unfortunately, its discontinuous feature induces some difficulties: the graphical representations of the copula may not be satisfactory from a visual and intuitive point of view. Moreover, there is no unique choice for building the inverse function of marginal functions. Finally, since the empirical copula estimator is not differentiable, it cannot, for example, be used to derive an estimate of the associated copula density or for optimization purposes. Studies have shown that a smoothed estimator may be preferable to the sample estimator. Firstly, smoothing reduces the random variation in the data, resulting in a more efficient estimator. Secondly, smoothing gives a smooth curve that displays some interesting features.*

Remark 3.3 1. *Corollary 3.1 remains valid when replacing the condition that the kernel function $k(\cdot)$ having compact support in (C.2) by another condition (C.4) which content is as follows*

(C.4). *There exists a sequence of positive real numbers a_n such that $a_n h$ tends to zero when n tends to infinity, and*

$$\sqrt{n} \int_{\{\|\mathbf{v}\| > a_n\}} |k(\mathbf{v})| d\mathbf{v} \rightarrow 0.$$

2. *Note that the conditions of Corollary 3.1 are grouped to control the deviations between the empirical copula process $\mathbb{A}_n(\cdot)$ and the smoothed empirical copula process $\widehat{\mathbb{A}}_n(\cdot)$.*

3.2 The law of iterated logarithm for empirical copula processes

From Theorem 2.1, we have almost surely

$$\limsup_{n \rightarrow \infty} \left\{ \left(\frac{n}{2 \log \log n} \right)^{1/2} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})| \right\} = \limsup_{n \rightarrow \infty} \frac{\sup_{\mathbf{u} \in [0,1]^d} |\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n)|}{(2n \log \log n)^{1/2}}. \quad (3.4)$$

We can state the following Corollary.

Corollary 3.4 *Under the same conditions of the Theorem 2.1, we have*

$$\limsup_{n \rightarrow \infty} \left\{ \left(\frac{n}{2 \log \log n} \right)^{1/2} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})| \right\} = \rho, \text{ a.s.}, \quad (3.5)$$

where

$$\rho^2 = \sup_{\mathbf{u} \in [0,1]^d} \text{Var}(\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, 1)).$$

Remark 3.5 *The result of Corollary 3.4 was obtained by Deheuvels (1979) (refer to Theorem 3.1) using a different method.*

Remark 3.6 *In the early 1970s there was considerable interest in multivariate rank-order statistics we may refer to Ruymgaart et al. (1972), Rüschendorf (1974, 1976) including the references therein. Such statistics are of the form*

$$\mathbf{R}_n = \frac{1}{n} \sum_{i=1}^n \mathcal{J} (F_{1n}(X_{1i}), \dots, F_{dn}(X_{di})),$$

where $\mathcal{J} : [0, 1]^d \rightarrow \mathbb{R}$, is measurable function satisfying some regularity conditons. The asymptotic normality of \mathbf{R}_n can be established under the weakest set of assumptions (see, Theorem 6 in Fermanian et al. (2004)) using Theorem 2.1.

4 Proofs

Proof of Theorem 2.1.

Consider the empirical processes defined, respectively, for $n \geq 1$, $\mathbf{u} \in [0, 1]^d$ and $0 \leq u_j \leq 1$, for $j = 1, \dots, d$, by

$$\alpha_n(\mathbf{u}) := n^{1/2}(\tilde{\mathbf{C}}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})), \quad (4.1)$$

$$\alpha_{jn}(u_j) := n^{1/2}\{G_{jn}(u_j) - u_j\}, \quad (4.2)$$

$$\beta_{jn}(u_j) := n^{1/2}\{G_{jn}^-(u_j) - u_j\}. \quad (4.3)$$

Recall the definition (1.8) of $\mathbb{A}_n(\cdot)$, the empirical process of copulas can be written, for $\mathbf{u} \in [0, 1]^d$, as follows

