

SPECTRAL THEOREM FOR MULTIPLIERS ON $L_\omega^2(\mathbb{R})$

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ABSTRACT. We study the spectrum $\sigma(M)$ of the multipliers M which commute with the translations in weighted spaces $L_\omega^2(\mathbb{R})$. For operators M in the algebra generated by the convolutions with $\phi \in C_c(\mathbb{R})$ we show that $\overline{\mu(\Omega)} = \sigma(M)$, where the set Ω is determined by the spectrum of the shift S and μ is the symbol of M . For the general multipliers M we establish that $\overline{\mu(\Omega)}$ is included in $\sigma(M)$. A generalization of these results is given for the weighted spaces $L_\omega^2(\mathbb{R}^k)$ where the weight ω has a special form.

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1. INTRODUCTION

In this paper we examine the spectrum of multipliers M in a weighted space $L_\omega^2(\mathbb{R})$. Our approach is based heavily on the existence of symbols for this class of operators and we show that the spectrum $\sigma(M)$ can be expressed by the symbol μ of M applied to a set Ω defined by the spectrum of the shift operator S . To announce our results we need some definitions.

A weight ω on \mathbb{R} is a non negative function on \mathbb{R} such that

$$\sup_{x \in \mathbb{R}} \frac{\omega(x+y)}{\omega(x)} < +\infty, \forall y \in \mathbb{R}.$$

Denote by $L_\omega^2(\mathbb{R})$ the space of measurable functions on \mathbb{R} such that

$$\int_{\mathbb{R}} |f(x)|^2 \omega(x)^2 dx < +\infty.$$

Let $C_c(\mathbb{R})$ be the set of continuous functions on \mathbb{R} with compact support. For a compact K of \mathbb{R} denote by $C_K(\mathbb{R})$ the subset of functions of $C_c(\mathbb{R})$ with support in K . The space $L_\omega^2(\mathbb{R})$ equipped by the norm

$$\|f\| = \left(\int_{\mathbb{R}} |f(x)|^2 \omega(x)^2 dx \right)^{\frac{1}{2}}$$

is a Banach space and $C_c(\mathbb{R})$ is dense in $L_\omega^2(\mathbb{R})$. We denote by \hat{f} or by $\mathcal{F}(f)$ the usual Fourier transform of $f \in L^2(\mathbb{R})$. Denote by S_x the operator of translation by x defined on $L_\omega^2(\mathbb{R})$ by

$$(S_x f)(t) = f(t-x), \forall t \in \mathbb{R}.$$

Let S (resp. S^{-1}) be the translation by 1 (resp. -1) on the space $L_\omega^2(\mathbb{R})$. Define the set

$$\Omega = \left\{ z \in \mathbb{C}, -\ln \rho(S^{-1}) \leq \operatorname{Im} z \leq \ln \rho(S) \right\},$$

where $\rho(A)$ is the spectral radius of A . For $\phi \in C_c(\mathbb{R})$ denote by M_ϕ the operator of convolution by ϕ on $L_\omega^2(\mathbb{R})$. Let \mathcal{A} (resp. \mathcal{B}) be the closed algebra generated by operators M_ϕ , for $\phi \in C_c(\mathbb{R})$

(resp. S_x , $x \in \mathbb{R}$) with respect to the topology of the operator norm. Denote by \widehat{A} the set of characters of a commutatif algebra A .

Definition 1. A bounded operator M on $L^2_\omega(\mathbb{R})$ is called a multiplier if

$$MS_x = S_x M, \forall x \in \mathbb{R}.$$

We will denote by \mathcal{M} the algebra of the multipliers on $L^2_\omega(\mathbb{R})$. First we treat the operators in the algebra \mathcal{A} working with the symbol of M defined in Section 2. Notice that if M_ϕ is the convolution by $\phi \in C_c(\mathbb{R})$, the symbol of M_ϕ is $\widehat{\phi}$. Our first result concerns the spectrum of M_ϕ .

