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Strong approximation of empirical copula processes by Gaussian processes

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Abstract. This paper investigates the problem of strong approximation of the empirical copula processes for arbitrary dimension, with continuous unknown margins. The idea of the proof is based on the results obtained in Deheuvels *et al.* (2006), and the theorem of strong approximation for an arbitrary distribution function proved in Csörgő and Horváth (1988). Using these results, we derive the normality for smoothed empirical copula processes and the L.I.L. for empirical copula processes.

AMS Subject Classifications: 62G30 ; 60F17

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1 Notations and Definitions

Let $\mathbf{X}_i = (X_{1i}, \dots, X_{di})$, $i = 1, 2, \dots$, be an independent replicæ of a d -dimensional random vector $\mathbf{X} \in \mathbb{R}^d$ with distribution function [df] $\mathbf{F}(x_1, \dots, x_d) := \mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d)$. Throughout this paper, we assume that the corresponding marginal df's $F_i(x_i) := \mathbb{P}(X_i \leq x_i)$, for $i = 1, \dots, d$ of \mathbf{X}_i are continuous. The quantile functions pertaining to $F_i(\cdot)$ for $i = 1, \dots, d$ are denoted, respectively, by $F_i^-(u_i) = \inf\{x_i : F_i(x_i) \geq u_i\}$, for $0 < u_i < 1$, $i = 1, \dots, d$. The copula function (see, e.g., Sklar (1959) and Deheuvels (1979, 1980, 1981b) among others) of $\mathbf{F}(\cdot)$ is defined as the distribution function

$$\mathbf{C}(u_1, \dots, u_d) = P(F_1(X_1) \leq u_1, \dots, F_d(X_d) \leq u_d), \text{ for every } \mathbf{u} \in [0, 1]^d$$

of the random vector $(F_1(X_1), \dots, F_d(X_d))$. For some historic notes, we refer to Schweizer (1991) and the recent survey conducted by Joe (1997) and Nelsen (2006). It's known that the copula associated with $\mathbf{F}(\cdot)$ is uniquely determined and is given by

$$\mathbf{C}(u_1, \dots, u_d) := \mathbf{F}(F_1^-(u_1), \dots, F_d^-(u_d)), \text{ for every } \mathbf{u} \in [0, 1]^d. \quad (1)$$

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Let's $\mathbb{1}_A(\cdot)$ be the indicator function of the set A . We define, for each $n \geq 1$, the empirical counterparts of $\mathbf{F}(\cdot)$, $F_1(\cdot), \dots, F_d(\cdot)$ and $F_1^-(\cdot), \dots, F_d^-(\cdot)$, respectively, by setting, for $j = 1, \dots, d$,

$$\mathbf{F}_n(x_1, \dots, x_d) := \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{1}\{X_{ji} \leq x_j\}, \quad \text{for } \mathbf{x} \in \mathbb{R}^d,$$

$$F_{jn}(x_j) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_{ij} \leq x_j\} = \mathbf{F}_n(1, \dots, x_j, \dots, 1), \quad \text{for } x_j \in \mathbb{R},$$

and

$$F_{in}^-(u_i) = \inf\{t_i \in \mathbb{R} : F_{in}(t_i) \geq u_i\}, \quad \text{for } u_i \in [0, 1].$$

In order to characterize the copula given in (1), we define the empirical copula function of $\mathbf{F}_n(\cdot)$ (or, equivalently, associated with $\mathbf{X}_1, \dots, \mathbf{X}_n$), as any copula $\mathbf{C}_n(\cdot)$, through the following identity

$$\mathbf{C}_n(u_1, \dots, u_d) := \mathbf{F}_n(F_{1n}^-(u_1), \dots, F_{dn}^-(u_d)), \quad \text{for } \mathbf{u} \in [0, 1]^d.$$

The empirical copulas were originally introduced by Deheuvels (1979, 1980, 1981b) under the name of empirical dependence functions. Under the assumption of margins independence. Deheuvels has extensively studied the empirical process of copula defined by

$$\gamma_n(\mathbf{u}) = n^{1/2}(\mathbf{C}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})), \quad \text{for } \mathbf{u} \in [0, 1]^d. \quad (2)$$

