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BRUNET-DERRIDA BEHAVIOR OF BRANCHING-SELECTION PARTICLE SYSTEMS ON THE LINE

JEAN BÉRARD, JEAN-BAPTISTE GOUÉRÉ

ABSTRACT. The term Brunet-Derrida behavior refers to the 1997 paper by E. Brunet and B. Derrida "Shift in the velocity of a front due to a cutoff" (see the reference within the paper), where it is shown, based on numerical simulations and heuristic arguments, that a certain branching-selection particle system on the line exhibits the following behavior: as N goes to infinity, the asymptotic velocity of the system with N particles converges to a limiting value at the surprisingly slow rate $(\log N)^{-2}$. In this paper, we consider a class of branching-selection particle systems on \mathbb{R} with N particles, defined through iterated branching-selection steps of the following type. During a branching step, each particle is replaced by two new particles, whose positions are shifted from that of the original particle by independently performing two random walk steps, according to some distribution p . During the selection step that follows, only the N rightmost particles are kept among the $2N$ particles obtained at the branching step, to form a new population of N particles. Under generic assumptions on p , it is shown that Brunet-Derrida behavior holds for the corresponding particle system. The proofs are based on ideas and results by R. Pemantle, and by N. Gantert, Y. Hu and Z. Shi, and rely on a comparison of the particle system with a family of N independent branching random walks killed below a linear space-time barrier. The results presented here both improve and generalize upon previous work by the first author of this paper, which was completed just before the results by Gantert, Hu and Shi became publicly available.

1. INTRODUCTION

In [5, 6], E. Brunet and B. Derrida studied a branching-selection particle system on \mathbb{Z} enjoying the following property: as the number N of particles in the system goes to infinity, the asymptotic speed of the population of particles in the system converges to its limiting value at an unexpectedly slow rate, of order $(\log N)^{-2}$. This behavior was established both by direct numerical simulation of the particle system, and by mathematically non-rigorous arguments of the following type: having a finite population of N particles instead of an infinite number of particles should be more or less equivalent, as far as the asymptotic speed is concerned, to introducing a cutoff value of $1/N$, in the deterministic equations that govern the time-evolution of the distribution of particles in the infinite-population limit. In turn, these equations can be viewed as discrete versions of the well-known F-KPP equations, and the initial problem is thus related to that of assessing the effect of introducing a small cutoff in F-KPP equations, upon the speed of the travelling wave solutions. In turn, this problem was studied by heuristic arguments and computer simulations (see [5, 6]), and rigorous mathematical results for this last problem have recently

been obtained, see [1, 2, 9]. Another approach is based on adding a small white-noise term with scale parameter $\sigma^2 = 1/N$ in the Fisher-KPP equation, see [7], and rigorous results have been derived for this model too, see [8, 13]. However, to our knowledge, no rigorous results dealing directly with a branching-selection particle system such as the one originally studied by Brunet and Derrida, are available (except the preliminary work [3] by the first author of this paper, the results of which are extended here).

The branching-selection particle systems we consider are similar (but not exactly identical) to the one considered by Brunet and Derrida in [5, 6]. To be specific, we consider a particle system with N particles on \mathbb{R} defined through the repeated application of branching and selection steps defined as follows:

- Branching: each of the N particles is replaced by two new particles, whose positions are shifted from that of the original particle by independently performing two random walk steps according to a given distribution p ;
- Selection: only the N rightmost particles are kept among the $2N$ obtained at the branching step, to form the new population of N particles.

Our assumptions on the random walk distribution p are listed below. For the most part, they correspond to those of the paper [11] by Gantert, Hu and Shi, with the simplifying feature that we consider only deterministic binary branching. Introduce the logarithmic moment generating function of p defined by

$$\Lambda(t) := \log \int \exp(tx) dp(x).$$

Here are the assumptions on p :

- (A1) The support of p is contained in $\mathbb{R}_+ = [0, +\infty[$;
- (A2) There exists $t > 0$ such that $\Lambda(t) < +\infty$;
- (A3) There exists $t^* \in]0, +\infty[$ such that $t^* \Lambda'(t^*) - \Lambda(t^*) = \log 2$.

Under these assumptions, let

$$\chi(p) := \frac{\pi^2}{2} t^* \Lambda''(t^*).$$

Then (see [11]) $0 < \chi(p) < +\infty$. A simple case for which these assumptions hold is the Bernoulli case $p = \alpha \delta_{+1} + (1 - \alpha) \delta_0$, when $\alpha \in]0, 1/2[$. This is precisely the case investigated in [3].

In Section 3, it is proved that, after a large number of iterated branching-selection steps, the displacement of the whole population of N particles is ballistic, with deterministic asymptotic speed $v_N(p)$, and that, as N goes to infinity, $v_N(p)$ increases to a limit $v_\infty(p)$, which turns out to be equal to $\Lambda'(t^*)$, and thus positive and finite. The main result concerning the branching-selection particle system is the following theorem:

Theorem 1. *Assume that (A1)-(A2)-(A3) hold. Then, as N goes to infinity,*

$$(1) \quad v_\infty(p) - v_N(p) \sim \chi(p) (\log N)^{-2}.$$

In [3], only the $(\log N)^{-2}$ order of magnitude of the difference $v_\infty(p) - v_N(p)$ was established. The recently released results by Gantert, Hu and Shi [11] on the

survival probability of a branching random walk killed below a linear space-time barrier allowed us to prove the much more precise result above.