Theorem 1. For $\phi \in C_c(\mathbb{R})$, we have

$$\sigma(M_\phi) = \overline{\widehat{\phi}(\Omega)}.$$

On the other hand, for every $M \in \mathcal{A}$ we obtain

Theorem 2. Let $M \in \mathcal{A}$ and let ν be the symbol of M . Then ν is a continuous function on Ω and we have

$$\sigma(M) = \overline{\nu(\Omega)}. \quad (1.1)$$

For general multipliers M we have a weaker result leading to the inclusion of $\mu(\Omega)$ in the spectrum of M .

Theorem 3. Let M be a multiplier on $L^2_\omega(\mathbb{R})$ and let μ be the symbol of M . Then we have

$$\overline{\mu(\Omega)} \subset \sigma(M). \quad (1.2)$$

We should mention that our results cannot be obtained by using known spectral mapping theorems since in general we have no spectral calculus for the operators in the algebra \mathcal{M} . For this reason we exploit the existence of symbols μ for such operators established in [6] for $L^2_\omega(\mathbb{R})$ and in [8] for $L^2_\omega(\mathbb{R}^k)$ assuming that the weight is a product of weights $\omega = \omega_1 \times \dots \times \omega_k$. The symbols μ are holomorphic in the set $\overset{\circ}{\Omega}$ and this plays a crucial role in our analysis since we may characterize the spectrum of M by the set $\mu(\Omega)$. For example if $T_t = e^{tS}$, then Theorem 3 yields the well known inclusion

$$e^{t\sigma(S)} \subset \sigma(e^{tS}).$$

On the other hand for the operators in \mathcal{M} it seems difficult to obtain an analog of (1.2) by using the techniques developed for C_0 -semi-groups and special Banach algebra (see [4], [3], [5], [1]).

2. PRELIMINARIES

First, we explain the link between the spectrum of S and the set where the symbol of a multiplier is defined. For $a \in \mathbb{R}$, denote by g_a the function $g_a(x) = g(x)e^{ax}$. In [6], we have established the following theorem.

Theorem 4. For every $M \in \mathcal{M}$, and for every $a \in I = [-\ln \rho(S^{-1}), \ln \rho(S)]$, we have

- 1) $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c(\mathbb{R})$.
- 2) There exists $\mu_{(a)} \in L^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} (Mf)(x) e^{ax} e^{-itx} dx = \mu_{(a)}(t) \int_{\mathbb{R}} f(x) e^{ax} e^{-itx} dx, \text{ a.e.}$$

i.e.

$$\widehat{(Mf)}_a = \mu_{(a)} \widehat{(f)}_a.$$

3) If $\overset{\circ}{I} \neq \emptyset$ then the function $\mu(z) = \mu_{(\text{Im } z)}(\text{Re } z)$ is holomorphic on $\overset{\circ}{\Omega}$.

Definition 2. Given $M \in \mathcal{M}$, if $\overset{\circ}{\Omega} \neq \emptyset$, we call symbol of M the function μ defined by

$$\mu(z) = \mu_{(\text{Im } z)}(\text{Re } z), \quad \forall z \in \overset{\circ}{\Omega}.$$

Moreover, if $a = -\ln \rho(S^{-1})$ or $a = \ln \rho(S)$, the symbol μ is defined for $z = x + ia$ by the same formula for almost all $x \in \mathbb{R}$.

We will say that $a \in \mathbb{R}$ verifies the property (P) if for every $M \in \mathcal{M}$ we have:

- (1) $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c(\mathbb{R})$
- (2) There exists $\mu_{(a)} \in L^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} (Mf)(x) e^{ax} e^{-itx} dx = \mu_{(a)}(t) \int_{\mathbb{R}} f(x) e^{ax} e^{-itx} dx, \quad a.e.$$

Denote by $\sigma(A)$ the spectrum of the operator A . First we have the following

Proposition 1. *We have*

$$\sigma(S) = \left\{ z \in \mathbb{C}, \frac{1}{\rho(S^{-1})} \leq |z| \leq \rho(S) \right\}$$

and the real a satisfies the property (P) if and only if $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$.

Theorem 4 implies that if $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$, then a satisfies the property (P).

