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## Slowly decaying states of three particles with resonant interactions

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**Abstract.** - We determine analytically how Efimov trimer states are modified by three-body losses within the model of Braaten and Hammer. We find a regime where the energies approach the positive real axis and the decay rates vanish.

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**Introduction.** – Motivated by nuclear physics, Efimov discovered that three particles with short-ranged two-body interactions of scattering length  $|a| = \infty$  can support an infinite series of weakly bound trimer states [1]. In contrast to systems of nucleons or of  $^4\text{He}$  atoms, ultracold gases of alkali metal atoms offer the possibility to tune  $a$  by using a Feshbach resonance, and thus to go more deeply into the limit  $|a| \rightarrow \infty$  where the Efimov effect sets in [2–14]. However the situation for alkali metal atoms is complicated by the existence of deeply bound dimer states: a weakly bound trimer necessarily decays into a deeply bound dimer and a free atom [6, 8]. This three-body loss process has prevented so far to produce weakly bound trimers experimentally. The first evidence for the existence of a weakly bound trimer state is the recent observation of a peak in the three-body loss rate from an ultracold atomic gas at large negative  $a$  [2]. Such peaks were predicted to occur for the values of  $a$  where the energy of an Efimov trimer reaches zero [5, 9]. The simplest description of three-body losses is the model of Braaten and Hammer, where three incoming low-energy unbound atoms are reflected back as three unbound atoms with a probability  $e^{-4\eta_*}$ , and recombine to a deeply bound dimer and an atom with a probability  $1 - e^{-4\eta_*}$ ;  $\eta_*$  being the so-called inelasticity parameter, which depends on the short-range details of the interaction potential and of its deeply bound states [3, 4, 9]. This model was used to obtain the decay rate of Efimov trimers in the regime of small inelasticity parameter  $\eta_* \ll 1$  [4, 8], as well as 3- and 4-body scattering properties [3, 4, 13] and the decay rate of efimovian 3-body states in a harmonic trap [15, 16]. This model is expected to become exact in the limit where  $|a|$  and the inverse relative momenta between atoms in the initial state are much larger than the range and effective range of the interaction potential [3, 4, 8, 9].

In this Letter, we determine analytically how Efimov trimers are modified for an arbitrary inelasticity parameter  $\eta_*$ . We find that the Efimov spectrum rotates counterclockwise in the complex plane by an angle proportional to  $\eta_*$ . When  $\eta_*$  reaches the critical value where this angle equals  $\pi$ , the discrete states disappear into the continuum. When  $\eta_*$  approaches

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this critical value from below, the energies approach the positive real axis, so that the decay rates tend to zero. Thus a large inelasticity parameter  $\eta_* \sim 1$  can paradoxically give rise to long-lived three-body states.

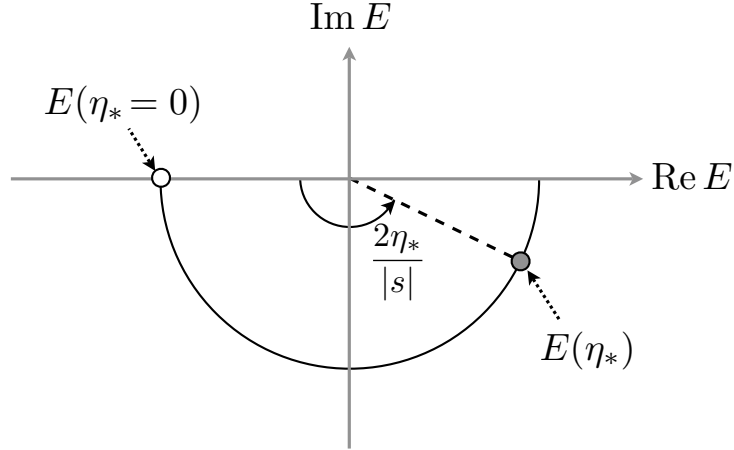


Fig. 1: Without three-body losses the inelasticity parameter  $\eta_* = 0$  and the energy of an Efimov state lies on the negative real axis. The effect of three-body losses is to rotate this energy in the complex plane by an angle  $2\eta_*/|s|$ , where  $|s| = 1.00624\dots$ . When this angle approaches  $\pi$ , the decay rate  $\Gamma = -2 \text{Im } E/\hbar$  vanishes.

