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SLOW DECAY OF GIBBS MEASURES WITH HEAVY TAILS

CYRIL ROBERTO

ABSTRACT. We consider Glauber dynamics reversible with respect to Gibbs measures with heavy tails. Spins are unbounded. The interactions are bounded and finite range. The self potential enters into two classes of measures, κ -concave probability measure and sub-exponential laws, for which it is known that no exponential decay can occur. We prove, using coercive inequalities, that the associated infinite volume semi-group decay to equilibrium polynomially and stretched exponentially, respectively. Thus improving and extending previous results by Bobkov and Zegarlinski.

1. INTRODUCTION

In the past decades, the study of functional inequalities deserved a lot of attention, not only on the side of theoretical probability and analysis, but also in statistical mechanics. This is due to the numerous fields of application: differential geometry, analysis of p.d.e., concentration of measure phenomenon, isoperimetry, trends to equilibrium in deterministic and stochastic evolutions...

The most popular functional inequalities are Poincaré and log Sobolev. Both are now well understood in many situations. We refer to [2], [20], [23], [24], [27], [29], [34], [36], [1] for an introduction.

Very recently, generalisations of Poincaré and log Sobolev inequalities were introduced and studied by probabilists and analysts. Let us mention weak Poincaré or super Poincaré inequalities, Orlicz-Poincaré or Orlicz-Sobolev inequalities, F -Sobolev inequalities, weighted Poincaré or weighted log Sobolev inequalities, modified log-Sobolev inequalities etc. To give a complete picture of the literature is out of reach. See [13, 21, 37, 12, 18, 5, 4, 7] (and references therein) for some of the most recent publications.

Few of those recent advances have been used so far in statistical mechanics, at the notable exception of [13, 40].

On the other hand, in the statistical mechanics community, progress have been done in the study of Poincaré and log Sobolev inequalities for large classes of models coming from the physics literature. Again, to give an updated list of publications is out of reach. Let us mention [26, 25, 8, 16, 15, 17].

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This paper intends to use advances in both communities in order to improve and extend some results of Bobkov and Zegarliniski [13] on the decay to equilibrium of some unbounded spins systems. We believe that the techniques coming more specifically from one community should be more largely exploited by the other one. This paper is one step in this direction.

If a lot of results are known for log-concave probability measures, not so much has been proved for measures with heavy tails (let us mention [33, 3, 12, 11, 13, 38]). In this paper the focus is on such measures with heavy tails (informally with tails larger than exponential) and our aim is to prove decay to equilibrium of unbounded spin systems in infinite dimension.

Now we introduce and discuss one of the main tool we shall use, namely, the weak Poincaré inequality. Consider for simplicity a one dimensional probability measure $d\nu = Z_V^{-1}e^{-V}$ with $(Z_V = \int e^{-V(u)}du < \infty)$. Then, a weak Poincaré inequality asserts that

$$(1) \quad \mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s \text{Osc}(f)^2 \quad \forall f : \mathbb{R} \rightarrow \mathbb{R}, \forall s > 0$$

where $\beta : (0, \infty) \rightarrow \mathbb{R}$ is a rate function associated to the weak Poincaré inequality and $\text{Osc}(f) = \sup f - \inf f$. In case when $\lim_0 \beta < \infty$, then the weak Poincaré inequalities reduce to the usual Poincaré inequalities. Most of the information is encoded in the behaviour of β near the origin. Moreover, note that $\mathbf{Var}_\nu(f) \leq \frac{1}{4} \text{Osc}(f)$, hence, only the values $s \in (0, 1/4)$ are relevant.

Weak Poincaré inequalities, in the form (1), have been introduced by Röckner and Wang [33]. However, inequalities with a free parapeter have a long story in analysis, see *e.g.* [32, 20, 28, 10, 9].

Using capacity techniques (Hardy type inequalities [30, 31, 6]) the best possible rate function for $V_p(x) = |x|^p$, $p \in (0, 1)$ was computed in [3]: $\beta(s) = c_p \log\left(\frac{2}{s \wedge 2}\right)^{2(\frac{1}{p}-1)}$. Also, in this case, it is known (see [1, Chapter 5]) that ν does not satisfies the usual Poincaré inequality. Equivalently there is no exponential decay to equilibrium. However, by standard differentiation (see Section 3), one gets that the semi-group $(\mathbf{S}_t)_{t \geq 0}$ associated to the one dimensional generator $L = \frac{d^2}{du^2} - V_p' \cdot \frac{d}{du}$ is stretched exponential decaying to equilibrium. More precisely,

$$\mathbf{Var}_\nu(\mathbf{S}_t f) \leq \frac{1}{c} e^{-ct^{p/(2-p)}} \text{Osc}(f)^2 \quad \forall f, \forall t > 0$$

for some constant $c = c(p)$ (the lack of smoothness of V_p at 0 is just little nuisance that one can easily handle).