Lemma 1. *If $a \in \mathbb{R}$ verifies the property (P), then $e^{a+ib} \in \sigma(S)$, for all $b \in \mathbb{R}$.*

Proof. Let $\alpha \in \mathbb{C}$ be such that $e^\alpha \notin \sigma(S)$. Then it is clear that $T = (S - e^\alpha I)^{-1}$ is a multiplier. Let $a \in \mathbb{R}$ verify the property (P). Then there exists $\nu_{(a)} \in L^\infty(\mathbb{R})$ such that

$$\widehat{(Tf)}_a = \nu_{(a)} \widehat{(f)}_a, \quad \forall f \in C_c(\mathbb{R}), \quad a.e.$$

Replacing f by $(S - e^\alpha I)g$, for $g \in C_c(\mathbb{R})$ we get

$$\widehat{(g)}_a(x) = \nu_{(a)}(x) \mathcal{F} \left([(S - e^\alpha I)g]_a \right)(x), \quad \forall g \in C_c(\mathbb{R}), \quad a.e.$$

and

$$\widehat{(g)}_a(x) = \nu_{(a)}(x) \widehat{g}_a(x) [e^{a-ix} - e^\alpha], \quad \forall g \in C_c(\mathbb{R}), \quad a.e.$$

Choosing a suitable $g \in C_c(\mathbb{R})$, we have

$$\nu_{(a)}(x) (e^{a-ix} - e^\alpha) = 1, \quad a.e.$$

Since $\nu_{(a)} \in L^\infty(\mathbb{R})$, we obtain that $\text{Re } \alpha \neq a$ and we conclude that

$$e^{a+ib} \in \sigma(S), \quad \forall b \in \mathbb{R}.$$

□

Taking into account Theorem 4 and Lemma 1, Proposition 1 follows directly.

3. SPECTRUM OF M_ϕ

In this section we characterize the spectrum of M_ϕ , for $\phi \in C_c(\mathbb{R})$ by using the results of the previous one. We recall that M_ϕ denotes the operator of convolution by $\phi \in C_c(\mathbb{R})$. We recall that Ω is the set

$$\Omega = \left\{ z \in \mathbb{C}, -\ln \rho(S^{-1}) \leq \operatorname{Im} z \leq \ln \rho(S) \right\}.$$

First notice that $0 \in \sigma(M_\phi)$. Indeed, suppose that M_ϕ is invertible. Then $M_\phi^{-1} \in \mathcal{M}$ and let μ be its symbol. For $a \in I$, we have

$$\mathcal{F}\left(M_\phi^{-1}(M_\phi f)_a\right) = \widehat{(f)}_a, \forall f \in C_c(\mathbb{R}).$$

Then, we get

$$\mu_{(a)}(x) \widehat{(\phi)}_a(x) \widehat{(f)}_a(x) = \widehat{(f)}_a(x), \forall f \in C_c(\mathbb{R}), \text{ a.e.}$$

and

$$\mu_{(a)}(x) \widehat{(\phi)}_a(x) = 1, \text{ a.e.}$$

Taking into account that $\mu_{(a)} \in L^\infty(\mathbb{R})$ and $\lim_{x \rightarrow +\infty} \widehat{(\phi)}_a(x) = 0$, we obtain a contradiction. Thus, we conclude that $0 \in \sigma(M_\phi)$.

Now, we establish the following

Lemma 2. *Let $\phi \in C_c(\mathbb{R})$. If $\lambda \in \sigma(M_\phi) \setminus \{0\}$ then there exists $\alpha \in \Omega$ such that*

$$\lambda = \hat{\phi}(\alpha).$$

Proof. Following [9], we have $\sigma(M_\phi) = \{\gamma(M_\phi), \gamma \in \widehat{\mathcal{M}}\}$. Let $\gamma \in \widehat{\mathcal{M}}$ be fixed such that $\lambda = \gamma(M_\phi)$. We have

$$M_\phi f = \int_{\mathbb{R}} S_x(f) \phi(x) dx$$

but we cannot deduce that

$$\gamma(M_\phi) = \int_{\mathbb{R}} \gamma(S_x) \phi(x) dx,$$

since we do not have the convergence of the Bochner integral $\int_{\mathbb{R}} S_x \phi(x) dx$ with respect to the operator norm. However, we claim that

$$\gamma(M_\psi) = \int_{\mathbb{R}} \frac{\gamma(M_\phi \circ S_x)}{\gamma(M_\phi)} \psi(x) dx, \forall \psi \in C_c(\mathbb{R}). \quad (3.1)$$