$$\begin{aligned} \mathbb{A}_n(\mathbf{u}) &= n^{1/2}(\tilde{\mathbf{C}}_n(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d)) - \mathbf{C}(u_1, \dots, u_d)) \\ &= \alpha_n(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d)) + n^{1/2}\{\mathbf{C}(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d)) \\ &\quad - \mathbf{C}(u_1, \dots, u_d)\} \\ &= \alpha_n(u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d)) \\ &\quad + n^{1/2}\{\mathbf{C}(u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d)) - \mathbf{C}(u_1, \dots, u_d)\} \\ &= \alpha_n(\mathbf{u}) + \{\alpha_n(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) - \alpha_n(\mathbf{u})\} \\ &\quad + n^{1/2}\{\mathbf{C}(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) - \mathbf{C}(\mathbf{u})\} \\ &= \alpha_n(\mathbf{u}) + \Delta_1(\mathbf{u}, n) + \Delta_2(\mathbf{u}, n), \end{aligned} \quad (4.4)$$

where $(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) = (u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d))$. We first evaluate the right side term $\Delta_2(\cdot, n)$ of (4.4). By differentiability assumption on $\mathbf{C}(\cdot)$, we can use Taylor's expansion

to obtain the following

$$\begin{aligned}\Delta_2(\mathbf{u}, n) &= \sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \sqrt{n} (G_{ni}^-(u_i) - u_i) \\ &\quad + \frac{\sqrt{n}}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 \mathbf{C}(\mathbf{u}^*)}{\partial u_i \partial u_j} (G_{ni}^-(u_i) - u_i) (G_{nj}^-(u_j) - u_j),\end{aligned}$$

where \mathbf{u}^* lies in the interior of the line segment joining $(G_{n1}^-(u_1), \dots, G_{nd}^-(u_d))$ and (u_1, \dots, u_d) . It follows from the definition of $\alpha_n(\cdot)$ in (4.1), for $u_j \in [0, 1]$, $j = 1, \dots, d$, that

$$\begin{aligned}\sqrt{n}(G_{ni}^-(u_i) - u_i) &= -\sqrt{n} (G_{ni}(G_{ni}^-(u_i)) - G_{ni}^-(u_i)) + \sqrt{n} (G_{ni}(G_{ni}^-(u_i)) - u_i) \\ &= -\alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) + \sqrt{n}(G_{ni}(G_{ni}^-(u_i)) - u_i).\end{aligned}$$

Using the fact that $|G_{ni}(G_{ni}^-(u_i)) - u_i| \leq 1/n$ and the Chung (1949)'s law of the iterated logarithm, one finds, almost surely,

$$\Delta_2(\mathbf{u}, n) = -\sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) + O(n^{-1/2} \log \log n),$$

uniformly in \mathbf{u} . It is well known since the work of Stute (1982), p. 99, that we have, almost surely,

$$\sup_{u_i \in [0,1]} |\alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) - \alpha_n(\mathbf{1}, u_i, \mathbf{1})| = O(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4}).$$

Then, it follows that uniformly in \mathbf{u} , almost surely,

$$\Delta_2(\mathbf{u}, n) = -\sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \alpha_n(\mathbf{1}, u_i, \mathbf{1}) + O(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4}),$$

as was observed by Stute (1984), p. 371. We next evaluate the term $\Delta_1(\cdot, n)$ in the right hand side of (4.4). Recall that $\Delta_1(\mathbf{u}, n)$ is the difference of $\alpha_n(\mathbf{u} + n^{-1/2} \beta_n(\mathbf{u}))$ and $\alpha_n(\mathbf{u})$. Let $w_n(\cdot)$ be the modulus of continuity of $\alpha_n(\cdot)$, defined by

$$w_n(\mathbf{a}) := \sup \left\{ \alpha_n(A) : A = \prod_{i=1}^d [u_i, v_i] \in [0, 1]^d, \text{ with } |[u_i, v_i]| = v_i - u_i \leq a_i, \forall i = 1, \dots, d \right\}.$$

We will make use of the following fact which is a particular case of Theorem 2.1, p. 367 of Stute (1984).

