Algebraic generality vs arithmetic generality in the controversy between C. Jordan and L. Kronecker (1874).

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HAL Id: hal-00340071
https://hal.science/hal-00340071v3

Preprint submitted on 1 Nov 2011

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Frédéric Brechenmacher ([1]).

Introduction ([2]).

Throughout the whole of 1874 Camille Jordan and Leopold Kronecker were quarrelling over two theorems. Although Jordan's canonical form theorem and Karl Weierstrass' elementary divisors theorem would be considered equivalent in regard with modern matrix theory, not only had these two theorems been stated independently between 1868 and 1870, they had also been lying within the distinct frameworks of two theories until some connections came to light, hence breeding the 1874 quarrel. ([3]) This controversy would eventually turn into an opposition over the algebraic or arithmetic nature of the "theory of forms". As we shall see in this paper, this opposition sheds light on two conflicting perspectives on "generality".

Because the 1874 debate over what should be a "truly general" approach to the theory of forms was related to disciplinary ideals, this episode raises issues on the larger scale of the nineteenth century on which mathematical disciplines had developed. As a matter of fact, in the series of essays he has devoted to the history of matrices, Thomas Hawkins has laid special emphasis on the "generality" of Weierstrass' theorem which he presented as a "keystone" of the history of Algebra. Because, Hawkins argued, "Weierstrass demonstrated more than theorems. He also demonstrated the possibility and desirability of a more rigorous approach to algebraic analysis that did not rest content with the prevailing tendency to reason vaguely in terms of the 'general case'". According to Hawkins, the elementary divisors theorem thus served as a paradigm of generality with the

[3] On the one hand, Jordan had stated a result for substitutions of linear groups, on the other hand, Weierstrass had stated a theorem in the framework of the theory of bilinear and quadratic forms. As for Kronecker, he would actually defend his reworking of Weierstrass' theorem by the introduction of what is nowadays designated as the invariant factors. See the appendix 1 for more details about the mathematics involved in the two theorems.
From the standpoint of linear algebra, the main notions we will consider in this paper, *i.e.* substitutions, bilinear forms, and quadratic forms can all be represented by matrices even though they point to various classes of equivalence, similarity or congruences of matrices. Connections between such notions were nevertheless one of the issues at stake in the 1874 controversy. At the time, quadratic forms could already be considered as particular bilinear forms with symmetric coefficients but connections between forms and groups of substitutions were more problematic even though both theories appealed to the characteristic equations of the linear systems involved.

consequence that "he [Weierstrass], more than anyone, was responsible for the emergence of the theory of matrices as a coherent, substantial branch of twentieth-century mathematics" ([4]).

In this paper, we shall seize the opportunity of the controversial debates of 1874 to access the various forms and meanings taken on by the "generality" attached to some organisations of knowledge which were used before the emergence of object-oriented disciplines. Our aim is therefore not to give a definition of the algebraic generality or of the arithmetic one but to pay special attention to the ways the actors were using such categories. Methodologically speaking we shall wonder about the two protagonists' perspectives on generality without focusing on the issues about the origins of abstract notions, theories, ways of reasoning and, more generally, structures, most authors have been dealing with when studying the history of linear algebra. A retrospective disciplinary identity has indeed often served as a lens for looking into the past, selecting relevant texts and authors, thereby giving structure to its own history while other identities that did not fit in this retrospective theoretical glance have stayed out of sight. The question especially arises as to the identities and significations taken on by the algebraic processes that have been developed within various frames - such as mechanics, arithmetic or geometry - and have passed from one to another before they would be considered as methods within unifying algebraic theories such as the post 1930's theory of matrices.

As a matter of fact, during the nineteenth century, algebraic processes such as the ones of manipulations of the "forms" of linear systems we will consider in this paper, were usually not identified to any method or notion which could be inscribed in a theoretical frame. They were nevertheless not limited to technical tools. As will be seen by looking through the prism of the 1874 controversy, processes such as the ones of Jordan's canonical reduction and Kronecker's invariants computations were resorting to some conflicting epistemic values related to internal philosophies of generality, such as "simplicity" and "abstraction" *vs* "effectivity" and "homogeneity" *vs* "formalism" and "genericity". We shall therefore designate here these processes as "practices" not only for the purpose of highlighting that they were not limited to technical tools but also for insisting that the identities these "practices" have been taking on must be considered as a problem. ([5])

One of the aims of this paper is indeed to investigate how the "generality" of such practices was resorting to both individual and collective aspects. As a matter of fact, even though the 1874 controversy was triggered by the opposition of two individual practices, the two perspectives on generality were also resorting to complex collectives involving both a long term shared history of investigations on a "general" *special* equation and two diverging lines of developments on forms of treatments of "special" equations in *general*.

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[4] (Hawkins 1977: 119).

[5] About the methodolical issues raised by questions of identities in the history of mathematics, see (Sinaceur 1991: 16) and (Goldstein 1995: 21).

1. A polemical "general" theory.

The controversy started in the winter 1873-1874 when two papers were successively read to the Academies of Paris and Berlin. ([6]) The quarrel was originally caused by Jordan's ambition to reorganise the theory of bilinear forms through what he designated as the "simple" notion of "canonical form." Jordan's December 1873 note was actually the first contribution to the theory of bilinear forms to be published out of Berlin. One of the issues at stake was therefore the organisation of a theory which used to be a local field of research limited to a few Berliners but in the process of turning into a more global theory.

In the 1870s, not only did many recent applications herald the major role the theory of bilinear forms would play in the following decades, this theory was also giving a new "homogeneous" and "general" treatment to different problems referring to various theories developed throughout the nineteenth century. ([7]) In the quotation below, Jordan alluded to geometry and Augustin Louis Cauchy's results on the principal axis of conics and quadrics (first question), to the arithmetic of quadratic forms relating especially to the works of Carl Gauss and Charles Hermite (second question), as well as to analytical mechanic and the solution given by Lagrange to systems $PY''+QY=0$ of linear differential equations with constant coefficients (third question) :

> It is known that there is an infinity of ways to reduce any bilinear polynomial
>
> $P= \Sigma A_{\alpha\beta}x_\alpha y_\beta \ (\alpha=1,2,...,n, \ \beta=1,2,...,n)$
>
> to the canonical form $x_1y_1+...+x_my_m$, by linear transformations applied to the two sets of variables $x_1,..,x_n, \ y_1,..,y_n$. Among the various questions of this kind that can be raised, we consider the following :
>
> 1. To reduce a bilinear polynomial P to a simple canonical form by orthogonal substitutions applied, some on the $x_1,...,x_n$, and some on the $y_1,...,y_n$.
>
> 2. To reduce P to a simple canonical form by any linear substitutions operating simultaneously on the x's and on the y's.
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> 3. To reduce simultaneously to a canonical form two polynomials P and Q by any linear substitutions applied on each of the two sets of variables respectively. ([8])

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[6] For a complete study of the 1874 controversy see (Brechenmacher 2007a).

[7] In modern parlance, bilinear forms played for a long time a role similar to the one matrices would be playing in the twentieth century linear algebra. After the 1870s, the theory of bilinear forms would play a key role on a global level thanks to its numerous applications in geometry (Klein 1868), the theory of quadratic forms (Kronecker 1874, Darboux 1874), and various problems related to systems of differential equations (Jordan 1871-1872), such as Fuchs equations (Hamburger 1872, Jordan 1874) or Pfaff's problem (Frobenius and Darboux 1875-1880).
See (Hawkins 1977) and (Brechenmacher 2006a).

[8] (Jordan 1873 : 7-11, translation F.B.). On sait qu'il existe une infinité de manières de ramener un polynôme bilinéaire $P= \Sigma A_{\alpha\beta}x_\alpha y_\beta \ (\alpha=1,2,...,n, \ \beta=1,2,...,n)$ à la forme canonique $x_1y_1+...+x_my_m$. . . par des transformations linéaires opérées sur les deux systèmes de variables $x_1,..,x_n, \ y_1,..,y_n$. Parmi les diverses questions de ce genre que l'on peut se proposer, nous considérons les suivantes : 1. Ramener un polynôme bilinéaire P à une forme canonique simple par des substitutions orthogonales opérées les unes sur $x_1,...,x_n$, les autres sur $y_1,...,y_n$. 2. Ramener P à une forme canonique simple par des substitutions linéaires quelconques opérées simultanément sur les x et

As we shall see in greater detail later, it was the *general* solutions both Weierstrass in 1868, and Jordan in 1871, had claimed to give to problems which had been tackled in the past that lead to the connection between the latter's two theorems. (^{9}) In order to investigate the issues of generality in the resulting controversy, it will therefore be customary to appeal to "scale games" between various scales of contexts and periods of times (^{10}).

Let first consider the local context of the theory of bilinear forms as it had developed in Berlin in the late 1860s. In 1866, two papers published by Elwin Christoffel and Kronecker had laid the foundations of a theory whose main problem was the characterisation of bilinear forms – given two forms $P=\Sigma A_{\alpha\beta}x_\alpha y_\beta$ and $P'=\Sigma B_{\alpha\beta}x_\alpha y_\beta$, find the necessary and sufficient conditions under which P can be transformed into P' by using linear substitutions. The main method was to look for invariants computed from the forms coefficients and which would be unaltered by linear transformations. It was actually the "general" resolution to the problem of the simultaneous transformations of two forms P and Q which would shortly become the main issue of the theory (*i.e.* the third problem in the list established by Jordan in the above quotation).

This problem challenged the "generality" of traditional polynomial methods. Pairs of forms *(P, Q)* could indeed be understood as polynomial forms $P+sQ$. The determinant $|P+sQ|$ is therefore a polynomial in *s*. But although this determinant provides a polynomial invariant, this invariant does not characterize completely the forms unless no multiple roots occur in the equation $|P+sQ|=0$ (*i.e.* what is nowadays designated as the characteristic equation). In a word, in case of multiple roots, non equivalent pairs of forms can be associated to a same invariant polynomial. (^{11}) The whole theory of bilinear forms, *i.e.* both its internal content and its usefulness in applications, revolved around the "general" resolution Weierstrass had given to this problem in 1868. On the one hand, this solution was considered as *general* because it worked whatever the multiplicity of the roots. On the other hand, this solution resorted to a *specific* method that appealed to the comparison of the algebraic factorization of the equation $|P+sQ|=0$ and of the decomposition of the determinant $|P+sQ|$ in sequences of minors (^{12}). In a sense, Weierstrass' elementary divisors gave to the theory both its generality and its specificity.

Jordan thus stroke at the core of the theory when he proposed in 1873 to reorganise the whole theory of bilinear forms by appealing to a method of canonical reduction which, he claimed, was "more general" than Weierstrass' invariant computations. Moreover, Jordan argued that "the problem of the simultaneous reduction of two functions P and Q is identical to the problem of the

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les *y*. 3. Ramener simultanément à une forme canonique deux polynômes *P* et *Q* par des substitutions linéaires quelconques, opérées isolément sur chacune des deux séries de variables.

^{9} For a comment from the standpoint of modern linear algebra, see the appendix I.

^{10} See (Revel 1996).

^{11} For instance, the bilinear forms *B(X,U)=ux+uy+vy* and the identity *I=ux+vy* both have a single eigenvalue equal to 1 but these two forms are not equivalent one two another.

^{12} Weierstrass solution was actually limited to the non singular case when $|P+sQ|$ did not vanish identically, another paper published by Kronecker in 1868 was devoted to the singular case $|P+sQ|=0$.

reduction of a linear substitution to its canonical form", (^{13}) and to the theorem he had stated in his 1870 *Traité des substitutions et des equations algébriques*:

> . . . the third [problem has already been dealt with] by M. Weierstrass ; but the solutions given by the eminent Berlin geometers are incomplete, so far as they left aside some exceptional cases which are nevertheless interesting. Their analysis is moreover quite difficult to follow – especially that of Mr Weierstrass. On the contrary, the new methods that we propose are extremely simple and hold no exceptions. . . The simultaneous reduction of two functions P and Q is a problem identical to the reduction of a linear substitution to its canonical form. (^{14})

In a paper he communicated to the Academy of Berlin in January, Kronecker rejected the whole theoretical organisation Jordan had put to the fore. Recalling that as soon as 1868 Weierstrass and he had organised the theory around the sole problem of the characterisation of pairs of forms, he questioned the relevance of Jordan's distinction between three problems of canonical reductions:

> In Mr *Jordan*'s memoir…, the solution of the first problem is not truly new, the solution of the second problem is missed, and that of the third one is not sufficiently grounded. We should add that actually the third problem includes the two others as particular cases, and that its complete solution stems from Mr *Weierstrass*' work of 1868, and can also be derived from my additional contribution to this work. Unless I am very much mistaken, there are serious grounds for questioning M. *Jordan*'s priority in the invention of his results, should they even be correct... (^{15})

During winter, Kronecker would develop his views on the organisation of the theory of bilinear forms in communicating monthly papers to the Academy of Berlin. Meanwhile, he and Jordan engaged a private correspondence with the aim of settling the quarrel of priority bred by the connections which had arisen between the canonical form and the elementary divisors theorems. (^{16})

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^{13} (Jordan 1873: 1487, translation F.B.).

^{14} (*Ibidem*). . . . le troisième [problème a déjà été traité] par M. Weierstrass ; mais les solutions données par les éminents géomètres de Berlin sont incomplètes, en ce qu'ils ont laissé de côté certains cas exceptionnels qui, pourtant, ne manquent pas d'intérêt. Leur analyse est en outre assez difficile à suivre, surtout celle de M. Weierstrass. Les méthodes nouvelles que nous proposons sont, au contraire, extrêmement simples et ne comportent aucune exception. . .La réduction simultanée de deux fonctions P et Q est un problème identique à celui de la réduction d'une substitution linéaire à sa forme canonique

^{15} (Kronecker 1874b: 1181, translation F.B.) Dans le Mémoire de M. *Jordan. . .,* la solution du premier problème n'est pas véritablement nouvelle ; la solution du deuxième est manquée, et celle du troisième n'est pas suffisamment établie. Ajoutons qu'en réalité ce troisième problème embrasse les deux autres comme cas particuliers, et que sa solution complète résulte du travail de M. *Weierstrass* de 1868 et se déduit aussi de mes additions à ce travail. Il y a donc, si je ne me trompe, de sérieux motifs pour contester à M. *Jordan* l'invention première de ses résultats, en tant qu'ils sont corrects. . .