Consider the application

$$\eta : C_c(\mathbb{R}) \ni \psi \longrightarrow \eta(\psi) = \gamma(M_\psi)$$

which is a continuous linear form on $C_c(\mathbb{R})$. Here $C_c(\mathbb{R})$ is equipped by the topology given by the inductive limit of $C_K(\mathbb{R})$, K being a compact subset of \mathbb{R} . Indeed, if K is a compact subset of \mathbb{R} and if $(\psi_n)_{n \in \mathbb{N}} \subset C_K(\mathbb{R})$ is a sequence uniformly convergent to $\psi \in C_K(\mathbb{R})$, for every $g \in L^2_\omega(\mathbb{R})$, we have

$$\|M_{\psi_n} g - M_\psi g\| \leq \int_K \|\psi_n - \psi\|_\infty \sup_{y \in K} \|S_y\| \|g\| dy$$

and so

$$\lim_{n \rightarrow \infty} \|M_{\psi_n} - M_\psi\| = 0.$$

We deduce that the application

$$C_c(\mathbb{R}) \ni \psi \longrightarrow M_\psi \in \mathcal{A}$$

is sequentially continuous and so it is continuous from $C_c(\mathbb{R})$ to \mathcal{A} . It follows that η is a continuous linear form on $C_c(\mathbb{R})$ and there exists a measure m (see [2], Chapter 3) such that

$$\eta(\psi) = \int_{\mathbb{R}} \psi(x) dm(x), \quad \forall \psi \in C_c(\mathbb{R})$$

and hence

$$\gamma(M_\psi) = \int \psi(x) dm(x).$$

This implies that for every $\psi \in C_c(\mathbb{R})$ we have

$$\begin{aligned} \gamma(M_\psi \circ M_\phi) &= \int_{\mathbb{R}} (\psi * \phi)(t) dm(t) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(x) \phi(t-x) dx \right) dm(t). \end{aligned}$$

Using Fubini theorem, we get

$$\begin{aligned} \gamma(M_\psi \circ M_\phi) &= \gamma(M_{\psi \circ \phi}) = \int \psi(x) \left(\int_{\mathbb{R}} \phi(t-x) dm(t) \right) dx \\ &= \int_{\mathbb{R}} \psi(x) \gamma(S_x \circ M_\phi) dx \end{aligned}$$

and this yields the claim (3.1). Consequently, we conclude that

$$\gamma(M_\psi) = \int_{\mathbb{R}} \psi(x) \gamma(S_x) dx, \quad \forall \psi \in C_c(\mathbb{R}). \quad (3.2)$$

Since $\gamma \in \widehat{\mathcal{M}}$, it is clear that $\gamma(S_x) \in \sigma(S_x)$ (see [9]). Set

$$\theta_\gamma(x) = \gamma(S_x) = \frac{\gamma(M_\phi \circ S_x)}{\gamma(M_\phi)}, \quad \forall x \in \mathbb{R}.$$

The application

$$x \longrightarrow S_x \circ M_\phi = M_{S_x(\phi)}$$

is continuous from \mathbb{R} into \mathcal{A} and we deduce that θ_γ is a continuous morphism from \mathbb{R} to \mathbb{C} . The function θ_γ verifies

$$\theta_\gamma(x+y) = \theta_\gamma(x)\theta_\gamma(y), \quad \forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}.$$

Therefore there exists $\alpha \in \mathbb{C}$ such that

$$\theta_\gamma(x) = e^{-i\alpha x}, \quad \forall x \in \mathbb{R}.$$

Applying (3.1) we get

$$\gamma(M_\phi) = \int_{\mathbb{R}} \phi(x) e^{-i\alpha x} dx = \hat{\phi}(\alpha).$$