**Efimov states without three-body losses.** – We start by reviewing Efimov’s solution of the three-body problem in the absence of three-body losses. We restrict for simplicity to the case of three identical bosonic particles, where the wavefunction  $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  is completely symmetric.<sup>1</sup> In the limit where the range of the interaction potential is negligible compared to  $|a|$  and to the typical de Broglie wavelength, the interaction can be replaced exactly by the zero-range model. An eigenstate of the zero-range model solves (i) the Schrödinger equation

$$-\frac{\hbar^2}{2m} \sum_{i=1}^3 \Delta_{\mathbf{r}_i} \psi = E \psi \quad (1)$$

when all interparticle distances  $r_{ij}$  are different from zero, and (ii) the Bethe-Peierls boundary condition

$$\exists A/ \psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \underset{r_{ij} \rightarrow 0}{=} \left( \frac{1}{r_{ij}} - \frac{1}{a} \right) A(\mathbf{R}_{ij}, \mathbf{r}_k) + O(r_{ij}) \quad (2)$$

in the limit  $r_{ij} \rightarrow 0$  where particles  $i$  and  $j$  approach each other while the position of their center of mass  $\mathbf{R}_{ij}$  and of the third particle  $\mathbf{r}_k$  are fixed. In what follows we consider the resonant case where the scattering length diverges,  $a = \infty$ . Equations (1,2) are then solved by Efimov’s Ansatz [1]

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = F(R) \left( 1 + \hat{P}_{13} + \hat{P}_{23} \right) \frac{1}{r\rho} \sin \left[ s \arctan \left( \frac{\rho}{r} \right) \right] \quad (3)$$

<sup>1</sup>The case of different masses and statistics can be included without difficulty: only the value of  $s$  is modified in the hyperradial problem [17].

where  $\hat{P}_{ij}$  exchanges particles  $i$  and  $j$ ,  $s \simeq i \cdot 1.00624$  is the only solution  $s \in i \cdot (0; +\infty)$  of the equation  $s \cos(s\pi/2) - 8/\sqrt{3} \sin(s\pi/6) = 0$ , the Jacobi coordinates are  $r = \|\mathbf{r}_2 - \mathbf{r}_1\|$  and  $\rho = \|2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2\|/\sqrt{3}$ , the hyperradius is  $R = \sqrt{(r^2 + \rho^2)}/2$ , and the hyperradial wavefunction  $F(R)$  solves the hyperradial Schrödinger equation

$$\left[ - \left( \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} \right) + \frac{s^2}{R^2} \right] F(R) = \frac{2m}{\hbar^2} E F(R). \quad (4)$$

In the limit  $R \rightarrow 0$  where all three particles approach each other, the attractive effective potential  $s^2/R^2$  diverges strongly, and it is necessary to impose a boundary condition on the hyperradial wavefunction  $F(R)$ , as first realized by Danilov [18]. This boundary condition is conveniently expressed as

$$F(R) \underset{R \rightarrow 0}{\propto} \left( \frac{R}{R_t} \right)^{-s} - \left( \frac{R}{R_t} \right)^s \quad (5)$$

where the three-body parameter  $R_t$  depends on short-range physics and is a parameter of the zero-range model. The solution of Eqs. (4,5) is given by the famous geometric series of Efimov trimers

$$E_n^0 = -\frac{2\hbar^2}{mR_t^2} \exp \left[ \frac{2}{|s|} \arg \Gamma(1+s) \right] \exp \left( n \frac{2\pi}{|s|} \right), \quad n \in \mathbb{Z}, \quad (6)$$

with a (here unnormalized) wavefunction

$$F(R) = K_s(\kappa R) \quad (7)$$

where  $K$  is a Bessel function and  $\kappa$  is defined by

$$E = -\frac{\hbar^2 \kappa^2}{2m} \quad (8)$$

with the determination  $\kappa > 0$  ensuring that  $F(R)$  decays exponentially for  $R \rightarrow \infty$ . Efimov's spectrum (6) is unbounded from below, in agreement with the Thomas effect [19] and with the fact that the spectrum for an interaction of finite range  $b$  coincides with Efimov's spectrum only in the limit  $|E| \ll \hbar^2/(mb^2)$  of weakly bound trimers.