^{16} Jordan's correspondence has been archived at the Ecole polytechnique. The part related to the correspondence has been edited in (Brechenmacher: 2006a).

Despite its function of scientific communication and although it would lead Jordan to recognise a partial anteriority of Kronecker and Weierstrass on some of his results as well as to grasp some tacit knowledge peculiar to the Berliners, ([17]) the correspondence would fail to reach agreement on a mathematical ground. On the one hand, Kronecker failed to bring Jordan round to his own ideas on the structure of the theory of bilinear forms. On the other hand Jordan did not succeed in convincing Kronecker of his "natural right" to claim the genuine originality of the distinction he had made between three kinds of canonical forms related to three types of groups of substitutions which, he said, "has rather brought to light than disparaged Weierstrass' result in highlighting the resolution it implicitly gave to a fundamental problem in linear substitutions theory which, to my opinion, is a much more fertile theory than the algebraic theory of forms of the second order". ([18])

In spring, the controversy would go public again. It would reach its climax with the publications of a train of notes and papers at the Academies of Paris and Berlin as well as in the *Journal de Liouville*. The quarrel of priority would then turn into an opposition over two theories (group theory *vs* the theory of forms) and two disciplines (Algebra vs Arithmetic) as well as over two practices (canonical reduction vs invariant computation) relating to conflicting philosophies of generality.

2. Weierstrass' theorem as marking a rupture in the history of general/generic reasonings.

As we shall see in this section, Kronecker associated Weierstrass' theorem with an ideal of generality. On the occasion of Kronecker's criticisms on the formal nature he imputed to Jordan's canonical reduction, the generality of Weierstrass' theorem would be presented as a rupture in the history of Algebra.

Taking as a starting point the "fault" of using as a denominator an algebraic expression that may vanish he picked out Jordan's December paper, Kronecker developed his views along the lines of an opposition between the uniform & the formal and the general & the homogeneous. ([19]) He especially blamed the uniform formal expressions that lose their meaning in some singular cases:

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[17] In (Brechenmacher: 2007a), we have shown how this correspondence highlighted some tacit knowledge peculiar to the Berliners. Until he received some detailed explanations from Kronecker, Jordan had some difficulties in understanding the properties of determinants Weierstrass and Kronecker had implicitly resorted to for computing invariants. Moreover, before the episode of the correspondence, Jordan did not understand how some papers published during the 1860s were connected one with another and to the emerging theory of bilinear forms. For instance, when he published his first note in December 1873, Jordan was unaware of the two papers in which Christoffel and Kronecker works had founded in 1866 the theory of bilinear forms and he did not realize the relation between Kronecker's 1868 memoir on "quadratic forms" and Weierstrass' 1868 paper despite the fact that these two papers had been conceived as the "two parts" of a same development and had thus been published successively in *Crelle's* Journal.
[18] (Jordan to Kronecker, January 1874, translation F.B.).
[19] Jordan promptly corrected this mistake which was of no consequence for his theoretical organisation, see (Brechenmacher 2006a: 689).

One is indeed used to discovering essentially new difficulties – especially in algebraic questions -, as soon as we free ourselves from the restriction to such cases one is accustomed to term general. As soon as one forces his way through the surface of this so called generality - which excludes any particularity-, one penetrates the true generality - which encompasses all singularities-, one generally finds the real difficulties of the study, but at the same time one finds the wealth of new viewpoints and phenomena which lie in its depths. ([20])

As shall be seen in greater details when we will return later to Kronecker's arithmetical agenda, the latter had a complex attitude toward Weierstrass' approach. On the one hand, Kronecker actually transformed completely the content of the elementary divisors theorem. But on the other hand, he nevertheless constantly presented Weierstrass' original statement as a model of "true generality". According to Kronecker, the general and homogeneous resolution the elementary divisors were giving to the problem of the characterisation of pairs of bilinear forms contrasted with the "inadequate results" of the "so called general" methods that had been sporadically developed "for over a century" in the particular (symmetric) case of pairs of quadratic forms. Kronecker indeed blamed these methods for the little attention they had given to the difficulties that might be caused by singularities such as multiple roots (or equal factors) occurring in the polynomial determinant $S=|P+sQ|$.

Kronecker was implicitly making reference here to the long run history of what was usually referred to as the "equation to the secular inequalities in planetary theory". His allusion to the "well known problem" of the consideration of pairs of quadratic forms, "which has been dealt with so often (although only occasionally) over the last century", was indeed a typical way to refer to a specific and well known collective of texts involving authors such as Joseph-Louis Lagrange, Pierre-Simon Laplace, Augustin Cauchy or Carl Gustav Jacobi. As Kronecker highlighted it, assigning specific values to the symbols involved in the general algebraic expressions these authors had dealt with raised difficulties as it may lead to $\frac{0}{0}$ expressions if common roots occured between S and its successive sub determinants ([21]). He therefore blamed these works of the past for their false "generality", *i.e.* for their focus on the *generic* case in which $S=0$ had no multiple roots.

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[20] (Kronecker 1874a: 367, translation F.B.). Denn man ist es gewohnt –zumal in algebraischen Fragen- wesentlich neue Schwierigkeiten anzutreffen, wenne man sich von der Beschränkung auf diejenigen Fälle losmachen will, welche man als die allgemeinen zu bezeichnen pflegt. Sobald man von der Oberfläche der sogenannten, jede Besonderheit ausschliessenden Allgemeinheit in das Innere der wahren Allgemeinheit eindringt, welche alle Singularitäten mit umfasst, findet man in der Regel erst die eigentlichen Schwierigkeiten der Untersuchung, zugleich aber auch die Fülle neuer Gesichtpunkte und Erscheinungen, welche sie in ihren Tiefen enthält.

[21] As Weierstrass had proven it in 1858, such problems would not happen in the symmetric case of pairs of quadratic forms which was the one related the equation to the secular inequalities. This problem could indeed be considered as a generalisation of the inertia law from usual quadratic forms with real coefficients to linear polynomials of quadratic forms. See the appendix II for more details.

On the opposite, Kronecker argued, when Weierstrass had dealt with such issues for the first time in 1858, the latter had developed a "truly general" approach in shifting the focus from the nature of the roots to an investigation of how the polynomial factorization of the equation $S=0$ paralleled the decomposition of the determinants extracted from $|P+sQ|$. To Kronecker's opinion, the approach Weierstrass had developed in 1858 for the particular (quadratic) case of the equation to the secular inequalities heralded the complete generality of the 1868 theorem on bilinear forms:

> This holds in the few algebraic questions which have been tackled completely and to the smallest details, such as the theory of networks of quadratic forms whose main features have been developed above. As long as one did not dare to dispense with the hypothesis that the determinant has only unequal factors, one can reach only inadequate results in the well known problem of the simultaneous transformations of two quadratic forms – a problem which has been dealt with so often (although only occasionally) over the last century - ; under this hypothesis, the true viewpoints on the investigation remained completely unacknowledged. Weierstrass' 1858 work dropped this hypothesis ; it already resulted in a higher insight and to a complete treatment of the case when only simple elementary divisors occur. But the general introduction of the notion of elementary divisor - of which only the first step was taken – occurred for the first time in Weierstrass' 1868 work, and it shed new light on the theory of networks for the case of an arbitrary, yet non vanishing, determinant. As I dropped this last restriction, and from this notion of elementary divisor, developed the more general notion of elementary network, the brightest light was shed on the new algebraic configurations, and at the same time by this complete treatment of the subject, the most valuable insights were reached on the theory of the higher invariants, as conceived in their true generality. ([22])

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[22] (*Ibid.*). Diess bewährt sich durchweg in den wenigen algebraischen Fragen, welche bis in alle ihre Einzelheiten vollständig durchgeführt sind, namentlich aber in der Theorie der Schaaren von quadratischen Formen, die oben in ihren Hauptzügen entwickelt worden ist. Denn so lange man es nicht wagte, die Voraussetzung fallen zu lassen, dass die Determinante nur ungleiche Factoren enthalten, gelangte man bei jener bekannten Frage der gleichzeitigen Transformation von zwei quadratischen Formen, welche seit einem Jahrhundert so vielfach, wenn auch meist blos gelegentlich, behandelt worden ist, nur zu höchst dürftigen Resultaten, und die wahren Gesichtpunkte der Untersuchung blieben gänzlich unerkannt. Mit dem Aufgeben jener Voraussetzung führte die *Weierstrass*'sche Arbeit vom Jahre 1858 schon zu einer höheren Einsicht und namentlich zu einer vollständigen Erledigung des Falles, in welchem nur einfache Elementartheiler vorhanden sind. Aber die allgemeine Einführung dieses Begriffes der Elementartheiler, zu welcher dort nur ein vorläufiger Schritt gethan war, erfolgte erst in der *Weierstrass*'schen Abhandlung vom Jahre 1868, und es kam damit ganz neues Licht in die Theorie der Schaaren für den Fall beliebiger, doch von Null verschiedener Determinanten. Als ich darauf auch diese letzte Beschränkung abstreifte und aus jenem Begriffe der Elementartheiler den allgemeineren der elementaren Schaaren entwickelte, verbreitete sich die vollste Klarheit über die Fülle der neu auftretenden algebraischen Gebilde, und bei dieser vollständigen Behandlung des Gegenstandes wurden zugleich die wertvollsten Einblicke in die Theorie der höheren, in ihrer wahren Allgemeinheit aufzufassenden Invarianten gewonnen.

Combining mathematical and historical arguments, Kronecker was the first to stress a history of what Thomas Hawkins would refer to in the 1970's as "generic reasoning" in algebra. For the latter, such a form of reasoning had played a key role since the development of symbolical algebra in Viète's works in the sixteenth century:

> The generality of the method of analysis had been viewed as its great virtue since its inception. Thus Viète stressed that the new method of analysis "does not employ its logic on numbers – which was the tediousness of the ancient analysts – but uses its logic through a logistic which in a new way has to do with species". . . . Analysis became a method for reasoning with, manipulating, expressions involving symbols with "general" values and a tendency developed to think almost exclusively in terms of the "general" case with little, if any, attention given to potential difficulties or inaccuracies that might be caused by assigning certain specific values to the symbols. Such reasoning with "general" expressions I shall refer to for the sake of brevity as *generic reasoning.* ([23])

But the lines of developments Hawkins actually discussed were nevertheless more specific than the whole domain of algebra. They actually coincided with the long run history Kronecker had alluded to in 1874. For both Kronecker and Hawkins, the tension generic/general was instrumental to the selection of a genealogy starting with the mechanical investigations of Lagrange and Laplace in the 18th century, involving Cauchy's 1829 memoir on the classification of conics and quadrics ([24]), and ending with Weierstrass' theorem. It was indeed by appealing to the rupture Kronecker attributed to Weierstrass' theorem that Hawkins argued that, although "historians writing on this subject have tended to emphasize the role [of] Arthur Cayley", who had developed a symbolical theory of matrices in 1858 ([25]), " there is much more to the theory of matrices – and to its history – than the formal aspect, *i.e.* the symbolical algebra of matrices. There is also a content ... the concept of an eigenvalue, the classification of matrices into types (symmetric, orthogonal, Hermitian, unitary, etc.), the theorems on the nature of the eigenvalues of the various types and, above all, those on the canonical (or normal) forms for matrices". ([26])

Moreover, this long run development of the theory of matrices was characterized as a progressive raise in the standards of rigor which eventually resulted in the rejection of the legitimacy of generic reasoning. As Hawkins highlighted it: ([27])

> . . . neither of them [Lagrange and Laplace] had pursued the study of the solutions of systems of linear differential equations with sufficient care to justify their claim [that the

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[23] (Hawkins 1977: 122).

[24] From the standpoint of modern algebra, Cauchy's 1829 memoir provided the first "general" proof that the eigenvalues of a symmetric matrix are real and that the corresponding quadratic form can be transformed into a sum of square terms (*i.e.* diagonalized) by means of an orthogonal transformation.

[25] Compare with the formal nature Kronecker attributed to Jordan's canonical form as will be detailed in the §5 below. On the various lines of developments in the history of matrices, see (Hawkins 1977) and (Brechenmacher 2006c & 2010).

[26] (Hawkins 1977: 1).

[27] See also R. Chorlay's article in this volume for issues about generality, genericity, arbitrariness, and rigour in the history of mathematical Analysis in the nineteenth century.

characteristic roots λ must be real]. They had no difficulty treating such a system when the characteristic roots are distinct, but their analysis of the case of multiple roots was inadequate. Given the generic tendency of their analytical methods, it is noteworthy that they considered the case at all. . . . Weierstrass' recognition of the questionable nature of their claims formed the starting with the prevailing tendency to reason vaguely in terms of the "general" case. ([28])

In a word, to both Kronecker's and Hawkins' views, Weierstrass' theorem marked a rupture in the history of Algebra. Not only did this theorem achieve a rigorous development as opposed to the generic nature of the reasonings of the past. It also resulted in a homogeneous solution as opposed to the specific arguments which had been developed to handle the singular cases which restricted the range of validity of general algebraic expressions. For instance, in a paper he had devoted to theta functions in 1866, Kronecker himself was still looking upon the occurrence of multiple roots as a "singular" case in which the "general" algebraic approach failed and for which it was customary to appeal to arguments specific to the context of theta functions. Arguments of the like usually aimed at claiming (falsely) that no multiple roots could occur in the context considered and that the algebraic expressions involved were therefore eventually fully general. Later on, Kronecker would nevertheless recognize that his reasoning was circular because he had appealed to the result he was actually aiming to prove.