Since $e^{-i\alpha} \in \sigma(S)$, an application of Proposition 1 yields

$$\frac{1}{\rho(S^{-1})} \leq e^{\text{Im } \alpha} \leq \rho(S)$$

which implies $\alpha \in \Omega$. This completes the proof of Lemma 2. \square

Proof of Theorem 1. We will prove that $\hat{\phi}(\Omega) \subset \sigma(M_\phi)$. Let $\lambda = \hat{\phi}(\alpha)$, with $\alpha \in \Omega$ and $a = \text{Im}(\alpha) \in [-\ln \rho(S^{-1}), \ln \rho(S)]$. Suppose that $\lambda \notin \sigma(M_\phi)$. Then, the operator $M_\phi - \lambda I$ is invertible and $(M_\phi - \lambda I)^{-1}$ is a multiplier on $L_\omega^2(\mathbb{R})$. Since $a \in [-\ln(\rho(S^{-1})), \ln(\rho(S))]$ there exists $\nu_{(a)} \in L^\infty(\mathbb{R})$ such that

$$\mathcal{F}\left(\left((M_\phi - \lambda I)^{-1}f\right)_a\right)(x) = \nu_{(a)}(x)\widehat{(f)_a}(x), \forall f \in C_c(\mathbb{R}), \text{ a.e.}$$

Replacing f by $(M_\phi - \lambda I)g$ which is obviously also in $C_c(\mathbb{R})$ if $g \in C_c(\mathbb{R})$, we get

$$\begin{aligned} \widehat{(g)_a}(x) &= \nu_{(a)}(x)\mathcal{F}\left(\left((M_\phi - \lambda I)g\right)_a\right)(x) \\ &= \nu_{(a)}(x)\left(\widehat{(\phi)_a}(x) - \lambda\right)\widehat{(g)_a}(x), \forall g \in C_c(\mathbb{R}), \text{ a.e.} \end{aligned}$$

Choosing a suitable $g \in C_c(\mathbb{R})$, we conclude that

$$1 = \nu_{(a)}(x)[\hat{\phi}(x + ia) - \lambda], \text{ a.e.}$$

Let $\text{Re } \alpha = x_0$ and let V be a neighborhood of x_0 such that $|\nu_{(a)}(x)| \leq m$, for all $x \in V$. Then, we have

$$|\hat{\phi}(x + ia) - \lambda| \geq \frac{1}{m},$$

which yields a contradiction for x suitably close to x_0 , because $\hat{\phi}(\alpha) = \hat{\phi}(x + ia) = \lambda$ and $\hat{\phi}$ is continuous. Thus $\lambda \in \sigma(M_\phi)$ and so $\hat{\phi}(\Omega) \subset \sigma(M_\phi)$. Lemma 2 implies $\sigma(M_\phi) \setminus \{0\} \subset \hat{\phi}(\Omega)$ and the proof of the theorem is complete. \square

4. GENERALIZATION

In this section we investigate the spectrum of a general multiplier on $L_\omega^2(\mathbb{R})$. We recall that the symbol μ of a multiplier M is in $\mathcal{H}^\infty(\mathring{\Omega})$ and it is essentially bounded on the boundary $\delta(\Omega)$ of Ω (see Theorem 4). Introduce the set

$$\mu(\Omega) = \{\mu(z) : z \in \Omega \setminus \mathcal{N}\},$$

\mathcal{N} being a set of measure zero.

Proof of Theorem 3 Assume that $\lambda \notin \sigma(M)$. Then $(M - \lambda I)^{-1}$ is a multiplier and for every $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$, we introduce the symbol $\nu_{(a)} \in L^\infty(\mathbb{R})$ such that

$$\mathcal{F}\left(\left((M - \lambda I)^{-1}f\right)_a\right)(x) = \nu_{(a)}(x)\widehat{(f)_a}(x), \forall f \in C_c(\mathbb{R}), \text{ a.e.}$$

Replacing f by $(M - \lambda I)g$ where $g \in C_c(\mathbb{R})$, we get

$$\begin{aligned} \widehat{(g)_a}(x) &= \nu_{(a)}(x)\mathcal{F}\left(\left((M - \lambda I)g\right)_a\right)(x) \\ &= \nu_{(a)}(x)(\mu_{(a)}(x) - \lambda)\widehat{(g)_a}(x), \text{ a.e.} \end{aligned}$$