In [13], Bobkov and Zegarliniski proved, using weak Poincaré inequalities, that some Gibbs measures in infinite volume (with self potential V_p) also satisfies a stretched exponential decay as above, but with a worst exponent. In this paper we shall prove the correct stretched exponential decay with the exponent $t^{p/(2-p)}$ not only for the potential V_p but also for a larger class of

potentials of sub-exponential type. Moreover, our technique, based on the bisection approach [29] together with the quasi factorisation property of the variance [8], applies also to potentials of the type $V = (1+\alpha) \log(1+|u|)$, $\alpha > 0$ leading to Cauchy type distributions and polynomial decay to equilibrium.

Note that there is a difficulty here with respect to the usual Poincaré and/or log Sobolev inequalities. Namely, weak Poincaré inequalities do not tensorise in general. In turn, there is no hope for a dimension free analysis, and one has to take care of the growing dimension (see Section 3 for a discussion about this fact).

The paper is organised as follows. The notations and the setting, in particular the Hamiltonian and the Gibbs measure we consider, are introduced in the next section. Section 3 is dedicated to the weak Poincaré inequalities, we recall few known facts and prove some perturbation properties. The results about the infinite volume Gibbs measure are collected and proved in Section 4. The main ingredients used in the proof of our theorems are postponed for the clarity of the exposition to the last two sections

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2. NOTATIONS AND SETTING.

2.1. The Configuration space. The *configuration space* we consider is $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ where $d \geq 1$ is an integer that denotes the dimension of the lattice \mathbb{Z}^d . Given $\Lambda \Subset \mathbb{Z}^d$ (*i.e.* Λ is a finite subset of \mathbb{Z}^d), we shall also deal with $\Omega_\Lambda = \mathbb{R}^\Lambda$. For any *configuration* $\sigma \in \Omega$, any site $x \in \mathbb{Z}^d$ and any $\Lambda \Subset \mathbb{Z}^d$, σ_x stands for the value of the configuration (or the spin) at x while σ_Λ is the configuration σ restricted to Λ . We denote by \mathcal{B}_Λ the σ -algebra of all Borell sets of Ω_Λ .

A function which is measurable with respect to \mathcal{B}_Λ with $\Lambda \Subset \mathbb{Z}^d$ is said to be *local*. For any smooth local function, we set $|||f||| = \sum_{x \in \mathbb{Z}^d} \|\nabla_x f\|_\infty$ where $\|\cdot\|_\infty$ is the sup norm and ∇_x denotes the derivative with respect to the variable σ_x .

The Euclidean distance on \mathbb{Z}^d is denoted by d . With a slight abuse of notation, for $a, b \in \mathbb{Z}$, we shall often set $[a, b]$ for $[a, b] \cap \mathbb{Z}$.

2.2. The Hamiltonian and the potentials. For any $\Lambda \Subset \mathbb{Z}^d$ the *Hamiltonian* $H_\Lambda : \Omega \rightarrow \mathbb{R}$ is given by

$$(2) \quad H_\Lambda(\sigma) = \sum_{x \in \Lambda} V(\sigma_x) + \frac{1}{T} \sum_{y: d(x,y) \leq r} W(\sigma_x - \sigma_y) \quad \forall \sigma \in \Omega$$

where $V, W : \mathbb{R} \rightarrow \mathbb{R}$ correspond respectively to the *self potential* and the *interaction potential* (pair potential). The parameter $T \in (0, \infty)$ is the *temperature* and $r \in \mathbb{N} \setminus \{0\}$ is the *range* of the interaction. For $\sigma, \tau \in \Omega$,

we also let $H_\Lambda^\tau(\sigma) := H_\Lambda(\sigma_\Lambda \tau_{\Lambda^c})$, where $\sigma_\Lambda \tau_{\Lambda^c}$ stands for the configuration equal to σ on Λ and to τ on Λ^c (the complement of Λ), and τ is called the *boundary condition*.

Remark 1. One could consider more general Hamiltonian H_Λ with *e.g.* infinite range interactions and/or interaction potentials W depending on the values of more than only two spins and/or depending on the sites etc. All the results below would hold in those more general settings, under specific assumptions. We make the choice of dealing with the Hamiltonian (2) for simplicity and for the clarity of the exposition.

Now we describe our assumptions on V and W . We collect in Hypothesis (H1) some smoothness conditions on V and W .

Hypothesis (H1). Given a self potential $V : \mathbb{R} \rightarrow \mathbb{R}$ and an interaction potential $W : \mathbb{R} \rightarrow \mathbb{R}$, we say that Hypothesis (H1) is satisfied if

- V is \mathcal{C}^1 and $\int_{\mathbb{R}} e^{-V(u)} du < \infty$;
- W is twice differentiable, $\|W\|_\infty < \infty$, $\|W'\|_\infty < \infty$ and $\|W''\|_\infty < \infty$.

The second assumption on V guarantees that $d\nu(u) = Z_V^{-1} e^{-V}$ (with $Z_V = \int e^{-V}$) defines a probability measure. The smoothness assumption about V will be needed when defining the Glauber dynamics.

On the other side, the assumptions about W will be useful when defining the infinite volume Gibbs measure.