In demonstrating the possibility of developing a truly general and homogeneous approach to the classification of pairs of quadratic and bilinear forms, the invariants introduced by Weierstrass had thus caused a rupture in Kronecker's mathematical works themselves. It is therefore customary to pay special attention to the involvement of the actors in the historical discourses they have been developing. This situation calls for a reconsideration of some classical categories of the historiography of mathematics. Epistemic values such as the ones of "rigour" or "generality" have indeed often played the role of structuring categories in the historiography of mathematical theories and disciplines. As has been already alluded to before, the tension between generic and general reasoning has been instrumental to the role Hawkins attributed to Weierstrass as marking a major stage in the history of the theory of matrices and, more generally, in the history of algebra:

> I would suggest that, insofar as anyone deserves the title of founder of the theory of matrices, it is Weierstrass. . . . His theory of elementary divisors provided a theoretical core, a substantial foundation, upon which to build. His work demonstrated the possibility of dealing by the methods of analysis with the non-generic case, thereby opening up a whole new world to mathematical investigation, a world that his colleagues and students proceeded to explore. . . . On motivational force common to the entire nineteenth century was a concern for a more rigorous level of reasoning in mathematics. . . . A concern for higher standards of reasoning was a driving force behind Weierstrass' work and also behind that of Cauchy and Dirichlet which preceded it and behind that of Kronecker and Frobenius whoch succeeded it. The rise of the theory of matrices was directly related to the

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[28] (Hawkins 1977 : 122-124).

fall of the generic approach to algebraic analysis. A concern for rigor did not mark the end of the creative development of the theory but its beginning. ([29])

It was however in the very special context of a controversy that Kronecker came to emphasise the generic nature of some algebraic reasoning of the past century. ([30]) The question therefore arises as to the potentially different views that were being held by other actors such as Jordan. For the purpose of investigating how the latter came to develop some connections between his researches on substitutions groups and the theory of forms we shall therefore develop a complementary approach to the work of Thomas Hawkins on the generic nature of algebraic reasoning in the long run.

3. An algebraic practice and some mechanical interpretations dating back to the time of Lagrange.

As we shall see here, Jordan's intervention in the theory of bilinear forms in 1873 was the consequence of a note the astronomer Antoine Yvon-Villarceau had addressed three years earlier to the geometers of the Academy of Paris. The latter had pointed out a mistake in a classical method dating back to Lagrange and which used to be considered as emblematic of the general mathematical treatment of some mechanical problems.

More precisely, Yvon-Villarceau questioned the method "for integrating the equations of a rotating solid body under the action of gravity" which had been "introduced by the illustrious author of the *Mécanique analytique* for the special case of the small oscillations of a loaded string whose equilibrium is slightly disturbed while one of its end remains in position." In 1766 Lagrange had devised a "general method" for the "general case" involving an arbitrary (finite) number of masses – as opposed to the particular case of a string loaded by two or three masses that had already been tackled by Jean d'Alembert. The generality here meant considering a system of n linear differential equations with constant coefficients.

Villarceau illustrated Lagrange's method in the case of a system of two differential equations as follows (with x_i functions of t and a_i constant coefficients) :([31])

$$\frac{d^2x_1}{dt}=a_1x_1+a_2x_2$$

$$\frac{d^2x_2}{dt}=a_2x_1+a_1x_2$$

The key point was to associate an algebraic equation of the second degree to the linear system by using the method of "elimination". This equation, that Cauchy had designated as the

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[29] (*Ibidem* : 157-159)

[30] On the construction of History by mathematical texts see (Goldstein 1995), (Dhombres 1998), (Cifoletti 1995), and (Brechenmacher 2006b).

[31] Note that in Villarceau's system *(a)*, the pair *(a,c)* of coefficients of u and s in the second row is the mirror image of the coefficients *(c,a)* in the first row. In (Brechenmacher: 2007b), we have shown that the symmetry property of mechanical systems originated in the specific practice Lagrange had devised in 1766 for the problems of small oscillations.

"characteristic equation" of the system has, in "general" (in the generic sense), two roots s and s'. The initial system can then be "reduced" to the two following independent equations ([32]):

$$\frac{d^2y_1}{dt}+sy_1=0$$

$$\frac{d^2y_2}{dt}+s'y_2=0$$

The initial problem can thus be solved by considering each of the above equations separately. The solutions x_1 and x_2 are given by linear combinations of expressions such as $y_1=sin(st+\varepsilon)$ and $y_2=sin(s't+\varepsilon')$. In the "general" case Lagrange had considered, *i.e.* for the case of a system of n equations with no symmetric property, a necessary condition for reducing the system to n independent equations was that the associated characteristic equation of the nth degree had n single roots.

As alluded to before, Villarceau's 1870 note aimed precisely at criticising a mechanical interpretation dating back to Lagrange for legitimating the algebraic resolution of the systems of differential linear equations. The presupposed mechanical stability (the oscillation had to remain small) had indeed usually been considered as having the consequence that only single roots could occur. Multiple roots were assumed to cause unbounded oscillations as the "time t would get out of the sinus" and solutions would take the form $y=tsin(st+\varepsilon)$. But, Villarceau argued, ([33])

> I claim that this condition is not necessary for the oscillations to remain small. . . . Here is a very simple case where equal roots occur in the characteristic equation: a homogeneous solid body of revolution oscillating around a point of its axis. It is plain to see without resorting to any computation, that the smallness of the oscillations is ascertained, provided that the initial oscillatory movement is small enough, and that, at the origin of the movement, the solid's centre of gravity is below its centre of suspension and not too far from the vertical axis passing through this point. ([34])

In 1870, Yvon-Villarceau therefore pointed to some serious « deficiencies » in the « general » resolution of problems of small oscillations in a similar way as Kronecker would blame in 1874 the "so called generality" of algebraic expressions. But the astronomer nevertheless did not aim at criticizing a type of generic reasoning. His purpose was to question a practice which consisted in combining some mechanical interpretations with the algebraic nature of the roots of a specific equation.

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[32] (Yvon-Villarceau 1870: 763, translation F.B.).

[33] From the standpoint of modern algebra, the stability of a system depends upon whether its matrix is diagonalisable or not. The inequality of the system's eigenvalues is a sufficient but not necessary condition. The mechanical systems studied by Lagrange are diagonalisable because they are symmetric.

[34] (*Ibidem*: 765). Je dis qu'il n'est pas nécessaire que cette condition soit remplie, pour que les petites oscillations se maintiennent. . . . Voici un cas très simple, auquel correspondent des racines égales de l'équation caractéristique : c'est celui d'un corps solide, homogène et de révolution, oscillant autour d'un point pris sur son axe de figure. Chacun comprendra sans recourir au calcul, que la petitesse des oscillations est assurée dans ce cas, si le centre de gravité est, à l'origine du mouvement, au-dessous du centre de suspension, à une petite distance de la verticale passant par ce point, et si le mouvement oscillatoire initial est suffisamment faible.

Although Villarceau's intervention had been stemming from mechanical concerns such as the application of Lagrange's method to the long term perturbations of the parameters determining the planetary orbits, it eventually brought up a theoretical question to the attention of the Academy's geometers. Because the occurrence of multiple roots was in no contradiction to mechanical stability and hence to the possibility of "reducing" a system of n equations to n single independent equations, the question arose as to the characterisation of such system which could be reduced to separate equations.

This question prompted the publication by Jordan of two notes in 1871 and 1872. In 1871 Jordan applied the canonical form he had introduced in 1870 for linear substitutions to the reduction of a "general" system of differential equations with constant coefficients:

$$\frac{dx_1}{dt} = a_{11}x_1 + \ldots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + \ldots + a_{2n}x_n$$

$$\ldots\ldots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + \ldots + a_{nn}x_n$$

He therefore gave a "form" to which such a system could be reduced whatever the multiplicity of the roots. When only single roots occur, this form is identical to the one given by Lagrange. But if a multiple root s occurs, the reduction of the system may involve the following kinds of expressions :

$$\frac{dy_1}{dt} = sy_1$$

$$\frac{dz_1}{dt} = sz_1 + y_1$$

$$\frac{du_1}{dt} = su_1 + z_1$$

$$\ldots$$

Such a reduced form can then be integrated directly. It yields solutions of the form $y_1 = e^{st}\psi(t)$ where $\psi(t)$ is a polynomial of degree r-1 (where r is the number of "variables" in the series $y_1, \ldots, w_1$) [35].

In 1872, Jordan devoted an additional paper to respond more specifically to the questions raised by Villarceau in proving that linear systems stemming from mechanical concerns can always be

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[35] Moreover, Jordan gave a characterization of the systems that can be reduced to a diagonal form by the necessary and sufficient condition that each characteristic root K of multiplicity μ has to be a root of each of the minors of order μ-1.
From the standpoint of Weierstrass' 1868 theorem, Jordan's condition is equivalent to the necessary and sufficient condition stating that two bilinear forms $P(x_i, y_i)$ and $Q(x_i, y_i)$ can be transformed simultaneously into sums of "square terms" (*i.e.* sx_iy_i) if and only if the elementary divisors of $pP+qQ$ are linear. If P and Q are real quadratic forms such that $pP+qQ$ is definite for some p and q, then the elementary divisors are linear and P and Q can be transformed simultaneously into sums of square terms.

reduced to *n* separate equations $\frac{dy_i}{dt} = sy_i$ because of their symmetric nature. In doing so, Jordan inscribed the initial mechanical question into the "more general" framework of the the theory of quadratic forms. Indeed, the coefficients of a symmetrical system define a quadratic form *Q* and the reduction of the system to its canonical form consists in considering simultaneously the forms *Q* and the form identity *I, i.e.* the polynomial form *Q+sI* ([36]).

In proving that the multiplicity of roots was of no relevance to the subject of mechanical stability, Jordan reached the same conclusion that Weierstrass had already been giving in 1858 (symmetric case) and in 1868. ([37]) It was thus thanks to a hundred-year old mechanical problem that a first connection between Jordan's and Weierstrass' theorems came to light. This connection was pointed out in 1873 by Meyer Hamburger who called Jordan's attention to the work of Weierstrass on bilinear forms. In 1873 Jordan pointed out that the transformation of forms could be seen as the composition of linear substitutions and eventually proved that his canonical reduction could be used to derive Weierstrass' theorem, with the result of prompting the ensuing controversy with Kronecker.

The 1874 controversy can therefore be considered from the perspective of the opposition of two different ends given to a shared history. It was indeed because of their capacity to give a general solution to some problems that had been handled throughout the eighteenth and nineteenth centuries that some identities between Jordan's and Weierstrass' theorems had arisen between 1870 and 1873. Both theorems were shedding some new light on the past as they were making some results of authors such as Lagrange or Cauchy appear incomplete because limited to what would be considered from now on as the special case in which only single roots occurred.

For the purpose of a deeper understanding of the role played by such a history in the quarrel, a bibliographic research has been carried out, by starting with the authors and texts Jordan and Kronecker referred to and by working out systematically the references that appeared successively in this paper chase. This methodology resulted in a network of texts covering the period 1766-1874 ([38]). A simplified representation of this collective organisation of texts is given in the annex n°1. The main knots in the entanglement of bibliographic references point to the mechanical work of Lagrange as well as to Cauchy's analytical geometry. The network can thus neither be identified to a theory nor to a discipline. What is then the coherence of this collective organisation of texts?

One of the main shared characteristics in this network of texts is the role of point of origin which is systematically attributed to Lagrange's solution of mechanical problems of "small oscillations". As has been already illustrated by Villarceau's 1870 note, this reference pointed especially to the association Lagrange had developed between the stability of a mechanical system and the

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[36] (Jordan 1872: 320).
[37] Although Weierstrass had already stated such a condition for the symmetric case in 1858, he devoted in 1875 to the general result Jordan had stated in 1872 a communication to the Academy of Berlin in which he applied his theorem of elementary divisors (making therefore no reference to Jordan).
[38] On the use of networks of texts for investigating collective organisations of knowledge, see (Goldstein 1999), (Goldstein and Schappacher 2007b) and (Brechenmacher 2006a & 2007a,b).

algebraic nature of the roots of the associated characteristic equations: the roots had to be "real, unequal and negative" for the oscillations to remain small. The validity of this conclusion had remained unquestioned until Weierstrass, and later Jordan, proved in 1858 and 1872 respectively, that the multiplicity of the roots had nothing to do with the stability of the system. Even though the network of texts stemmed from a specific problem, this problem was thus considered to have been solved by Lagrange right from the beginning until two ends would be given to the network. This collective organisation of texts can therefore not be considered as having been subtended by some attempts to solve a given problem ([39])

In the 1770s, Lagrange's method for deciding of the stability of mechanical systems had been extended to the description of the "secular inequalities" of the parameters determining the planetary orbits ([40]). The algebraic nature of the roots of the associated equation was therefore linked to the issue of the stability of the solar system. In 1789 Laplace had attempted to give a general proof of the stability of the solar system and had highlighted that the reality of the roots occurring in Lagrange's approach was related to the property of symmetry of the coefficients of the system. After this episode, it was the *special* nature of the associated equation that would eventually give to the network its coherence. Amongst the many texts whose interest for celestial mechanics went no further than the identification of this equation, we may cite Cauchy's 1829 "Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planetes", James Joseph Sylvester's 1851 "On the equation to the Secular Inequalities in the Planetary Theory", or Hermite's 1857 "Mémoire sur l'équation à l'aide de laquelle, etc.",

It was therefore the special nature of the equation to the secular inequalities that gave to the network its main coherence. Moreover, as has been shown in a paper devoted to a detailed analysis of this collective of texts ([41]), the reference to the "equation to etc." was resorting to a specific "algebraic practice". The operatory process this practice resorted to, as detailed in the appendix II, supported analogies through which meanings were extended from a domain to another ([42]). It was for instance thanks to the equation to the secular inequalities that Cauchy developed in 1829 a formal analogy between various problems such as mechanical oscillations, the rotation of a solid body or the classification of conics and quadrics. ([43])

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[39] From the standpoint of the 1930s modern algebra, the different problems appearing in the discussion would be considered as belonging to the theory of matrices and consisting in the reduction of a pair *(A,B)* of matrices in *(D,I)* where *A* is symmetric, *B* is definite symmetric, *D* is diagonal and *I* the identity matrix such as, for instance the linear differential systems with constant coefficients *BY'=AY* which are related to the eigenvalue problem *AX=λBX*.
For more details, see (Gantmacher 1959 : 311).
[40] (Lagrange 1781 : 125).
[41] (Brechenmacher 2007b). The algebraic process related to this practice is detailed in the appendix II.
[42] Supporting analogies by operatory processes was a common way to legitimate extensions of meanings in the 19th century. See (Durand Richard 1996 & 2008).
[43] See (Cauchy 1829 : 173). From the standpoint of modern algebra, Cauchy was interested in the transformation of a quadratic form in three variables into a sum of squares. This problem also arose in the mathematical analysis of the rotational motion of a rigid body as studied by Lagrange

Even though it was never explicitly identified to a method *per se*, the practice attached to the "equation to etc." both developed and handed down through the network within different methods belonging to various theoretical frameworks. It especially played a key role in several crucial developments in algebra and number theory, such as the algebraic proof Sylvester and Hermite gave to Sturm's theorem, and which questioned the relationships between the functions of analysis, the equations of algebra and the quadratic forms of number theory ([44]). Related as it was to a special equation, the practice of the "equation etc." had nevertheless progressively taken on a specific algebraic identity which eventually gave rise to a shared algebraic culture. Reflecting on the connections between the theories involved with the equation to the secular inequalities was a topic especially popular for the emerging communities of teachers of mathematics which revolved around journals such as the *Nouvelles annales de mathématiques* or the *Cambridge Mathematical Journal* in the 1840s-1850s.