The rest of the paper is organized as follows. In Section 2, we provide the precise notations and definitions that are needed in the sequel. Section 3 contains a discussion of various elementary properties of the model we consider. Section 4 collects the results from [11] that are used in the sequel. Section 5 contains the proof of the upper bound part of Theorem 1, while Section 6 contains the proof of the lower bound part. Section 7 discusses the Bernoulli case when Assumption (A3) is not met.

2. NOTATIONS AND DEFINITIONS

It is convenient to represent finite populations of particles by finite counting measures on \mathbb{R} . For all $N \geq 1$, let \mathcal{C}_N denote the set of finite counting measures on \mathbb{R} with total mass equal to N , and \mathcal{C} the set of all finite counting measures on \mathbb{R} .

For $\nu \in \mathcal{C}$, the total mass of ν (i.e. the number of particles in the population it describes) is denoted by $M(\nu)$. We denote by $\max \nu$ and $\min \nu$ respectively the maximum and minimum of the (finite) support of μ . We also define the diameter $d(\nu) := \max \nu - \min \nu$. Given two positive measures μ, ν on \mathbb{R} , we use the notation \prec to denote the usual stochastic order: $\mu \prec \nu$ if and only if $\mu([x, +\infty[) \leq \nu([x, +\infty[)$ for all $x \in \mathbb{R}$. In particular, $\mu \prec \nu$ implies that $M(\mu) \leq M(\nu)$, and it is easily seen that, if $\mu = \sum_{i=1}^{M(\mu)} \delta_{x_i}$ and $\nu = \sum_{i=1}^{M(\nu)} \delta_{y_i}$, with $x_1 \geq \dots \geq x_{M(\mu)}$ and $y_1 \geq \dots \geq y_{M(\nu)}$, $\mu \prec \nu$ is equivalent to $M(\mu) \leq M(\nu)$ and $x_i \leq y_i$ for all $i \in \llbracket 1, M(\mu) \rrbracket$.

In the sequel, we use the notation $(X_n^N)_{n \geq 0}$ to denote a Markov chain on \mathcal{C}_N whose transition probabilities are given by the branching-selection mechanism with N particles, and which starts at a deterministic value $X_0^N \in \mathcal{C}_N$. We assume this Markov chain is defined on a reference probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$.

3. ELEMENTARY PROPERTIES OF THE MODEL

As a first quite elementary property, note that, from Assumption (A1), $\mathbb{E}(\max X_n^N) < +\infty$ for all $n \geq 0$, whatever the (deterministic) initial condition $X_0^N \in \mathcal{C}_N$.

3.1. Estimates on the diameter.

Proposition 1. *Let $u_N := \lceil \frac{\log N}{\log 2} \rceil + 1$. For all $N \geq 1$, all initial population $X_0^N \in \mathcal{C}_N$, and all $n \geq u_N$, $d(X_n^N)$ is stochastically dominated by u_N times the maximum of $2Nu_N$ i.i.d. random variables with distribution p .*

Proof. Consider $n \geq u_N$, and let $y = \max X_{n-u_N}^N$. Assume first that $\min X_k^N < y$ for all $n+1-u_N \leq k \leq n$. Since, due to (A1), all the random walk steps that are performed during branching steps are ≥ 0 , this implies that all the particles descended by branching from a particle located at y at time $n-u_N$, are preserved by the selection steps performed from $X_{n-u_N}^N$ to X_n^N . Since there are $2^{u_N} > N$ such particles at time n , this is a contradiction. As a consequence, we know that there must be an index $n+1-u_N \leq k \leq n$ such that $\min X_k^N \geq y$. Since, by (A1),

$t \mapsto \min X_t^N$ is non-decreasing, we deduce that $\min X_n^N \geq y$. Now, let K denote the maximum of the $2Nu_N$ branching random walk steps that are performed between time $n - u_N$ and time u_N . We see from the definition of y that $\max X_n^N \leq y + u_N K$. As a consequence, $d(X_n^N) = \max X_n^N - \min X_n^N \leq u_N K$. \square

The following corollary is then a straightforward consequence, in view of Assumption (A2).

Corollary 1. *For all $N \geq 1$ and all initial population $X_0^N \in \mathcal{C}_N$, $\lim_{n \rightarrow +\infty} n^{-1} d(X_n^N) = 0$, both with probability one and in $L^1(\mathbb{P})$.*

3.2. Monotonicity properties.

Proposition 2. *For all $1 \leq N_1 \leq N_2$, and $\mu \in \mathcal{C}_{N_1}$, $\nu \in \mathcal{C}_{N_2}$ such that $\mu \prec \nu$, there exists a coupling $(Z_n^{N_1}, T_n^{N_2})_{n \geq 0}$ between two versions of the branching-selection particle system, with N_1 and N_2 particles respectively, such that $Z_0^{N_1} := \mu$, $T_0^{N_2} := \nu$, and $Z_n^{N_1} \prec T_n^{N_2}$ for all $n \geq 0$.*