Fact 1. Let $\{a_n\}_{n \geq 1}$ be a sequence in $(0, 1)$ such as $a_n \downarrow 0$, as $n \rightarrow \infty$, and

$$i) n a_n^d \uparrow \infty, \quad ii) n a_n^d / \log n \rightarrow \infty, \quad iii) \log(1/a_n) / \log \log n \rightarrow \infty.$$

Then, we have, almost surely,

$$\lim_{n \rightarrow \infty} \{2a_n^d \log(1/a_n^d)\}^{-1/2} w_n(a_n, \dots, a_n) = 1.$$

An application of the Chung (1949) law of the iterated logarithm shows that, for each $j = 1, \dots, d$, almost surely,

$$\limsup_{n \rightarrow \infty} \left\{ (\log \log n)^{-1/2} \sup_{0 \leq u \leq 1} |\beta_{jn}(u_j)| \right\} = 2^{-1/2}. \quad (4.5)$$

From (4.5), we have almost surely for all $j = 1, \dots, d$ and n large enough,

$$\sup_{0 \leq u_j \leq 1} |n^{-1/2} \beta_{jn}(u_j)| \leq \frac{(\log \log n)^{1/2}}{n^{1/2}} \leq \frac{(\log n)^{2/d}}{n^{1/d}} := a_n,$$

an application of Fact 1 shows that, as $n \rightarrow \infty$, we have, almost surely,

$$\sup_{\mathbf{u} \in [0,1]^d} |\Delta_1(\mathbf{u}, n)| \leq w_n(a_n) = O(n^{-1/2}(\log n)^{3/2}).$$

The next fact, due to Csörgő and Horváth (1988), p. 102, provides a strong approximation result appropriate for our need.

Fact 2. On a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, it is possible to define $\{\alpha_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$, jointly with a sequence of Gaussian process $\{\mathcal{K}_{\mathbf{C}}(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d, t \geq 0\}$, in such a way that, as $n \rightarrow \infty$, almost surely,

$$\sup_{\mathbf{u} \in [0,1]^d} |\sqrt{n}\alpha_n(\mathbf{u}) - \mathcal{K}_{\mathbf{C}}(\mathbf{u}, n)| = O(n^{1/2-1/(4d)}(\log n)^{3/2}). \quad (4.6)$$

Therefore, by the triangle inequality,

$$\begin{aligned} \sup_{\mathbf{u} \in [0,1]^d} |\sqrt{n}\mathbb{A}_n(\mathbf{u}) - \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n)| &\leq \sup_{\mathbf{u} \in [0,1]^d} |\sqrt{n}\alpha_n(\mathbf{u}) - \mathcal{K}_{\mathbf{C}}(\mathbf{u}, n)| + w_n(a_n) \\ &+ \sum_{j=1}^d \left| \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j} \right| \sup_{0 \leq u_j \leq 1} |\sqrt{n}\alpha_n(\mathbf{1}, u_j, \mathbf{1}) - \mathcal{K}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n)| \\ &+ O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}), \\ &\leq O(n^{1/2-1/(4d)}(\log n)^{3/2}) + O(n^{-1/2}(\log n)^{3/2}) \\ &+ O(n^{1/2-1/(4d)}(\log n)^{3/2}) + O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}), \end{aligned}$$

then, we have almost surely, as $n \rightarrow \infty$,

$$\sup_{\mathbf{u} \in [0,1]^d} |\sqrt{n}\mathbb{A}_n(\mathbf{u}) - \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n)| = O(n^{1/2-1/(4d)}(\log n)^{3/2})$$

and thus the proof of Theorem 2.1 is complete. \square

Proof of Corollary 3.1.