Further, set $U_{ji} := F_j(X_{ji})$ for $i = 1, \dots, n$, $j = 1, \dots, d$ and $\mathbf{U}_i := (U_{1i}, \dots, U_{di})$. Thus, the random vectors \mathbf{U}_i are distributed according to the copula $\mathbf{C}(\cdot)$, and the sequence U_{ji} of random variables are independent and uniformly distributed on $(0, 1)$. For each $n \geq 1$, $0 \leq u_j \leq 1$ and $1 \leq j \leq d$, set

$$\tilde{\mathbf{C}}_n(u_1, \dots, u_d) := \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{1}\{U_{ji} \leq u_j\}, \quad \text{and} \quad G_{jn}(u_j) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{U_{ij} \leq u_j\}.$$

Consider the empirical processes defined, respectively, for $n \geq 1$, $\mathbf{u} \in [0, 1]^d$ and $0 \leq u_j \leq 1$, for $j = 1, \dots, d$, by

$$\alpha_n(\mathbf{u}) := n^{1/2}(\tilde{\mathbf{C}}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})), \quad (3)$$

$$\alpha_n^j(u_j) := n^{1/2}\{G_{jn}(u_j) - u_j\}, \quad (4)$$

$$\beta_{jn}(u_j) := n^{1/2}\{G_{jn}^-(u_j) - u_j\}. \quad (5)$$

Then, with $X \stackrel{d}{=} Y$ which means that X and Y have the same distribution, we have

$$\mathbf{F}_n(x_1, \dots, x_d) \stackrel{d}{=} \tilde{\mathbf{C}}_n(F_1(x_1), \dots, F_d(x_d))$$

and

$$(F_{1n}(x_1), \dots, F_{dn}(x_d)) \stackrel{d}{=} (G_{1n}(F_1(x_1)), \dots, G_{dn}(F_d(x_d))).$$

From the definition of $F_{in}^-(\cdot)$, it follows that

$$G_{in}^-(u_i) \stackrel{d}{=} F_i\{F_{in}^-(u_i)\},$$

and we also have

$$\mathbf{C}_n(u_1, \dots, u_d) \stackrel{d}{=} \tilde{\mathbf{C}}_n(G_{1n}^-(u_1), \dots, G_{dn}^-(u_d)).$$

Since the copula function associated with $\mathbf{F}_n(\cdot)$ is not unique, it is convenient to investigate the *modified empirical copula process*, which is, in turn, uniquely defined by

$$\gamma_n(\mathbf{u}) = n^{1/2}(\tilde{\mathbf{C}}_n(G_{1n}^-(u_1), \dots, G_{dn}^-(u_d)) - \mathbf{C}(u_1, \dots, u_d)), \text{ for } \mathbf{u} \in [0, 1]^d. \quad (6)$$

If the margins are known, Neuhaus (1971) and Bickel and Wichura (1971) showed

$$\gamma_n(\mathbf{u}) \xrightarrow{d} \mathbb{B}_{\mathbf{C}}(\mathbf{u}) \quad (7)$$

for $J_1^{(d)}$ -topology on D_d as $n \rightarrow \infty$, where $B_{\mathbf{C}}$ is a continuous Gaussian process with

$$\mathbb{E}(B_{\mathbf{C}}(\mathbf{u})) = 0, \quad \mathbb{E}(B_{\mathbf{C}}(\mathbf{u})B_{\mathbf{C}}(\mathbf{v})) = \mathbf{C}(\mathbf{u} \wedge \mathbf{v}) - \mathbf{C}(\mathbf{u})\mathbf{C}(\mathbf{v})$$

where $\mathbf{u} \wedge \mathbf{v} = (\min(u_1, v_1), \dots, \min(u_d, v_d))$. For the precise definition of the space D_d and the $J_1^{(d)}$ -topology, and for more multivariate empirical processes, we refer to Einmahl (1987). Fermanian *et al.* (2004), show that the weak convergence of $\gamma_n(\cdot)$ to $\mathbb{B}_{\mathbf{C}}^*(\cdot)$ (in the bivariate case) holds on $[0, 1]^2$ when $\mathbf{C}(\cdot)$ has continuous partial derivatives on $[0, 1]^2$, where

$$\mathbb{B}_{\mathbf{C}}^*(\mathbf{u}) = B_{\mathbf{C}}(\mathbf{u}) - B_{\mathbf{C}}(u_1, 1) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_1} - B_{\mathbf{C}}(1, u_2) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_2}, \quad \mathbf{u} \in [0, 1]^2.$$