Proof. Consider an i.i.d. family $(\varepsilon_{n,i,j})_{n \geq 0, i \in \llbracket 1, N_2 \rrbracket, j=1,2}$ with common distribution p . For all $n \geq 0$, define the branching-selection particle system $(Z_n^{N_1})_{n \geq 0}$ by $Z_0^{N_1} := \mu$, selection steps consisting in keeping only the N_1 rightmost particles, and branching steps defined as follows. Starting from $Z_{n,k}^{N_1} := \sum_{i=1}^{N_1} \delta_{x_i}$, with $x_1 \geq \dots \geq x_{N_1}$, let the new population be $\sum_{(i,j) \in \llbracket 1, N_1 \rrbracket \times \{1,2\}} \delta_{x_i + \varepsilon_{n,i,j}}$. Then define $(T_n^{N_2})_{n \geq 0}$ by $T_0^{N_2} := \nu$, selection steps consisting in keeping only the N_2 rightmost particles, and the following branching steps. Starting from $T_{n,k}^{N_2} := \sum_{i=1}^{N_2} \delta_{y_i}$, with $y_1 \geq \dots \geq y_{N_2}$, let the new population be $\sum_{(i,j) \in \llbracket 1, N_2 \rrbracket \times \{1,2\}} \delta_{y_i + \varepsilon_{n,i,j}}$. Now assume that, for a given $n \geq 0$, $Z_n^{N_1} \prec T_n^{N_2}$. With the above notations, we have that $x_i \leq y_i$, whence $x_i + \varepsilon_{n,i,j} \leq y_i + \varepsilon_{n,i,j}$, for all $1 \leq i \leq N_1$ and $j = 1, 2$. It follows that a sub-population of size $2N_1$ of the population obtained from $T_n^{N_2}$ by branching, dominates (in the sense of \prec) the whole population, of size $2N_1$, obtained by branching from $Z_n^{N_1}$. It is then quite easy to deduce that $Z_{n+1}^{N_1} \prec T_{n+1}^{N_2}$. The result follows by induction since we assume that $\mu \prec \nu$. \square

Proposition 3. *There exists $v_N(p) \in \mathbb{R}_+$ such that, with probability one, and in $L^1(\mathbb{P})$, whatever the initial population $X_0^N \in \mathcal{C}_N$,*

$$\lim_{n \rightarrow +\infty} n^{-1} \min X_n^N = \lim_{n \rightarrow +\infty} n^{-1} \max X_n^N = v_N(p).$$

Proof. Note that, in view of Corollary 1, if the two limits in the above statement exist, they must be equal. Moreover, owing to the translation invariance of our particle system, and to Proposition 2, we see that it is enough to prove the result when $X_0^N = N\delta_0$.

Consider an i.i.d. family $(\varepsilon_{k,i,j})_{k \geq 0, i \in \llbracket 1, N \rrbracket, j=1,2}$ with common distribution p . For all $n \geq 0$, denote by $(W_{n,k}^N)_{k \geq 0}$ a branching-selection system starting at $W_{n,0}^N := N\delta_0$, whose selection steps consist in keeping only the N rightmost particles, and whose branching steps are specified as follows. Starting from $W_{n,k}^N := \sum_{i=1}^N \delta_{x_i}$, with $x_1 \geq \dots \geq x_N$, let the new population be $\sum_{(i,j) \in \llbracket 1, N \rrbracket \times \{1,2\}} \delta_{x_i + \varepsilon_{n+k,i,j}}$. Observe

that $(W_{0,n}^N)_{n \geq 0}$ has the same distribution as $(X_n^N)_{n \geq 0}$. Then, observe that for all $n, m \geq 0$, $\max W_{n+m,0}^N \leq \max W_{0,n}^N + \max W_{n,m}^N$, by an argument similar to the proof of Proposition 2. Moreover, for each $d \geq 1$, the random variables $(W_{dn,d}^N)_{n \geq 0}$ form an i.i.d. family, and for any given k , the distribution of $W_{n,k}^N$ clearly does not depend on n . It is routine to check that the other hypotheses of Kingman's subadditive ergodic theorem hold (see e.g. [10]), so we deduce that $\lim_{n \rightarrow +\infty} n^{-1} \max X_n^N$ exists both a.s. and in $L^1(\mathbb{P})$. A similar argument, using superadditivity instead of subadditivity, holds for $\min X_n^N$. \square

Proposition 4. *The sequence $(v_N(p))_{N \geq 1}$ is non-decreasing.*

Proof. Consequence of the fact that, when $N_1 \leq N_2$, $N_1 \delta_0 \prec N_2 \delta_0$, and of the monotonic coupling property given in Proposition 2. \square

We can deduce from the above proposition that there exists $v_\infty(p) \in \mathbb{R}_+$ such that $\lim_{N \rightarrow +\infty} v_N(p) = v_\infty(p)$. A consequence of the proof of Theorem 1 below is that $v_\infty(p)$ is in fact equal to the number $v(p) := \Lambda'(t^*)$, which is > 0 from our assumptions on p .