Choosing a suitable $g \in C_c(\mathbb{R})$, we get

$$1 = \nu_{(a)}(x)(\mu_{(a)}(x) - \lambda), \text{ a.e.}$$

Given $x \in \mathbb{R}$, satisfying

$$|\mu_{(a)}(x)| \leq \|\mu_{(a)}\|_\infty \text{ and } |\nu_{(a)}(x)| \leq \|\nu_{(a)}\|_\infty, \quad (4.1)$$

it is clear that if $\lambda = \mu_{(a)}(x) = \mu(x + ia)$ we obtain a contradiction. For every x for which (4.1) holds and for $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$ we deduce that

$$\mu_{(a)}(x) = \mu(x + ia) \in \sigma(M).$$

According to Theorem 4, ν and μ are holomorphic on $\overset{\circ}{\Omega}$, and so $\nu_{(a)}$ and $\mu_{(a)}$ are continuous on \mathbb{R} , for every $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$. Consequently, (4.1) holds for almost every x and the proof is complete. \square

Notice that ν may not be continuous on the boundary of Ω . Indeed, for $\omega = 1$, let $h \in L^\infty(\mathbb{R})$ be a function which is not continuous on \mathbb{R} . Define the operator H on $L^2(\mathbb{R})$ by the formula

$$Hf = \mathcal{F}^{-1}(h\hat{f}).$$

Then H is a multiplier on $L^2(\mathbb{R})$, but its symbol h is not continuous on $\Omega = \mathbb{R}$.

Now we present one example. Suppose that χ is a complex Borel measure such that

$$\int_{-\infty}^{+\infty} \|S_t\| d|\chi|(t) < +\infty. \quad (4.2)$$

Then the operator M_χ defined by the formula

$$M_\chi(f) = \int_{-\infty}^{+\infty} S_t(f) d\chi(t), \quad \forall f \in L^2_\omega(\mathbb{R}),$$

is obviously a multiplier on $L^2_\omega(\mathbb{R})$. The condition (4.2) implies that

$$\int_{\mathbb{R}} e^{ax} d|\chi|(x) < +\infty, \quad \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)],$$

so the integral

$$\int_{\mathbb{R}} e^{-i\alpha x} d\chi(x)$$

converges for all $\alpha \in \Omega$. Clearly, the symbol of M_χ , defined in Section 2, becomes

$$\widehat{\chi}(\alpha) = \int_{\mathbb{R}} e^{-i\alpha x} d\chi(x).$$

The results of the previous sections imply

$$\overline{\widehat{\chi}(\sigma(S))} \subset \sigma(M_\chi) \quad (4.3)$$

The inclusion (4.3) has been established by other methods in [1] and [4]. We see that the inclusion (4.3) may also be obtained by the tools of our paper.

Now we give an illustration of Theorem 3. Let $T_t = e^{tM}$, where $M \in \mathcal{M}$. It is clear that the group $(T_t)_{t \in \mathbb{R}}$ is included in \mathcal{M} . For $t \in \mathbb{R}$, $a \in I$ and $f \in C_c(\mathbb{R})$, we have

$$(\widehat{e^{tM}f})_a(y) = \sum_{k \geq 0} \frac{t^k (\widehat{M^k f})_a(y)}{k!}$$

$$= \sum_{k \geq 0} \frac{t^k}{k!} (\mu_{(a)}(y))^k \widehat{(f)_a}(y) = e^{t\mu_{(a)}(y)} \widehat{(f)_a}(y) \quad a.e.,$$

where $\mu_{(a)}$ is the symbol of M . Then the symbol of T_t is the function $z \rightarrow e^{t\mu(z)}$. Now suppose that $M = S$. Then we have $\mu_{(a)}(y) = e^{-(iy-a)}$ and $\mu(z) = e^{-iz}$, $\forall z \in \Omega$. Following Theorem 3, we obtain $\{e^{te^{-iz}}, z \in \Omega\} \subset \sigma(T_t)$ and

$$e^{t\sigma(S)} \subset \sigma(e^{tS}), \quad \forall t \in \mathbb{R}. \quad (4.4)$$

The inclusion (4.4) follows also from a classical result about the spectrum of a semi-group (see [4], [3]).