First, observe that the difference between the *empirical copula process* $\mathbb{A}_n(\cdot)$ and the smoothed empirical copula process $\widehat{\mathbb{A}}_n(\cdot)$ can be controlled as follows. We have

$$\begin{aligned}
\widehat{\mathbb{A}}_n(\mathbf{u}) &= \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k \left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}} \right) \mathbf{C}_n(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right) \\
&= \left(\frac{1}{h} \int_{[0,1]^d} k \left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}} \right) \sqrt{n} (\mathbf{C}_n(\mathbf{v}) - \mathbf{C}(\mathbf{v})) \right) d\mathbf{v} \\
&\quad + \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k \left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}} \right) \mathbf{C}(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right) \\
&= \left(\frac{1}{h} \int_{[0,1]^d} k \left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}} \right) \mathbb{A}_n(\mathbf{v}) \right) d\mathbf{v} \\
&\quad + \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k \left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}} \right) \mathbf{C}(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right). \tag{4.7}
\end{aligned}$$

Thus, one can derive the following

$$\begin{aligned}
\sup_{\mathbf{u} \in [0,1]^d} |\widehat{\mathbb{A}}_n(\mathbf{u}) - \mathbb{A}_n(\mathbf{u})| &\leq \sup_{\mathbf{u} \in [0,1]^d} \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{h^{1/d}}, \frac{u_i}{h^{1/d}} \right]} (\mathbb{A}_n(\mathbf{u} - h^{1/d}\mathbf{v}) - \mathbb{A}_n(\mathbf{u})) k(\mathbf{v}) d\mathbf{v} \right| \\
&\quad + \sup_{\mathbf{u} \in [0,1]^d} |\mathbb{A}_n(\mathbf{u})| \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{h^{1/d}}, \frac{u_i}{h^{1/d}} \right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| \\
&\quad + \sqrt{n} \sup_{\mathbf{u} \in [0,1]^d} \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{h^{1/d}}, \frac{u_i}{h^{1/d}} \right]} (\mathbf{C}(\mathbf{u} - h^{1/d}\mathbf{v}) - \mathbf{C}(\mathbf{u})) k(\mathbf{v}) d\mathbf{v} \right| \\
&\quad + \sqrt{n} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}(\mathbf{u})| \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{h^{1/d}}, \frac{u_i}{h^{1/d}} \right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| \\
&:= \nabla_{1;n} + \nabla_{2;n} + \nabla_{3;n} + \nabla_{4;n}. \tag{4.8}
\end{aligned}$$

We first evaluate $\nabla_{3;n}$ in the right side of (4.8). Under conditions (F.1), (C.1)-(C.3) and using Taylor expansion of order s we get

$$\nabla_{3;n} = \frac{h^{s/d}}{s!} \sqrt{n} \sup_{\mathbf{u} \in [0,1]^d} \left| \int \sum_{j_1 + \dots + j_d = s} u_1^{j_1} \dots u_d^{j_d} \frac{\partial^s \mathbf{C}(\mathbf{u} - h_n \theta \mathbf{v})}{\partial u_1^{j_1} \dots \partial u_d^{j_d}} K(\mathbf{v}) d\mathbf{v} \right|,$$

where $\theta = (\theta_1, \dots, \theta_d)$ and $0 < \theta_i < 1$. Thus a straightforward application of Lebesgue dominated convergence theorem gives

$$n^{-1/2} h^{-(s/d)} \nabla_{3;n} = \frac{1}{k!} \sup_{\mathbf{u} \in [0,1]^d} \left| \sum_{j_1 + \dots + j_d = s} \frac{\partial^s \mathbf{C}(\mathbf{u})}{\partial u_1^{j_1} \dots \partial u_d^{j_d}} \int u_1^{j_1} \dots u_d^{j_d} K(\mathbf{v}) d\mathbf{v} \right|. \tag{4.9}$$

