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A semantics for obligations

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Abstract

We attack two problems here:

First, we want to combine a modal accessibility relation with a preferential relation in a modal language. This problem is solved by adding a complementary modality.

Second, we want to characterize obligations syntactically and semantically. This problem is solved syntactically by suitable disjunctions of conjunctions - but not negations. Semantically, we define a Hamming neighbourhood, and show that both descriptions are equivalent.

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1 Introduction

Our main contributions are, in the opinion of the authors:

- (1) A semantical and syntactical characterization of obligations, which avoids the paradoxa due to arbitrary right weakening (Ross paradox).
 The semantical characterization is by suitable neighbourhoods of the set of ideal cases, where as many obligations as possible are satisfied. The syntactical characterization is by suitable disjunctions of conjunctions. Such disjunctions describe exactly above neighbourhoods.
- (2) The introduction of a “negative” relation (mR_n iff it is not the case that mRn), which allows full combination of usual modal logic with a preference relation, as it allows to describe exactly the models reachable from m . In particular, properties like rankedness can now be expressed for sets of reachable models.

In a certain way, both ideas are solutions to the problem of unwanted weakening:

- Obligations are formulas which describe Hamming neighbourhoods (but not necessarily their weakenings),
- the formula true in all R -successors of m , but in no R_- -successor of m describes exactly the R -successors of m .

The latter is used to describe central coherence properties of nonmonotonic logics.

Our approach works for absolute and conditional obligations, which can be hard, i.e. without exceptions, or soft, i.e. allowing exceptions, and also for contrary-to-duty obligations.

Whether the precise definition of the Hamming neighbourhood is the one which corresponds best to our intuitions about obligations can only be decided by a careful comparison of both, and is certainly beyond the scope of this article. We feel, however, quite confident, that the basic idea to avoid arbitrary weakening by choosing suitable

subsets is correct, and that the idea of choosing some kind of neighbourhood of the ideal cases is correct too: Obligations are there to steer us in some way towards the morally ideal, and neighbourhoods make certain that we are not too far off.

The formal details are all either trivial, or we can build upon earlier research.

2 Obligations as Hamming neighbourhoods

2.1 Introduction, main definitions

We will assume that there are no background assumptions, i.e. everything which is considered important is said explicitly.

2.1.1 The ideal case and obligations

We split the task of defining obligations in two parts:

First, we define the ideal case, and then we define obligations as suitable neighbourhoods of the ideal cases. They will correspond to disjunctions of conjunctions, where the conjunctions have non-empty intersections with the set of ideal cases, and the whole disjunction covers the ideal cases. (See Fact 2.1 (page 4) .)

For the ideal case, we will adopt our usual notation:

If X is a set, $\mu(X)$ will be the subset of ideal cases, and if T is a theory, $\overline{\overline{T}}$ will be the theory defining the ideal cases of T . \overline{T} will be the set of classical consequences of T .

We will have some trivial properties:

$$\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$$

$$\overline{\overline{\overline{T}}} = \overline{\overline{T}}$$

$$T \subseteq \overline{\overline{T}}$$

(these two will not apply here) and a decisive property:

$$\text{if } T' \vdash T, \text{ and } \text{Con}(\overline{\overline{T}}, T'), \text{ then } \overline{\overline{T'}} = \overline{\overline{\overline{\overline{T}} \cup T'}},$$

which guarantees rankedness on the semantic side, rationality on the proof theoretical side.

We will not defend rankedness intuitively, it seems to be adopted more or less universally in the case of obligations.

Thus, obligations are certain, but usually not all, consequences of the ideal cases, the latter are defined by a ranked structure. Obligations themselves have certain proof theoretic and semantic properties.

Conversely, given a set of obligations, ideal cases are those upon which we cannot improve, i.e. there is no other case which satisfies all the obligations of the first, plus some additional ones. This does not mean that it has to fulfill all, we just cannot do better.

Note that our approach works for hard (i.e. without exceptions) obligations, but also for obligations with exceptions, and also for arbitrary subsets of the universe. In the infinite case, the proof theoretic property has to be adapted, the semantic property (via Hamming neighbourhood) goes through as is.

The motivation for our approach will be given in Section 2.2 (page 5) .

2.1.2 The Hamming-Neighbourhood

Definition 2.1

Given a propositional language \mathcal{L} defined by the set $v(\mathcal{L})$ of propositional variables, let \mathcal{L}_\wedge be the set of all (finite) consistent conjunctions of elements from $v(\mathcal{L})$ or their negations. Thus, $p \wedge \neg q \in \mathcal{L}_\wedge$ if $p, q \in v(\mathcal{L})$, but $p \vee q, \neg(p \wedge q) \notin \mathcal{L}_\wedge$. Finally, let $\mathcal{L}_{\vee\wedge}$ be the set of all (finite) disjunctions of formulas from \mathcal{L}_\wedge . (As we will later not consider all formulas from \mathcal{L}_\wedge , this will be a real restriction.)

Given a set of models M for a finite language \mathcal{L} , define $\phi_M := \bigwedge\{p \in v(\mathcal{L}) : \forall m \in M.m(p) = v\} \wedge \bigwedge\{\neg p : p \in v(\mathcal{L}), \forall m \in M.m(p) = f\} \in \mathcal{L}_\wedge$. (If there are no such p , set $\phi_M := TRUE$.)

This is the strongest $\phi \in \mathcal{L}_\wedge$ which holds in M .

Definition 2.2

Let $\Sigma := \{\sigma : I \rightarrow A\}$, the set of sequences from an index set I in A .

(In our intended application, Σ will be the set of propositional models of some language.)

Define for $\sigma, \sigma', \sigma'' \in \Sigma$

$\langle \sigma, \sigma', \sigma'' \rangle$ iff $\forall i \in I(\sigma(i) = \sigma''(i) \rightarrow \sigma'(i) = \sigma(i))$ (intuitively, σ' is between σ and σ''), and

$[\sigma, \sigma''] := \{\sigma' \in \Sigma : \langle \sigma, \sigma', \sigma'' \rangle\} = \{\sigma' : \forall i \in I(\sigma(i) = \sigma''(i) \rightarrow \sigma'(i) = \sigma(i))\}$, intuitively, this is the interval between σ and σ'' .

Let $X \subseteq Y \subseteq X' \subseteq \Sigma$, then Y is a Hamming neighbourhood of X in X' iff $\forall y \in Y \exists x \in X.[x, y] \cap X' \subseteq Y$.

By abuse of language, we will also call a theory or formula a Hamming neighbourhood of another theory or formula, iff the corresponding model sets are.

Fact 2.1

(1) If x, y are models, then $[x, y] = M(\phi_{\{x, y\}})$.

(2) If $X \subseteq X' \subseteq \Sigma$, then X and X' are Hamming neighbourhoods of X in X' .

(3) If $X \subseteq Y_j \subseteq X' \subseteq \Sigma$ for $j \in J$, and all Y_j are Hamming neighbourhoods of X in X' , then so is $\bigcup\{Y_j : j \in J\}$. This is not necessarily true for intersections.

(4) Let \mathcal{L} be a finite propositional language, $\phi \vdash \phi'$, then ψ with $\phi \vdash \psi \vdash \phi'$ is a Hamming neighbourhood of ϕ in ϕ' iff ψ is equivalent to some $\bigvee\{\phi_k \in \mathcal{L}_\wedge : k \in K\} \wedge \phi'$ such that

$Con(\phi_k, \phi)$ for all $k \in K$.

Proof

(1) and (2) are trivial.

(3) Unions: trivial.

Intersections: Consider the language defined by p, q, r, s , $X' := M_{\mathcal{L}}$.

Let $X := \{pq \neg r \neg s, \neg p \neg q r s\}$, $x := pqrs$. Then $X \cup \{x, pq r \neg s, pq \neg r s\}$ and $X \cup \{x, \neg p q r s, p \neg q r s\}$ both are Hamming neighbourhoods, but their intersection $X \cup \{x\}$ is not.

(4) For simplicity, we forget about ϕ' .

“ \Rightarrow ”: Let ψ be a Hamming neighbourhood of ϕ , $m \models \psi$. Then there is $m' \models \phi$ s.t. $[m, m'] = M(\phi_{\{m, m'\}}) \subseteq M(\psi)$. Take the disjunction of all such $\phi_{\{m, m'\}}$.

“ \Leftarrow ”: By $\phi \vdash \psi$, $M(\phi) \subseteq M(\psi)$. Let $m \in M(\bigvee\{\phi_k : k \in K\}) = \bigcup M(\{\phi_k : k \in K\})$, $\phi_k \in \mathcal{L}_\wedge$. Let $m \in M(\phi_k)$, where $\phi_k \in \mathcal{L}_\wedge$, with $M(\phi_k) \cap M(\phi) \neq \emptyset$, let $m' \in M(\phi_k) \cap M(\phi)$. Then $M(\phi_{\{m, m'\}}) = [m, m'] \subseteq M(\phi_k)$.

□

The lack of closure under intersections (which is possible only when the original obligations are inconsistent with the set of situations considered) is certainly a drawback of our definition.

One might therefore consider a modified definition:

Definition 2.3

Let Y be s.t. $X \subseteq Y \subseteq X'$. Y is called a Hamming neighbourhood-2 of X in X' iff

$\forall y \in Y \forall x \in X(x \text{ is closest to } y \text{ (in the Hamming distance)} \Rightarrow [x, y] \cap X' \subseteq Y)$.

In our example, both $[pq \neg r \neg s, pqrs]$ and $[\neg p \neg q r s, pqrs]$ would have to be included in the neighbourhood. It

is, of course, the existential quantifier which caused problems in the original definition. The new definition guarantees closure under (arbitrary) intersections and unions.

Which one is the better definition can probably only be decided by our intuition about obligations, so it is up for discussion.

Definition 2.4

Let $\emptyset \neq X \subseteq Y \subseteq X'$ where X is the set of ideal situation in X' . Then Y is an obligation in X' iff $Y \neq X'$ and Y is a Hamming neighbourhood of X in X' .

2.2 Discussion and motivation

An obligation is an asymmetrical formula. Do not kill means something like: it is better not to kill than to kill. In ordinary logic, ϕ and $\neg\phi$ are symmetrical, there is nothing which distinguishes ϕ over $\neg\phi$, or vice versa. In obligations, there is. This is a fundamental property, which can also be found in orders, planning (we move towards the goal or not), reasoning with utility (is ϕ or $\neg\phi$ more useful?), and probably others, like perhaps the Black Raven paradox.

As ordinary logic does not have this asymmetry, ordinary logic cannot be used without modification in said situations. A famous example is the Ross paradox: If you have the obligation to post the letter, you have the obligation to post or burn the letter. The reasoning is by weakening, justified in ordinary logic, if ϕ holds in a situation, then so does $\phi \vee \psi$. But if we take the obligation to post or burn seriously, we can satisfy it by burning the letter, an obvious nonsense.

It is not the “or” itself which is the problem. For instance, in case of an accident, to call an ambulance or to help the victims by giving first aid is a perfectly reasonable obligation. It is the negation of the obligation not to burn the letter which is the problem.

Obviously, the negation turns the asymmetry of obligations around. It is better not to burn the letter than to burn it, and we have added in above paradox via “or” a “negative” obligation.

The Ross paradox as formulated above is peculiar as “burn” implies “not post”, we modify it now to independent obligations. The first is “post the letter”, the second: “water the flowers”.

Obviously, “post the letter or do not water the flowers” is not a good obligation on the basis of the original obligations. “Post the letter and water the flowers” is a good obligation on this basis, “post the letter or water the flowers” also is a good obligation, when we satisfy it, we do something good, whatever part of the disjunct we satisfy; not the ideal, but something. We might e.g. not have time to do both, but we can at least satisfy one obligation.

2.2.1 Basic idea

This is then our basic idea:

If α and β are obligations, then so will be $\alpha \wedge \beta$ and $\alpha \vee \beta$, but not anything involving $\neg\alpha$ or $\neg\beta$. (In a non-trivial manner, leaving aside tautologies and contradictions which have to be considered separately.)

To summarize: “and” and “or” preserve the asymmetry, “not” does not, therefore we can combine obligations using “and” and “or”, but not “not”. Thus, a reasonable notion of derivation of obligations will work with \wedge and \vee , but not with \neg .

2.2.2 The semantical side

As shown above, our semantics does not reflect faithfully the above idea, as Hamming neighbourhoods are not closed under intersections. Whether our definition is the right one can only be shown by careful analysis of our intuitions about obligations.

We think, however, that some such notion of neighbourhood is a good candidate for a semantics:

- (1) It must not be that the system of suitable sets is closed under supersets, otherwise we have closure under arbitrary right weakening, and thus the Ross paradox.