5. SPECTRUM OF $M \in \mathcal{A}$

Now we examine the spectrum of multipliers forming a larger class than those of operators M_ϕ . We will show that if $M \in \mathcal{A}$ and ν is the symbol of M , then ν is a continuous function on Ω and we have

$$\sigma(M) = \overline{\nu(\Omega)}.$$

Proof of Theorem 2. First, we show that ν is continuous on Ω . Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence of $C_c(\mathbb{R})$ such that $\lim_{n \rightarrow +\infty} \|M_{\phi_n} - M\| = 0$. The construction of $\nu_{(a)}$ for $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$, given in [6], defines $\nu_{(a)}$ as the limit of $\widehat{(\psi_n)_a}$ with respect to the weak topology of $L^2(\mathbb{R})$, where $(\psi_n)_{n \in \mathbb{N}}$ is a special sequence in $C_c(\mathbb{R})$ such that $(M_{\psi_n})_{n \in \mathbb{N}}$ converges to M with respect to the strong operator topology. Using the same argument as in [6], we get

$$\nu_{(a)}(x) = \lim_{n \rightarrow \infty} \widehat{(\phi_n)_a}(x)$$

with respect to the weak topology of $L^2(\mathbb{R})$. For fixed $a \in [-\ln \rho(S^{-1}), \ln \rho(S)]$ and $x \in \mathbb{R}$ there exists $\gamma \in \widehat{\mathcal{M}}$ (see section 2) such that

$$\widehat{(\phi)_a}(x) = \gamma(M_\phi), \quad \forall \phi \in C_c(\mathbb{R}).$$

This implies that for every $k \in \mathbb{N}$, $n \in \mathbb{N}$, we have

$$|\widehat{(\phi_n)_a}(x) - \widehat{(\phi_k)_a}(x)| = |\gamma(M_{\phi_n}) - \gamma(M_{\phi_k})| \leq \|M_{\phi_n} - M_{\phi_k}\|. \quad (5.1)$$

Since $(M_{\phi_n})_{n \in \mathbb{N}}$ converges to M in \mathcal{A} , we conclude that $\widehat{(\phi_n)_a}$ converges uniformly on \mathbb{R} to a continuous function $\mu_{(a)}$ and that there exists a constant C such that

$$|\widehat{(\phi_n)_a}(x)| \leq C, \quad \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)], \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}.$$

Moreover, we have $\lim_{x \rightarrow +\infty} \mu_{(a)}(x) = 0$. We obtain that $\widehat{(\phi_n)_a}$ converges to $\mu_{(a)}$ with respect to the weak topology of $L^2(\mathbb{R})$. Thus we can identify the symbol of M with the function μ defined by

$$\mu(x + ia) = \mu_{(a)}(x), \quad \forall a \in [-\ln \rho(S^{-1}), \ln \rho(S)], \quad \forall x \in \mathbb{R}.$$

Taking into account (5.1), it is clear that μ is continuous on Ω .

Given $\lambda \in \sigma(M) \setminus \{0\}$, there exists $\gamma \in \widehat{\mathcal{M}}$ such that $\lambda = \gamma(M)$. Then we have

$$\lambda = \lim_{n \rightarrow \infty} \gamma(M_{\phi_n}).$$

According to the results in section 2, there exists $\alpha \in \Omega$ such that

$$\gamma(M_{\phi_n}) = \widehat{\phi_n}(\alpha), \quad \forall n \in \mathbb{N}.$$

Consequently, we have

$$\lambda = \lim_{n \rightarrow \infty} \gamma(M_{\phi_n}) = \lim_{n \rightarrow \infty} \widehat{\phi_n}(\alpha) = \mu(\alpha).$$

Since μ is equal to the symbol ν of M , we conclude that

$$\sigma(M) \setminus \{0\} \subset \nu(\Omega).$$

Taking into account the result of the previous section, the proof is complete. \square

6. SPECTRUM OF MULTIPLIERS ON $L^2_\omega(\mathbb{R}^k)$

Theorems 1-3 can be generalized for multipliers on $L^2_\omega(\mathbb{R}^k)$, $k > 1$, where ω is a weight on \mathbb{R}^k satisfying the following condition

$$\omega = \omega_1 \times \dots \times \omega_k,$$

where $\omega_1, \dots, \omega_k$ are weights on \mathbb{R} . From now on, ω denotes a weight on \mathbb{R}^k having the above form.