Then we have by condition (C.1) and (C.3)

$$\nabla_{3;n} = O(n^{1/2} h^{s/d}) = o(1). \tag{4.10}$$

By Theorem 2.1 and almost sure continuity of the Gaussian process $\mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n)$ we have

$$\begin{aligned}\nabla_{1;n} &\leq \sup_{\mathbf{u}, \mathbf{v} \in [0,1]^d} \sup_{|\mathbf{u}-\mathbf{v}| \leq h} |\mathbb{A}_n(\mathbf{v}) - \mathbb{A}_n(\mathbf{u})| \left| \int K(\mathbf{v}) d\mathbf{v} \right| \\ &= o_P(1)O(1) = o_P(1).\end{aligned}\tag{4.11}$$

We next evaluate $\nabla_{2;n}$ in the right side of (4.8). We have $\sup_{\mathbf{u} \in [0,1]^d} |\mathbb{A}_n(\mathbf{u})| = O_P(1)$ and as n tends to infinity (by condition (C.2))

$$\sqrt{n} \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{h^{1/d}}, \frac{u_i}{h^{1/d}} \right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| = o(1),$$

then

$$\nabla_{2;n} = o_P(1).$$

Using the same arguments we obtain

$$\nabla_{4;n} = o_P(1).$$

Reorganizing the above results, one finds

$$\sup_{\mathbf{u} \in [0,1]^d} |\widehat{\mathbb{A}}_n(\mathbf{u}) - \mathbb{A}_n(\mathbf{u})| = o_P(1).\tag{4.12}$$

Now the Corollary follows from (4.12) in conjunction with Theorem 2.1, and the fact that

$$\begin{aligned}\sup_{\mathbf{u} \in [0,1]^d} \left| \widehat{\mathbb{A}}_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n) \right| &\leq \sup_{\mathbf{u} \in [0,1]^d} |\widehat{\mathbb{A}}_n(\mathbf{u}) - \mathbb{A}_n(\mathbf{u})| \\ &\quad + \sup_{\mathbf{u} \in [0,1]^d} \left| \mathbb{A}_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathcal{K}_{\mathbf{C}}^*(\mathbf{u}, n) \right|.\end{aligned}$$

Thus the proof of Corollary is complete. □

References

- Adler, R. J. (1990). *An introduction to continuity, extrema, and related topics for general Gaussian processes*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA.
- Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula methods in finance*. Wiley Finance Series. John Wiley & Sons Ltd., Chichester.

- Chung, K.-L. (1949). An estimate concerning the Kolmogoroff limit distribution. *Trans. Amer. Math. Soc.*, **67**, 36–50.
- Csörgő, M. (1979). Strong approximations of the Hoeffding, Blum, Kiefer, Rosenblatt multivariate empirical process. *J. Multivariate Anal.*, **9**(1), 84–100.
- Csörgő, M. and Horváth, L. (1988). A note on strong approximations of multivariate empirical processes. *Stochastic Process. Appl.*, **28**(1), 101–109.
- Csörgő, M. and Horváth, L. (1993). *Weighted approximations in probability and statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester. With a foreword by David Kendall.
- Csörgő, M. and Révész, P. (1981). *Strong approximations in probability and statistics*. Probability and Mathematical Statistics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York.
- Csörgő, S. and Hall, P. (1984). The Komlós-Major-Tusnády approximations and their applications. *Austral. J. Statist.*, **26**(2), 189–218.
- Cui, S. and Sun, Y. (2004). Checking for the gamma frailty distribution under the marginal proportional hazards frailty model. *Statist. Sinica*, **14**(1), 249–267.
- DasGupta, A. (2008). *Asymptotic theory of statistics and probability*. Springer Texts in Statistics. New York, NY: Springer..
- Deheuvels, P. (1979). La fonction de dépendance empirique et ses propriétés. Un test non paramétrique d'indépendance. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, **65**(6), 274–292.
- Deheuvels, P. (1980). Nonparametric test of independence. In *Nonparametric asymptotic statistics (Proc. Conf., Rouen, 1979) (French)*. Vol. 821 (J. P. Raoult, Ed.), *Lecture Notes in Math.*, pages 95–107. Springer, Berlin.
- Deheuvels, P. (1981). Multivariate tests of independence. In *Analytical methods in probability theory (Oberwolfach, 1980)*. Vol. 861 (D. Dugué, E. Lukacs and V. K. Rohatgi, Eds.), *Lecture Notes in Math.*, pages 42–50. Springer, Berlin.