Notions like “big subset” or “small exception sets” are closed under supersets, so they are not suitable.

A system of neighbourhoods is not necessarily closed under supersets.

- (2) Obligations express something like an approximation to the ideal case where all obligations (if possible, or, as many as possible) are satisfied, so we try to be close to the ideal. If we satisfy an obligation, we are (relatively) close, and stay so as long as the obligation is satisfied.

The notion of neighbourhood expresses the idea of being close, and containing everything which is sufficiently close.

Behind “containing everything which is sufficiently close” is the idea of being in some sense convex. Thus, “convex” is another basic notion to be investigated. (See here also the investigation of “between” in [Sch04].)

- (3) The Hamming distance is a natural measure of distance of situations.

Thus, we have not only an abstract semantics (some kind of neighbourhood), but also a concrete way to construct it.

The situation is thus similar to the one in preferential structures, where we have the abstract semantics of big subsets, and the concrete semantics of a preferential relation.

Thus, one can hope that the development of both - abstract and concrete semantics - will prove similarly fruitful in future.

2.2.3 Description vs. obligation and different types of obligations

This is perhaps the right place to point out the following differences.

When we describe the situations where obligations are satisfied or not, usual logic is adequate, e.g. when we have posted the letter, then it holds that the letter is posted or the flowers are not watered, but this is a description, and not an obligation. The logic of descriptions is ordinary logic, the logic of obligations is different.

We turn to different types of obligations.

- (1) Hard obligations.

So far, our obligations are “hard” i.e. without exceptions, as in the Ten Commandments. You should not kill.

- (2) Soft obligations.

Many obligations have exceptions. Consider the following example: you are in a library. Of course, you should not pour water on a book. But if the book has caught fire, you should pour water on it to prevent worse damage. In stenographic style these obligations read: “Do not pour water on books”. “If a book is on fire, do pour water on it.” It is like “birds fly”, but “penguins do not fly”, “soft” or nonmonotonic obligations, which have exceptions, which are not formulated in the original obligation, but added as exceptions.

- (3) Conditional obligations.

Conditional obligations are valid only in certain circumstances. “Help the wounded” applies only in case of an accident or so. We could have formulated the library obligation also without exceptions: “When you are in a library, and the book is not on fire, do not pour water on it.” “When you are in a library, and the book is on fire, pour water on it.” This formulation avoids exceptions. Conditional obligations behave like restricted quantifiers: they apply in a subset of all possible cases.

- (4) Contrary-to-duty obligations.

Contrary-to-duty obligations are about different degrees of fulfillment. If you should ideally not have any fence, but are not willing or able to fulfill this obligation (e.g. you have a dog which might stray), then you should at least paint it white to make it less conspicuous.

This is also a conditional and in our formulation a soft obligation. Conditional, as it specifies what has to be done if there is a fence, and soft as we said “you should have no fence” - but there are exceptions, which are detailed in the next part.

The new aspect in contrary-to-duty obligations is the different degree of fulfillment.

We will not treat contrary-to-duty obligations here, as they do not seem to have any import on our basic ideas and solutions.

2.3 Elaboration of the basic idea

2.3.1 A consequence relation for obligations

Our aim is to find an intuitively acceptable notion of derivation for obligations. In particular, all original obligations should again be derivable.

We recall that the basic problem seems to be negation, which turns the asymmetry of obligations on its head.

We look at the different cases. Let U be the universe we consider (basically the set of all models), and \mathcal{O} be the set of obligations.

We leave aside the problem of trivial obligations (which hold everywhere or nowhere, their exclusion can always be added as a condition).

Moreover, we will interchangeably consider obligations as formulas or as the model sets they define.

- (1) Strict (i.e. without exceptions) obligations.

\mathcal{O} defines an order on U , by

$m < m'$ iff $\forall O \in \mathcal{O}(m' \in O \rightarrow m \in O)$ and $\exists O \in \mathcal{O}(m \in O, m' \notin O)$, and

$m \sim m'$ iff $\forall O \in \mathcal{O}(m' \in O \leftrightarrow m \in O)$

(This will be made official below in Definition 2.5 (page 9) .)

- (1.1) We consider the full set U , and assume that the set of obligations is consistent, i.e. that it is possible to satisfy them all simultaneously. (If not, we proceed as in the conditional case $U' \subseteq U$ below.)

We define: $X \subseteq U$ is a derived obligation iff

X is (corresponds to) a disjunction of conjunctions from \mathcal{O} .

We write $\mathcal{D}(\mathcal{O})$ for this set of disjunctions of conjunctions. Note that this does NOT include negations of obligations! (This is the asymmetry.)

We will see below (in Section 2.4 (page 8)) that this is equivalent to:

$X \subseteq U$ is an obligation iff it is downward closed under above order. Obviously, this is not symmetrical either.

- (1.2) Let $U' \subseteq U$

Again, the closure condition is adequate.

Consider now the first criterion: As it might not be possible to satisfy all obligations together (U' might exclude this), we should admit as elementary obligations only elements from $\mathcal{D}(\mathcal{O})$, which cover the set of best (by above order) $m \in U'$, i.e. $O \in \mathcal{D}(\mathcal{O})$ s.t. $\mu(U') \subseteq O$, where $\mu(U')$ is the set of best elements of U' . In addition, we require the conjunctions to have non-empty intersection with the set of best cases.

This is no basic problem, as we respect again the asymmetry, and do not admit any negations.

- (2) Soft obligations, i.e. obligations with exceptions.

Example: The considerate assassin:

Normally, one should not offer a cigarette to someone, out of respect for his health. But the considerate assassin might do so nonetheless, on the cynical reasoning that the victim's health is going to suffer anyway:

(1) One should not kill, $\neg k$.

(2) One should not offer cigarettes, $\neg o$.

(3) The assassin should offer his victim a cigarette before killing him, if k , then o .

Here, globally, $\neg k$ and $\neg o$ is best, but among k -worlds, o is better than $\neg o$. The set of obligations is $\mathcal{O} = \{\neg k, \neg o\}$.

Let this situation be represented by a ranked model, with its order, i.e. $\neg k \wedge \neg o < \neg k \wedge o < k \wedge o < k \wedge \neg o$.

(2.1) We consider the full set U .

Consider again $\mathcal{D}(\mathcal{O})$. This time, this cannot correspond to a downward closed set (in the representing ranked structure), because of the exceptions. Example: $\neg o$ is in \mathcal{D} , but $k \wedge o$ is better than $k \wedge \neg o$. Downward closure will only hold for “most” cases, but not for all.

(2.2) We consider $U' \subset U$.

This time, what is an obligation in U , need not be an obligation in U' any more. In our example, $\neg o$ is a global obligation, but within k , o is the obligation.

So we have to extract first the new obligations.

(2.2.1) Simple case: the old obligations or their negation hold in all best models of U' , denote this again $\mu(U')$.

In this case, this, i.e. elements of \mathcal{O} or their negations, determine the new set of obligation, say \mathcal{O}' , and we continue with \mathcal{O}' instead of \mathcal{O} , i.e. consider $\mathcal{D}(\mathcal{O}')$ defined from \mathcal{O}' as $\mathcal{D}(\mathcal{O})$ was defined from \mathcal{O} .

Attention: in our example, $\neg o$ is NOT in $\mathcal{D}(\mathcal{O}')$, but o is.

So what is fundamentally new here is that the negation of a global obligation can now be a local obligation. But this is to be expected, as we are precisely dealing with obligations with exceptions.

(2.2.2) More complicated case: The old obligations or their negations do not hold everywhere in $\mu(U')$. In this case, we have to consider disjunctions of old obligations or their negations, which cover $\mu(U')$.

In addition, we require the conjunctions to have non-empty intersection with the set of best cases.

Remark 2.2

A still more complicated case: The language of obligations is not uniform, i.e. there are subsets $V \subseteq U$ where obligations are defined, which are not defined in $U - V$.

Take again $U' \subseteq U$, consider $\mu(U')$. For $x \in \mu(U')$, several languages may be applicable. We consider the obligations o or their negations $\neg o$ in all languages applicable to a , and which hold in a . We do this for all $a \in \mu(U')$. We then form again disjunctions of such obligations which form a cover of $\mu(U')$.

We will not pursue this case here any further.

2.4 Derived obligations in the hard case

2.4.1 Intuition

A family of obligations is for us a family of (propositional) formulas, like $\neg kill$ which is read as the obligation not to kill.

We will work semantically, so a family \mathcal{O} of obligations is just a family of subsets of a given base set U , $\mathcal{O} \subseteq \mathcal{P}(U)$.

It is well known that obligations behave differently from usual formulas. We can weaken a formula, and it is still true if it was so before, but the Ross paradox shows that this is not the case for obligations. So the deduction of true formulas must behave differently from the deduction of obligations.

Before we give a definition, we describe some postulates derived obligations should satisfy.

Let then \mathcal{O} a family of obligations, i.e. subsets of U .

(1) If x and x' are in the same subset $\mathcal{O}' \subseteq \mathcal{O}$ of obligations, then an obligation derived from \mathcal{O} should not separate them. More precisely, if $x \in \mathcal{O}' \Leftrightarrow x' \in \mathcal{O}'$, and D is a derived obligation, then $x \in D \Leftrightarrow x' \in D$.

Example: If the only obligation is not to kill, then it should not be derivable not to kill and to eat spaghetti.

(2) If x is in a derived obligation, x' is not, then it should not be the case that x' is better than x in the following sense: x' is in $\mathcal{O} \in \mathcal{O}$ if x is, and there is some $\mathcal{O} \in \mathcal{O}$ in which x' is, but not x .

Example: If $\neg kill$ and $\neg steal$ are the only obligations, then $\neg kill \wedge \neg steal$ is better than $\neg kill \wedge steal$, so if $\neg kill \wedge steal$ is in the derived obligation, then so should be $\neg kill \wedge \neg steal$.

This excludes the Ross paradox: The situation where the letter is not posted but is not burnt either is better than $\neg posted \wedge burnt$, but excluded. (This example works, of course, with background assumptions, so is better to use the following example: One should post the letter, and should water the plants. The Ross paradox is then: one should post the letter or not water the plants. The $posting \wedge \neg water$ is better than $\neg posting \wedge \neg water$, and again excluded.)

(3) Every original obligation should also be a deduced obligation, corresponding to $\alpha, \beta \vdash \alpha$.

(4) A derived obligation should not be trivial, i.e. neither empty nor U .

This last property is not very important from an algebraic point of view, and easily satisfiable, so we will not give it too much importance.

Inspired by (1) and (2), we define for a given family \mathcal{O} over U :

Definition 2.5

(1) Define for $x, x' \in U$:

$$x \sim x' :\Leftrightarrow \forall O \in \mathcal{O} (x \in O \Leftrightarrow x' \in O)$$

$$x \leq x' :\Leftrightarrow \forall O \in \mathcal{O} (x' \in O \Rightarrow x \in O)$$

$$x < x' :\Leftrightarrow x \leq x', \text{ and } x \not\sim x'$$

(2) $X \subseteq U$ is closed in U wrt. \mathcal{O} iff

$$\forall x \in X, x' \in U (x' \leq x \rightarrow x' \in X).$$

Definition 2.6

We define $\mathcal{O} \vdash D$ for $D \subseteq U$ iff D is closed in U wrt. \mathcal{O} and call D a derived obligation of \mathcal{O} .

Fact 2.3

Let $O \in \mathcal{O}$, then $\mathcal{O} \vdash O$.

Proof

If $x \in O, x' \notin O$, then $x' \not\leq x$. \square

Fact 2.4

(1) Let $D \subseteq U' \subseteq U$, D closed in U wrt. \mathcal{O} , then D is also closed in U' wrt. \mathcal{O} .

(2) Let $D \subseteq U' \subseteq U$, D closed in U' wrt. \mathcal{O} , U' closed in U wrt. \mathcal{O} , then D is closed in U wrt. \mathcal{O} .

(3) Let $D_i \subseteq U'$ be closed, all $i \in I$, then so are $\bigcup \{D_i : i \in I\}$ and $\bigcap \{D_i : i \in I\}$.

Proof

(1) Trivial.

(2) Let $x \in D \subseteq U'$, $x' \leq x$, then $x' \in U'$ by prerequisite, so $x' \in D$ again by prerequisite.

(3) Trivial.

\square

Definition 2.7

Let \mathcal{O} over U be given.

$X \subseteq U$ is (wi) (for union of intersections) iff there is a family $\mathcal{O}_i \subseteq \mathcal{O}$, $i \in I$ s.t. $X = \bigcup \{ \bigcap \mathcal{O}_i : i \in I \}$.

Fact 2.5

Let \mathcal{O} over U be given. Then

- (1) X is closed
- (2) X is (ui)

are equivalent.