3.3. Coupling with a family of N branching random walks. Let $\text{BRW}_1, \dots, \text{BRW}_N$ denote N independent branching random walks, each with value zero at the root, deterministic binary branching, and i.i.d. displacements with common distribution p along each edge. For $1 \leq i \leq N$, and $n \geq 0$, let $\text{BRW}_i(n)$ denote the set of vertices of BRW_i located at depth n in the tree, and let $\mathcal{T}_n^N := \text{BRW}_1(n) \cup \dots \cup \text{BRW}_N(n)$. For every n , fix an a priori (i.e. depending only on the tree structure, not on the random walk values) total order on \mathcal{T}_n^N . We now define by induction a sequence $(G_n^N)_{n \geq 0}$ of subsets such that, for each $n \geq 0$, G_n^N is a random subset of \mathcal{T}_n^N with exactly N elements. First, let us set $G_0^N := \mathcal{T}_0^N$. Then, given $n \geq 0$ and G_n^N , let H_n^N denote the subset of \mathcal{T}_{n+1} formed by the children of the vertices in G_n^N . Then, define G_{n+1}^N as the subset of H_n^N formed by the N vertices that are associated with the largest values of the underlying random walk (breaking ties by using the a priori order on \mathcal{T}_n^N). It is now quite obvious that the sequence formed by the (random) empirical distributions attached to the sequence of subsets G_0^N, \dots, G_n^N , has exactly the same distribution as X_0^N, \dots, X_n^N , when started from $X_0^N := N\delta_0$. Thus, we can take for our reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the one on which $\text{BRW}_1, \dots, \text{BRW}_N$ are defined, and let X_n^N be equal to the empirical distribution associated with the subset G_n^N , and so obtain a coupling between $(X_n^N)_{n \geq 0}$ (with $X_n^N = N\delta_0$) and the N branching random walks $\text{BRW}_1, \dots, \text{BRW}_N$.

4. RESULTS ON THE BRANCHING RANDOM WALK KILLED BELOW A LINEAR SPACE-TIME BARRIER

Throughout this section and the following, we use the notation BRW to denote a generic branching random walk with value zero at the root, deterministic binary branching, and i.i.d. displacements with common distribution p along each edge. We use the notation $\Phi(u)$ to denote the value of the random walk associated with

a vertex $u \in \text{BRW}$. Let us start with the following definition, adapted from [14]. Given $v \in \mathbb{R}$ and $m \geq 1$, we say that a vertex $u \in \text{BRW}$ is (m, v) -good if there exists an infinite descending path $u =: u_0, u_1, \dots, u_m$ such that $\Phi(u_i) - \Phi(u_0) \geq vi$ for all $i \in \llbracket 0, m \rrbracket$. Similarly, we say that u is (∞, v) -good if there exists a descending path $u =: u_0, u_1, \dots$ such that $\Phi(u_i) - \Phi(u_0) \geq vi$ for all $i \in \llbracket 0, +\infty \rrbracket$.

With this terminology, the main result in [11] can be stated as follows, remembering that $\chi(p) := \frac{\pi^2}{2} t^* \Lambda''(t^*)$.

Theorem 2. (Theorem 1.2 in [11]) *Let $\rho(\infty, \epsilon)$ denote the probability that the root of BRW is $(\infty, v(p) - \epsilon)$ -good. Then, as ϵ goes to zero,*

$$\rho(\infty, \epsilon) = \exp\left(-\left[\frac{\chi(p) + o(1)}{\epsilon}\right]^{1/2}\right).$$

We shall need a result which, although not stated explicitly in [11], appears there as an intermediate step in the proof of the upper bound part of Theorem 1.2.

Theorem 3. (Proof of the upper bound part of Theorem 1.2 in [11]) *Let $\rho(m, \epsilon)$ denote the probability that the root of BRW is $(m, v(p) - \epsilon)$ -good. For any $0 < \beta < \chi(p)$, there exists $\theta > 0$ such that, for all large m ,*

$$\rho(m, \epsilon) \leq \exp\left(-\left[\frac{\chi(p) - \beta}{\epsilon}\right]^{1/2}\right), \text{ with } \epsilon := \theta/m^{2/3}.$$

5. THE UPPER BOUND

The arguments used here in the proof of the upper bound, combine ideas from the paper [14] by Pemantle, which deals with the closely related question of obtaining complexity bounds for algorithms that seek near optimal paths in branching random walks, and the estimate on $\rho(m, \epsilon)$ from the paper [11], by Gantert, Hu and Shi. In fact, the proof given below is basically a rewriting of the proof of the lower complexity bound in [14] in the special case of algorithms that do not jump, with the following slight differences: we are dealing with N independent branching random walks being explored in parallel, rather than a single branching random walk; we consider possibly unbounded random walk steps; that we use the estimate in [11] instead of the cruder one derived in [14].

We start with an elementary result adapted from [14].

Lemma 1. (Adapted from Lemma 5.2 in [14].) *Let $0 < v_1 < v_2$, $n \geq 1$, $m \in \llbracket 1, n \rrbracket$, and let $0 =: x_0, \dots, x_n$ be a sequence of real numbers such that $0 \leq x_{i+1} - x_i \leq K$ for all $i \in \llbracket 0, n-1 \rrbracket$. Let $I := \{i \in \llbracket 0, n-m \rrbracket; x_j - x_i \geq v_1(j-i) \text{ for all } j \in \llbracket i, i+m \rrbracket\}$. If $x_n \geq v_2 n$, then $\#I \geq \frac{v_2 - v_1}{K - v_1} \frac{n}{m} - K/(K - v_1)$.*

Since Lemma 1 admits so short a proof, we give it below for the sake of completeness, even though it is quite similar to that in [14].