Given $\phi \in C_c(\mathbb{R}^k)$, the Fourier transform $\widehat{\phi}$ is defined on \mathbb{C}^k . Set

$$e_m = (e_{m,1}, \dots, e_{m,k}),$$

where $e_{m,i} = 0$, if $m \neq i$ and $e_{m,m} = 1$. For $m = 1, \dots, k$, let S_m be the translation by e_m defined on $L^2_\omega(\mathbb{R}^k)$. Introduce

$$\mathcal{U} = \{z = (z_1, \dots, z_k) \in \mathbb{C}^k, \operatorname{Im} z_i \in [-\ln \rho(S_i^{-1}), \ln \rho(S_i)], \text{ for } i = 1, \dots, k\}.$$

We have (see [7] and [8]) the following representation theorem for multipliers on $L^2_\omega(\mathbb{R}^k)$.

Theorem 5. *Let M be a multiplier on $L^2_\omega(\mathbb{R}^k)$. Then there exists $\nu \in L^\infty(\mathcal{U})$ such that*

$$\int_{\mathbb{R}^k} (Mf)(x) e^{-i\langle x, z \rangle} dx = \nu(z) \int_{\mathbb{R}^k} f(x) e^{-i\langle x, z \rangle} dx,$$

for all $z \in \overset{\circ}{\mathcal{U}}$ and for almost every $z \in \delta(\mathcal{U})$.

Given a multiplier M on $L^2_\omega(\mathbb{R}^k)$, we call symbol of M the function ν introduced in the previous theorem. Moreover, in [8] the following result was established.

Proposition 2. *We have $z = (z_1, \dots, z_k) \in \mathcal{U}$ if and only if*

$$e^{-iz_m} \in \sigma(S_m), \text{ for } m = 1, \dots, k.$$

The set \mathcal{U} is related to the joint spectrum of S_1, \dots, S_k . Let \mathfrak{A} be a commutative Banach algebra with unit I . We recall the following

Definition 3. *The joint spectrum $\sigma_s(A_1, \dots, A_k)$ of the operators $A_1, \dots, A_k \in \mathfrak{A}$ is the set $\{(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k, \sum_{m=1}^k (A_m - \alpha_m I) J_m \text{ is not invertible in } \mathfrak{A}, \forall (J_1, \dots, J_k) \in \mathfrak{A}^k\}$.*

In general $\sigma_s(A_1, \dots, A_k) \neq \sigma(A_1) \times \dots \times \sigma(A_k)$ and the determination of $\sigma_s(A_1, \dots, A_k)$ is a quite difficult problem. However, in the spaces $L^2_\omega(\mathbb{R}^k)$, we have the equality

$$\sigma_s(S_1, \dots, S_k) = \sigma(S_1) \times \dots \times \sigma(S_k)$$

(see ([8])). Using Theorem 5, Proposition 2 and the arguments in Section 3, we obtain the following

Theorem 6. For $\phi \in C_c(\mathbb{R}^k)$, we have

$$\sigma(M_\phi) = \overline{\hat{\phi}(\mathcal{U})}.$$

Moreover, repeating the arguments in Sections 4-5, we obtain

Theorem 7. Let $M \in \mathcal{A}$ and let ν be the symbol of M . Then ν is a continuous function on \mathcal{U} and we have

$$\sigma(M) = \overline{\nu(\mathcal{U})}.$$

Let $M \in \mathcal{M}$ and let ν be its symbol. Then

$$\overline{\nu(\mathcal{U})} \subset \sigma(M).$$

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