Deheuvels (1980, 1981a) described the limiting behavior $\gamma_n(\cdot)$ in the case of independence of margins. In more general case the reader is referred to Gaenssler and Stute (1987). Recently, Deheuvels (2008) derive optimal rates for strong approximation of empirical copula processes by sequences of Gaussian processes. In the last reference, a full characterization of empirical copulas is provided. We can refer to Deheuvels (2008) and the references therein concerning the invariance principles for copulas.

In the present paper, we are concerned with strong approximations of the empirical copula processes $\{\gamma_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$, based upon $\mathbf{X}_1, \dots, \mathbf{X}_n$, by a single Gaussian process. Denote by $\{\mathbb{K}_{\mathbf{C}}(\mathbf{u}, t); \mathbf{u} \in [0, 1]^d, t \geq 0\}$ a Gaussian process, with continuous sample paths, mean zero, and covariance function given by

$$\mathbb{E}(\mathbb{K}_{\mathbf{C}}(\mathbf{u}, s)\mathbb{K}_{\mathbf{C}}(\mathbf{v}, t)) = (s \wedge t) \{\mathbf{C}(\mathbf{u} \wedge \mathbf{v}) - \mathbf{C}(\mathbf{u})\mathbf{C}(\mathbf{v})\}, \text{ for } \mathbf{u}, \mathbf{v} \in [0, 1]^d \text{ and } s, t \geq 0.$$

Note that the process $\{\mathbb{K}_{\mathbf{C}}(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d, t \geq 0\}$ is called a Kiefer process. For each $n > 0$ and $u_j \in [0, 1]$, for $j = 1, \dots, d$, set

$$\begin{aligned} \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n) &= \mathbb{K}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathbb{K}_{\mathbf{C}}(1, \dots, 1, u_j, 1, \dots, 1, n) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j} \\ &=: \mathbb{K}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathbb{K}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n) \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j}, \quad \mathbf{u} \in [0, 1]^d. \end{aligned} \quad (8)$$

The remaining of our paper is organized as follows. In the next section we will give our main result concerning the strong approximation of empirical copula processes by a Gaussian processes, which is stated in Theorem 2.1 below. In section 3 we will give some applications of Theorem 2.1, more precisely we will give the limit law of smoothed empirical copula and the law of iterated logarithm for the empirical copula processes. The proof of these results will be sketched in the section 4.

2 Results

We may now state our strong approximation theorem as follows.

Theorem 2.1 *Assume that $\mathbf{C}(\cdot)$, associated with $\mathbf{F}(\cdot)$, is twice continuously differentiable on $(0, 1)^d$, and the second derivative be continuous on $[0, 1]^d$. On suitable probability space, we may define the empirical copula processes $\{\gamma_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d; n > 0\}$ in combination with a Gaussian process $\{\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d; t \geq 0\}$, in such a way that, almost surely as $n \rightarrow \infty$*

$$\sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n} \gamma_n(\mathbf{u}) - \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)| = O\left(n^{1/2-1/(4d)} (\log n)^{3/2}\right), \quad (9)$$

where $\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, t)$ is defined in (8).

The proof of Theorem 2.1 is given in Section 4.