Proof

(2) \Rightarrow (1) :

By Fact 2.3 (page 9) and Fact 2.4 (page 9) (3).

(1) \Rightarrow (2) :

If there is $m \in X$, $m \notin O$ for all $O \in \mathcal{O}$, then $X = U$, take $\mathcal{O}_i := \emptyset$.

For $m \in X$ let $\mathcal{O}_m := \{O \in \mathcal{O} : m \in O\}$. Let $X' := \bigcup\{\bigcap \mathcal{O}_m : m \in X\}$.

$X \subseteq X'$: trivial, as $m \in X \rightarrow m \in \bigcap \mathcal{O}_m \subseteq X'$.

$X' \subseteq X$: Let $m' \in \bigcap \mathcal{O}_m$ for some $m \in X$. It suffices to show that $m' \leq m$. $m' \in \bigcap \mathcal{O}_m = \bigcap\{O \in \mathcal{O} : m \in O\}$, so for all $O \in \mathcal{O}$ $m \in O \rightarrow m' \in O$.

□

Corollary 2.6

Every derived obligation is a classical consequence of the original set of obligations.

Definition 2.8

Let \mathcal{O} be over U .

(1) Let $\mathcal{O}' \subseteq \mathcal{O}$. Define for $m \in U$ and $\delta : \mathcal{O}' \rightarrow 2 = \{0, 1\}$

$m \models \delta := \forall O \in \mathcal{O}' (m \in O \Leftrightarrow \delta(O) = 1)$

(2) \mathcal{O} is independent iff $\forall \delta : \mathcal{O} \rightarrow 2. \exists m \in U. m \models \delta$.

Obviously, independence does not inherit downward to subsets of U .

Definition 2.9

Let \mathcal{O} be over U .

$\mathcal{D}(\mathcal{O}) := \{X \subseteq U : \forall \mathcal{O}' \subseteq \mathcal{O} \forall \delta : \mathcal{O}' \rightarrow 2$

$((\exists m, m' \in U, m, m' \models \delta, m \in X, m' \notin X) \Rightarrow (\exists m'' \in X. m'' \models \delta \wedge m'' < m'))\}$

This property expresses that we can satisfy obligations independently: If we respect O , we can, in addition, respect O' , and if we are hopeless kleptomaniacs, we may still not be a murderer. If $X \in \mathcal{D}(\mathcal{O})$, we can go from $U - X$ into X by improving on all $O \in \mathcal{O}$, which we have not fixed by δ , if δ is not too rigid.

Example 2.1

The Ross paradox is not in $\mathcal{D}(\mathcal{O})$.

Proof

Suppose we have the alphabet p, q and the obligations $\{p, q\}$, then $D := p \vee \neg q \notin \mathcal{D}(\mathcal{O})$: Consider $dom(\delta) := \{p\}$, $\delta(p) = 0$. Let $m \models \neg p \wedge q$, $m' \models \neg p \wedge \neg q$, then $m, m' \models \delta$, $m \notin D$, $m' \in D$. But going from m to m' improves no $O \in \mathcal{O}$. (This is the only way to go into D respecting δ .) □

Fact 2.7

Let \mathcal{O} be over U and independent. Then

- (1) X is closed
- (2) $X \in \mathcal{D}(\mathcal{O})$

are equivalent.

Proof

(1) \Rightarrow (2) :

Let X be closed, and $\mathcal{O}' \subseteq \mathcal{O}$, $\delta : \mathcal{O}' \rightarrow 2$, $m, m' \models \delta$, $m \in X$, $m' \notin X$. Let m'' be s.t. $m'' \models \delta$, and for all $O \in \mathcal{O} - \text{dom}(\delta)$ $m'' \in O$. This exists by independence. Then $m'' \leq m'$, but also $m'' \leq m$, so $m'' \in X$. Suppose $m'' \sim m'$, then $m' \leq m''$, so $m' \in X$, contradiction, so $m'' < m'$.

(2) \Rightarrow (1) :

Let $X \in \mathcal{D}(\mathcal{O})$, but let X not be closed. Thus, there are $m \in X$, $m' \leq m$, $m' \notin X$.

Case 1: Suppose $m' \sim m$. Let $\delta_m : \mathcal{O} \rightarrow 2$, $\delta_m(O) = 1$ iff $m \in O$. Then $m, m' \models \delta_m$, and there cannot be any $m'' \models \delta_m$, $m'' < m'$, so $X \notin \mathcal{D}(\mathcal{O})$.

Case 2: $m' < m$. Let $\mathcal{O}' := \{O \in \mathcal{O} : m \in O \Leftrightarrow m' \in O\}$, $\text{dom}(\delta) = \mathcal{O}'$, $\delta(O) := 1$ iff $m \in O$ for $O \in \mathcal{O}'$. Then $m, m' \models \delta$. If there is $O \in \mathcal{O}$ s.t. $m' \notin O$, then by $m' \leq m$ $m \notin O$, so $O \in \mathcal{O}'$. Thus for all $O \notin \text{dom}(\delta)$, $m' \in O$. But then there is no $m'' \models \delta$, $m'' < m'$, as m' is already optimal among the n with $n \models \delta$.

□

Corollary 2.8

Let \mathcal{O} be over U and independent. Then

- (1) X is closed
- (2) $X \in \mathcal{D}(\mathcal{O})$
- (3) X is *(ui)*

are equivalent.

□

Note that all above definitions and results - with the exception of Corollary 2.6 (page 10) - are independent from logic, and U might as well be a set of developments as the set of models of a classical propositional language. Moreover, no assumption of finiteness was made.

We should not close under inverse \wedge , i.e. if $\phi \wedge \phi'$ is an obligation, we should not conclude that ϕ and ϕ' separately are obligations, as the following example shows.

Example 2.2

Let p stand for: post letter, b : burn letter, s : strangle grandmother.

Consider now $\phi \wedge \phi'$, where $\phi = p \vee (\neg p \wedge b)$, $\phi' = p \vee (\neg p \wedge \neg b \wedge s)$. $\phi \wedge \phi'$ is equivalent to p - though it is perhaps a bizarre way to express the obligation to post the letter. ϕ leaves us the possibility to burn the letter, and ϕ' to strangle the grandmother, and neither seem good obligations. □

This is particularly important in the case of soft obligations, as we see now, when we try to apply the rules of preferential reasoning to obligations.

One of the rules of preferential reasoning is the (*OR*) rule:

$$\phi \sim \psi, \phi' \sim \psi \Rightarrow \phi \vee \phi' \sim \psi.$$

Suppose we have $\phi \sim \psi' \wedge \psi''$, and $\phi' \sim \psi'$. We might be tempted to split $\psi' \wedge \psi''$ - as ψ' is a “legal” obligation, and argue: $\phi \sim \psi' \wedge \psi''$, so $\phi \sim \psi'$, moreover $\phi' \sim \psi'$, so $\phi \vee \phi' \sim \psi'$. The following example shows that this is not always justified.

Example 2.3

Consider the following obligations for a physician:

Let ϕ' imply that the patient has no heart disease, and if ϕ' holds, we should give drug A or (not drug A , but drug B), abbreviated $A \vee (\neg A \wedge B)$. (B is considered dangerous for people with heart problems.)

Let ϕ imply that the patient has heart problems. Here, the obligation is $(A \vee (\neg A \wedge B)) \wedge (A \vee (\neg A \wedge \neg B))$, equivalent to A .

The false conclusion would then be $\phi' \sim A \vee (\neg A \wedge B)$, and $\phi \sim A \vee (\neg A \wedge B)$, so $\phi \vee \phi' \sim A \vee (\neg A \wedge B)$, so in both situation we should either give A or B , but B is dangerous in “one half” of the situations.

□

3 Our system in a modal framework

3.1 Preferential reasoning in a modal framework

The essential rule characterizing ranked structures is

$$(RatM) \phi \sim \psi, \phi \not\sim \neg\psi' \Rightarrow \phi \wedge \psi' \sim \psi.$$

The language of obligations has usually the flavour of modal languages, whereas the language describing preferential structures is usually different in decisive aspects.

If we accept that the description of obligations is suitably given by ranked structures, then we have ready characterizations available. So our task will be to adapt them to fit reasonably well into a modal logic framework. We discuss this now.

We will suppose that we have an entry point u into the structure, from which all models are visible through relation R , with modal operators \Box and \Diamond . R is supposed to be transitive.

The first hurdle is to express minimality in modal terms. Boutilier and Lamarre have shown how to do it, see [Bou90a] and [Lam91]. It was criticized in [Mak93], but this criticism does not concern our approach as we use different relations for accessibility and minimization.

We introduce a new modal operator working with the minimality relation, say we call the relation R' , and the corresponding operators \Box' and \Diamond' . Being a minimal model of α can now be expressed by $m \models \alpha \wedge \neg\Diamond'\alpha$.

So $\alpha \sim \beta$ reads: $u \models \Box((\alpha \wedge \neg\Diamond'\alpha) \rightarrow \beta)$.

(*RatM*) is translated to

$$u \models \Box\left((\phi \wedge \neg\Diamond'\phi) \rightarrow \psi\right) \wedge \Diamond\left(\phi \wedge \neg\Diamond'\phi \wedge \psi'\right) \rightarrow \Box\left((\phi \wedge \psi' \wedge \neg\Diamond'(\phi \wedge \psi')) \rightarrow \psi\right).$$

The second hurdle is to handle subsets defined by accessibility from a given model m . In above example, all was done from u , with formulas. But we also have to make sure that we can handle expressions like “in all best models among those accessible from m ϕ holds”. The set of all those accessible models corresponds to some ϕ_m , and then we have to choose the best among them. In particular, we have to make sure that the axioms of our system hold not only for the models of some formulas seen from u , but also when those formulas are defined by the set of models accessible from some model m .

Let $R(m) := \{n : mRn\}$, and $\mu(X)$ be the minimal models of X .

Suppose we want to say now: If mRm' (so $R(m') \subseteq R(m)$ by transitivity), and $R(m') \cap \mu(R(m)) \neq \emptyset$, then $\mu(R(m')) = R(m') \cap \mu(R(m))$. How can we express this with modal formulas? If we write $m \models \Box\phi$, then we

know that ϕ holds everywhere in $R(m)$, but ϕ might not be precise enough to describe $R(m)$, e.g. ϕ might be TRUE.

We introduce an auxiliary modal relation R_- with operators \Box_- and \Diamond_- s.t. mR_-m' iff $\text{not}(mRm')$. (If R is not reflexive, R_- will not be either, and we change the definition accordingly.)

We can now characterize $R(m)$ by $\phi_m : m \models \Box\phi_m \wedge \neg\Diamond_- \phi_m$.

We can now express that ϕ holds in the minimal models of $R(m)$ by

$$m \models (\Box\phi_m \wedge \neg\Diamond_- \phi_m) \wedge \Box((\phi_m \wedge \neg\Diamond'\phi_m) \rightarrow \phi).$$

Finally, we can express e.g. (AND)

$$\alpha \sim \phi, \alpha \sim \phi' \Rightarrow \alpha \sim \phi \wedge \phi'$$

in the case where α is defined by some $R(m)$ as follows:

$$u \models \Box \left((\Box\phi_m \wedge \neg\Diamond_- \phi_m) \wedge \Box((\phi_m \wedge \neg\Diamond'\phi_m) \rightarrow \phi) \wedge \Box((\phi_m \wedge \neg\Diamond'\phi_m) \rightarrow \phi') \rightarrow \Box((\phi_m \wedge \neg\Diamond'\phi_m) \rightarrow \phi \wedge \phi') \right).$$

The (OR) rule will use some ϕ_m and $\phi_{m'}$, and then work with $\phi_m \vee \phi_{m'}$.

An alternative idea First, it is not sufficient just to write $m \models \Box((\neg\Diamond'TRUE) \rightarrow \phi)$ to express that ϕ holds in all models minimal among those accessible from m , as, with R' , we might leave $R(m)$, and will find this way only those models, which are globally minimal and accessible from m .

This leads to the following idea: Reactive structures allow us to modify the structure while traversing it. So we could, when we come from m , cut all R' -links leading outside $R(m)$, and would now see the minimal elements of $R(m)$. We would thus generate relations R'_m etc., for each m . Above $m \models \Box((\neg\Diamond'TRUE) \rightarrow \phi)$ would then be replaced by $m \models \Box((\neg\Diamond'_m TRUE) \rightarrow \phi)$.

There are, however, problems, like how to express $M(\phi) \subseteq R(m)$, so we adopt our above approach.

3.2 Our system

3.2.1 A two-step approach

Our approach was composed of two steps:

- (1) Find the ideal situations
- (2) Construct obligations.