This shared algebraic culture had nevertheless remained limited to a periodical form until monographs and textbooks would be devoted to the theory of bilinear forms in the 1880s. Until the development of theoretical frameworks such as the ones Jordan and Kronecker were quarrelling about in 1874, it was therefore above all a historical identity that characterised the algebraic identity of the practice of the equation to the secular etc. It was indeed systematically by appealing to a corpus of earlier texts in periodical publications that authors pointed to the specific nature of this practice. For instance, in 1829, it was by appealing to the mechanical works of Lagrange and Laplace that Cauchy identified the specific algebraic practice he connected with his method for determining the principal axis of conics and quadrics. In doing so, Cauchy transcended the framework of analytical geometry he was originally interested in, with the result of "generalising" to n variables an analytical method originally devised for two or three variables.

As we have seen in the second paragraph of this paper, making reference to the corpus of the equation to the secular etc. in similar way as Cauchy's 1829 reference, would still play a key role when the issue of generality would be debated in the 1874 controversy. This way of referring to a shared history was actually a part of the specificity of the algebraic practice of the "equation etc.". It delimitated a shared subject by delimitating a long-term discussion between texts. As we shall see in greater detail in the next paragraph, in the absence of any monograph or theoretical synthesis, this discussion was a mode of legitimating some "extensions to the general" by appealing to analogies supported by an operative process. It is therefore customary to look more closely at how such a traditional way to delimitate a shared algebraic culture related to a legitimate generality would later on be torn apart by two disciplines: Algebra and Arithmetic.

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in the 18th century. For a description of Cauchy's work on the problem of the rotation of a solid body in connection to Lagrange's analytic reformulation of the solution given by Euler, see (Hawkins 1975 :18).

[44] See (Sinaceur 1991).

4. On generality and the algebraic status of a polynomial practice.

As we shall see here, from the outset of our corpus of texts to its two ends in Weierstrass' and Jordan's papers, it was an ambition of generality which was driving authors on making reference on a discussion they consequently joined and enriched themselves.

It was already with the aim of generalising d'Alembert's investigations of a swinging string loaded with three masses to the "oscillations of an unspecified system of bodies" that Lagrange had been working out in 1766 the polynomial process at the origin of the discussion. Because of the generality he attributed to his description of a motion that Daniel Bernouilli had regarded as too irregular to be treated by analytic methods, [45] Lagrange made the problem of small oscillations come out first among the examples of applications he gave of the "general principles" of his *Mécanique analytique*.

Generality was then the main impetus for the development of a discussion on the qualitative nature of characteristic roots. It nevertheless took on changing meanings between 1766 and 1874. To Lagrange's mind, the fact that his method would fail if multiple roots should occur did not restrict any of its generality. The method was, indeed, resorting to implicit mechanical representations. It was because the oscillations of a swinging string loaded with n masses could be mechanically represented as a combination of independent oscillations of n strings loaded with a single mass, that linear differential systems were thought to be representable as combinations of independent equations. In Lagrange's method algebraic roots could not be dissociated from their mechanical representations as periods of oscillations and the occurrence of multiple roots was therefore (wrongly) believed to be contradictory to the existence of n independent oscillations. Multiple roots would thus be contradictory to the prerequisites that had been made on the bounded nature of the oscillations: because the problem concerned a swinging string, the displacements of the masses from the vertical must consequently remain small, and such would not be the case if the solutions contained exponential $e^{\delta t}$ which would increase to infinity.

The stakes in the implication of Lagrange's conclusion –the roots have to be real and unequal because the oscillations have to remain bounded – changed when the method was generalised to the secular inequalities in planetary theory. The stability of the solar system could not be taken for granted and Lagrange therefore pointed out that "it would be difficult, perhaps impossible, to determine the roots of the equation in general" for it would mean demonstrating the reality and inequality of the roots of a very general n^{th} degree polynomial while these roots can not be expressed by radicals in general as soon as n is greater than five. [46] Lagrange worked out an effective computation for a system of four planets. He determined that the roots of the associated fourth degree equation are real, negative and unequal. Lagrange thus came to the conclusion that "one may wonder whether, by changing the values [of the masses of the planets], equal or imaginary roots may occur. For removing all doubt, it would be necessary to prove, in general, that the roots of the equation are always real and unequal, whatever the values of the masses. That is easy when the mutual action of only two planets is considered simultaneously, since the equation is only of the second degree, but this equation becomes more and more complicated and

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[45] See (Truesdell 1960 : 156).
[46] See (Lagrange 1766 : 538).

higher [in degree] as the number of planets increases". ([47]) Laplace did not stay content with this numerical computation and his aim of devising a "fully general" demonstration that would not depend upon the approximate values assigned to the masses of the planets brought him to engage in the discussion of the "equation to etc."

As has already been pointed out before, Cauchy's 1829 intervention was caused by the latter's ambition to generalise to *n* variables a method he had devised for two or three variables in a geometric framework. The polynomial practice peculiar to the systems of *n* equations related to the equation to the secular inequalities was then translated in terms of determinants. ([48]) But this general perspective raised new issues as some algebraic expressions involved in Cauchy's method may take a $\frac{0}{0}$ value in case of multiple roots. ([49]) The occurrence of multiple roots now appeared as a singular case limiting the range of validity of an algebraic expression. It thus seemed necessary to introduce some particular methods for this singular case. Following d'Alembert and Lagrange, Cauchy thus initially went out of algebra by appealing to the specific argument that consisted in making the roots unequal by the use of infinitesimal quantities. But Cauchy did not stay content with this situation. He indeed blamed polynomial methods for being overburdened with singular cases. In the aim of developing a fully homogeneous resolution, he then appealed to the calculus of residues and to complex analysis. ([50])

The change of perspective on generality induced by this ideal of homogeneity impulsed further developments involving Jacobi, Sylvester, Hermite, Borchardt, and eventually Weierstrass. ([51]) In 1858, Weierstrass eventually gave a general, homogeneous, and algebraic solution to the problem in arguing that:

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[47] (Lagrange 1784 : 316, translation T. Hawkins)

[48] See (Hawkins 1977 : 125).

[49] For this reason, Cauchy's proof of the reality of the roots of the equation to the secular inequalities was valid only if no successive equations $S_i(x)$ relating to the successive determinants extracted from $S(x)$ have a root in common. From the standpoint of modern algebra, the existence of the orthogonal transformation that diagonalizes Cauchy's quadratic form depends upon the reality of the eigenvalues (*i.e.* the roots of the characteristic equation) as well as upon the non existence of multiple roots.

[50] Since its origin in 1826, the calculus of residues had been introduced by Cauchy as a way to deal with problems caused by multiple roots in generic algebraic expressions. See (Dahan Dalmedico 1992 : 197) and (Brechenmacher 2007b).

[51] See (Hawkins 1977 : 128-133) for some descriptions of the works of Jacobi and Borchardt and of the proof given by Dirichlet in 1846 (which would become an appendix to the third edition of Lagrange's *Mécanique Analytique* in 1853) to the fact that a state of equilibrium in a conservative mechanical system is stable if the potential function assumes a strict maximum value. Sylvester's works of 1850-1852 would lead him to introduce the notions of "matrix" and 'minors" (in which Darboux would see in 1874 an origin of Weierstrass' elementary divisors). See (Brechenmacher 2006c). The methods developed by Sylvester and Cayley in the context of the development of the theory of invariants would be invested by Hermite in an arithmetical framework (quadratic forms, decomposition in four squares).

However, it does not appear that special attention has been given to the particular circumstances that arise when the roots of the equations *f(s)=0* are not all different ; and the difficulties which they present – of which I was made aware by a question to be discussed more fully later – do not seem to have been properly cleared up. At first, I also believed that this would not be possible without extensive discussions in view of the large number of different cases that can occur. It seemed all the more desirable to me to show that the solution to the problem given by the above-named mathematicians could be modified in such a way that it does not at all matter whether some of the quantities s_1, s_2, ... , s_n are equal. […] After Lagrange had given the form of the integrals, and shown how

their arbitrary constants are by the initial values of x_1, $\frac{dx_1}{dt}$ etc., he argued, on the basis

of the conditions that must hold – namely that x_1, always remain infinitely small if they initially were –, to the effect that the characteristic equation must not have multiple roots, otherwise, inside the integral, there would appear a member that could become arbitrary large over time. The same statement happens to be repeated by Laplace, when he dealt with the secular variations of the planets in the *Mécanique céleste* ; and, as far as I know, this statement has been repeated again by all the other authors treating this subject, if they ever mentioned the case of equal roots (which, for instance, Poisson does not). But this is not justified […] and the [same condition] can hold without all the roots of the equation *f(s)=0* being different one from the other, if only the function Ψ stays negative and if its determinant does not vanish ; since, in fact, one has repeatedly dealt with particular cases of the above equations (in which this condition is not satisfied), but without having found any elements of the sort described above.([52])

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[52] (Weierstrass 1858 : 244, translation F.B.). Dagegen scheint es nicht, als ob den eigentümlichen Umständen, die eintreten, wenn die Wurzeln der Gleichung *f(s)*=0 nicht alle von einander verschieden sind, besondere Beachtung geschenkt, und die Schwierigkeit, die sich alsdann darbieten, und auf die ich bei einer nachher näher zu besprechenden Frage aufmerksam geworden bin, schon gehörig aufgeklärt sein. Auch glaubte ich anfangs, es würde dies bei der großen Zahl verschiedener Fälle, die vorkommen können, nicht ohne weitläufige Erörterungen möglich sein. Umso erwünschter war es mir, zu finden, dass sich die von den genannten Mathematikern gegebene Lösung der Aufgabe in einer Weise modifizieren lässt, bei der ein ganz gleichgültig ist, ob unter den Größen s_1, s_2, ...s_n gleiche vorkommen oder nicht. […] Nachdem Lagrange die Form der Integral angegeben und gezeigt hat, wie die willkürlichen Constanten derselben durch die

Anfangswerthe von x_1, $\frac{dx_1}{dt}$, u.s.w. bestimmt werden, führt er unter den Bedingungen die erfüllt

sein müssen, damit x_1, $\frac{dx_1}{dt}$ stets unendlich klein bleiben, wenn sie es ursprünglich sind, auch die

an, dass die genannte Gleichung keine gleiche Wurzeln haben dürfe, weil sonst in den Integralen Glieder vorkommen würden, die mit der Zeit beliebig gross werden könnten. Dieselbe Behauptung findet sich bei Laplace wiederholt, da wo er in der *Mécanique céleste* die Säcular-Störungen der Planeten behandelt, und ebenso, so viel mir bekannt ist, bei allen übrigen diesen Gegenstand behandelnden Autoren, wenn sie überhaupt den Fall der gleichen Wurzeln erwähnen,

We have seen that some ambitions of generality had been strongly linked to the development of the network of the "equation to etc." since its origin. But we have seen also that these ambitions of generality had been originally resorting to the algebraic identity of a practice attached to a special equation. Recall that Algebra was not considered at the time as an autonomous research discipline. ([53]) It was actually because it was considered as algebraic that the practice attached to the "equation etc." was not formalized further than something that could be done with a special equation. The same kind of attitude can be illustrated with other *special* types of equations such as the binomial equation, the modular equation, etc. ([54]) It was precisely because of this non formal algebraic identity that the practice attached to the "equation to the etc." circulated between theories and supported analogies and "generalisations" ([55]).

After the 1830s, this equation appeared as an archetype of algebraic generality. This algebraic generality especially appealed to authors such as Jacobi or Sylvester. When the latter had aimed at stating a purely algebraic proof of Sturm's theorem, he had indeed started by investigating the "equation to etc." ([56]) In this context, the algebraic practice attached to this equation was connected to the theory of quadratic forms as it had stemmed from Gauss' "higher arithmetic". More precisely, it was discussed in connection to the inertia law of quadratic forms. In the context of his works on Sturm theorem, Hermite introduced a distinction between the traditional "arithmetical theory of forms" and the "algebraic theory of forms" related to the equation of the secular inequalities. The first concerns single quadratic forms with integer coefficients. These forms are characterized by the sum of squares they can be reduced to (the inertia law). The second concerns pairs of quadratic forms with real coefficients whose characterisation can thus be understood as a generalisation of the inertia law ([57]).