Remark 1 In the particular case of independence, i.e., $\mathbf{C}(\mathbf{u}) = \prod_{i=1}^d u_i$ the process $\{\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n) : \mathbf{u} \in [0, 1]^d; n \geq 0\}$ is equal to

$$\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n) =: \mathbb{K}_{\mathbf{C}}(\mathbf{u}, n) - \sum_{j=1}^d \mathbb{K}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n) \prod_{i \neq j}^d u_i, \quad \mathbf{u} \in [0, 1]^d,$$

with mean zero and covariance functions

$$\mathbb{E}(\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, s) \mathbb{K}_{\mathbf{C}}^*(\mathbf{v}, t)) = (s \wedge t) \left\{ \prod_{i=1}^d (u_i \wedge v_i) + (d-1) \prod_{i=1}^d u_i v_i - \sum_{i=1}^d (u_i \wedge v_i) \prod_{i \neq j}^d u_i v_j \right\}$$

where $\mathbf{u}, \mathbf{v} \in [0, 1]^d$ and $s, t \geq 0$. For more details the reader may refer to Csörgő (1979).

3 Applications

3.1 Smoothed empirical copula processes

We define smoothed estimators $\widehat{\mathbf{C}}_n(\cdot)$ of $\mathbf{C}(\cdot)$ by

$$\widehat{\mathbf{C}}_n(\mathbf{u}) = \frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u} - \mathbf{v}}{h^{1/d}}\right) \mathbf{C}_n(\mathbf{v}) d\mathbf{v} \text{ for } \mathbf{u} \in [0, 1]^d, \quad (10)$$

where $k(\cdot)$ is a kernel function and $h = h(n)$ is the smoothing parameter. Similarly to the previous section, we define the smoothed empirical copulas process by

$$\widehat{\gamma}_n(\mathbf{u}) := \sqrt{n}(\widehat{\mathbf{C}}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})) \text{ for } \mathbf{u} \in [0, 1]^d. \quad (11)$$

Assume $\mathbf{C}(\cdot)$ fulfills the following condition:

(F.1). $\mathbf{C}(\cdot)$ has a bounded s th derivative.

For our result we suppose that h and $k(\cdot)$ satisfy the following regularity conditions:

(C.1). $h = h(n) \rightarrow 0$, $nh \rightarrow \infty$ and $\sqrt{nh^{s/d}} \rightarrow 0$ as $n \rightarrow \infty$;

(C.2). $k(\cdot)$ is a continuous density function and compactly supported;

(C.3). $k(\cdot)$ is of order s .

Corollary 3.1 *Assume that (F.1) and (C.1)-(C.3) hold. Then, on suitable probability space, we may define the empirical copula processes $\{\widehat{\gamma}_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d; n > 0\}$ in combination with a Gaussian process $\{\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d; t \geq 0\}$, in such a way that, as $n \rightarrow \infty$*

$$\sup_{\mathbf{u} \in [0,1]^d} |\widehat{\gamma}_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)| = o_P(1). \quad (12)$$

The proof of Corollary 3.1 is given in Section 4.

Remark 2 1. *Corollary 3.1 remains valid when replacing the condition that the kernel function $k(\cdot)$ having compact support in (C.2) by*

(C.4). *There exists sequences of real numbers a_n such that $a_n h$ tends to zero when n tends to infinity, and*

$$\sqrt{n} \int_{\{\|\mathbf{v}\| > a_n\}} |k(\mathbf{v})| d\mathbf{v} \rightarrow 0.$$

2. *Note that conditions of Corollary 3.1 are grouped to control the deviations between classical and smoothed empirical copulas processes.*

3.2 The law of iterated logarithm for empirical copula processes

From Theorem 2.1, we have almost surely

$$\limsup_{n \rightarrow \infty} \left\{ \left(\frac{n}{2 \log \log n} \right)^{1/2} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})| \right\} = \limsup_{n \rightarrow \infty} \frac{\sup_{\mathbf{u} \in [0,1]^d} |\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)|}{(2n \log \log n)^{1/2}}. \quad (13)$$

According to Wichura (1973), we have almost surely, as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \frac{\sup_{\mathbf{u} \in [0,1]^d} |\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)|}{(2n \log \log n)^{1/2}} \leq d + 1. \quad (14)$$

Combining (13) and (14), we can state the following Corollary.

Corollary 3.2 *Under the same conditions of the Theorem 2.1, we have*

$$\limsup_{n \rightarrow \infty} \left\{ \left(\frac{n}{2 \log \log n} \right)^{1/2} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}_n(\mathbf{u}) - \mathbf{C}(\mathbf{u})| \right\} \leq d + 1. \quad (15)$$

Remark 3 The result of Corollary 3.2 was obtained by Deheuvels (1979) (refer to Theorem 3.1) using a different method.