We will follow this also in the representation construction.

Let the language be finite.

Let then a set of obligations be given. They can be soft, and relative, i.e. conditional. Given T , write $\overline{\overline{T}}$ for the set of obligations for T , and ϕ_T for their conjunction, i.e. $\phi_T := \bigwedge \overline{\overline{T}}$.

We can construct a ranked model iff

$$T' \vdash T \text{ and } \text{Con}(\phi_T, T') \Rightarrow (\phi_{T'} \leftrightarrow \phi_T \wedge \bigwedge T').$$

This is a direct consequence of Proposition 5.9 (page 30) and Fact 5.6 (page 28) below.

(Closure under classical logic is not necessary, any generating system is sufficient.)

We denote by \Vdash the syntactic construction of obligations, i.e. given T , ϕ_T , $\phi_T \Vdash \psi$ iff $\phi_T \vdash \psi$ and ψ is a disjunction of conjunctions ψ_i s.t. $\text{Con}(\psi_i, \phi_T)$.

We denote by \models the semantic construction of obligations, i.e. given T , ϕ_T , $\phi_T \models \psi$ iff ψ describes a Hamming neighbourhood of ϕ_T .

It was shown in Fact 2.1 (page 4) that \Vdash and \models are equivalent.

If the obligations we set out with were not only elementary ones, then they have to be of the required form already. If all obligations have the required syntactic form, we will find them again. If the obligations were all which had this form, our system is complete.

The only thing still to do is to translate it into a modal language.

Recall that in our modal context formulas may be given implicitly, by the models accessible from some model m , which we translated into

$$\phi_m \text{ by } m \models \Box\phi_m \wedge \neg\Diamond_{-}\phi_m.$$

To describe rankedness, we have to consider all possible cases, i.e. T and T' can be given by some formula (the language is finite), or one or both are given implicitly by the models accessible from some m .

ϕ_T is then the conjunction as above, and will e.g. be given by the obligations conditional to T , as seen from the entry point u , $u \models (o/T)$, so ϕ_T is the conjunction of all such o . Derived obligations are defined as above from ϕ_T and T .

If T is some ϕ_m , obligations are global obligations seen from m , $m \models o$, ϕ_T is their conjunction, and derived obligations are defined as above from ϕ_T and ϕ_m . (Conditional obligations (o/ψ) seen from m are equivalent to $(o/\psi \wedge \phi_m)$ seen from u .)

To express rankedness, we need to express T' , which may be defined implicitly, we use again ϕ_m .

Thus, we do not even need to express rankedness by a new relation, R' , as outlined above, following Boutelier, but we can, of course. What we really need in the modal context is the negative relation R_{-} , with \Box_{-} and dual \Diamond_{-} .

We summarize:

- (1) Rankedness is described by the semantic condition

$$X \subseteq Y \rightarrow \mu(Y) = \mu(X) \cap Y$$

and the syntactic condition

$$T' \vdash T \text{ and } \text{Con}(\phi_T, T') \Rightarrow (\phi_{T'} \leftrightarrow \phi_T \wedge \bigwedge T').$$

- (2) T, T' are either given, or described by ϕ_m , using R and R_{-} (or \Box and \Box_{-} etc.).
- (3) $\mu(T)$ is described by the conjunction of obligations valid in T , ϕ_T .
- (4) Derived obligations are described syntactically by suitable disjunctions of conjunctions of basic obligations, semantically by Hamming neighbourhoods of ϕ_T .

A full system is obtained by enumerating all the possible combinations how T and T' might be defined.

The representation result is obtained by putting the one for ranked structures together with the equivalence of Hamming neighbourhoods and suitable disjunctions of conjunctions.

4 Comparison to other systems, deontic paradoxa

4.1 Comparison to CJ

4.1.1 Introduction

The present paper was inspired by the important paper [CJ02] of J. Carmo and A. Jones on contrary-to-duties.

In that paper Carmo and Jones present a logical system designed to solve many of the current puzzles of contrary-to-duties. They propose a system with the connectives the unary $O(B)$ and the binary $O(B/A)$ and using these connectives give a detailed case analysis of several contrary-to-duty paradoxes.

D. Gabbay, in his paper [Gab08] proposed a reactive Kripke semantics approach to contrary-to-duties and made use of the Carmo and Jones paper to draw upon examples and analysis. Gabbay promised in his paper a critical analysis of the Carmo–Jones approach and a comparison with his own paper. Meanwhile Gabbay and Schlechta developed the reactive and hierarchical approach to conditionals [GS08d] as well as a general road map paper for preferential semantics [GS08c] and armed with this new arsenal of methods (Carmo–Jones paper was written 10 years ago), we believe we can do justice to the Carmo–Jones paper and view it against the background of current methods.

4.1.2 The Carmo-Jones system

To fix our notation etc, the following is the Carmo-Jones system, regarded formally as a logical system with axioms and semantics as proposed by Carmo-Jones. (We take the liberty to change notation slightly, and will sometimes call the system CJ system.)

Alphabet:

classical propositional logic, with 5 additional modal operators:

\Box_a with dual \Diamond_a - the actually necessary/possible

\Box_p with dual \Diamond_p - the potentially necessary/possible

$O(./.)$ a dyadic deontic operator

$O_a(.)$ monadic deontic operator: actual obligations

$O_p(.)$ monadic deontic operator: potential obligations

Semantics:

A model $\mathcal{M} = \langle W, av, pv, ob, V \rangle$ where

- (1) $W \neq \emptyset$
- (2) V an assignment function
- (3) $av : W \rightarrow \mathcal{P}(W)$ (the actually accessible worlds) such that
 - (3-a) $av(w) \neq \emptyset$
- (4) $pv : W \rightarrow \mathcal{P}(W)$ (the potentially accessible worlds) such that
 - (4-a) $av(w) \subseteq pv(w)$
 - (4-b) $w \in pv(w)$
- (5) $ob : \mathcal{P}(W) \rightarrow \mathcal{P}(\mathcal{P}(W))$ - the ‘‘morally good’’ sets
 - (5-a) $\emptyset \notin ob(X)$
 - (5-b) if $Y \cap X = Z \cap X$, then $Y \in ob(X) \Leftrightarrow Z \in ob(X)$
 - (5-c) if $Y, Z \in ob(X)$, then $Y \cap Z \in ob(X)$
 - (5-d) if $Y \subseteq X \subseteq Z$, $Y \in ob(X)$, then $(Z - X) \cup Y \in ob(Z)$

Validity in w is defined (for fixed \mathcal{M}) inductively as follows ($M(\phi)$ is the set of points where ϕ holds):

$w \models p :\Leftrightarrow w \in V(p)$

the usual conditions for classical connectives

$w \models \Box_a \phi :\Leftrightarrow av(w) \subseteq M(\phi)$

$w \models \Box_p \phi :\Leftrightarrow pv(w) \subseteq M(\phi)$

$w \models O(\phi/\psi) :\Leftrightarrow M(\phi) \cap M(\psi) \neq \emptyset$ and $\forall X (X \subseteq M(\psi), X \cap M(\phi) \neq \emptyset \Rightarrow M(\phi) \in ob(X))$

$w \models O_a \phi :\Leftrightarrow M(\phi) \in ob(av(w))$ and $av(w) \cap M(\neg\phi) \neq \emptyset$

$w \models O_p \phi :\Leftrightarrow M(\phi) \in ob(pv(w))$ and $pv(w) \cap M(\neg\phi) \neq \emptyset$

Axiomatics

(A) \Box_a and \Box_p

(1) \Box_p is a normal modal operator of type KT

(2) \Box_a is a normal modal operator of type KD

(3) $\Box_p \phi \rightarrow \Box_a \phi$

(B) Characterisation of $O(./.)$

(4) $\neg O(\perp/\psi)$

(5) $O(\phi/\psi) \wedge O(\phi'/\psi) \rightarrow O(\phi \wedge \phi'/\psi)$

- (6) $O(\phi/\psi) \rightarrow O(\phi/\phi \wedge \psi)$ (SA1)
- (7) If $\vdash \psi \leftrightarrow \psi'$, then $\vdash O(\phi/\psi) \leftrightarrow O(\phi/\psi')$
- (8) If $\vdash \psi \rightarrow (\phi \leftrightarrow \phi')$, then $\vdash O(\phi/\psi) \leftrightarrow O(\phi'/\psi)$
- (C) Relationship between $O(./.)$ and \Box_p
- (9) $\Diamond_p O(\phi/\psi) \rightarrow \Box_p O(\phi/\psi)$
- (10) $\Diamond_p(\psi \wedge \psi' \wedge \phi) \wedge O(\phi/\psi) \rightarrow O(\phi/\psi \wedge \psi')$ (SA2)
- (D) Characterization of O_a and O_p
- (11) $O_a \phi \wedge O_a \psi \rightarrow O_a(\phi \wedge \psi)$
 $O_p \phi \wedge O_p \psi \rightarrow O_p(\phi \wedge \psi)$
- (E) Relationships between O_a (O_p) and \Box_a (\Box_p)
- (12) $\Box_a \phi \rightarrow (\neg O_a \phi \wedge \neg O_a \neg \phi)$
 $\Box_p \phi \rightarrow (\neg O_p \phi \wedge \neg O_p \neg \phi)$
- (13) $\Box_a(\phi \leftrightarrow \psi) \rightarrow (O_a \phi \leftrightarrow O_a \psi)$
 $\Box_p(\phi \leftrightarrow \psi) \rightarrow (O_p \phi \leftrightarrow O_p \psi)$
- (F) Relationships between $O(./.)$, O_a (O_p) and \Box_a (\Box_p)
- (14) $O(\phi/\psi) \wedge \Box_a \psi \wedge \Diamond_a \phi \wedge \Diamond_a \neg \phi \rightarrow O_a \phi$
 $O(\phi/\psi) \wedge \Box_p \psi \wedge \Diamond_p \phi \wedge \Diamond_p \neg \phi \rightarrow O_p \phi$
- (15) $O(\phi/\psi) \wedge \Diamond_a(\phi \wedge \psi) \wedge \Diamond_a(\psi \wedge \neg \phi) \rightarrow O_a(\psi \rightarrow \phi)$
 $O(\phi/\psi) \wedge \Diamond_p(\phi \wedge \psi) \wedge \Diamond_p(\psi \wedge \neg \phi) \rightarrow O_p(\psi \rightarrow \phi)$

4.1.3 Methodological discussion

We believe that Carmo and Jones important insight was that to solve contrary-to-duty and other Deontic paradoxes we need a wider family of operators capable of describing a wider context surrounding the problematic paradoxes. We agree with this view wholeheartedly. Gabbay's papers [Gab08] and [Gab08a] use reactive semantics to create such a context and the present paper will use hierarchical modality to create essentially the same context.

It would be useful to describe the methodology we use.

Viewed formally, we have here a logical system CJ proposed by Carmo-Jones and a proposed semantics $\mathcal{M}(CJ)$ for it, intended to be applied to the contrary-to-duties application area CTD.

How can we evaluate (improve on) such a system?

Let us list the methodological parameters involved.

The semantics proposed must be compatible with the intended application. This means that the spirit of the semantics must correspond to the application.

We explain by an example. Consider modal logic S4 and assume we are trying to apply it to the analysis of the tenses of natural language.

The phrase "A is true from now on" can be modelled by $\Box A$.

The phrase "John is reading now" i.e. the progressive tense can also be modelled as $\Box(\text{John is reading})$.

Both examples give rise to modal S4. However the Kripke accessibility relation for S4 is the semantics suitable for the "from now on" linguistic construction, while the McKinsey-Tarski open intervals semantics for S4 is more suitable for the analysis of the progressive. (Sentences A are assigned intervals $W(A)$ and $\Box A$ is read as the topological interior of $W(A)$.)

Carmo-Jones indeed offer an analysis of the compatibility of their system in Section 6 of their paper. We will examine that.

Soundness and completeness We ask whether the system is sound and complete for the semantics. If not, what axioms do we need to add to the system or what changes do we propose to the system to obtain correspondence? We will find that CJ is not complete for the proposed semantics.

4.1.4 Discrepancies inside the CJ system

A closer look at semantics and proof theory reveals a certain asymmetry in the treatment of unary vs. binary obligations, and elsewhere:

- (1) Unary obligations are dependent on accessibility relations av and pv , binary ones are not. As a consequence, unary obligations depend on the world we are in, binary ones do not.
- (2) Unary obligations must not be trivial, i.e. the contrary must be possible, binary ones can be trivial.
- (3) (And perhaps deepest) Binary obligations postulate additional properties of the basic choice function ob (which makes it essentially ranked), unary obligations need only basic properties (essentially corresponding to a not necessarily smooth preferential relation). This property is put into the validity condition, and not into rules as one would usually expect.
- (4) In the validity condition for $O(B/A)$ we have $X \subseteq M(A)$ and $X \cap M(B) \neq \emptyset$, in the syntactic condition (SA2) we have $\diamond(A \wedge B \wedge C) \wedge O(C/B) \vee O(C/A \wedge B)$. These two coincide only if \diamond is consistency - i.e. the underlying relation is the trivial universal one.
- (5) Semantic condition 5-d) gives essentially the condition for a preferential structure, an analogue on the syntactical side is missing - see Example 4.1 (page 17) below, which shows that the axioms are not complete for the semantics.