Proof of Lemma 1. (Adapted from [14].) Consider a sequence $0 =: x_0, \dots, x_n$ as in the statement of the lemma. Let then $\tau_0 := 0$, and, given $\tau_i \leq n$, define inductively $\tau_{i+1} := \inf\{j \in \llbracket \tau_i + 1, n \rrbracket; x_j < x_{\tau_i} + v_1(j - \tau_i) \text{ or } j = \tau_i + m\}$, with the convention

that $\inf \emptyset = n + 1$. Now color the vertices x_k , $k \in \llbracket 0, n - 1 \rrbracket$ according to the following rules: if $x_{\tau_{i+1}} \geq x_{\tau_i} + v_1(\tau_{i+1} - \tau_i)$ and $\tau_{i+1} \leq n$, then $x_{\tau_i}, \dots, x_{\tau_{i+1}-1}$ are colored red. Note that this yields a segment of m red vertices, and that τ_i then belongs to I . Otherwise, $x_{\tau_i}, \dots, x_{\tau_{i+1}-1}$ are colored blue. Let V_{red} (resp. V_{blue}) denote the number of red (resp. blue) vertices in x_0, \dots, x_{n-1} . Then decompose the value of x_n into the contributions of the steps $x_{k+1} - x_k$ performed from red and blue vertices respectively. On the one hand, the contribution of red vertices is $\leq KV_{red}$. On the other hand, the contribution of blue vertices is $\leq V_{blue} \times v_1 + Km$, where the m is added to take into account a possible last segment colored in blue only because it has reached the index n . Writing that $n = V_{red} + V_{blue}$, we deduce that $v_2 n \leq KV_{red} + v_1(n - V_{red}) + Km$, so that $V_{red} \geq \frac{v_2 - v_1}{K - v_1} n - Km / (K - v_1)$. Then use the fact that at least V_{red}/m vertices belong to I . \square

In [14], Lemma 1 and an estimate of the type given by Theorem 3 are used in combination with an elaborate second moment argument. In the present context, the following quite simple first moment argument turns out to be sufficient.

Proof of the upper bound part of Theorem 1. Assume that $X_0^N = N\delta_0$. Let $\beta > 0$ and $\theta > 0$ be as in Theorem 3. Then let $\lambda > 0$, and define

$$m := \left\lceil \theta^{3/2} \left(\frac{(1 + \lambda)(\log N)}{(\chi(p) - \beta)^{1/2}} \right)^3 \right\rceil,$$

and $\epsilon := \theta/m^{2/3}$, so that, by Theorem 3,

$$(2) \quad \rho(m, \epsilon) \leq N^{-(1+\lambda)} \text{ for all large } N.$$

Then let $0 < \zeta < 1$ and define $v_2 := v(p) - (1 - \zeta)\epsilon$ and $v_1 := v(p) - \epsilon$.

Let also $n := \lfloor N^\xi \rfloor$ for some $0 < \xi < \lambda$. Now consider $\kappa > 0$, and let $K := \kappa \log(2Nn)$. Consider the maximum of the random walk steps performed during the branching steps of $(X_k^N)_{k \geq 0}$ between time 0 and time n . There are $2Nn$ such steps, so that, by assumption (A2), there exists a value of κ such that the probability that this maximum is larger than or equal to K is less than $(2Nn)^{-2008}$ for all large enough N . Now denote by B_n the number of vertices in $\text{BRW}_1 \cup \dots \cup \text{BRW}_N$ that are $(m, v(p) - \epsilon)$ -good (each with respect to the BRW_i it belongs to) and belong to $G_0^N \cup \dots \cup G_n^N$ (G_n^N is defined in Section 4). Observe that, with our definitions, for N large enough, $\frac{v_2 - v_1}{K - v_1} \frac{n}{m} - K / (K - v_1) > 0$. As a consequence, using Lemma 1, we see that, for N large enough, the event $\max X_n^N \geq v_2 n$ implies that either there exists a random walk step between time 0 and n which is $\geq K$, or $B_n \geq 1$. Using the union bound and the above estimate, we deduce that

$$(3) \quad \mathbb{P}(\max X_n^N \geq v_2 n) \leq (2Nn)^{-2008} + \mathbb{P}(B_n \geq 1).$$

On the other hand, B_n can be written as

$$(4) \quad B_n := \sum_{u \in \text{BRW}_1 \cup \dots \cup \text{BRW}_N} \mathbf{1}(u \text{ is } (m, v(p) - \epsilon)\text{-good}) \mathbf{1}(u \in G_0^N \cup \dots \cup G_n^N).$$

Now observe that, by definition, for a vertex u at depth ℓ , the event $u \in G_0^N \cup \dots \cup G_n^N$ is measurable with respect to the random walk steps performed up to depth ℓ , while the event that u is $(m, v(p) - \epsilon)$ -good is measurable with respect to the random walk steps performed starting from depth $\geq \ell$. As a consequence, these two events are independent. Since the total number of vertices in $G_0^N \cup \dots \cup G_n^N$ is equal to $N(n+1)$, we deduce from (2) and (4) that $E(B_n) \leq N(n+1)N^{-(1+\lambda)}$. Using Markov's inequality, we finally deduce from (3) that

$$(5) \quad \mathbb{P}(\max X_n^N \geq v_2 n) \leq (2Nn)^{-2008} + (n+1)N^{-\lambda}.$$

Now start with the obvious inequality, valid for all $t \geq 0$, $\exp(t \max X_n^N) \leq \sum_{u \in \mathcal{T}_n^N} \exp(t\Phi(u))$. Taking expectations, we deduce that $\mathbb{E}(\exp(t \max X_n^N)) \leq 2^n \exp(n\Lambda(t))$. Using the definition of t^* , we then obtain that

$$(6) \quad \mathbb{E}(\exp(t^*(\max X_n^N - v(p)n))) \leq 1.$$

Using (6) and the fact that, for all large enough x , $x \leq \exp(t^*x/2008)$, we deduce that, for all $b > 0$, and all large enough n ,

$$(7) \quad \mathbb{E}[\max X_n^N \mathbf{1}(\max X_n^N \geq (v(p) + b)n)] \leq \exp(-\frac{2007}{2008}t^*bn) + v(p)n \exp(-t^*bn).$$