As a matter of fact, Hermite, and later Weierstrass, had turned Lagrange's initial concerns into a theorem about "transformations" and "forms". The latter two terms had thus been given explicit mathematical definitions in the "higher arithmetic" of quadratic forms. In contrast, the uses of the term "form" in connection with the "equation to etc." had traditionally pointed to various and mostly implicit meanings. For Lagrange and Laplace, the existence of "integrable forms" for the differential systems of small oscillations was induced from mechanical interpretations. It was therefore not by appealing to "transformations" that independent equations were associated to the initial linear system but by the computation of the systems' mechanical parameters through the

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was z.B. bei Poisson nicht geschieht. Aber sie ist nicht begründet. […], wenn nur die Function Ψ stets negativ bleibt, und ihre Determinante nicht Null ist, was stattfinden kann, ohne dass die Wurzeln der Gleichung $f(s) = 0$ alle von einander verschieden sind ; wie man denn auch wirklich besondere Fälle der obige Gleichungen, bein denen diese Bedingung nicht erfüllt ist, mehrfach behandelt und doch keine Glieder von der angegebene Beschaffenheit gefunden hat.

[53] In the teaching of mathematics, the academic discipline of Algebra was usually considered as more elementary than analysis. See (Brechenmnacher and Ehrhardt 2010).

[54] See (Goldstein and Schappacher 2007b) and (Brechenmacher 2011).

[55] For more details about the circulation of algebraic practices, see (Brechenmacher 2010).

[56] See (Sinaceur 1991).

[57] See the appendix II for some more mathematical details.

characteristic equation. ([58]) In Cauchy's 1829 paper, the "transformations of homogeneous functions" were related to geometrical meanings which supported some processes of changes of rectangular systems of coordinates.

To the different meanings and representations the term "form" had taken on in connection with the ambitions of generality of Lagrange, Laplace and Cauchy, would succeed mathematical theories whose subject would be a "fully general" characterization of "forms". Should such an issue belong to arithmetic or algebra? In 1874, Jordan and Kronecker were referring to a shared history related to a specific practice they had in common. We have seen that this practice consisted in investigating pairs of bilinear or quadratic forms (*P, Q)* by making use of the polynomial decomposition of $S=|P+sQ|$. As we shall see in the following section, some disciplinary ideals on algebra and arithmetic nevertheless induced conflicting perspectives on the generality of the theory of "forms".

5. Arithmetic generality vs algebraic generality.

In 1874, Kronecker opposed the true generality of the arithmetical nature of the theory of forms to the formal nature he attributed to the algebraic group theoretical approach of Jordan. Recall that Jordan had designated as canonical forms three different algebraic expressions related to the operations of three kinds of groups of substitutions. For this reason, he was accused of resorting to a notion without any "general relevance" nor "objective content". According to Kronecker, Jordan had therefore mixed up the "formal aspects" of some "means of action" (canonical forms) with the "true subject of investigation" and its "content".

Although "normal forms" similar to Jordan's canonical form were being used by Kronecker himself; to the latter's opinion, resorting to such algebraic expressions was legitimate provided that they would remain in their relative places of "methods" as opposed to the "notions" relating to the "other disciplines" - such as Arithmetic - it was Algebra's duty to serve. ([59]) On the contrary, the reification of algebraic methods into notions and theoretical issues would lead to mistake a mere "formal" development for a "general" and "uniform" presentation. Kronecker thus mocked Jordan's claims for the greater simplicity and generality of his canonical reduction as a naïve simplism which contented itself with the illusionary generality of the uniformity of a formal development.

> Should such general expressions be found, one should in every case be able to justify
> calling all of them canonical forms on the basis of their generality and simplicity ; but if

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[58] Even though it might seem natural nowadays to wonder about those matrices which, because of their multiple eigenvalues, might not be transformed into some diagonal forms, this question was actually irrelevant to the ways the terms "forms" and "transformations" were being considered at the times of Lagrange or Cauchy. On the "mathematical interpretation of the essential mechanical concepts" in Lagrange's analytical mechanic see (Panza 1992 : 205).

[59] For instance, in order to prove that two non singular pairs of bilinear forms can be transformed one into the other, Weierstrass has proven in 1868 that both forms can be linearly transformed to what Kronecker referred to as a "normal form" similar to Jordan's canonical form. But Kronecker would not state any theorem about such normal forms which were not the purpose of his investigations. On Weierstrass' 1868 proof see (Hawkins 1977) and (Brechenmacher 2006a).

one does not want to stick to the purely formal viewpoint which is often put the fore in the more recent Algebra – certainly not for the greatest advantage of the true knowledge -, one shall not omit to derive the correction of these canonical forms on the basis of inner grounds. Truly, these so called canonical or normal forms are determined only by the orientation of the study, but they should not be seen as the aim of the research… One should not at all be surprised if for an exposition both uniform and completely general such as is found in the above mentioned work [of Jordan], some new principles turn out to be necessary ; and on the contrary it would be amazing if in accordance with Jordan's claims ("The new methods we are proposing are, on the contrary, extremely simple…" "A very simple discussion shows that one can transform …."), the simplest means were sufficient. [60]

The purpose of the reorganisation Kronecker devised in 1874 for the theory of bilinear forms was to give a truly arithmetical foundation to some various results that had been obtained in the 1860's. [61] Although Kronecker had already been implicitly referring to the legacy of the works of Gauss and Hermite on the arithmetic of quadratic forms in 1866 – as when he had preferred to make use of the term "form" to name what others would designate as a function (Weierstrass 1858) or as a "polynomial" (Jordan 1873) -, his monthly communications to the Academy of Berlin during the winter of 1874 were aiming at an explicit generalisation of the arithmetic notion of "equivalence classes" from forms to networks of forms. "As an application of Arithmetic notions to Algebra", two bilinear forms or two networks of bilinear forms were designated as "equivalent" and as belonging to a same "class" when one could be linearly transformed into another. [62]

Some disciplinary ideals were coming along with this arithmetic orientation of the theory. These ideals expressed themselves in the criticisms Kronecker made of Jordan's statement that a

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[60] (Kronecker 1874a: 367, translation F.B.). Nachträglich, wenn dergleichen allgemeine Ausdrücke gefunden sind, dürfte die Bezeichnung derselben als canonische Formen allenfalls durch ihre Allgemeinheit und Einfachheit motiviert werden können ; aber wenn man nicht bei den bloss formalen Gesichtspunkten stehen bleiben will, welche –gewiss nicht zum Vortheil der wahren Erkenntnis- in der neueren Algebra vielfach in den Vordergrund getreten sind, so darf man nicht unterlassen, die Berechtigung der aufgestellten canonischen Formen aus inneren Gründen herzuleiten. In Wahrheit sind überhaupt die so genannten canonischen oder Normalformen lediglich durch die Tendenz der Untersuchung bestimmt und daher nur als Mittel, nicht aber als Zweck der Forschung anzusehen. . . . Dass sich aber für eine zugleich einheitliche und ganz allgemeine Entwickelung, wie sie in der ben erwähten Arbeit gegeben ist, gewisse neue Principien als nöthig erwiesen, kann durchaus nicht befremden, und es wäre im Gegentheil zu verwundern, wenn wirklich den *Jordan*'schen Behauptungen gemäss ("Les méthodes nouvelles que nous proposons sont, au contraire extrêmement simples..." "On voit par une discussion très simple, que l'on peut transformer...") die allereinfachsten Mittel dazu ausreichen sollten.

[61] According to Kronecker, this arithmetization ambition was stemming from discussions with E. Kummer. On Kummer's ideal numbers see J. Boniface's paper in this volume.

[62] Two families of bilinear forms $s\Phi$-Ψ and $s\Phi'$-Ψ' are equivalent if one can be transformed into the other by (possibly different) non singular linear transformations of the x and y variables (where $\Phi = \sum_{i,j=1}^{n} A_{ij} x_i x_j$ and $\Psi = \sum_{i,j=1}^{n} B_{ij} x_i x_j$).

sufficient condition for two forms to be equivalent was the identity of their canonical forms. According to Kronecker, despite being true, this proposition had to be rejected because it did not state any effective process for deciding of the equivalence. Jordan's reduction indeed appeals to an algebraic decomposition of the characteristic determinant for which no effective process can be given "in general" as soon as the polynomial degree exceeds five. It thus had to be distinguished from the "immediate possibility afforded by the theoretical criteria of equivalence to set a complete system of invariants" effectively computed from the form's coefficients. Kronecker supported his claims by introducing a new system of invariants to replace Weierstrass' ones. The elementary divisors indeed appealed to the resolution of general polynomial equations exactly as Jordan's canonical form did. But in 1874 Kronecker assigned the name of "elementary divisors" to the new system of invariants he introduced. These new invariants were introduced as the result of the arithmetical process for computing the *g.c.d.*'s of the successive minors extracted from the polynomial determinant $|A+sB|$. ([63]) Appealing to the legacy of Gauss, Kronecker then opposed the effectiveness of such a process to the formal nature of explicit algebraic formulas such as Jordan's canonical form:

> In the arithmetical theory of forms, one must certainly be satisfied by the indication of a procedure for deciding of the question of the equivalence, and this problem was indeed formulated explicitly in this way too (cf. Gauss : *Disquitiones arithmeticae*, *Sectio* V. . .). The procedure itself is here also based on the transformation to reduced forms: but it must not be forgotten that, in the arithmetic theory, these [reduced forms] have a completely different meaning than the one they have in the Algebra. Indeed, there, the invariants of the equivalent forms are, by their very nature, number-theoretic functions of the coefficients; it is thus not surprising that such [invariants], can be directly defined, although not explicitly but only described as the final result of arithmetic operations ; for much the same is true with most concepts of arithmetic, e.g. even the simple notion of greatest common divisor. ([64])

The ideal of effectiveness, which the historiography has been usually linking to Kronecker's 1882 arithmetic theory of algebraic magnitudes, ([65]) had thus already been strongly expressed on the

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[63] See the appendix I for more details about the invariants Kronecker introduced in 1874.

[64] (Kronecker 1874c : 415, translation F.B.). "In der arithmetischen Theorie der Formen muss man sich freilich mit der Angabe eines Verfahrens zur Entscheidung der Frage der Aequivalenz begnügen und das betreffende Problem wird deshalb auch ausdrücklich in dieser Weise formuliert (cf. Gauss : Disquitiones arithmeticae, Sectio V. . .) Das Verfahren selbst beruht auch dort auf dem Uebergange zu reducirten Formen : doch ist dabei nicht zu übersehen, dass denselben in den arithmetischen Theorien eine ganz andere Bedeutung zukommt als in der Algebra. Da nämlich die Invarianten äquivalenter Formen dort ihrer Natur nach nur zahlentheoretische Functionen der Coëfficienten sind, so kann es nicht befremden, wenn dieselben zwar direct definiert aber nicht explicite sondern nur als Endresultate arithmetischer Operationen dargestellt werden können ; denn ganz ähnlich verhält es sich mit den meisten arithmetischer Begriffen, z.B. schon mit jenem einfachsten Begriffe des grössten gemeinsamen Theilers."

[65] As shall be seen in greater details later, the arithmetical properties of polynomials played an important role in Kronecker's 1850-1870 work on the solvability of equations. These properties

occasion of the 1874 controversy when Kronecker blamed "literal expressions" such as Jordan's canonical form.

Throughout the 1874 controversy, Jordan was retorting to Kronecker's assaults by claiming the greater generality and simplicity of his method. Way off the naïve simplism caricatured by Kronecker, Jordan's ideal of simplicity was linked to a practice of "reduction" of "general problems" into chains of sub problems. It supported a criticism of Kronecker's 1868 characterization of singular pairs of bilinear forms as having failed to find the "true reduced forms" which had to be simplest links of the chain of reductions with no possibility of further simplification. ([66])

Jordan's practice of reduction originated in his researches on groups of substitutions in the 1860's for the purpose of a *general* investigation of the *special* types of equations that could be solved by radicals. In order to handle the generality of this problem, Jordan had developed a "machinery" to reduce the types of groups of substitutions attached to the equations considered from the general to the special ([67]). The investigation of general solvable groups was therefore being reduced to the analysis of some successive particular groups such as the "transitive", "primitive", "linear" or "symplectic" groups. Among others, the linear group - and its properties such as the theorem stating the reduction of linear substitutions to their "simplest" or "canonical" forms -, had "originated" from the practice of reduction Jordan had made use of in his 1860's investigations. ([68])

When Jordan responded to Yvon-Villarceau in 1871, he appealed to his practice of reduction for bringing general systems of linear equations down to the sequence of "simplest forms" corresponding to the decomposition of the characteristic equation into its simplest (linear) factors. Because it required the resolution of a general algebraic equation, Jordan's canonical reduction

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would later be essential in Kronecker's 1882 arithmetic theory of algebraic magnitudes and his conception on a "Rationalsbereich". This theory was indeed based on polynomial forms as an alternative to Dedekind's fields.

[66] See (Jordan 1874b : 13).

[67] (Dieudonné 1970 : 168)

[68] In order to characterize which primitive equations are solvable by radicals, Galois had asserted that the degree of such equations is of he form p^n, p prime and that the corresponding group G of permutations has to be a solvable subgroup of the linear group. In contrast with his predecessors who often concentrated upon projective linear substitutions, Jordan made the consideration of homogeneous linear substitutions fundamental in his 1867 investigations on the determination of all the irreducible equations of a given degree that were solvable by radicals. It was for the purpose of establishing the three general types of solvable subgroups of the group of linear substitutions in two variables that he stated in 1868 the three kinds of "canonical forms" such a linear substitution S can be reduced to, depending of the nature of the roots of $|S-kI| \equiv 0|p]$. In 1870 the general canonical form theorem for n variables played a major role in Jordan's method for building up solvable groups from their composition series. This method especially involved determining linear substitutions that commute with a given substitution S. For a detailed analysis of the role played by canonical forms in Jordan's investigations on solvable groups as well as on the evolution of the role played by linear groups between 1870 and 1900 see (Brechenmacher 2006a & 2011).

nevertheless showed a formal nature. It did actually not give to the astronomers any practical resolution of the problem.

In the 1870s, it was nevertheless thanks to the practices –such as the canonical reduction – he had originally devised for group theory that Jordan succeeded in extending the range of his investigations to subjects such as differential equations (1871-1878), the theory of forms (1872-1875) as well as arithmetic and number theory (1874-1881). These practices did not come alone in the applications and Kronecker's criticisms highlight some of the algebraic ideals - such as simplicity and abstraction - which were walking along with them.