4.1.5 Incompleteness of the CJ system

Example 4.1

Let \mathcal{L} be defined by p, q , $W := M_{\mathcal{L}}$ be the set of its models.

Let $m_1 \models p \wedge q$, $m_2 \models p \wedge \neg q$, $M_1 := \{m_1\}$, $M_2 := \{m_2\}$.

We write $M(A)$ for the set of models of A .

Set $ob(M_1) := \{M \subseteq M_{\mathcal{L}} : M_1 \subseteq M\}$, $ob(M_2) := \{M \subseteq M_{\mathcal{L}} : M_2 \subseteq M\}$, $ob(M) := \emptyset$ for all other M .

Let $av(w) := pv(w) := W$ for all $w \in W$, i.e. both are defined by wRw' for all w, w' .

Thus, $O_a = O_p$, there is only one \Box , etc., and $M \models_w \Box A$ iff A is a tautology.

$M \models_w OA$ will never hold, as $av(w) = W$, and $ob(W) = \emptyset$.

$M \models_w O(B/A)$ is independent from w , so we write just $M \models O(B/A)$.

Suppose $M \models O(B/A)$ holds, then $M(A) \cap M(B) \neq \emptyset$, and thus $M(B) \in ob(M(A))$. So A has to be (equivalent to) $p \wedge q$ or $p \wedge \neg q$. But the only subsets of $M(A)$ are then \emptyset and $M(A)$, and we have $O(\phi/p \wedge q)$ iff $\vdash p \wedge q \rightarrow \phi$, and $O(\phi/p \wedge \neg q)$ iff $\vdash p \wedge \neg q \rightarrow \phi$. No other $O(A/B)$ hold.

We check the axioms (page 293-294) of [CJ02]:

1-5 are trivial.

6. is trivial, as $M \models O(B/A)$ implies $\vdash A \rightarrow B$.

7. is trivial.

8. Let $O(A/C)$, $\vdash C \rightarrow (A \rightarrow B)$, then $\vdash C \rightarrow A$, so $\vdash C \rightarrow B$, so $O(B/C)$.

9. trivial.

10. If $O(C/B)$ and $Con(A, B, C)$, then $\vdash B \rightarrow A$, as B is complete, so $\vdash A \wedge B \leftrightarrow B$.

11. is void.

12.-13. trivial

14. If $O(B/A)$, then $\neg \Box A$.

15. If $O(B/A)$, then $\vdash A \rightarrow B$, so $\diamond(A \wedge \neg B)$ is impossible.

Thus, our example satisfies the CJ axioms.

If the system were to satisfy 5-d), then $M(p) = \{m_1, m_2\} \in ob(M(p))$, and we would have $O(p/p)$:

First, $M(p) \cap M(p) \neq \emptyset$. We then have to consider $X = M(p)$, M_1, M_2 . But $M(p) \in ob(M_1) \cap ob(M_2) \cap ob(M(p))$, thus $O(p/p)$ holds.

□

4.1.6 Simplifications of the CJ system

We make now some simplifications which will help us to understand this article.

(1) We assume the language is finite, thus we will not have any problems with non-definable model sets - see e.g. [GS08c] for an illustration of what can happen otherwise.

(2) We assume that $ob(X) \subseteq \mathcal{P}(X)$. This is justified by the following fact, which follows immediately from the system of CJ, condition 5-b):

Fact 4.1

If $A \in ob(X)$, $A \subseteq X$, $B \subseteq W - X$, then $A \cup B \in ob(X)$. Conversely, if $A \in ob(X)$, then $A \cap X \in ob(X)$.

Thus, what is outside X , does not matter, and we can concentrate on the inside of X . (Of course, the validity condition has then to be modified, $M(B) \in ob(X)$ will be replaced by: There is $X' \in ob(X)$, $X' = M(B) \cap X$.)

By 5-c), ob is closed under finite intersection, by overall finiteness, there is thus a smallest (by \subseteq) $A \in ob(X)$. We call this $\mu(X)$. Thus, $\mu(X) \subseteq X$, which is condition $(\mu \subseteq)$. (If the language is not finite, we would have to work with the limit version. As we work with formulas only, this would not present a fundamental problem.)

Let $X \subseteq Z$, $Y := \mu(X)$, then by 5-d) $((Z - X) \cup Y) \in ob(Z)$, so $\mu(Z) \subseteq ((Z - X) \cup Y)$, or $\mu(Z) \cap X \subseteq \mu(X)$, which is condition (μPR) .

We thus have that μ satisfies $(\mu \subseteq)$ and (μPR) , and we know that this suffices for a representation by preferential structures - see e.g. [Sch92] and Section 5 (page 21) .

Thus, the basic choice function ob is preferential for unary O .

Note that $(\mu \subseteq) + (\mu PR)$ imply (μOR) : $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$ - see [GS08c] and Section 5 (page 21) .

When we look now at the truth conditions for O_a and O_i , we see that we first go to the accessible worlds - $av(w)$ or $pv(w)$ - and check whether $\mu(av(w)) \subseteq M(A)$ respectively $\mu(pv(w)) \subseteq M(A)$ (and whether $\neg A$ is possible). Thus, in preferential terms, whether $av(w) \vdash A$, but $av(w) \not\vdash A$.

The case of $O(B/A)$ is a bit more complicated and is partly dissociated from O_a and O_i .

We said already above that $O(B/A)$ is independent from av and pv , and from w .

Second, and more importantly, the condition for $O(B/A)$ implies a converse of (μOR) or (μPR) :

- (1) Setting $X := M(A)$, we have $\mu(M(A)) \subseteq M(B)$,
- (2) as all sets are definable, we can choose B s.t. $\mu(M(A)) = M(B)$,
- (3) for $X \subseteq M(A)$, we have - using (2) - $\mu(X) \subseteq \mu(M(A)) \cap X$ if $X \cap \mu(M(A)) \neq \emptyset$.

We thus have - if $O(B/A)$ holds - together with (μPR) that $(\mu =)$ holds, i.e.

$X \subseteq Y, X \cap \mu(Y) \neq \emptyset \Rightarrow \mu(X) = \mu(Y) \cap X$.

By 5-a) $\mu(X) \neq \emptyset$, so $(\mu \emptyset)$ holds, too, and by [Sch04], see also [GS08c] and Section 5 (page 21) , we know that such μ can be represented by a ranked smooth structure where all elements occur in one copy only.

Thus, the basic choice function ob is ranked for binary $O(B/A)$.

4.1.7 Suggested modifications of the CJ system

- (1) We assume finiteness (see above)

- (2) We work with the smallest element of $ob(X)$ (see above)
- (3) We use only one accessibility relation (or operation) a . This is justified, as we are mainly interested in formal properties here.
- (4) We make both O and $O(./.)$ dependent on a . So validity of $O(./.)$ depends on w , too.
This eliminates one discrepancy between O and $O(./.)$.
- (5) We allow both O and $O(./.)$ to be trivial. We could argue here philosophically, e.g.: if you are unable to kill your grandmother, should you then not any longer be obliged not to kill her? (No laws for jail inmates?) But we do this rather by laziness, to simplify the basic machinery.
This eliminates a second discrepancy.
- (6) We take rankedness as a basic condition for ob , so it does not depend any more on validity of some $O(./.)$.

4.1.8 Translation of CJ to our system and differences

We have to define av , pv , ob , $O(\phi/\psi)$, $O_a(\phi)$, $O_p(\phi)$ in our system.

As av/pv O_a/O_p do not differ substantially, we just treat av and O_a .

Let av be expressed by R . Then $av(m) = R(m) = M(\phi_m)$.

$ob(X) = \mu(X)$, where $\mu(X)$ is described by $\bigwedge \overline{T}$, if $X = M(T)$, i.e. the conjunction of all obligations which hold in X .

$O_a(\phi)$ iff ϕ is an obligation which holds in $R(m)$, i.e. iff ϕ describes a Hamming environment of $\bigwedge \overline{\phi_m}$.

We also make $O(\phi/\psi)$ dependent on m and R :

$O(\phi/\psi)$ iff ϕ is an obligation which holds in $R(M) \cap M(\psi)$, i.e. iff ϕ describes a Hamming environment of $\bigwedge (\overline{\phi_m} \wedge \psi)$.

Probably the main conceptual difference between CJ and GS is that we do not only give a semantics for the ideal case (which is implicit in the CJ system: a ranked structure), but also a semantics (and syntactic derivation rules) for the obligations themselves, which protects them against arbitrary weakening of the ideal situation.

4.1.9 Comments on CJ Section 6

- We do not quite understand the derived obligation to kill and offer a cigarette. We think this should rather be: $O(\neg kill)$, $O(\neg offer)$, $O(offer/kill)$.
- P. 317, violation of $O(B/A)$, a better definition seems to be:
 m violates $O(B/A)$ iff in m holds:
 $\diamond^-(O(B/A) \wedge \diamond(A \wedge B)) \wedge A \wedge \neg B$
(\diamond^- is the inverse relation).
In other words: in some antecedent, $O(B/A)$ was postulated, and A and B were possible, but now (i.e. in m) $A \wedge \neg B$ holds. (We can strengthen: $m \models \square(A \wedge \neg B)$.)
- We also think that temporal developments and intentions should better be coded explicitly, as implicit coding often leads to counterintuitive results. It is not our aim to treat such aspects here.

4.2 Comments on Meyer et al.

These are comments on the article [MDW94], Section 2.1, “Some well-known paradoxes”, there. Note that $F(\phi)$ stands for “ ϕ is forbidden”, which is equivalent to $O(\neg\phi)$, see the following section there.

- It seems that 1., 2., 5. all have the same origin, i.e. unrestricted weakening.
- 4. should be $F(\phi)$, but $F(\neg\psi/\phi)$

- 6. should read $O(\phi) \wedge O(\psi/\phi) \wedge O(\neg\psi/\neg\phi) \wedge \Box\neg\phi$ - and there is nothing very wrong with it.
- 7. should be $F(\phi) \wedge O(\psi/\phi)$ etc. - there is nothing wrong with it
- 8. seems out of the scope of our discussion
- 9. also seems to be better written as conditional obligations.
- 12. should read $O(K\phi/\phi)$

4.3 Comparison to the \mathcal{A} -ranked semantics of [GS08d]

We point out here the main points of [CJ02], [GS08d], and the present article, which differentiate them from the others.

- The Carmo-Jones article
 - (1) It contains much material on motivation, and discussion of examples and paradoxa.
 - (2) It gives an account of the differences between describing situations and valid obligations.
 - (3) It presents a descriptive semantics.
 - (4) It puts the operators in the object language and uses a modal logic language, as usual in the field.
- The Gabbay-Schlechta article on \mathcal{A} -ranked semantics
 - (1) It contains a relatively exhaustive semantics for the ideal cases in contrary-to-duty obligations.
 - The \mathcal{A} -ranked semantics allows us to express that a whole hierarchy of obligations (if ... possible, then ...; if not, but ..., then ...; ...) is satisfied, i.e. the agent “does his best”. This hierarchy is directly built into the semantics, which is a multi-layered, semi-ranked structure, which can also be re-used in other contexts.
 - The article contains a sound and complete characterization of the semantics with full proofs.
 - The language is that of usual nonmonotonic logics, i.e. rules are given in the meta-language.
 - (2) Paradoxa like the Ross paradoxon are not treated at all, we only treat the ideal case, and not individual obligations.
 - (3) The additional accessibility relation is added without changing the overall language to a modal flavour.
- The present article
 - (1) As motivation, we essentially ruminate the Ross paradoxon and the polite assassin example.
 - (2) Thus, our main interests are in weakening the weakening of classical logic, and to admit obligations with exceptions. This is discussed quite in extenso.
 - (3) We work with a ranked structure describing ideal situations as usual, but go further from those ideal situations to reason about individual obligations.
Our approach is thus a 2-step process, first finding the ideal cases, and then looking at derivable obligations.
 - (4) We present a neighbourhood semantics and a corresponding syntactic characterization for individual obligations. The neighbourhood is a neighbourhood of the ideal cases, so we need to know them. The (rather trivial) correspondence between syntax and semantics is shown. Possible shortcomings (lack of closure of the neighbourhoods, changes of language) are discussed and solutions sketched.
 - (5) We fully integrate the underlying logic for the ideal cases in a modal framework, using an idea by Boutelier and Lamarre, and extending it with a complementary relation to precisely characterize the successor sets.