4 Proofs

4.1 Proof of theorem 2.1

For readability of this paper, we recall some facts that we will need for our demonstration.

Fact 1. The processes $\mathbb{K}_{\mathbf{C}}^*(\cdot, n)$, $\mathbb{K}_{\mathbf{C}}^{(j)}(\cdot, n)$ and $\mathbb{K}_{\mathbf{C}}^{(i)}(\cdot, n)$, for all $1 \leq i \neq j \leq d$, are independent.

For convenience, we will denote the sup-norm of a bounded function f , defined on $I = [0, 1]$ or $I = [0, 1]^d$, by $\|f\| = \sup_{x \in I} |f(x)|$. The next fact, due to Kiefer (1970) and Bahadur (1966) provides a L.I.L. of Bahadur-Kiefer process (see also, Deheuvels and Mason (1990)).

Fact 2. We have, for $j = 1, \dots, d$, with probability 1,

$$\limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log \log n)^{-1/4} \|\alpha_n^j + \beta_{jn}\| = 2^{1/4},$$

Recall the definition (6) of \mathbb{G}_n , the empirical process of copulas can be written, for $\mathbf{u} \in [0, 1]^d$, as

follows

$$\begin{aligned}
\gamma_n(\mathbf{u}) &= n^{1/2}(\tilde{\mathbf{C}}(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d)) - \mathbf{C}(u_1, \dots, u_d)) \\
&= \alpha_n \left(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d) \right) + n^{1/2} \left\{ \mathbf{C} \left(G_{1,n}^-(u_1), \dots, G_{d,n}^-(u_d) \right) \right. \\
&\quad \left. - \mathbf{C}(u_1, \dots, u_d) \right\} \\
&= \alpha_n \left(u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d) \right) \\
&\quad + n^{1/2} \left\{ \mathbf{C} \left(u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d) \right) - \mathbf{C}(u_1, \dots, u_d) \right\} \\
&= \alpha_n(\mathbf{u}) + \left\{ \alpha_n(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) - \alpha_n(\mathbf{u}) \right\} \\
&\quad + n^{1/2} \left\{ \mathbf{C}(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) - \mathbf{C}(\mathbf{u}) \right\} \\
&= \alpha_n(\mathbf{u}) + \Delta_1(\mathbf{u}, n) + \Delta_2(\mathbf{u}, n). \tag{16}
\end{aligned}$$

where $(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u})) = (u_1 + n^{-1/2}\beta_{1n}(u_1), \dots, u_d + n^{-1/2}\beta_{dn}(u_d))$. We first evaluate the right side term $\Delta_2(\mathbf{u}, n)$ of (16). By differentiability assumption on $\mathbf{C}(\cdot)$, we can use Taylor's expansion to obtain the following

$$\begin{aligned}
\Delta_2(\mathbf{u}, n) &= \sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \sqrt{n}(G_{ni}^-(u_i) - u_i) \\
&\quad + \frac{\sqrt{n}}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 \mathbf{C}(\mathbf{u}^*)}{\partial u_i \partial u_j} (G_{ni}^-(u_i) - u_i)(G_{nj}^-(u_j) - u_j),
\end{aligned}$$

where \mathbf{u}^* lies in the interior of the line segment joining $(G_{n1}^-(u_1), \dots, G_{nd}^-(u_d))$ and (u_1, \dots, u_d) . It follows from the definition of $\alpha_n(\cdot)$ in (3), for $u_j \in [0, 1]$, $j = 1, \dots, d$, that

$$\begin{aligned}
\sqrt{n}(G_{ni}^-(u_i) - u_i) &= -\sqrt{n}(G_{ni}(G_{ni}^-(u_i)) - G_{ni}^-(u_i)) + \sqrt{n}(G_{ni}(G_{ni}^-(u_i)) - u_i) \\
&= -\alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) + \sqrt{n}(G_{ni} \circ G_{ni}^-(u_i) - u_i).
\end{aligned}$$