Now, observe that $\mathbb{E}(n^{-1} \max X_n^N)$ is bounded above by

$$v_2 + (v(p) + b)\mathbb{P}(\max X_n^N \geq v_2 n) + \mathbb{E}[\max X_n^N \mathbf{1}(\max X_n^N \geq (v(p) + b)n)].$$

Choosing b large enough, we deduce from (5), (7), and the definition of v_2 , that, for all large enough N , $\mathbb{E}(n^{-1} \max X_n^N) \leq (v(p) - (1 - \zeta)\epsilon) + o((\log N)^{-2})$. Using subadditivity, we easily obtain that $v_N(p) \leq (v(p) - (1 - \zeta)\epsilon) + o((\log N)^{-2})$. Now, as N goes to infinity, $\epsilon \sim \frac{\chi(p) - \beta}{(1 + \lambda)^2} (\log N)^{-2}$. Since these estimates are true for arbitrarily small β , λ and ζ , the result follows. \square

6. THE LOWER BOUND

The proof of the lower bound on the convergence rate of $v_N(p)$ to $v_\infty(p)$ that was given in [3] was in some sense a rigorous version of the heuristic argument of Brunet and Derrida according to which we should compare the behavior of the particle system with N particles, with a version of the infinite population limit modified by a cutoff value $1/N$. The proof given here relies upon a direct comparison with branching random walks, using the fact that, above the threshold induced by the selection steps, the behavior of our branching-selection particle system is exactly that of a branching random walk.

Consider $0 < \lambda < 1$ and let $\epsilon := \frac{\chi(p)}{((1-\lambda)\log N)^2}$. With this choice of ϵ , as N goes to infinity, Theorem 2 yields that

$$(8) \quad \rho(\infty, \epsilon) = N^{-(1-\lambda)+o(1)}.$$

Now consider the event A that the root is not $(\infty, v(p) - \epsilon)$ -good in any of $\text{BRW}_1, \dots, \text{BRW}_N$. Since these branching random walks are independent, we see that $\mathbb{P}(A) = (1 - \rho(\infty, \epsilon))^N$, so that, using (8),

$$(9) \quad \mathbb{P}(A) \leq \exp(-N^{\lambda+o(1)}).$$

Remember the notation $u_N := \lceil \frac{\log N}{\log 2} \rceil + 1$, let $0 < \eta < 1$, and let $m := \lceil \frac{v(p)u_N}{\eta\epsilon} \rceil$ and $n := m + u_N$. Consider a vertex u at depth m in a branching random walk BRW, and assume that $\Phi(u) \geq (v(p) - \epsilon)m$. With our definition of m and n , we see that, for all descending path $u =: u_m, \dots, u_n$, $\Phi(u_i) \geq (v(p) - \epsilon(1 + \eta))i$ for all $i \in \llbracket m, n \rrbracket$, using only the fact that the support of p is contained in \mathbb{R}_+ . Now, by definition, on the event A^c , there exists at least one descending path u_0, \dots, u_m in one of $\text{BRW}_1, \dots, \text{BRW}_N$, such that $\Phi(u_i) \geq (v(p) - \epsilon)i$ for all $i \in \llbracket 1, m \rrbracket$. The previous observation allows us to deduce that, in this case, there are at least 2^{u_N} distinct descending paths of the form $root =: u_0, \dots, u_n$ such that $\Phi(u_i) \geq (v(p) - \epsilon(1 + \eta))i$ for all $i \in \llbracket 1, n \rrbracket$.

Let $\delta := \epsilon(1 + \eta)$. Define the event $B := \{\min(X_i^N) < (v(p) - \delta)i \text{ for all } i \in \llbracket 1, n \rrbracket\}$, and assume that $B \cap A^c$ occurs. From the definition of our branching-selection system and the above argument, we conclude that there must be at least 2^{u_N} distinct vertices in the set G_n^N , which is a contradiction since $2^{u_N} > N$. As a consequence, $B \cap A^c = \emptyset$, so that $B \subset A$. From (9), we thus obtain that

$$(10) \quad \mathbb{P}(B) \leq \exp(-N^{\lambda+o(1)}).$$

To exploit this bound, we use the following result.

Proposition 5. *For all N large enough,*

$$v_N(p) \geq (v(p) - \delta)(1 - n\mathbb{P}(B)).$$

Proof. We re-use the notations of the proof of Proposition 3. Start with $\Gamma_0 := 0$ and $J_0 := 0$, and $i := 0$. Given $i \geq 0$, Γ_i and J_i , let $L_{i+1} := \inf\{k \in \llbracket 1, n \rrbracket; \min(W_{\Gamma_i, k}^N) \geq (v(p) - \delta)k\}$, with the convention that $\inf \emptyset := n$. Then let $\Gamma_{i+1} := \Gamma_i + L_{i+1}$, and let $J_{i+1} := J_i + \min(W_{\Gamma_i, L_{i+1}}^N)$.