5 Definitions and proofs

Definition 5.1

We use \mathcal{P} to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \rightarrow \bigcup\{X_i : i \in I\}, \forall i \in I. g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X , and V the set-theoretic universe we work in - the class of all sets. Given a set of pairs \mathcal{X} , and a set X , we denote by $\mathcal{X} \upharpoonright X := \{\langle x, i \rangle \in \mathcal{X} : x \in X\}$. When the context is clear, we will sometime simply write X for $\mathcal{X} \upharpoonright X$. (The intended use is for preferential structures, where x will be a point (intention: a classical propositional model), and i an index, permitting copies of logically identical points.)

$A \subseteq B$ will denote that A is a subset of B or equal to B , and $A \subset B$ that A is a proper subset of B , likewise for $A \supseteq B$ and $A \supset B$.

Given some fixed set U we work in, and $X \subseteq U$, then $\mathcal{C}(X) := U - X$.

If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X , we say that \mathcal{Y} satisfies

(\cap) iff it is closed under finite intersections,

(\bigcap) iff it is closed under arbitrary intersections,

(\cup) iff it is closed under finite unions,

(\bigcup) iff it is closed under arbitrary unions,

(\mathcal{C}) iff it is closed under complementation,

($-$) iff it is closed under set difference.

We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.

We make ample and tacit use of the Axiom of Choice.

Definition 5.2

We work here in a classical propositional language \mathcal{L} , a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ , etc., theories T, S , etc.

$v(\mathcal{L})$ will be the set of propositional variables of \mathcal{L} .

$M_{\mathcal{L}}$ will be the set of (classical) models for \mathcal{L} , $M(T)$ or M_T is the set of models of T , likewise $M(\phi)$ for a formula ϕ .

$\mathcal{D}_{\mathcal{L}} := \{M(T) : T \text{ a theory in } \mathcal{L}\}$, the set of definable model sets.

Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \mathcal{D}_{\mathcal{L}}$, $\mathcal{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.

An operation $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called definability preserving, (dp) or (μdp) in short, iff for all $X \in \mathcal{D}_{\mathcal{L}} \cap \mathcal{Y}$ $f(X) \in \mathcal{D}_{\mathcal{L}}$.

We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ - as needed for theory revision - with the obvious meaning.

\vdash will be classical derivability, and

$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under \vdash .

$Con(\cdot)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is classical consistent, likewise for $Con(T)$.

$Con(T, T')$ will stand for $Con(T \cup T')$, etc.

Given a consequence relation \vdash , we define

$\overline{\overline{T}} := \{\phi : T \vdash \phi\}$.

(There is no fear of confusion with $\overline{\overline{T}}$, as it just is not useful to close twice under classical logic.)

$T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.

If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m)$, $m \in M_{\mathcal{L}}$. (\models will usually be classical validity.)

Definition 5.3

We introduce here formally a list of properties of set functions on the algebraic side, and their corresponding logical rules on the other side.

Recall that $\overline{T} := \{\phi : T \vdash \phi\}$, $\overline{\overline{T}} := \{\phi : T \sim \phi\}$, where \vdash is classical consequence, and \sim any other consequence.

We show, wherever adequate, in parallel the formula version in the left column, the theory version in the middle column, and the semantical or algebraic counterpart in the right column. The algebraic counterpart gives conditions for a function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, where U is some set, and $\mathcal{Y} \subseteq \mathcal{P}(U)$.

Precise connections between the columns are given in Proposition 5.2 (page 24)

When the formula version is not commonly used, we omit it, as we normally work only with the theory version.

A and B in the right hand side column stand for $M(\phi)$ for some formula ϕ , whereas X, Y stand for $M(T)$ for some theory T .

Basics		
(AND) $\phi \sim \psi, \phi \sim \psi' \Rightarrow$ $\phi \sim \psi \wedge \psi'$	(AND) $T \sim \psi, T \sim \psi' \Rightarrow$ $T \sim \psi \wedge \psi'$	Closure under finite intersection
(OR) $\phi \sim \psi, \phi' \sim \psi \Rightarrow$ $\phi \vee \phi' \sim \psi$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ OR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(wOR) $\phi \sim \psi, \phi \vdash \psi \Rightarrow$ $\phi \vee \phi' \sim \psi$	(wOR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ wOR) $f(X \cup Y) \subseteq f(X) \cup Y$
(disjOR) $\phi \vdash \neg \phi', \phi \sim \psi,$ $\phi' \sim \psi \Rightarrow \phi \vee \phi' \sim \psi$	(disjOR) $\neg \text{Con}(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μ disjOR) $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(LLE) Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \sim \psi \Rightarrow$ $\phi' \sim \psi$	(LLE) $\overline{\overline{T}} = \overline{\overline{T'}} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$	trivially true
(RW) Right Weakening $\phi \sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $\phi \sim \psi'$	(RW) $T \sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $T \sim \psi'$	upward closure
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ is classically closed	trivially true
(SC) Supraclassicality $\phi \vdash \psi \Rightarrow \phi \sim \psi$	(SC) $\overline{\overline{T}} \subseteq \overline{\overline{T}}$	($\mu \subseteq$) $f(X) \subseteq X$
(REF) Reflexivity $T \cup \{\alpha\} \sim \alpha$		
(CP) Consistency Preservation $\phi \sim \perp \Rightarrow \phi \vdash \perp$	(CP) $T \sim \perp \Rightarrow T \vdash \perp$	($\mu \emptyset$) $f(X) = \emptyset \Rightarrow X = \emptyset$
		($\mu \emptyset \text{fin}$) $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\phi \cup \phi'}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{T \cup T'}}$	(μ PR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$
		(μ PR')
		$f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $T \sim \alpha; T \cup \{\alpha\} \sim \beta \Rightarrow$ $T \sim \beta$	(CUT) $T \subseteq \overline{\overline{T'}} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	(μ CUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$

Cumulativity		
(<i>CM</i>) Cautious Monotony $\phi \sim \psi, \phi \sim \psi' \Rightarrow$ $\phi \wedge \psi \sim \psi'$	(<i>CM</i>) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	(μ <i>CM</i>) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or (<i>ResM</i>) Restricted Monotony $T \sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \sim \beta$		(μ <i>ResM</i>) $f(X) \subseteq A \cap B \Rightarrow f(X \cap A) \subseteq B$
(<i>CUM</i>) Cumulativity $\phi \sim \psi \Rightarrow$ $(\phi \sim \psi' \Leftrightarrow \phi \wedge \psi \sim \psi')$	(<i>CUM</i>) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	(μ <i>CUM</i>) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	($\subseteq \supseteq$) $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	($\mu \subseteq \supseteq$) $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality		
(<i>RatM</i>) Rational Monotony $\phi \sim \psi, \phi \not\sim \neg\psi' \Rightarrow$ $\phi \wedge \psi' \sim \psi$	(<i>RatM</i>) $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'}} \cup T$	(μ <i>RatM</i>) $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	(<i>RatM</i> =) $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}} \cup T$	(μ =) $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	(<i>Log</i> =') $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{T'}} \cup T$	(μ =') $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
	(<i>Log</i>) $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}},$ or $\overline{\overline{T'}},$ or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	(μ) $f(X \cup Y)$ is one of $f(X), f(Y)$ or $f(X) \cup f(Y)$
	(<i>Log</i> \cup) $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	($\mu \cup$) $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	(<i>Log</i> \cup') $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	($\mu \cup'$) $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
		($\mu \in$) $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$

(*PR*) is also called infinite conditionalization - we choose the name for its central role for preferential structures (*PR*) or (μ *PR*).

The system of rules (*AND*) (*OR*) (*LLE*) (*RW*) (*SC*) (*CP*) (*CM*) (*CUM*) is also called system *P* (for preferential), adding (*RatM*) gives the system *R* (for rationality or rankedness).

Roughly: Smooth preferential structures generate logics satisfying system *P*, ranked structures logics satisfying system *R*.

A logic satisfying (*REF*), (*ResM*), and (*CUT*) is called a consequence relation.

(*LLE*) and (*CCL*) will hold automatically, whenever we work with model sets.

(*AND*) is obviously closely related to filters, and corresponds to closure under finite intersections. (*RW*) corresponds to upward closure of filters.

More precisely, validity of both depend on the definition, and the direction we consider.

Given f and ($\mu \subseteq$), $f(X) \subseteq X$ generates a principal filter: $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition: If $X = M(T)$, then $T \sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (*AND*) and (*RW*) are then trivial.

Conversely, we can define for $X = M(T)$

$$\mathcal{X} := \{X' \subseteq X : \exists \phi (X' = X \cap M(\phi) \text{ and } T \sim \phi)\}.$$

(*AND*) then makes \mathcal{X} closed under finite intersections, (*RW*) makes \mathcal{X} upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete \mathcal{X} by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.

Alternatively, we can define

$$\mathcal{X} := \{X' \subseteq X : \bigcap \{X \cap M(\phi) : T \sim \phi\} \subseteq X'\}.$$

(*SC*) corresponds to the choice of a subset.

(*CP*) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.

(*PR*) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ , T' for ψ , and $\phi \wedge \psi \vdash \sigma$, so $\phi \vdash \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.

(*CUM*) (whose more interesting half in our context is (*CM*)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold.

Fact 5.1

The following table is to be read as follows: If the left hand side holds for some function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, and the auxiliary properties noted in the middle also hold for f or \mathcal{Y} , then the right hand side will hold, too - and conversely.

Basics			
(1.1)	(μPR)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu PR')$
(1.2)		\Leftarrow	
(2.1)	(μPR)	$\Rightarrow (\mu \subseteq)$	(μOR)
(2.2)		$\Leftarrow (\mu \subseteq) + (-)$	
(2.3)		$\Rightarrow (\mu \subseteq)$	(μwOR)
(2.4)		$\Leftarrow (\mu \subseteq) + (-)$	
(3)	(μPR)	\Rightarrow	(μCUT)
(4)	$(\mu \subseteq) + (\mu \subseteq \supseteq) + (\mu CUM) +$ $(\mu RatM) + (\cap)$	$\not\Rightarrow$	(μPR)
Cumulativity			
(5.1)	(μCM)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu ResM)$
(5.2)		\Leftarrow (infin.)	
(6)	$(\mu CM) + (\mu CUT)$	\Leftrightarrow	(μCUM)
(7)	$(\mu \subseteq) + (\mu \subseteq \supseteq)$	\Rightarrow	(μCUM)
(8)	$(\mu \subseteq) + (\mu CUM) + (\cap)$	\Rightarrow	$(\mu \subseteq \supseteq)$
(9)	$(\mu \subseteq) + (\mu CUM)$	$\not\Rightarrow$	$(\mu \subseteq \supseteq)$
Rationality			
(10)	$(\mu RatM) + (\mu PR)$	\Rightarrow	$(\mu =)$
(11)	$(\mu =)$	\Rightarrow	(μPR) ,
(12.1)	$(\mu =)$	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu =')$,
(12.2)		\Leftarrow	
(13)	$(\mu \subseteq), (\mu =)$	$\Rightarrow (\cup)$	$(\mu \cup)$,
(14)	$(\mu \subseteq), (\mu \emptyset), (\mu =)$	$\Rightarrow (\cup)$	$(\mu \parallel), (\mu \cup'), (\mu CUM)$,
(15)	$(\mu \subseteq) + (\mu \parallel)$	$\Rightarrow (-)$ of \mathcal{Y}	$(\mu =)$,
(16)	$(\mu \parallel) + (\mu \in) + (\mu PR) +$ $(\mu \subseteq)$	$\Rightarrow (\cup) + \mathcal{Y}$ contains singletons	$(\mu =)$,
(17)	$(\mu CUM) + (\mu =)$	$\Rightarrow (\cup) + \mathcal{Y}$ contains singletons	$(\mu \in)$,
(18)	$(\mu CUM) + (\mu =) + (\mu \subseteq)$	$\Rightarrow (\cup)$	$(\mu \parallel)$,
(19)	$(\mu PR) + (\mu CUM) + (\mu \parallel)$	\Rightarrow sufficient, e.g. true in $\mathbf{D}_{\mathcal{L}}$	$(\mu =)$.
(20)	$(\mu \subseteq) + (\mu PR) + (\mu =)$	$\not\Rightarrow$	$(\mu \parallel)$,
(21)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel)$	$\not\Rightarrow$ (without $(-)$)	$(\mu =)$
(22)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel) +$ $(\mu =) + (\mu \cup)$	$\not\Rightarrow$	$(\mu \in)$ (thus not representability by ranked structures)

Proposition 5.2

The following table is to be read as follows:

Let a logic \vdash satisfy (*LLE*) and (*CCL*), and define a function $f : \mathbf{D}_{\mathcal{L}} \rightarrow \mathbf{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp) , and $\overline{\overline{T}} = Th(f(M(T)))$.