Using the fact that $|G_{ni} \circ G_{ni}^-(u_i) - u_i| \leq 1/n$ and the Chung (1949)'s law of the iterated logarithm, one finds, almost surely,

$$\Delta_2(\mathbf{u}, n) = -\sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) + O(n^{-1/2}(\log \log n)^{1/2}).$$

Further more, from Stute (1982), we have, almost surely,

$$\sup_{u_i \in [0,1]} |\alpha_n(\mathbf{1}, G_{ni}^-(u_i), \mathbf{1}) - \alpha_n(\mathbf{1}, u_i, \mathbf{1})| = O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}).$$

Then, it follows that, almost surely,

$$\Delta_2(\mathbf{u}, n) = -\sum_{i=1}^d \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_i} \alpha_n(\mathbf{1}, u_i, \mathbf{1}) + O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}),$$

in agreement with the result of Stute (1984), p. 371. We next evaluate the term $\Delta_1(\mathbf{u}, n)$ in the right hand side of (16). Recall that $\Delta_1(\mathbf{u}, n)$ is the difference of $\alpha_n(\mathbf{u} + n^{-1/2}\beta_n(\mathbf{u}))$ and $\alpha_n(\mathbf{u})$. Let $w_n^*(\cdot)$ be the modulus of continuity of $\alpha_n(\cdot)$, defined by

$$w_n^*(\mathbf{a}) := \sup \left\{ \alpha_n(A) : A = \prod_{i=1}^d [u_i, v_i] \in [0, 1]^d, \text{ with } |[u_i, v_i]| = v_i - u_i \leq a_i, \forall i = 1, \dots, d \right\},$$

We will make use of the following fact which is a particular case of Theorem 2.1, p. 367 of Stute (1984).

Fact 3. Let $\{a_n\}_{n \geq 1}$ be a sequence in $(0, 1)$ such as $a_n \downarrow 0$, as $n \rightarrow \infty$, and

$$i) na_n^d \uparrow \infty, \quad ii) na_n^d / \log n \rightarrow \infty, \quad iii) \log(1/a_n) / \log \log n \rightarrow \infty$$

Then, with probability 1, we have

$$\lim_{n \rightarrow \infty} \left\{ 2a_n^d \log(1/a_n^d) \right\}^{-1/2} w_n^*(a_n, \dots, a_n) = 1.$$

An application of the Chung (1949) law of the iterated logarithm shows that, for each $j = 1, \dots, d$, almost surely,

$$\limsup_{n \rightarrow \infty} \left\{ (\log \log n)^{-1/2} \sup_{0 \leq u \leq 1} |\beta_{jn}(u_j)| \right\} = 2^{-1/2}. \quad (17)$$

From (17), we have almost surely for all $j = 1, \dots, d$ and all large n

$$\sup_{0 \leq u_j \leq 1} |n^{-1/2} \beta_{jn}(u_j)| \leq \frac{(\log \log n)^{1/2}}{n^{1/2}} \leq \frac{(\log n)^{2/d}}{n^{1/d}} := a_n,$$

an application of fact 3 shows that, as $n \rightarrow \infty$, we have, almost surely,

$$\sup_{\mathbf{u} \in [0, 1]^d} |\Delta_1(\mathbf{u}, n)| \leq w_n^*(a_n) = O\left(n^{-1/2}(\log n)^{3/2}\right).$$

The next fact, due to Csörgő and Horváth (1988), p. 102, provides a strong approximation result appropriate for our need.