As we have seen, Jordan's canonical forms were a part of a broader practice of reduction of *general* problems to a chain of *special* ones. The same practice was used by Jordan for the reduction of compound groups to simple groups, *i.e.* what is now designated as the Jordan-Hölder theorem. ([69]) In a detailed study of the collective form of references to the works of Jordan in the nineteenth century, we have shown that this form of treatment of the tension between the general and the special must be understood in a much broader context which involves especially the long-terme legacy of both Évariste Galois and Louis Poinsot ([70]).

In the twentieth century, Jordan's *Traité* has often been described by the historiography as the book which unfolded the true group theoretical nature of Galois' works, thereby giving to group theory the status of an autonomous theory. Such a presentation nevertheless raises many problems when compared to a few facts. First, what Jordan explicitly introduced as the "*Théorie de Galois*" was referring to the note "Sur *la théorie des nombres"* and not to the much celebrated "*Mémoire sur les conditions de résolubilité des equations par radicaux*". For Jordan, Galois' theory was therefore concerned with the "number theoretical imaginaries" of equations of congruences such as $x^{p^\nu} \equiv x (mod p)$ *i.e.* what is nowadays referred to as Galois fields. Second, this presentation was very coherent with the main issue Jordan addressed in the *Préface* of his *Traité*, *i.e.* the "classification and transformation of irrational quantities". As a matter of fact, it was in the third section of his book, entitled "*Livre III. Des irrationnelles*", that Jordan would expose the results he attributed to Galois' "*Mémoire*".

The first part of the *Traité* was actually devoted to the notion of congruence that had been introduced by Gauss as a fundamental notion of his "higher arithmetic." It readily focused on the case of higher congruences which had been studied by Galois. ([71]) Jordan insisted on the "analogies" between the equations of congruences and the "ordinary theory of equations". For instance, if a higher congruence breaks up into linear factors, the symmetric functions of the roots are rational functions of the coefficients. But, Jordan added, as in the case of ordinary equations, and as "Galois was the first to show it", the latter theorem can be extended to its "full generality"

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[69] Moreover, the fact that Jordan had also stated a false theorem on the chain reduction of systems of imprimitivity illustrates that the latter's practice of reduction was not limited to processes such as the one of canonical reduction. See (Neumann 2006) and (Brechenmacher 2006a: 195).
[70] (Brechenmacher 2011).
[71] Higher congruences had also been studied in detail by Gauss who nevertheless never published his work. On the "missing section eight" of the *Disquitiones arithmeticae*, see (Frei 2007).

by the consideration of "imaginary numbers" (^{72}). As was common at the time, and contrary to later axiomatic approaches, it was thus by appealing to the analogy carried on by a process that Jordan legitimated the introduction of Galois number theoretical imaginaries (^{73}). But Jordan went further in legitimating Galois imaginaries by considering the *special* case of the binomial (cyclotomic) equation $x^{p-1} \equiv 1$ *(mod. p)* (with *p* a prime number) as both an *application* and a *model* for the *general* considerations of the "Galois theory". The whole *Traité* was actually structured by Galois imaginaries and linear substitutions in a complex tension between both the *general* / the *special* and the *applied* / the *model*. Both tensions ran all along the book and were instrumental to the way Jordan presented his own work as a mere comment of the works of Galois.

This situation highlights that it was actually the question of the legitimacy of the *general* methods and notions for dealing with the characterization of the "irrationals" that were defined by *special* equations which was at the core of the opposition between Jordan and Kronecker. Before the 1880s, Galois's works had been mainly commented in the broad framework of the problem of the "irrationals" (^{74}). The impossibility to solve by radicals *general* algebraic equations of degree greater than four highlighted the necessity to characterize the *special* nature of the irrational quantities and functions defined by both algebraic equations and differential equations involving algebraic expressions. From the 1830s to the 1850s, the problem of the nature of the irrationals involved elliptic (or abelian) functions (and therefore complex analysis) in the framework of the field of research of the *arithmetic algebraic analysis* as it has been described by Catherine Goldstein and Norbert Schappacher. (^{75}) Most of these perspectives were actually considered in the *Traité* even though Jordan's own contributions focused on the problem of the solvability by radicals which he reformulated in the framework of groups of substitutions. This situations sheds light on how the 1874 quarrel was both resorting to individual specificities and to collective lines of developments which had diverged since the 1850s from shared histories such as the algebraic practice attached to "the equation to the etc." or, on a larger scale, from the field of arithmetic algebraic analysis (^{76}).

In the 1860's, Jordan had reshaped what Caroline Ehrhardt has designated as the "collective memory" of Galois in publishing the "comments on Galois" which Joseph Liouville had promised in 1846 (^{77}). It is well known that Jordan focused on the use Galois had made of the concept of group but it is less well known that this reading of Galois took place in the legacy of what Louis Poinsot had designated as the "Theory of order". As has been shown by Jenny Boucard, the Theory of order especially supported a faith in a "general" theory, in the sense of higher considerations on the very essence of mathematics, and which connected algebra, number theory, and geometry. This theory had been especially introduced by Poinsot as having an analogous

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^{72} (Jordan 1870: 7).
^{73} See (Durand-Richard 1996 & 2008).
^{74} (Brechenmacher 2011).
^{75} (Goldstein and Schappacher 2007a&b).
^{76} As shall be seen later, on a more global scale these diverging lines threatened the unity of the field of research of arithmetic algebraic analysis, see (Goldstein and Schappacher, 2007b: 97).
^{77} See (Ehrhardt 2007).

relation to algebra as Gauss' higher arithmetic to usual arithmetic or the *analysis situs* to geometry ([78]).

The generality of the theory of order was modeled on the special properties of the cyclotomic equation $x^{p-1}=1$. The roots of such an equation could indeed be generated by the powers of a single one of them, ξ (a primitive root):

$$\xi, \xi^2, \xi^3, \dots, \xi^{p-1}$$

But the list of the roots could also be reordered by considering the primitive root of an equation of congruence on their marks, *i.e.* the equation $x^{p-1} \equiv 1$ *(mod. p)* that, as we have seen, Jordan would discuss later in connection to the "theory of Galois" ([79]):

$$\xi_0 = \xi, \xi_1 = \xi^g, \xi_2 = \xi^{g^2}, \dots, \xi^{g^{p-1}} \quad .$$

In the first case, the roots would be indexed so that it would be possible to turn one of them into the next on the list by "moving forward" *i.e.* by a substitution $|z \; z+1|$. In the second case, the substitutions would correspond to the successive arithmetical powers $|z \; g^q z|$ operating on the marks of the roots. This reordering of the roots, as "if in a circle" as had commented Poinsot in 1808, played key role on the way Gauss had proved that the equations $x^{p-1}=1$ could be solved by radical. The proof of Gauss was commented by Poinsot as a method of decomposition of the roots of the equation into "groups". Given a cyclotomic equation of degree $p-1=qr$, its roots can be reindexed by the consideration of equations of congruences to be decomposed into q cyclic "groups" of a same number of r roots which can be made to move "forward" cyclically by the substitution $|z \; z+1|$, while the "groups" themselves could be permuted one with the other by cyclic operations analogous to the way the powers of a primitive root generated the whole set of roots of a binomial equation, *i.e.* by substitutions $|z \; g^q z|$ From the combinations of the latter two kinds of substitutions would "originate", as would say Jordan, the "linear groups" $|z \; az+b|$ studied by Galois. ([80]) The latter had especially designated the method described above as "the method of decomposition of M. Gauss" by which an "imprimitive group" of order qr could be decomposed into q primitive groups of the same order r. As has been shown by Catherine Goldstein and Norbert Schappacher, the *special* cases Galois had considered, - especially the binomial equation and the modular equation - can clearly be understood as *models* which "informed Galois about what he had to formulate in the general theory" ([81]).

In Jordan's *Traité,* a complex circular system of generalizations and applications made the decomposition of imprimitive groups into primitive groups appear as the first step of a completely general chain of reduction of a compound group into simple ones. The crucial notion of linear substitution was for instance introduced in the *Livre II* as "originating" from a *generalization* of

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[78] See (Boucard 2011).

[79] It is not the place here to detail the mathematics involved here. See (Frei 2007), (Neumann 2007), and (Boucard 2011). This connection between algebra and number theory would play key role in Galois' introduction of the number theoretical imaginaries *(i.e.* finite fields) and in the specificity of Jordan's later approach on linear groups and Galois fields. See (Brechenmacher 2011).

[80] Nowadays, such a group would be designated as an affine group. Neither Galois nor Jordan made any difference between linear and affine groups.

[81] (Goldstein and Schappacher 2007a: 34).

the *special* case of the cyclic substitutions associated to binomial congruences studied in the *Livre I*. Later on, when the notion of a group of an equation had been introduced in the *Livre III*, the origin of the linear group could be considered as a *model* for the *generalization* of cyclotomic equations to the metacyclic equations Galois had considered. ([82]) The same could be said about the relationship between the cyclicity of primitive roots and the notion of group of an equation: *special cases* such as binomial equations and cyclic groups or metacyclic equations and linear groups were both *models* for the *general* theory and *applications* of it. In the *Livre IV*, the problem at the "origin" of the linear group would ultimately appear as a *model* for the method of chain reduction and it was in this context that Jordan would make use of his canonical reduction of linear substitutions.

In a word, the *Traité* was structured by the consideration of *special* cases as *model cases* for successive *generalizations* and this chain of generalization would itself later be considered as both the *model* for and the *application* of a *general* theory. This form of treatment of the tension between the special/ the general, the applied/the model, therefore played key role for legitimating Jordan's approach. As a matter of fact, the only theorem Jordan explicitly attributed to himself – and not to Galois - crystallized this tension into a *problem* and a *method*. The problem was the one of the classification of *general* finite solvable groups into *special* categories; the method was the chain reduction from the general to the simple. As later adoptions of terminologies such as the ones of "general linear groups" and "special linear group" would exemplify it, the crystallization of this tension would play a key role in the emergence of the theory of finite groups.

Since the mid 1860's, Jordan had attributed to groups of substitutions the ontological nature of an essential theory, differing from the rest of mathematics, he had previously given to Poinsot's Theory of order. In his *Traité,* he eventually claimed that his method for dealing with both general and special groups was a higher standpoint on the crucial problem of the "characterization and the classification of the irrationals". Given the fact that Kronecker was well known to be involved in the latter problem, it is actually likely that the purpose of Jordan when he concluded the *Préface* of his treaty by insisting on his difficulties to read Kronecker's works, was to avoid to recognize explicitly that claiming that the whole theory of the irrationals came from Galois was a direct challenge to Kronecker ([83]).

Already in the 1850's, Kronecker had developed a specific approach to equations he claimed to be faithful to Gauss' legacy. But Kronecker had little interest for Galois' conceptual considerations and was much more interested in explicit expressions, such as the ones Abel had given to the roots

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[82] The equations considered by Galois were introduced by Jordan as generalizing cyclotomic equations in three different ways. First, they were defined as irreducible equations of prime degree *p* whose all the roots could be expressed rationally by two of them, an obvious generalization of the case of cyclotomic (and more generally abelian equations) whose roots are rational functions of one of them. Second, their groups were constituted of substitutions of the form $|x\ ax+\alpha|$, *i.e.* the kind of linear groups that had been introduced in the *Livre II* as "originating" from the consideration of the substitutions conjugated to the cyclic groups attached to cyclotomic equations. Third, particular cases of Galois' equation were given by equations defined by $x^p-A=0$, *i.e.* obvious generalizations of the binomial equations.
[83] (Jordan 1870: VIII).

of the quintic when proving the impossibility to solve it by radicals. The statement that the roots of abelian equations with integer coefficients can be expressed as rational functions of the roots of unity of cyclotomic equations, *i.e.* what is nowadays designated as the Kronecker-Weber theorem, was explicitly considered by Kronecker as aiming at separating the domains of algebra and of the theory of number in the investigation of the "essence" of the quantities associated to algebraic equations.

As was made clear in the text Joseph Alfred Serret added in 1854 to the second edition of his *Algèbre supérieure,* Kronecker considered the theory of equations as resorting to the investigation of the "true nature" of irrational quantities by substituting to the impossible problem of expressing the roots of a general equation by an algebraic function of the coefficients of this equation, the problem of finding the most general functions of given quantities which would satisfy to an equation of a given degree whose coefficients are rational functions of the same given quantities ([84]). This problem would be explored throughout Kronecker's works on both the theory of forms and on the complex multiplication of elliptic functions. In this context, Kronecker developed the concept of "equations with affects" to deal with special equations, such as the ones which could be solved by radicals. He therefore did not resort to the concept of Galois' group of an equation and it would have been surprising if he had been enthusiastic about Jordan's claim that all the special characteristics of equations were mirrored by their associated groups.

As is illustrated by the 1874 controversy, Kronecker and his former student Eugen Netto were nevertheless among the most attentive readers of Jordan ([85]). But when Kronecker published in 1882 his influential *Grundzüge einer arithmetischen Theorie der algebraischen Grössen,* the dissociation of the "foundational" arithmetical nature of "algebraic quantities" from the treaties on substitutions Netto published the same year was clearly opposed to Jordan's claim that groups of substitutions were a higher standpoints on the theory of irrationals. In contrast wirth the formal nature Kronecker associated to the algebraic "groups of substitutions", the latter aimed at building a "concrete", *i.e.* effective, arithmetical theory of algebraic quantities funded on the notion of "forms" *i.e.* of rational functions of n variables ([86]).

It is well known that Kronecker's *Grundzüge* was opposed to some recently published works of Dedekind on algebraic number fields and ideals ([87]). But Kronecker also presented his work as a natural outgrowth of life long concerns for the arithmetical grounds of algebraic methods. He especially insisted on the notion of "domain of rationality" which limited the field of legitimated quantities to the ones that may be expressed effectively as rational functions of a given list of quantities. In this framework, expressions such as Jordan's canonical forms could not be expressed in general as opposed to the invariants Kronecker had introduced by rational operations appealing to *g.c.d.*s computations.