**Effect of three-body losses.** – We now determine how Efimov's result is modified by three-body losses. Again, the three-body problem with zero-range interactions is given by (1,2), and thanks to Efimov's Ansatz (3) we reduce it to the radial Schrödinger equation (4). Following Braaten and Hammer [3,4,8] we model three-body losses by generalizing the boundary condition (5) to

$$F(R) \underset{R \rightarrow 0}{\propto} \left( \frac{R}{R_t} \right)^{-s} - e^{-2\eta_*} \left( \frac{R}{R_t} \right)^s \quad (9)$$

where the inelasticity parameter  $\eta_*$  depends on the details of the finite-range interactions which one wishes to model.<sup>2</sup> The physical meaning of (9) is that the outgoing wave  $\propto R^s$  has an amplitude which is smaller than the amplitude of the ingoing wave  $\propto R^{-s}$  by a factor  $e^{-2\eta_*}$ , i.e. three ingoing atoms are reflected with a probability  $e^{-4\eta_*}$  and are lost by three-body recombination with a probability  $1 - e^{-4\eta_*}$ .

In the absence of losses ( $\eta_* = 0$ ), the boundary condition (9) reduces to Efimov's boundary condition (5), and one recovers Efimov's spectrum (6):

$$E_n(\eta_* = 0) = E_n^0. \quad (10)$$

<sup>2</sup>E.g. for  $^{133}\text{Cs}$  atoms near the  $-11$  G Feshbach resonance, the fit of the theoretical result of [9] to the experimental data of [2] gives  $\eta_* \simeq 0.06$  [2] and  $R_t \simeq 30$  nm [2,15].

In presence of losses ( $\eta_* > 0$ ), we solve the hyperradial problem (4,9) with the additional boundary condition that  $F(R)$  must decay quickly enough at infinity.<sup>3</sup> Using the known properties of Bessel functions [20], we obtain the energies

$$E_n(\eta_*) = \exp\left(i \frac{2\eta_*}{|s|}\right) E_n(\eta_* = 0), \quad (11)$$

i. e. the spectrum is rotated in the complex plane counterclockwise around the origin by an angle  $2\eta_*/|s|$ . The result (11) only holds if the angle  $2\eta_*/|s| < \pi$ , i. e. for  $\eta_* < \eta_{*c}$  with

$$\eta_{*c} = \frac{\pi|s|}{2} = 1.5806\dots, \quad (12)$$

while for  $\eta_* > \eta_{*c}$  there is no normalisable solution.

The wavefunction is still given by Eqs. (7,8), now with the determination

$$\text{Re } \kappa > 0 \quad (13)$$

of the sign of  $\kappa$ , which ensures that the wavefunction is normalisable since

$$F(R) \underset{R \rightarrow \infty}{\propto} \frac{e^{-\kappa R}}{\sqrt{R}}. \quad (14)$$

The decay rate

$$\Gamma \equiv -\frac{2}{\hbar} \text{Im } E \quad (15)$$

is given by

$$\hbar\Gamma = 2 \sin\left(\frac{2\eta_*}{|s|}\right) |E(\eta_* = 0)|. \quad (16)$$

In the limit of small losses  $\eta_* \ll 1$  we recover the known result [8]

$$\hbar\Gamma \simeq \frac{4\eta_*}{|s|} |E(\eta_* = 0)|. \quad (17)$$

A surprising effect occurs when  $\eta_*$  approaches the critical value  $\eta_{*c}$  from below: the energies approach the opposite of the energies of the Efimov states without losses

$$E(\eta_*) \underset{\eta_* \rightarrow \eta_{*c}}{\longrightarrow} |E(\eta_* = 0)|, \quad (18)$$

so that the decay rates tend to zero. Consistently, the sizes of the states diverge: since the imaginary part of  $\kappa$  tends to a positive value and its real part tends to  $0^+$ , the behavior (14) of the wavefunction at large  $R$  is an ingoing wave with a *slowly* decaying envelope.

This intriguing effect occurs within the effective low-energy model of Braaten and Hammer. Thus it should also occur for finite-range interactions supporting deeply bound dimers, provided the two-body interaction potential is tuned in such a way that  $\eta_*$  is slightly below  $\eta_{*c}$ ; this could be checked numerically using the methods of [5, 6, 12].

Experimentally, it is rather unlikely to find a Feshbach resonance close to which  $\eta_*$  is slightly below  $\eta_{*c}$ . However this regime may become accessible if interatomic interactions are tuned using an additional control parameter, e. g. an electric field [21].

\* \* \*

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<sup>3</sup>More precisely the normalisation condition is  $\int_0^\infty dR R |F(R)|^2 < \infty$ , since the quantity  $\int d\mathbf{r}_1 d\mathbf{r}_2 |\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)|^2$ , which does not depend on  $\mathbf{r}_3$ , has to be finite.

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