Fact 4. On a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, it is possible to define $\{\alpha_n(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$, jointly with a sequence of Gaussian process $\{\mathbb{K}_{\mathbf{C}}(\mathbf{u}, t) : \mathbf{u} \in [0, 1]^d, t \geq 0\}$, in such a way that, as $n \rightarrow \infty$, almost surely,

$$\sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n}\alpha_n(\mathbf{u}) - \mathbb{K}_{\mathbf{C}}(\mathbf{u}, n)| = O\left(n^{1/2-1/(4d)}(\log n)^{3/2}\right). \quad (18)$$

Therefore, by the triangle inequality,

$$\begin{aligned} \sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n}\gamma_n(\mathbf{u}) - \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)| &\leq \sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n}\alpha_n(\mathbf{u}) - \mathbb{K}_{\mathbf{C}}(\mathbf{u}, n)| + w_n^*(a_n) \\ &+ \sum_{j=1}^d \left| \frac{\partial \mathbf{C}(\mathbf{u})}{\partial u_j} \right| \sup_{0 \leq u_j \leq 1} |\sqrt{n}\alpha_n(\mathbf{1}, u_j, \mathbf{1}) - \mathbb{K}_{\mathbf{C}}^{(j)}(\mathbf{1}, u_j, \mathbf{1}, n)| \\ &+ O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}), \\ &\leq O\left(n^{1/2-1/(4d)}(\log n)^{3/2}\right) + O\left(n^{-1/2}(\log n)^{3/2}\right) \\ &+ O\left(n^{1/2-1/(4d)}(\log n)^{3/2}\right) + O\left(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}\right), \end{aligned}$$

then, we have almost surely, as $n \rightarrow \infty$,

$$\sup_{\mathbf{u} \in [0,1]^d} |\sqrt{n}\gamma_n(\mathbf{u}) - \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)| = O\left(n^{1/2-1/(4d)}(\log n)^{3/2}\right)$$

and thus the proof of Theorem 2.1 is complete. \square

4.2 Proof of corollary 3.1

First, observe that the difference between the *modified empirical copula process* $\gamma_n(\cdot)$ and the smoothed empirical copula process $\hat{\gamma}_n(\cdot)$ can be controlled as follows. We have

$$\begin{aligned} \hat{\gamma}_n(\mathbf{u}) &= \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u}-\mathbf{v}}{h^{1/d}}\right) \mathbf{C}_n(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right) \\ &= \left(\frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u}-\mathbf{v}}{h^{1/d}}\right) \sqrt{n}(\mathbf{C}_n(\mathbf{v}) - \mathbf{C}(\mathbf{v})) d\mathbf{v} \right. \\ &\quad \left. + \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u}-\mathbf{v}}{h^{1/d}}\right) \mathbf{C}(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right) \right) \\ &= \left(\frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u}-\mathbf{v}}{h^{1/d}}\right) \gamma_n(\mathbf{u}) d\mathbf{v} \right. \\ &\quad \left. + \sqrt{n} \left(\frac{1}{h} \int_{[0,1]^d} k\left(\frac{\mathbf{u}-\mathbf{v}}{h^{1/d}}\right) \mathbf{C}(\mathbf{v}) d\mathbf{v} - \mathbf{C}(\mathbf{u}) \right) \right). \end{aligned} \quad (19)$$

Thus, one can derive the following

$$\begin{aligned} |\hat{\gamma}_n(\mathbf{u}) - \gamma_n(\mathbf{u})| &\leq \left| \int_{\prod_{i=1}^d \left[\frac{u_{i-1}}{h^{1/d}}, \frac{u_i}{h^{1/d}}\right]} (\gamma_n(\mathbf{u} - h^{1/d}\mathbf{v}) - \gamma_n(\mathbf{u})) k(\mathbf{v}) d\mathbf{v} \right| \\ &\quad + \sup_{\mathbf{u} \in [0,1]^d} |\gamma_n(\mathbf{u})| \left| \int_{\prod_{i=1}^d \left[\frac{u_{i-1}}{h^{1/d}}, \frac{u_i}{h^{1/d}}\right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| \\ &\quad + \sqrt{n} \left| \int_{\prod_{i=1}^d \left[\frac{u_{i-1}}{h^{1/d}}, \frac{u_i}{h^{1/d}}\right]} (\mathbf{C}(\mathbf{u} - h^{1/d}\mathbf{v}) - \mathbf{C}(\mathbf{u})) k(\mathbf{v}) d\mathbf{v} \right| \\ &\quad + \sqrt{n} \sup_{\mathbf{u} \in [0,1]^d} |\mathbf{C}(\mathbf{u})| \left| \int_{\prod_{i=1}^d \left[\frac{u_{i-1}}{h^{1/d}}, \frac{u_i}{h^{1/d}}\right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| \\ &:= \xi_{1;n}(\mathbf{u}) + \xi_{2;n}(\mathbf{u}) + \xi_{3;n}(\mathbf{u}) + \xi_{4;n}(\mathbf{u}). \end{aligned} \quad (20)$$