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[84] Kronecker *in* (Serret 1854 : 561).
[85] The publication by Netto of a new proof of the chain reduction in 1874 was closed to turn into another quarrel. See (Brechenmacher 2007a).
[86] (Kronecker 1882: 34).
[87] On this topic, see epecialy (Goldstein and Schappacher 2007), (Corry 1996), (Edwards, 1980), (Edwards, Neumann, Punkert 1982).

We have seen that in 1874 Kronecker had opposed the ideal of effectivity of a form of arithmetic generality to the excessive generality of formal algebraic expressions. In 1882, this ideal of effectivity was at the core of the notion of "rationality" that laid the ground to the arithmetic theory of algebraic quantities. Moreover, Kronecker explicitly opposed his notion of equations with affects to the use of the notion of "Galois groups ". The *affect* of an equation lies in the manner in which, as a result of particular relations among the coefficients or the roots, certain resolvent functions of the roots would break up into irreducible factors on a given rational domain ([88]). When discussing the approaches of Abel and Galois, Kronecker claimed that, while Abel had stayed within the boundaries of a given rational domain by considering "concrete" rational functions of the roots of a given special equation, Galois had "abstracted" from the "problem of special equations" irrational groups whose importance "was only theoretical". For Kronecker, Galois had then "escaped freely" in the abstraction of a problem that Abel had "found and developped" *i. e.* the characterisation of the "classes of equations" corresponding to a given domain of rationality. From this "algebraic question" emerged "one the most interesting problem" in number theory which Kronecker strived at unveling the true "arithmetical nature" ([89]).

Conclusion.

We shall now come to some conclusions that may be drawn from the conflicting perspectives on generality relating to the two practices opposed by Jordan and Kronecker in 1874. In a word, while Jordan, on the one hand, criticized the lack of generality of Kronecker's invariant computations because they did not reduce pairs of forms to their simplest expression, Kronecker considered Jordan's canonical form as a "formal notion" with no "objective meaning" which therefore failed to reach a true generality. What was truly general for the one was exactly what was falsely general for the other.

It was in the first place the "general" solution they claimed to have achieved for different problems that had been handled in the past by authors such as Lagrange, Laplace, Cauchy or Hermite that had prompted some connections between Jordan's canonical reduction and Weierstrass' elementary divisors. The reference to a common history therefore played a key role in the controversy. Not only did the "equation to the secular inequalities in the planetary theory" play a major role in identifying a specific shared practice that consisted in expressing the solutions of linear equations as polynomial factors of their characteristic equation. But Kronecker also stressed a history of what T. Hawkins would later refer to as the "generic reasoning" in the eighteenth-nineteenth centuries algebra when he blamed traditional algebraic practices for their tendency to focus on the generic case with little attention given to the difficulties that may be caused by assigning specific values to algebraic symbols.

Jordan's canonical forms could nevertheless not be charged with such an indictment of "so called generality", and we have actually seen that both Jordan and Kronecker criticized the form of

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[88] In general, symmetric functions of the roots are rational functions of the coefficients. In case of equations with affects, certain non symmetric irreducible functions of the roots are also rationally known.

[89] (*Ibidem*: 58).

treatment of generality attached to the traditional practice mentioned above. Both aimed at setting the grounds of the "theory of forms" on new forms of generality. The issue of generality could thus not be dissociated from the one of the organisation of knowledge. As opposed to a traditional way to legitimate generalisations from a domain of knowledge to another by resorting to the analogies supported by operative processes, Jordan and Kronecker both aimed at inserting what used to be considered as different problems within a single "general" problem of "transformations" of pairs of "forms". But the two mathematicians did not only disagree on the nature of this theoretical organization but also on both the forms of generality and on the treatments of the general they were advocating.

On the one hand, in the algebraic organisation Jordan gave to the theory, transformations resulted from the action of some linear groups of substitutions. In order to achieve "general results" on forms, underlying substitutions had to be reduced to their "simplest canonical forms" depending on the nature of the linear group the substitutions were belonging to. This practice of reduction was both the result of the individual approach developed by Jordan on groups of substitutions in the 1860s and of the long run local legacies of both the "essential generality" of Poinsot's theory of order – in the sense of something transcending disciplinary borders - and of Galois' "general" approach to model cases special equations – in the sense of a conceptual abstract oriented treatment of general special types of equations.

On the other hand, for Kronecker, the algebraic nature of Jordan's approach prevented it from reaching any general theoretical level. As Kronecker insisted, one shall not mistake algebraic methods for the "general notions" relating to arithmetic it was algebra's duty to serve. Methods such as groups of substitutions, he argued, were relative to the orientations one would develop in his researches and had therefore no inherent signification. Faking such methods for notions would only lead to formal and falsely general developments. Moreover, Kronecker blamed the way Jordan had dealt with the general and the special by "reducing" a general problem to a chain of simplest ones. Because the way Jordan had applied it to the reduction of linear substitutions to their canonical forms resorted to the determination of the roots of general algebraic equations, Kronecker condemned the "false generality" and the "formal" nature of Jordan's explicit formula of canonical reduction because of its non effectivity. On the opposite, Kronecker appealed directly to the legacy of Gauss on behalf of his claim that the theory of forms should be considered as belonging to arithmetic and should consequently focus on the characterisation of equivalence classes in establishing arithmetical invariants thanks to some effective procedures such as *g.c.d.*s computations.

As we have seen in this paper, the collectives to which the conflicting values of generality of Jordan and Kronecker appealed to where quite complex. Even though at the down of World War I, nationalistic discourses would oppose the "French style of thinking" - as exemplified by the Theory of order in the legacy of Jordan and Galois - to the German Algebra ([90]), it is not possible to analyse the 1874 quarrel as a direct echo of the war of 1870 or in the frame of some antagonisms between France and Germany or Paris and Berlin. As a matter of fact, when the controversy reached its climax in the spring of 1874, Gaston Darboux published a paper on the

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[90] (Brechenmacher 201?).

"algebraic theory of forms" which aimed at putting to the fore the legacy of Hermite and which orientation was much closer to Kronecker's one than to Jordan's one ([91]). But the fact that Jordan's and Kronecker's values of generality cannot be inscribed in any obvious social, institutional or national category does not mean that these values were purely individual.

We have seen that both mathematicians appealed to what used to be a shared algebraic culture at a time when Algebra was not an autonomous discipline. This shared history took the form a network of mainly periodical publications. For this reason, it referred to a way of treatment of generality which was neither reflexively identified to a method nor to a theory. What we have designated as a "practice" took on an identity which could not be dissociated from the network of texts in which it circulated. But the controversy was not only the consequence of the two different ends Jordan and Weierstrass had given to this quite globally shared history. It actually opposed two more local collective lines of developments on the *general* theory of the irrationals defined by *special* equations. We shall conclude this paper in highlighting that the 1874 quarrel actually proceeded more from the divergence of these two lines than from their meeting. In this sense, in parallel to the broad phenomenon of universalization of new ideals of rigours which would pay attention to singularities and would separate homogeneity from genericity, various other forms of treatment of the general emerged and circulated on more local scales.

In two influent papers he published between 1878 and 1880, Georg Frobenius structured the organization of the theory of forms for the next fifty years. In doing so, he proposed a synthesis between the notions put to the fore by Jordan and Kronecker. But at the deeper level of the processes he favoured, Frobenius followed Kronecker in aiming at building a rational theory of forms based on effective computations of invariants. In this context, Jordan's canonical form lost its status of a theorem and appeared as a mere consequence of a more "general approach" in the specific case in which "use is made of the complex number introduced by Galois". ([92])

But despite the influence of Frobenius' theory, Jordan's approach would keep circulating on a model close to the one we have seen in the case of the network of "the equations to etc." As a matter of fact, Jordan's canonical form would circulate in between methods and theories for a few decades. While Frobenius' theory characterized forms by the computation of polynomial invariants, Jordan's reduction "transformed" dynamically a given form to its simplest expression. The representation Jordan had developed for linear substitutions allowed to "see" the simultaneous decomposition into some sub groups of the indices on which a substitution operated and of the substitution itself ([93]). In contrast with the static nature of invariant computations, the representation Jordan appealed to for reducing general problems to chains of simpler problems

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[91] On Hermite's theory of forms and on Hermite's program for characterizing irrational quantities, see (Goldstein 2007).

[92] (Frobenius 1879: 483, translation T. Hawkins 1977: 153);
For almost half a century, Jordan's canonical form would only be considered as a theorem in a specific network of texts involving especially American and French works on linear groups and Galois fields. See (Brechenmacher 2011).

[93] From the standpoint of linear algebra, this corresponds to a decomposition of a vector space under the action of an operator into a sum of a stable subspaces. See appendix n° 1 as well as (Brechenmacher 2006a: 167-187).

was inducing some dynamic ways of thinking about "transformations", "reductions", or "decompositions". This representation depicted how a general problem could be reduced into a chain of simpler problems. It would later be the basis of the 1930s method of matrix decomposition such as in the example below:
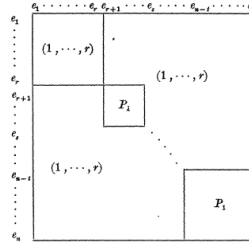
"Jordan's canonical form theorem" would indeed be referred to as a central result in most of the treaties on the "theory of matrices" of the 1930's. But this "general" theorem would actually be connected with two kinds of canonical forms: ([94])

$$A = \begin{Vmatrix} \lambda_1 & 1 & 0 & & & & & & 0 \\ 0 & \lambda_1 & 0 & & & & & & \\ & & \lambda_2 & 1 & 0 & & & & \\ & & 0 & \lambda_2 & 1 & & & & \\ & & 0 & & \lambda_2 & 0 & 0 & 0 & \\ & & & 0 & \lambda_3 & 0 & 0 & & \\ & & & & & \lambda_4 & 1 & \\ & & & & & & \lambda_4 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & 0 & \dots & & \dots & 0 & -\alpha_8 \\ 1 & 0 & \dots & & \dots & & -\alpha_7 \\ 0 & 1 & 0 & \dots & \dots & & -\alpha_6 \\ \dots & 0 & 1 & 0 & 0 & \dots & \\ & \dots & 0 & 1 & 0 & 0 & \dots & \dots \\ & & \dots & 0 & 1 & 0 & 0 & \dots \\ & & & & 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & \dots & & \dots & 0 & 1 & -\alpha_1 \end{Vmatrix}$$

On the one hand the Jordan canonical matrix A would be considered as the "simplest" form for the maximal decomposition of a matrix. On the other hand the "rational canonical form" B would be obtained as the result of effective procedures. The very basis of the 1930s "general" theorem would therefore be funded on an articulation of the two points of views which had been opposed in 1874 ([95]).

Between 1874 and 1930, a tension between canonical forms and invariants would play a major role in the complex history of the practices that would come to give a universal dimension to the operatory processes attached to the matrix pictorial representations ([96]). On the fringe of the treatment of generality in the predominant theory of bilinear forms, some authors such as Jordan, Henri Poincaré, Eduard Weyr, Theodor Molien, Kurt Hensel, William Burnside, Leonard Dickson or Léon Autonne would handle general problems through some practices of reductions to canonical forms, thereby developing some operatory processes on imagery representations. Studying how these practices were fitting in some networks of texts raises some issues about disciplines and communities formations, evolutions and connections in connection with some the ambitions of "generalisations" which, as has been portrayed in this paper for the case of the

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[94] The two matrices given in this example both relate to the minimal polynomial $\lambda^8 + \alpha_1\lambda^7 + \dots + \alpha_7\lambda + \alpha_8 = (\lambda - \lambda I)^2(\lambda - \lambda_2)^3(\lambda - \lambda_3)(\lambda - \lambda_4)$.
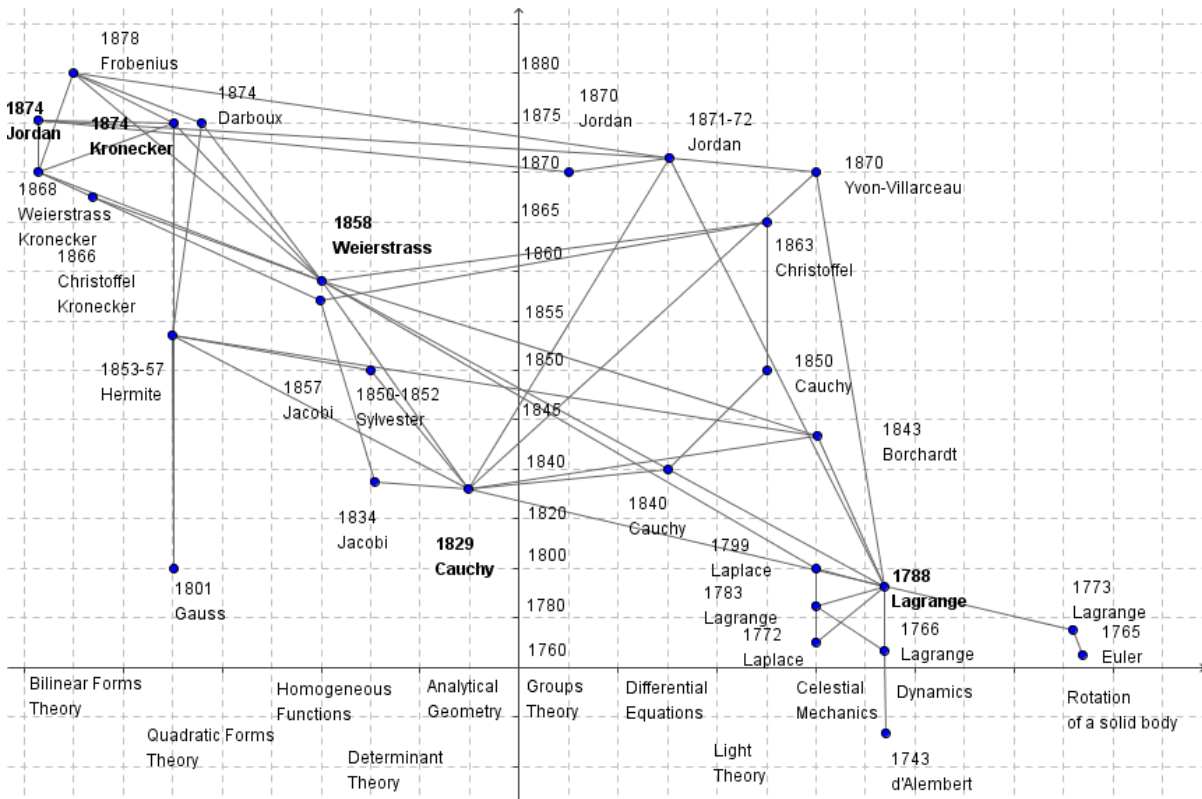[95] See for instance (Aitken and Turnbul 1932 : 1).
[96] See (Brechenmacher : 2010).

network of the discussion on the equation to the secular inequalities of planetary theory, are of particular importance for the history of linear algebra.