This construction may be describes as follows: starting from a reference position J_i and a reference time index Γ_i , run the original branching-selection process from $N\delta_{J_i}$, until either n steps have been performed or the minimum value in the population of particles exceeds the reference position by an amount of at least $(v(p) - \delta)$ times the number of steps performed since the beginning of the run. Then set J_{i+1} to the minimum current position, and Γ_i to the current time, and repeat the procedure with $i := i + 1$. Using an argument similar to the proof of Proposition 2, it is then quite easy to deduce that,

$$(11) \quad \text{for all } i \geq 0, \min X_{\Gamma_i}^N \geq J_i.$$

Now, clearly, the sequences $(\Gamma_{i+1} - \Gamma_i)_{i \geq 0}$ and $(J_{i+1} - J_i)$ are i.i.d. The common distribution of the $\Gamma_{i+1} - \Gamma_i$ is that of L , while the common distribution of the $J_{i+1} - J_i$ is that of $\min X_L^N$, where L is the random variable $L := \inf\{i \in \llbracket 1, n \rrbracket; \min(X_i^N) \geq (v(p) - \delta)i\}$, with the convention that $\inf \emptyset := n$. From the law of large numbers and Proposition 3, we have that, $\lim_{i \rightarrow +\infty} i^{-1} \min X_{\Gamma_i}^N = v_N(p)\mathbb{E}(L)$, while the law of large numbers and (11) imply that $\liminf_{i \rightarrow +\infty} i^{-1} \min X_{\Gamma_i}^N \geq \mathbb{E}(\min X_L^N)$. We conclude that $v_N(p) \geq \frac{\mathbb{E}(\min X_L^N)}{\mathbb{E}(L)}$. Now, by definition, $\min X_L^N \geq (v(p) - \delta)L\mathbf{1}(B^c)$, so that $\mathbb{E}(\min X_L^N) \geq (v(p) - \delta)(\mathbb{E}(L) - \mathbb{E}(L\mathbf{1}(B)))$. Using the fact that $1 \leq L \leq n$, we obtain that $\frac{\mathbb{E}(\min X_L^N)}{\mathbb{E}(L)} \geq (v(p) - \delta)(1 - n\mathbb{P}(B))$. \square

Proof of the lower bound part in Theorem 1. In view of Proposition 5, we deduce that $v_N(p) \geq (v(p) - (1 + \eta)\epsilon)(1 - n\mathbb{P}(B))$. Then, from (10), we see that

$$v_N(p) \geq v(p) - \frac{\chi(p)(1 + \eta)}{(1 - \lambda)^2}(\log N)^{-2} + o((\log N)^{-2}).$$

Since λ and η can be taken arbitrarily small, the conclusion follows. \square

Remark 1. *Observe that the proof does not in fact require a lower bound for $\rho(\infty, \epsilon)$, but rather for $\rho(m, \epsilon)$, and note that $m \propto (\log N)^3$ while $\epsilon \propto (\log N)^{-2}$. Similarly, for the upper bound proved in the previous section, an upper bound over $\rho(m, \epsilon)$ with $m \propto (\log N)^3$ and $\epsilon \propto (\log N)^{-2}$ was used.*

7. THE BERNOULLI CASE WHEN $1/2 \leq \alpha < 1$

In the Bernoulli case $p = \alpha\delta_1 + (1 - \alpha)\delta_0$, with $\alpha = 1/2$ or $1/2 < \alpha < 1$, the behavior turns out to be quite different from Brunet-Derrida, as stated in the following theorems.

Theorem 4. *For $\alpha = 1/2$, there exists $0 < C_*(p) \leq C^*(p) < +\infty$ such that, for all large N ,*

$$(12) \quad C_*(p)N^{-1} \leq v_\infty(p) - v_N(p) \leq C^*(p)N^{-1}.$$

Theorem 5. *For $\alpha > 1/2$, there exists $0 < C_*(p) \leq C^*(p) < +\infty$ such that, for all large N ,*

$$(13) \quad C_*(p)N \leq -\log(v_\infty(p) - v_N(p)) \leq C^*(p)N.$$

In the case $1/2 \leq \alpha < 1$, it turns out that $v_\infty(p) = v(p) = 1$.

7.1. Upper bound when $\alpha = 1/2$. It is easily checked that, for all $m \geq 0$, the number of particles at position exactly m after m steps, that is, $X_m^N(m)$, is stochastically dominated by the total population at the m -th generation of a family of N independent Galton-Watson trees, with offspring distribution binomial(2, 1/2). This corresponds to the critical case of Galton-Watson trees, and the probability that such a tree survives up to the m -th generation is $\leq cm^{-1}$ for some constant $c > 0$ and all large m . As a consequence, for large enough m , $\mathbb{P}(X_m^N(m) \geq 1) \leq \mathbb{E}(X_m^N(m)) \leq cNm^{-1}$.

On the other hand, we have by definition that $m^{-1}\mathbb{E}\max(X_m^N) \leq 1 - \frac{1}{m}\mathbb{P}(X_m^N(m) = 0)$. Choosing $m := AN$, where $A \geq 1$ is an integer, we see that, for large N , $m^{-1}\mathbb{E}\max(X_m^N) \leq 1 - 1/AN(1 - c/A)$. The upper bound in (12) follows by choosing $A > c$.

7.2. Lower bound when $p = 1/2$. Given $m \geq 1$, define $U := \inf\{n \in \llbracket 1, m \rrbracket; X_n^N(n) \leq 2N/3\}$, with the convention that $\inf \emptyset := m$. Let D denote the event that $\min X_U^N < U - 1$.