- Deheuvels, P. (2009). A multivariate Bahadur-Kiefer representation for the empirical copula process. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, **364**, 120–147.
- Fermanian, J.-D., Radulović, D., and Wegkamp, M. (2004). Weak convergence of empirical copula processes. *Bernoulli*, **10**(5), 847–860.
- Frees, E. W. and Valdez, E. A. (1998). Understanding relationships using copulas. *N. Am. Actuar. J.*, **2**(1), 1–25.
- Komlós, J., Major, P., and Tusnády, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **32**, 111–131.
- Gaenssler, P. and Stute, W. (1987). *Seminar on empirical processes*, volume 9 of *DMV Seminar*. Birkhäuser Verlag, Basel.
- Joe, H. (1997). *Multivariate models and dependence concepts*, volume 73 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London.
- McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative risk management*. Princeton Series in Finance. Princeton University Press, Princeton, NJ. Concepts, techniques and tools.
- Moore, D. S. and Spruill, M. C. (1975). Unified large-sample theory of general chi-squared statistics for tests of fit. *Ann. Statist.*, **3**, 599–616.
- Nelsen, R. B. (2006). *An introduction to copulas*. Springer Series in Statistics. Springer, New York, second edition.
- Philipp, W. and Pinzur, L. (1980). Almost sure approximation theorems for the multivariate empirical process. *Z. Wahrsch. Verw. Gebiete*, **54**(1), 1–13.
- Piterbarg, V. I. (1996). *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI. Translated from the Russian by V. V. Piterbarg, Revised by the author.
- Rüschendorf, L. (1974). On the empirical process of multivariate, dependent random variables. *J. Multivariate Anal.*, **4**, 469–478.
- Rüschendorf, L. (1976). Asymptotic distributions of multivariate rank order statistics. *Ann. Statist.*, **4**(5), 912–923.

- Rüschendorf, L. (2009). On the distributional transform, Sklar's theorem, and the empirical copula process. *J. Statist. Plann. Inference*, **139**(11), 3921–3927.
- Ruymgaart, F. (1973). *Asymptotic Theory for Rank Tests for Independence, MC Tract 43*. Ph.D. thesis, Amsterdam: Mathematisch Instituut.
- Ruymgaart, F. H., Shorack, G. R., and van Zwet, W. R. (1972). Asymptotic normality of nonparametric tests for independence. *Ann. Math. Statist.*, **43**, 1122–1135.
- Shorack, G. R. and Wellner, J. A. (1986). *Empirical processes with applications to statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.
- Sklar, A. (1973). Random variables, joint distribution functions, and copulas. *Kybernetika (Prague)*, **9**, 449–460.
- Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, **8**, 229–231.
- Stute, W. (1982). The oscillation behavior of empirical processes. *Ann. Probab.*, **10**, 86–107.
- Stute, W. (1984). The oscillation behavior of empirical processes: The multivariate case. *Ann. Probab.*, **12**, 361–379.
- Tsukahara, H. (2005). Semiparametric estimation in copula models. *Canad. J. Statist.*, **33**(3), 357–375.
- van der Vaart, A. W. and Wellner, J. A. (1996). *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York. With applications to statistics.
- Wichura, M. J. (1973). Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *Ann. Probability*, **1**, 272–296.