If \vdash satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for \Rightarrow hold, too - f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is a function, with $D_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic \vdash by $\overline{\overline{T}} := Th(f(M(T)))$, then \vdash satisfies (LLE) and (CCL). If f satisfies (μdp) , then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for \Leftarrow hold, too - \vdash will satisfy the property in the left hand side.

If “formula” is noted in the table, this means that, if one of the theories (the one named the same way in Definition 5.3 (page 21)) is equivalent to a formula, we do not need (μdp) .

Basics			
(1.1)	(OR)	\Rightarrow	(μOR)
(1.2)		\Leftarrow	
(2.1)	(disjOR)	\Rightarrow	$(\mu disjOR)$
(2.2)		\Leftarrow	
(3.1)	(wOR)	\Rightarrow	(μwOR)
(3.2)		\Leftarrow	
(4.1)	(SC)	\Rightarrow	$(\mu \subseteq)$
(4.2)		\Leftarrow	
(5.1)	(CP)	\Rightarrow	$(\mu \emptyset)$
(5.2)		\Leftarrow	
(6.1)	(PR)	\Rightarrow	(μPR)
(6.2)		$\Leftarrow (\mu dp) + (\mu \subseteq)$	
(6.3)		\Leftarrow without (μdp)	
(6.4)		$\Leftarrow (\mu \subseteq)$ T' a formula	
(6.5)	(PR)	\Leftarrow T' a formula	$(\mu PR')$
(7.1)	(CUT)	\Rightarrow	(μCUT)
(7.2)		\Leftarrow	
Cumulativity			
(8.1)	(CM)	\Rightarrow	(μCM)
(8.2)		\Leftarrow	
(9.1)	(ResM)	\Rightarrow	$(\mu ResM)$
(9.2)		\Leftarrow	
(10.1)	$(\subseteq \supseteq)$	\Rightarrow	$(\mu \subseteq \supseteq)$
(10.2)		\Leftarrow	
(11.1)	(CUM)	\Rightarrow	(μCUM)
(11.2)		\Leftarrow	
Rationality			
(12.1)	(RatM)	\Rightarrow	$(\mu RatM)$
(12.2)		$\Leftarrow (\mu dp)$	
(12.3)		\Leftarrow without (μdp)	
(12.4)		\Leftarrow T a formula	
(13.1)	(RatM =)	\Rightarrow	$(\mu =)$
(13.2)		$\Leftarrow (\mu dp)$	
(13.3)		\Leftarrow without (μdp)	
(13.4)		\Leftarrow T a formula	
(14.1)	(Log =')	\Rightarrow	$(\mu =')$
(14.2)		$\Leftarrow (\mu dp)$	
(14.3)		\Leftarrow without (μdp)	
(14.4)		$\Leftarrow T$ a formula	
(15.1)	(Log)	\Rightarrow	(μ)
(15.2)		\Leftarrow	
(16.1)	(Log \cup)	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu \cup)$
(16.2)		$\Leftarrow (\mu dp)$	
(16.3)		\Leftarrow without (μdp)	
(17.1)	(Log \cup')	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu \cup')$
(17.2)		$\Leftarrow (\mu dp)$	
(17.3)		\Leftarrow without (μdp)	

Definition 5.4

Fix $U \neq \emptyset$, and consider arbitrary X . Note that this X has not necessarily anything to do with U , or \mathcal{U} below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V - where V is the set theoretical universe we work in.

(A) Preferential models or structures.

(1) The version without copies:

A pair $\mathcal{M} := \langle U, \prec \rangle$ with U an arbitrary set, and \prec an arbitrary binary relation on U is called a preferential model or structure.

(2) The version with copies:

A pair $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ with \mathcal{U} an arbitrary set of pairs, and \prec an arbitrary binary relation on \mathcal{U} is called a preferential model or structure.

If $\langle x, i \rangle \in \mathcal{U}$, then x is intended to be an element of U , and i the index of the copy.

We sometimes also need copies of the relation \prec , we will then replace \prec by one or several arrows α attacking non-minimal elements, e.g. $x \prec y$ will be written $\alpha : x \rightarrow y$, $\langle x, i \rangle \prec \langle y, i \rangle$ will be written $\alpha : \langle x, i \rangle \rightarrow \langle y, i \rangle$, and finally we might have $\langle \alpha, k \rangle : x \rightarrow y$ and $\langle \alpha, k \rangle : \langle x, i \rangle \rightarrow \langle y, i \rangle$, etc.

(B) Minimal elements, the functions $\mu_{\mathcal{M}}$

(1) The version without copies:

Let $\mathcal{M} := \langle U, \prec \rangle$, and define

$$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg \exists x' \in X \cap U. x' \prec x\}.$$

$\mu_{\mathcal{M}}(X)$ is called the set of minimal elements of X (in \mathcal{M}).

Thus, $\mu_{\mathcal{M}}(X)$ is the set of elements such that there is no smaller one in X .

(2) The version with copies:

Let $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ be as above. Define

$$\mu_{\mathcal{M}}(X) := \{x \in X : \exists \langle x, i \rangle \in \mathcal{U}. \neg \exists \langle x', i' \rangle \in \mathcal{U} (x' \in X \wedge \langle x', i' \rangle \prec \langle x, i \rangle)\}.$$

Thus, $\mu_{\mathcal{M}}(X)$ is the projection on the first coordinate of the set of elements such that there is no smaller one in X .

Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of minimal elements of X in the structure. If the context is clear, we will also write just μ .

We sometimes say that $\langle x, i \rangle$ “kills” or “minimizes” $\langle y, j \rangle$ if $\langle x, i \rangle \prec \langle y, j \rangle$. By abuse of language we also say a set X kills or minimizes a set Y if for all $\langle y, j \rangle \in \mathcal{U}$, $y \in Y$ there is $\langle x, i \rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x, i \rangle \prec \langle y, j \rangle$.

\mathcal{M} is also called injective or 1-copy, iff there is always at most one copy $\langle x, i \rangle$ for each x . Note that the existence of copies corresponds to a non-injective labelling function - as is often used in nonclassical logic, e.g. modal logic.

We say that \mathcal{M} is transitive, irreflexive, etc., iff \prec is.

Note that $\mu(X)$ might well be empty, even if X is not.

Definition 5.5

We define the consequence relation of a preferential structure for a given propositional language \mathcal{L} .

(A)

(1) If m is a classical model of a language \mathcal{L} , we say by abuse of language

$$\langle m, i \rangle \models \phi \text{ iff } m \models \phi,$$

and if X is a set of such pairs, that

$$X \models \phi \text{ iff for all } \langle m, i \rangle \in X \text{ } m \models \phi.$$

(2) If \mathcal{M} is a preferential structure, and X is a set of \mathcal{L} -models for a classical propositional language \mathcal{L} , or a set of pairs $\langle m, i \rangle$, where the m are such models, we call \mathcal{M} a classical preferential structure or model.

(B)

Validity in a preferential structure, or the semantical consequence relation defined by such a structure:

Let \mathcal{M} be as above.

We define:

$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e. $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

\mathcal{M} will be called definability preserving iff for all $X \in \mathbf{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \mathbf{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\mathbf{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 5.6

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, \mathcal{Y} will be $\mathbf{D}_{\mathcal{L}}$.)

A preferential structure \mathcal{M} is called \mathcal{Y} -smooth iff for every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X . More precisely:

(1) The version without copies:

If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:

If $x \in X \in \mathcal{Y}$, and $\langle x, i \rangle \in \mathcal{U}$, then either there is no $\langle x', i' \rangle \in \mathcal{U}, x' \in X, \langle x', i' \rangle \prec \langle x, i \rangle$ or there is $\langle x', i' \rangle \in \mathcal{U}, \langle x', i' \rangle \prec \langle x, i \rangle, x' \in X$, s.t. there is no $\langle x'', i'' \rangle \in \mathcal{U}, x'' \in X$, with $\langle x'', i'' \rangle \prec \langle x', i' \rangle$.

When considering the models of a language \mathcal{L} , \mathcal{M} will be called smooth iff it is $\mathbf{D}_{\mathcal{L}}$ -smooth; $\mathbf{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set \mathcal{Y} is, the stronger the condition \mathcal{Y} -smoothness will be.

Fact 5.3

Let \prec be an irreflexive, binary relation on X , then the following two conditions are equivalent:

(1) There is Ω and an irreflexive, total, binary relation \prec' on Ω and a function $f : X \rightarrow \Omega$ s.t. $x \prec y \leftrightarrow f(x) \prec' f(y)$ for all $x, y \in X$.

(2) Let $x, y, z \in X$ and $x \perp y$ wrt. \prec (i.e. neither $x \prec y$ nor $y \prec x$), then $z \prec x \rightarrow z \prec y$ and $x \prec z \rightarrow y \prec z$.

□

Definition 5.7

We call an irreflexive, binary relation \prec on X , which satisfies (1) (equivalently (2)) of Fact 5.3 (page 27), ranked. By abuse of language, we also call a preferential structure $\langle X, \prec \rangle$ ranked, iff \prec is.

Fact 5.4

If \prec on X is ranked, and free of cycles, then \prec is transitive.

Proof

Let $x \prec y \prec z$. If $x \perp z$, then $y \succ z$, resulting in a cycle of length 2. If $z \prec x$, then we have a cycle of length 3. So $x \prec z$. □

Remark 5.5

Note that $(\mu =')$ is very close to $(\text{Rat}M)$: $(\text{Rat}M)$ says: $\alpha \sim \beta, \alpha \not\sim \neg\gamma \Rightarrow \alpha \wedge \gamma \sim \beta$. Or, $f(A) \subseteq B, f(A) \cap C \neq \emptyset \rightarrow f(A \cap C) \subseteq B$ for all A, B, C . This is not quite, but almost: $f(A \cap C) \subseteq f(A) \cap C$ (it depends how many B there are, if $f(A)$ is some such B , the fit is perfect).

Fact 5.6

In all ranked structures, $(\mu \subseteq)$, $(\mu =)$, (μPR) , $(\mu =')$, $(\mu \parallel)$, $(\mu \cup)$, $(\mu \cup')$, $(\mu \in)$, $(\mu RatM)$ will hold, if the corresponding closure conditions are satisfied.

Proof

$(\mu \subseteq)$ and (μPR) hold in all preferential structures.

$(\mu =)$ and $(\mu =')$ are trivial.

$(\mu \cup)$ and $(\mu \cup')$: All minimal copies of elements in $f(Y)$ have the same rank. If some $y \in f(Y)$ has all its minimal copies killed by an element $x \in X$, by rankedness, x kills the rest, too.

$(\mu \in)$: If $f(\{a\}) = \emptyset$, we are done. Take the minimal copies of a in $\{a\}$, they are all killed by one element in X .

$(\mu \parallel)$: Case $f(X) = \emptyset$: If below every copy of $y \in Y$ there is a copy of some $x \in X$, then $f(X \cup Y) = \emptyset$. Otherwise $f(X \cup Y) = f(Y)$. Suppose now $f(X) \neq \emptyset$, $f(Y) \neq \emptyset$, then the minimal ranks decide: if they are equal, $f(X \cup Y) = f(X) \cup f(Y)$, etc.

$(\mu RatM)$: Let $X \subseteq Y$, $y \in X \cap f(Y) \neq \emptyset$, $x \in f(X)$. By rankedness, $y \prec x$, or $y \perp x$, $y \prec x$ is impossible, as $y \in X$, so $y \perp x$, and $x \in f(Y)$.

□

The following table summarizes representation by preferential structures.

“singletons” means that the domain must contain all singletons, “1 copy” or “ ≥ 1 copy” means that the structure may contain only 1 copy for each point, or several, “ $(\mu \emptyset)$ ” etc. for the preferential structure mean that the μ -function of the structure has to satisfy this property.