Using an argument similar to the proof of Proposition 5, with U and D in place of L and B respectively, we deduce that

$$(14) \quad v_N(p) \geq 1 - \frac{1}{\mathbb{E}(D)} - m\mathbb{P}(D).$$

The lower bound in (12) is then a consequence of the two following claims.

Our first claim is that there exists $c > 0$ such that $\mathbb{P}(D) \leq \exp(-cN)$ for all large N . Let $N_k^\ell(x)$ denote the number of particles that are born from a particle at position x during the branching step that is applied to X_k^N , and that are located at position $x + \ell$. perform a random walk step equal to from the proof of Lemma ??, and choose δ small enough so that $(1 - \delta)4N/3 > N$. It is easily seen that $D \subset \{N_{U-1}^1(U-1) \leq (1 - \delta)2N/3\} \cup \{N_{U-1}^0(U-1) \leq (1 - \delta)2N/3\}$. Now, by definition, one has that $X_{U-1}^N(U-1) \geq 2N/3$, so that, by a standard large deviations bound for binomial random variables (see e.g. [12]), conditional on X_{U-1}^N , the probabilities of $N_{U-1}^1(U-1) \leq (1 - \delta)2N/3$ and $N_{U-1}^0(U-1) \leq (1 - \delta)2N/3$ are both $\leq \exp(-c(\delta)N)$. The bound on $\mathbb{P}(D)$ follows.

Our second claim is that, for small enough $\epsilon > 0$, with $m := \lfloor \epsilon N \rfloor$, there exists $c(\epsilon) > 0$ such that $\mathbb{E}(D) \geq c(\epsilon)N$ for all large N . To prove it, introduce the Markov chains $(V_k)_{k \geq 0}$ and $(Z_k)_{k \geq 0}$ on \mathbb{N} , defined as follows. First, $V_0 \in \mathbb{N}$, and, given V_0, \dots, V_k , the next term V_{k+1} is the minimum of N and of a random variable with a binomial($2V_k, 1/2$) distribution. On the other hand, $Z_0 \in \mathbb{N}$, and, given Z_0, \dots, Z_k , the distribution of Z_{k+1} is binomial($2Z_k, 1/2$). Observe that the sequence $(X_n^N(n))_{n \geq 0}$ is a version of V started at $V_0 := N$. Now, it is easily seen that, given two starting points $x, y \in \mathbb{N}$ such that $x \leq y$, one can couple two versions of V starting from x and y respectively, in such a way that the version starting from y is always above the version starting from x . As a consequence, U stochastically dominates the random variable $T := \inf\{n \in \llbracket 1, m \rrbracket; V_n \leq 2N/3\}$ (again with $\inf \emptyset := m$), where V is started at $V_0 := \lfloor 3N/4 \rfloor$. Then observe that the distributions of V and Z started with $V_0 := Z_0 := \lfloor 3N/4 \rfloor$, considered up to the hitting time of $\llbracket N, +\infty \rrbracket$, coincide. As a consequence, the probabilities of the events $\{\sup_{k \in \llbracket 0, m \rrbracket} |V_k - \lfloor 3N/4 \rfloor| \geq N/16\}$ and $\{\sup_{k \in \llbracket 0, m \rrbracket} |Z_k - \lfloor 3N/4 \rfloor| \geq N/16\}$ coincide, and the first of these two events implies that $T = m$. Now, $(Z_k)_{k \geq 0}$ is a martingale, so that, by Doob's maximal inequality, $P\left(\sup_{k \in \llbracket 0, m \rrbracket} |Z_k - \lfloor 3N/4 \rfloor| \geq N/16\right) \leq E(Z_m - \lfloor 3N/4 \rfloor)^2 (N/16)^{-2}$. Then, it is easily checked from the definition that $E(Z_{k+1}^2 | Z_k) = Z_k^2 + Z_k/2$, and, using again the fact that $(Z_k)_{k \geq 0}$ is a martingale, we deduce that $E(Z_m - \lfloor 3N/4 \rfloor)^2 \leq mN/2$. As a consequence, we see that, choosing $\epsilon > 0$ small enough, we can ensure that $P\left(\sum_{k \in \llbracket 0, \lfloor \epsilon N \rfloor \rrbracket} |Z_k - \lfloor 3N/4 \rfloor| \geq N/16\right) \leq 1/2008$ for all large N . For such an ϵ , and all N large enough, we thus have that $\mathbb{P}(U = m) \geq P(T = m) \geq 1/2008$. The conclusion follows.

7.3. Upper and lower bound when $1/2 < \alpha < 1$. As for the upper bound, observe that asking all the $2N$ particles generated during the branching step to remain at the position from which they are originated has a probability equal to at most $(1 - \alpha)^{2N}$, so that $\mathbb{E}(\max X_n) \leq n(1 - (1 - \alpha)^{2N})$. As for the lower bound,

observe that, starting from N particles at a site, the number of particles generated from these during a branching step and that perform $+1$ random walk steps has a binomial($2N, \alpha$) distribution, whose expectation is $2pN$, with $2\alpha > 1$. Using a standard large deviations bound for binomial random variables, we see that the probability for this number to be less than N is $\leq \exp(-cN)$ for some $c > 0$. As a consequence, $\mathbb{E}(\min X_n) \geq n(1 - \exp(-cN))$.

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