Annex 1.

The network of the equation to the secular inequalities on planetary theory.

1878 Frobenius
1874 Jordan
1874 Kronecker
1874 Darboux
1880
1870 Jordan
1871-72 Jordan
1875
1870
1868 Weierstrass Kronecker
1866 Christoffel Kronecker
1870 Yvon-Villarceau
1865
1858 Weierstrass
1860
1863 Christoffel
1853-57 Hermite
1855
1850
1857 Jacobi
1850-1852 Sylvester
1850 Cauchy
1845
1843 Borchardt
1840
1834 Jacobi
1829 Cauchy
1820
1840 Cauchy
1801 Gauss
1800
1799 Laplace
1783 Lagrange
1788 Lagrange
1773 Lagrange
1780
1772 Laplace
1766 Lagrange
1765 Euler
1760

Bilinear Forms Theory | Quadratic Forms Theory | Homogeneous Functions | Analytical Geometry | Determinant Theory | Groups Theory | Differential Equations | Light Theory | Celestial Mechanics | 1743 d'Alembert | Dynamics | Rotation of a solid body

Appendix I.

More about Jordan's canonical form and Weierstrass' elementary divisors.

From the standpoint of the post 1930s linear algebra, Jordan's canonical form theorem is equivalent to the elementary divisors theorem.

The one gives a canonical form to which all similar matrices can be reduced to by similarity transformations. The other provides a set of polynomials which are invariants for similarity transformations and therefore characterize similarity classes.

- *Similarity of matrices.*

Both theorems provide a characterisation of the classes of similarities of matrices with coefficients belonging to an algebraically closed field: two matrices *A* and *B* are similar if and only if there exists an invertible matrix *U* such that

$$U^{-1}AU=B$$

We give below some examples of the relationships between three Jordan's canonical forms and three polynomial decompositions of the characteristic polynomial

$$|A-\lambda I|=(\lambda-1)^2(\lambda-2)^3(\lambda-3).$$

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An elementary divisor $(\lambda - a)^k$ corresponds to a k by k Jordan block : ([97])

Jordan's canonical forms.	$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ & & & & & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & 0 & & \\ & & & 2 & 1 & \\ & & & & 2 & \\ & & & & & 3 \end{pmatrix}$
Elementary divisors.	$(\lambda - 1), (\lambda - 1),$ $(\lambda - 2), (\lambda - 2), (\lambda - 2),$ $(\lambda - 3)$	$(\lambda - 1), (\lambda - 1),$ $(\lambda - 2)^3,$ $(\lambda - 3)$	$(\lambda - 1), (\lambda - 1),$ $(\lambda - 2), (\lambda - 2)^2,$ $(\lambda - 3)$

- *Equivalence of pairs of matrices.*

Similarity is nevertheless not the only class of equivalence which can be considered on matrices as is illustrated by the classification Jordan gave of three problems of reduction to canonical forms in his paper of December 1873. From the standpoint of modern algebra, the canonical form $x_1y_1 + ... + x_my_m$ characterizes the equivalence classes of square matrices for the "equivalence relation"

$$ARB \Leftrightarrow \exists U,V \in GL_n(\mathbb{C}), \ UAV = B$$

The first problem concerns the similarity relation of symmetric matrices by orthogonal matrices

$$ARB \Leftrightarrow \exists U \in O((\mathbb{C}), \ U^{-1}AV = B$$

The second problem relates to the congruence relation for square matrices

$$ARB \Leftrightarrow \exists U \in GL_n((\mathbb{C}), \ {}^tUAU = B$$

The third problem focuses on the equivalence of pairs of matrices *(A, B)* which is of importance for systems of differential equations with constant coefficients

$$AY'' + BY = 0.$$

Jordan's claims that the canonical form he had introduced for classes of similarity of substitutions can be used for pairs of bilinear forms was based on the fact that the problem of the *classes of similarity* of matrices is equivalent to the one of the *equivalence classes* of (non singular) pairs of matrices, ([98]) *i.e.* linear polynomial of matrices such as *A-λI*: two such matrices *A-λI* and *B-λI* are equivalent if and only if there exists two invertible matrices *U* and *V* such that *U(A-λI)V=B-λI* .

Given a pair of bilinear forms *sP-Q*, *|P|≠0*, applying Jordan's theorem to the linear substitution *QP⁻¹* shows that there exists a non singular substitution *U* such that *U⁻¹(QP⁻¹)U=J* where *J* is in canonical form. Thus *H(sP-Q)K=sI-J*, where *H=U⁻¹* and *K=P⁻¹U*. The family of forms *sP-Q* can thus be reduced to its canonical form *sI-J*.

- *Kronecker's invariant factors.*

In nowadays formulation, the "invariant factors" Kronecker introduced in 1874 give a method for deciding of the equivalence of (non singular) pairs of matrices which is valid on a principal ring (such as the ring of integers or of polynomials with integer coefficients) whereas Jordan's

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[97] See (Gantmacher 1959).
[98] See (Gantmacher 1959) for a proof.

canonical form and Weierstrass' elementary divisors are only valid on algebraic closed field such as the field of complex numbers.

Consider a family of bilinear forms $sP-Q$ and let $S(s)$ denote the determinant $|sP-Q|$. Let $S_1(s)$ be the greatest common divisor of all the first minors of $S(s)$ (which are polynomials in s). Similarly, $S_2(s)$ is defined as the greatest common divisor of all the second minors of $S(s)$ and so on.

Then $S_i(s)$ divides $S_{i-1}(s)$ and if $E_i(s)$ denotes the polynomial $S_{i-1}(s)/S_i(s)$ then $E_i(s)$ divides $E_{i-1}(s)$. Thus, $S(s)$ differs from the product of the $E_i(s)$ by a constant.

Let now imagine to consider the situation in an algebraic closed field so that each polynomial expression can be broken up in linear factors. Let $s_1, s_2, ..., s_k$ bet the distinct roots of $S(s)$, then

$$E_i(s) = c_i \prod_{j=1}^{n} (s - s_j)^{m_{ij}}$$

where c_i is constat and the m_{ij} are positive integers or zero. Each factor $e_{ij} = (s - s_j)^{m_{ij}}$ with $m_{ij}>0$ is what Weierstrass had called an elementary divisor of $S(s)$. In the case of the consideration of an algebraic close field, Kronecker's invariants are therefore equivalent to Weierstrass' elementary divisors.

Appendix II.
On the practice of the equation to the secular inequalities in planetary theory.

In a paper devoted to an extensive study of the network of the "equation to etc.", ([99]) it has been shown that the special nature of the equation to the secular inequalities was closely related to a specific process for expressing the solutions of symmetric linear systems:

$$\frac{dx_1}{dt} = a_1 x_1 + ... + l_1 x_n$$

$$\frac{dx_2}{dt} = a_2 x_1 + ... + l_2 x_n$$

$$.......$$

$$\frac{dx_n}{dt} = a_n x_1 + ... + l_n x_n$$

by polynomial quotients of the determinant S of the system and its sub determinants P_{1i} : ([100])

$$(*) \frac{\dfrac{P_{1i}}{S}(x)}{x - s_j}$$

The systems $(x_i^{s_j})_{1 \le i \le n}$ of solutions of the above systeù related to a root s_j of the characteristic equation were indeed expressed by:

$$x_i^{s_j} = \frac{\dfrac{P_{1i}}{S}(s_j))}{x - s_j},$$

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[99] (Brechenmacher 2008).
[100] *i.e.* what we would now call the minors of the matrix of the linear system lying on the intersection of the first row and i^{th} column subsets

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Such expressions would thus lead to $\frac{0}{0}$ expressions if common roots should occur between S and its successive sub determinants. In 1858, Weierstrass had proven that, in the quadratic case, each root of multiplicity p of $S(x)=0$ had to be a root of $P_{1i}(x)=0$ of a multiplicity at least equal to p-1. In the symmetric case, the expression (*) is therefore valid regardless of the multiplicity of roots. One of the impulse of the emergence of the theory of bilinear forms in the 1860s was the generalisation of such kinds of problems to the non symmetric case.

Expressions such as (*) were at first considered as involving some equations obtained by elimination methods. After some authors such as Cauchy or Jacobi had based their approach on determinants, (*) gradually came to be considered as involving the successive sub-determinants $\frac{\Delta_{i+1}}{\Delta_i}$ extracted from S. In this framework, Weierstrass' 1858 theorem as well as some previous works by Hermite can be understood as a generalization of the inertia law for a quadratic form A to a pair of quadratic forms (A, I).

According to the inertia law as it had been introduced by Hermite and Sylvester in the 1850's, it is possible to transform a quadratic form A into a sum of squares

$$A = \Delta_{n-1} X_1{}^2 + \frac{\Delta_{n-2}}{\Delta_{n-1}} X_2{}^2 + \ldots + \frac{\Delta}{\Delta_1} X_n{}^2,$$

where $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_{n-1}, 1$ were the principal minors of A.

In 1858, Weierstrass stated that it is possible to transform simultaneously two quadratic forms $(P = \sum_{i=1}^{n} A_{ij} x_i x_j \; Q = \sum_{i=1}^{n} B_{ij} x_i x_j)$- P being definite positive - into sums of square terms $(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} s_i X_i^2)$ by means of what we would designate today as orthogonal substitutions.

In 1874 G. Darboux explicitly generalised the inertia law to polynomials of quadratic forms $A+sI$ ([101]). Darboux's paper was published next to Jordan's controversial one in *Liouville's Journal* ([102]). Both proposed new demonstrations of Weierstrass' theorems: while Jordan focused on the 1868 result on pairs of bilinear forms, Darboux worked out new perspectives on the 1858 theorem on quadratic forms in developing the methods introduced by Hermite and Sylvester in the 1850's.

From the standpoint of modern algebra the process attached to the practice of the "equation to etc." could be considered as a method giving the general polynomial expressions of the eigenvectors of a symmetric matrix A. Such expressions are indeed given by the columns of the matrix of cofactors computed from the polynomial matrix A-λI. This formulation nevertheless induces some anachronisms not only because it resorts to modern notions or theories but also because it is implicitly related to some methods of "transformations" (which include some geometrical analogy such as the "symmetry" property of mechanical systems) which were

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[101] (Darboux 1874 : 367). Darboux was merely concern with the geometry of surfaces and his method was inserted by Gundelfinger in the third edition of Hesse's analytical geometry. For a detailed description of this method see (Drach and Meyer 1907)

[102] (Jordan 1874a)

extraneous to the practices used during the discussion of the "equation to etc.". As a matter of fact, when he gave an integrable "form" to his systems, Lagrange did not appeal to a method of "transformation" but to the computation of some mechanical parameters (the proper periods). Moreover, the symmetric property of mechanical systems was a consequence of the process devised by Lagrange for a direct computation of the solutions relating to the initial conditions from the characteristic equation. From the standpoint of modern algebra, this process can be considered as making use of dual orthogonality.

Given a symmetric matrix *A*, the coordinates of its eigenvectors are given by not equal to zero columns of the adjoint matrix of the characteristic matrix *A-xI* (*i.e.* the matrix of cofactors). Consider, for instance, the matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The characteristic equation and minors are given by:

$S=-x(3-x)(1-x)$, $P_{11}(x)=(1-x)(2-x)-1$, $P_{12}(x)=(1-x)$ et $P_{13}=-1$.

The expression $((1-x)(2-x)-1, 1-x, -1)$ gives the polynomial coordinates of any eigenvector (while $|S|$ gives the square of its norm), *e.g.* for the eigenvalue $s_1=1$, $\dfrac{S}{x-1}=x(3-x)$,

$$x_1^{s_1} = \frac{P_{11}}{\frac{S}{x-1}}(1) = -\frac{1}{2}, x_2^{s_1} = \frac{P_{12}}{\frac{S}{x-1}}(1) = 0, x_3^{s_1} = \frac{P_{11}}{\frac{S}{x-1}}(1) = -\frac{1}{2},$$

The coordinates of the normed eigenvector relating to the eigenvalue 1 are *($1/\sqrt{2}$, 0, $1/\sqrt{2}$)*.
Similarly for the eigenvalues $s_2=0$,

$$x_1^{s_2} = \frac{1}{3}, x_2^{s_2} = \frac{1}{3}, x_3^{s_2} = -\frac{1}{3}$$

and $s_3=3$,

$$x_1^{s_3} = \frac{1}{6}, x_2^{s_3} = \frac{2}{6}, x_3^{s_3} = -\frac{1}{6}$$

The corresponding normed vectors are $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$ et *($1/\sqrt{6}$, $2/\sqrt{6}$, $-1/\sqrt{6}$)*.
Interpretation for quadratic forms : la form associated to *A* in the canonical basis of $\mathring{A}^3$,

$$A(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 + x_2^2 + 2x_2x_3 + x_3^2 = 1.X_1^2 + 0.X_2^2 + 3.X_3^2$$

with

$$(X_1, X_2, X_3) = (x_i^{s_j})_{i,j=1,2,3} A(x_i^{s_j})_{i,j=1,2,3}^{-1}$$

where $(x_i^{s_j})_{i,j=1,2,3}$ is the change of orhonormal basis matrix.

$$(x_i^{s_j})_{i,j=1,2,3} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

Interpretation for a (symmetric) bilinear form:

$A(x,y) = x_1y_1-x_1y_2-y_1x_2+x_2y_2+x_2y_3+y_3x_2+x_3y_3=1.X_1Y_1+0.X_2Y_2+3.X_3Y_3.$

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