μ -function		Pref. Structure		Logic
$(\mu \subseteq) + (\mu PR)$	\Leftarrow	general	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR)$
	\Rightarrow		\Leftarrow	
			$\not\Leftarrow$ without (μdp) $\not\Leftarrow$ without (μdp)	
$(\mu \subseteq) + (\mu PR)$	\Leftarrow	transitive	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR)$
	\Rightarrow		\Leftarrow	
			$\not\Leftarrow$ without (μdp) \Leftrightarrow without (μdp)	
$(\mu \subseteq) + (\mu PR) + (\mu CUM)$	\Leftarrow	smooth	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR) + (CUM)$
	$\Rightarrow (U)$		$\Leftarrow (U)$	
			$\not\Leftarrow$ without (μdp)	
$(\mu \subseteq) + (\mu PR) + (\mu CUM)$	\Leftarrow	smooth+transitive	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR) + (CUM)$
	$\Rightarrow (U)$		$\Leftarrow (U)$	
			$\not\Leftarrow$ without (μdp) \Leftrightarrow without (μdp)	
$(\mu \subseteq) + (\mu =) + (\mu PR) + (\mu =') + (\mu \parallel) + (\mu \cup) + (\mu \cup') + (\mu \in) + (\mu RatM)$	\Leftarrow	ranked, ≥ 1 copy		
	$\not\Leftarrow$	ranked		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset)$	$\Leftrightarrow, (U)$	ranked, 1 copy + $(\mu \emptyset)$		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset)$	$\Leftrightarrow, (U)$	ranked, smooth, 1 copy + $(\mu \emptyset)$		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset fin) + (\mu \in)$	$\Leftrightarrow, (U)$, singletons	ranked, smooth, ≥ 1 copy + $(\mu \emptyset fin)$		
$(\mu \subseteq) + (\mu PR) + (\mu \parallel) + (\mu \cup) + (\mu \in)$	$\Leftrightarrow, (U)$, singletons	ranked ≥ 1 copy	$\not\Leftarrow$ without (μdp)	$(RatM), (RatM =), (Log \cup), (Log \cup')$
			$\not\Leftarrow$ without (μdp)	any “normal” characterization of any size

Definition 5.8

Let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a preferential structure. Call \mathcal{Z} $1 - \infty$ over Z , iff for all $x \in Z$ there are exactly one or infinitely many copies of x , i.e. for all $x \in Z$ $\{u \in \mathcal{X} : u = \langle x, i \rangle \text{ for some } i\}$ has cardinality 1 or $\geq \omega$.

Lemma 5.7

Let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a preferential structure and $f : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ with $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be represented by \mathcal{Z} , i.e. for $X \in \mathcal{Y}$ $f(X) = \mu_{\mathcal{Z}}(X)$, and \mathcal{Z} be ranked and free of cycles. Then there is a structure \mathcal{Z}' , $1 - \infty$ over Z , ranked and free of cycles, which also represents f .

Proof

We construct $\mathcal{Z}' = \langle \mathcal{X}', \prec' \rangle$.

Let $A := \{x \in Z : \text{there is some } \langle x, i \rangle \in \mathcal{X}, \text{ but for all } \langle x, i \rangle \in \mathcal{X} \text{ there is } \langle x, j \rangle \in \mathcal{X} \text{ with } \langle x, j \rangle \prec \langle x, i \rangle\}$,

let $B := \{x \in Z : \text{there is some } \langle x, i \rangle \in \mathcal{X}, \text{ s.t. for no } \langle x, j \rangle \in \mathcal{X} \langle x, j \rangle \prec \langle x, i \rangle\}$,

let $C := \{x \in Z : \text{there is no } \langle x, i \rangle \in \mathcal{X}\}$.

Let $c_i : i < \kappa$ be an enumeration of C . We introduce for each such c_i ω many copies $\langle c_i, n \rangle : n < \omega$ into \mathcal{X}' , put all $\langle c_i, n \rangle$ above all elements in \mathcal{X} , and order the $\langle c_i, n \rangle$ by $\langle c_i, n \rangle \prec' \langle c_{i'}, n' \rangle : \Leftrightarrow (i = i' \text{ and } n > n') \text{ or } i > i'$. Thus, all $\langle c_i, n \rangle$ are comparable.

If $a \in A$, then there are infinitely many copies of a in \mathcal{X} , as \mathcal{X} was cycle-free, we put them all into \mathcal{X}' . If $b \in B$, we choose exactly one such minimal element $\langle b, m \rangle$ (i.e. there is no $\langle b, n \rangle \prec \langle b, m \rangle$) into \mathcal{X}' , and omit all other elements. (For definiteness, assume in all applications $m = 0$.) For all elements from A and B , we take the restriction of the order \prec of \mathcal{X} . This is the new structure \mathcal{Z}' .

Obviously, adding the $\langle c_i, n \rangle$ does not introduce cycles, irreflexivity and rankedness are preserved. Moreover, any substructure of a cycle-free, irreflexive, ranked structure also has these properties, so \mathcal{Z}' is $1 - \infty$ over Z , ranked and free of cycles.

We show that \mathcal{Z} and \mathcal{Z}' are equivalent. Let then $X \subseteq Z$, we have to prove $\mu(X) = \mu'(X)$ ($\mu := \mu_{\mathcal{Z}}, \mu' := \mu_{\mathcal{Z}'}$).

Let $z \in X - \mu(X)$. If $z \in C$ or $z \in A$, then $z \notin \mu'(X)$. If $z \in B$, let $\langle z, m \rangle$ be the chosen element. As $z \notin \mu(X)$, there is $x \in X$ s.t. some $\langle x, j \rangle \prec \langle z, m \rangle$. x cannot be in C . If $x \in A$, then also $\langle x, j \rangle \prec' \langle z, m \rangle$. If $x \in B$, then there is some $\langle x, k \rangle$ also in \mathcal{X}' . $\langle x, j \rangle \prec \langle x, k \rangle$ is impossible. If $\langle x, k \rangle \prec \langle x, j \rangle$, then $\langle z, m \rangle \succ \langle x, k \rangle$ by transitivity. If $\langle x, k \rangle \perp \langle x, j \rangle$, then also $\langle z, m \rangle \succ \langle x, k \rangle$ by rankedness. In any case, $\langle z, m \rangle \succ' \langle x, k \rangle$, and thus $z \notin \mu'(X)$.

Let $z \in X - \mu'(X)$. If $z \in C$ or $z \in A$, then $z \notin \mu(X)$. Let $z \in B$, and some $\langle x, j \rangle \prec' \langle z, m \rangle$. x cannot be in C , as they were sorted on top, so $\langle x, j \rangle$ exists in \mathcal{X} too and $\langle x, j \rangle \prec \langle z, m \rangle$. But if any other $\langle z, i \rangle$ is also minimal in \mathcal{Z} among the $\langle z, k \rangle$, then by rankedness also $\langle x, j \rangle \prec \langle z, i \rangle$, as $\langle z, i \rangle \perp \langle z, m \rangle$, so $z \notin \mu(X)$. \square

To prove our main result, we first show an abstract result:

We give a generalized abstract nonsense result, taken from [LMS01], which must be part of the folklore:

Lemma 5.8

Given a set X and a binary relation R on X , there exists a total preorder (i.e. a total, reflexive, transitive relation) S on X that extends R such that

$$\forall x, y \in X (xSy, ySx \Rightarrow xR^*y)$$

where R^* is the reflexive and transitive closure of R .

Proof

Define $x \equiv y$ iff xR^*y and yR^*x . The relation \equiv is an equivalence relation. Let $[x]$ be the equivalence class of x under \equiv . Define $[x] \preceq [y]$ iff xR^*y . The definition of \preceq does not depend on the representatives x and y chosen. The relation \preceq on equivalence classes is a partial order. Let \leq be any total order on these equivalence classes

that extends \preceq . Define xSy iff $[x] \leq [y]$. The relation S is total (since \leq is total) and transitive (since \leq is transitive) and is therefore a total preorder. It extends R by the definition of \preceq and the fact that \leq extends \preceq . Suppose now xSy and ySx . We have $[x] \leq [y]$ and $[y] \leq [x]$ and therefore $[x] = [y]$ by antisymmetry. Therefore $x \equiv y$ and xR^*y . \square

Proposition 5.9

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$ be closed under finite unions. Then $(\mu \subseteq)$, $(\mu \emptyset)$, $(\mu =)$ characterize ranked structures for which for all $X \in \mathcal{Y}$ $X \neq \emptyset \rightarrow \mu_{<}(X) \neq \emptyset$ hold, i.e. $(\mu \subseteq)$, $(\mu \emptyset)$, $(\mu =)$ hold in such structures for $\mu_{<}$, and if they hold for some μ , we can find a ranked relation $<$ on U s.t. $\mu = \mu_{<}$. Moreover, the structure can be chosen \mathcal{Y} -smooth.

Proof

Completeness:

Note that by Fact 5.1 (page 24) (3) + (4) $(\mu \parallel)$, $(\mu \cup)$, $(\mu \cup')$ hold.

Define aRb iff $\exists A \in \mathcal{Y}(a \in \mu(A), b \in A)$ or $a = b$. R is reflexive and transitive: Suppose aRb , bRc , let $a \in \mu(A)$, $b \in A$, $b \in \mu(B)$, $c \in B$. We show $a \in \mu(A \cup B)$. By $(\mu \parallel)$ $a \in \mu(A \cup B)$ or $b \in \mu(A \cup B)$. Suppose $b \in \mu(A \cup B)$, then $\mu(A \cup B) \cap A \neq \emptyset$, so by $(\mu =)$ $\mu(A \cup B) \cap A = \mu(A)$, so $a \in \mu(A \cup B)$.

Moreover, $a \in \mu(A)$, $b \in A - \mu(A) \rightarrow \neg(bRa)$: Suppose there is B s.t. $b \in \mu(B)$, $a \in B$. Then by $(\mu \cup)$ $\mu(A \cup B) \cap B = \emptyset$, and by $(\mu \cup')$ $\mu(A \cup B) = \mu(A)$, but $a \in \mu(A) \cap B$, *contradiction*.

Let by Lemma 5.8 (page 29) S be a total, transitive, reflexive relation on U which extends R s.t. $xSy, ySx \rightarrow xRy$ (recall that R is transitive and reflexive). Define $a < b$ iff aSb , but not bSa . If $a \perp b$ (i.e. neither $a < b$ nor $b < a$), then, by totality of S , aSb and bSa . $<$ is ranked: If $c < a \perp b$, then by transitivity of S cSb , but if bSc , then again by transitivity of S aSc . Similarly for $c > a \perp b$.

$<$ represents μ and is \mathcal{Y} -smooth: Let $a \in A - \mu(A)$. By $(\mu \emptyset)$, $\exists b \in \mu(A)$, so bRa , but (by above argument) not aRb , so bSa , but not aSb , so $b < a$, so $a \in A - \mu_{<}(A)$, and, as b will then be $<$ -minimal (see the next sentence), $<$ is \mathcal{Y} -smooth. Let $a \in \mu(A)$, then for all $a' \in A$ aRa' , so aSa' , so there is no $a' \in A$ $a' < a$, so $a \in \mu_{<}(A)$.

\square

References

- [Bou90a] C.Boutilier, "Conditional Logics of Normality as Modal Systems", AAAI 1990, Boston, p.594
- [CJ02] J.Carmo, A.J.I.Jones, "Deontic logic and contrary-to-duties", in: Handbook of Philosophical Logic, Vol. 8, D.Gabbay, F.Guenther eds., pp. 265-343, Kluwer, 2002
- [GS08c] D. Gabbay, K.Schlechta, "Roadmap for preferential logics", hal-00311941, arXiv 0808.3073, and submitted
- [GS08d] D. Gabbay, K.Schlechta, "A theory of hierarchical consequence and conditionals", hal-00311937, arXiv 0808.3072, and submitted
- [Gab08] D.M.Gabbay, "Reactive Kripke models and contrary to duty obligations", DEON-2008, Deontic Logic in Computer Science, R.v.d.Meyden, L.v.d.Torre eds., LNAI 5076, pp 155-173, Springer 2008
- [Gab08a] D.M.Gabbay, "Reactive Kripke models and contrary to duty obligations", to appear in: Journal of applied logic, special issue on Deon-2008
- [LMS01] D.Lehmann, M.Magidor, K.Schlechta: "Distance Semantics for Belief Revision", Journal of Symbolic Logic, Vol.66, No. 1, March 2001, p. 295-317

- [Lam91] P.Lamarre: "S4 as the conditional logic of nonmonotonicity", KR 91, p.357-367
- [MDW94] J.J.Ch.Meyer, F.P.M.Dignum, R.J.Wieringa, "The paradoxes of deontic logic revisited: a computer science perspective", University of Utrecht, NL, Dept. Comp.Sc., Tech.Rep., 1994
- [Mak93] D.Makinson: "Five faces of minimality", *Studia Logica* 52 (1993), p. 339-379
- [Sch04] K.Schlechta: "Coherent systems", Elsevier, Amsterdam, 2004
- [Sch92] K.Schlechta: "Some results on classical preferential models", *Journal of Logic and Computation*, Oxford, Vol.2, No.6 (1992), p. 675-686