We first evaluate $\xi_{3;n}(\cdot)$ in the right side of (20). Under conditions (F.1), (C.1)-(C.3) and using Taylor expansion of order s we get

$$|\xi_{3;n}(\mathbf{u})| = \frac{h_n^{s/d}}{s!} \sqrt{n} \left| \int \sum_{k_1 + \dots + k_d = s} u_1^{k_1} \dots u_d^{k_d} \frac{\partial^s \mathbf{C}(\mathbf{u} - h_n \theta \mathbf{v})}{\partial u_1^{k_1} \dots \partial u_d^{k_d}} K(\mathbf{v}) d\mathbf{v} \right|,$$

where $\theta = (\theta_1, \dots, \theta_d)$ and $0 < \theta_i < 1$. Thus a straightforward application of Lebesgue dominated convergence theorem gives

$$n^{-1/2} h_n^{-(s/d)} |\xi_{3;n}(\mathbf{u})| = \frac{1}{k!} \left| \sum_{k_1 + \dots + k_d = s} \frac{\partial^s \mathbf{C}(\mathbf{u})}{\partial u_1^{k_1} \dots \partial u_d^{k_d}} \int u_1^{k_1} \dots u_d^{k_d} K(\mathbf{v}) d\mathbf{v} \right|. \quad (21)$$

Then we have by condition (C.1) and (C.3)

$$|\xi_{3;n}(\mathbf{u})| = O(n^{1/2} h_n^{s/d}) = o(1). \quad (22)$$

By Theorem 2.1 and almost sure continuity of the Gaussian process $\mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n)$ we have

$$\begin{aligned} \xi_{1;n}(\mathbf{u}) &\leq \sup_{\mathbf{u}, \mathbf{v} \in [0,1]^d} \sup_{|\mathbf{u}-\mathbf{v}| \leq h_n} |\gamma_n(\mathbf{v}) - \gamma_n(\mathbf{u})| \left| \int K(\mathbf{v}) d\mathbf{v} \right| \\ &= o_P(1) O(1) = o_P(1). \end{aligned} \quad (23)$$

We next evaluate $\xi_{2;n}(\cdot)$ in the right side of (20). We have $\sup_{\mathbf{u} \in [0,1]^d} |\gamma_n(\mathbf{u})| = O_P(1)$ and as n tends to infinity (by condition (C.2))

$$\sqrt{n} \left| \int_{\prod_{i=1}^d \left[\frac{u_i-1}{n^{1/d}}, \frac{u_i}{n^{1/d}} \right]} k(\mathbf{v}) d\mathbf{v} - 1 \right| = o(1),$$

then $\xi_{2;n}(\mathbf{u}) = o_P(1)$. Using the same arguments we obtain $\xi_{4;n}(\mathbf{u}) = o_P(1)$. Reorganizing the above results, one finds

$$\sup_{\mathbf{u} \in [0,1]^d} |\widehat{\gamma}_n(\mathbf{u}) - \gamma_n(\mathbf{u})| = o_P(1). \quad (24)$$

Now the Corollary follows from (24) in conjunction with Theorem 2.1, and the fact that

$$\begin{aligned} \sup_{\mathbf{u} \in [0,1]^d} \left| \widehat{\gamma}_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n) \right| &\leq \sup_{\mathbf{u} \in [0,1]^d} |\widehat{\gamma}_n(\mathbf{u}) - \gamma_n(\mathbf{u})| \\ &\quad + \sup_{\mathbf{u} \in [0,1]^d} \left| \gamma_n(\mathbf{u}) - \frac{1}{\sqrt{n}} \mathbb{K}_{\mathbf{C}}^*(\mathbf{u}, n) \right|. \end{aligned}$$

and thus the proof of Corollary is complete. \square

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