More specifically, the self potentials $V : \mathbb{R} \rightarrow \mathbb{R}$ we shall consider enter into two classes of examples: κ -concave probability measures (a notion introduced by Borell [14], see [11] for a comprehensive introduction) and sub-exponential like laws. More precisely, given any convex function $U : \mathbb{R} \rightarrow (0, \infty)$, we shall consider either $V = (1 + \alpha) \log U$ with $\alpha > 0$ (κ -concave case with $\kappa = -1/\alpha$) or $V = U^p$, $p > 0$ (sub-exponential like laws).

The corresponding probability measure $d\nu(u) = Z_V^{-1} e^{-V(u)} du$ on \mathbb{R} (with $Z_V = \int e^{-V(u)} du$) reads respectively as

$$d\nu(u) = \frac{1}{Z_\alpha U(u)^{1+\alpha}} du \quad \text{and} \quad d\nu(u) = Z_p^{-1} e^{-U(u)^p} du.$$

Prototypes are respectively the Cauchy distributions ($U(u) = 1 + |u|$ or equivalently $V(u) = (1 + \alpha) \log(1 + |u|)$) and the sub-exponential laws ($U(u) = |u|$ or equivalently $V(u) = |u|^p$):

$$(3) \quad d\nu(u) = \frac{\alpha}{2(1 + |u|)^{1+\alpha}} du \quad \text{and} \quad d\nu(u) = \frac{e^{-|u|^p}}{2\Gamma(1 + \frac{1}{p})} du$$

for $\alpha > 0$ and $p \in (0, 1]$. In both examples the measure ν has tails larger than exponential. For the sub-exponential law, in order to fulfil Hypothesis (H1) one could consider *e.g.* $U(x) = \sqrt{1 + x^2}$ to avoid differentiability trouble in 0.

2.3. The Gibbs measures. The *finite volume Gibbs measure* in $\Lambda \Subset \mathbb{Z}^d$ at temperature T and boundary condition τ is given by

$$(4) \quad \mu_\Lambda^\tau(d\sigma) = (Z_\Lambda^\tau)^{-1} \exp\{-H_\Lambda^\tau(\sigma)\} \prod_{x \in \Lambda} d\sigma_x \times \delta_{\Lambda^c, \tau}(d\sigma)$$

where $\delta_{\Lambda^c, \tau}$ is the Dirac probability measure on Ω_{Λ^c} which gives mass 1 to the configuration τ and Z_Λ^τ is the proper normalisation factor. We denote with $\mu_\Lambda^\tau(f)$ the expectation of f with respect to μ_Λ^τ , while $\mu_\Lambda(f)$ denotes the functions $\tau \mapsto \mu_\Lambda^\tau(f)$. For any Borel set $X \subset \Omega$ we set $\mu_\Lambda(X) := \mu_\Lambda(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function on X . We write $\mu_\Lambda(f, g)$ to denote the covariance (with respect to μ_Λ) of f and g and $\mathbf{Var}_{\mu_\Lambda}(f) = \mu_\Lambda(f, f)$ for the variance of f under μ_Λ .

The family of measures (4) satisfies the DLR compatibility conditions: for all Borell sets $X \subset \Omega$

$$\mu_\Delta(\mu_\Lambda(X)) = \mu_\Delta(X) \quad \forall \Lambda, \Delta \Subset \mathbb{Z}^d \text{ such that } \Lambda \subset \Delta.$$

If in addition of Hypothesis (H1), T is large enough, then (see [22, Proposition (8.8)]) the Dobrushin's uniqueness condition is satisfied. Hence there exists a unique infinite volume Gibbs measure μ satisfying $\mu(\mu_\Lambda(X)) = \mu(X)$ for $\Lambda \Subset \mathbb{Z}^d$ and any Borell set $X \subset \Omega$. Moreover (see [22, Remark (8.26)] together with Corollary (8.32)) (if T is large enough) there exist constants $D = D(r, T, \|W\|_\infty)$ and $m = m(r, T, \|W\|_\infty)$ such that for any $\Lambda \Subset \mathbb{Z}^d$, it holds

$$(5) \quad |\mu_\Lambda^\tau(f, g)| \leq D |\Delta_f| |\Delta_g| \|f\|_\infty \|g\|_\infty e^{-m d(\Delta_f, \Delta_g)}$$

for any boundary condition τ , any bounded local functions f and g with support Δ_f and Δ_g satisfying $\Delta_f, \Delta_g \subset \Lambda$. Here $|\cdot|$ stands for the Lebesgue measure on \mathbb{Z}^d . Inequality (5) is known as the *strong mixing condition*. Note that the previous argument does not depend on the self potential V .

In the sequel, we will always assume the following:

Hypothesis (H2). Given the potentials V and W , the temperature T , we say that Hypothesis (H2) is satisfied if there exists a unique infinite volume Gibbs measure μ and if the strong mixing condition (5) holds true.

In particular, by the argument above, if (H1) is satisfied then (H2) is also satisfied as soon as T is large enough, whatever the choice of V .

2.4. The dynamics. The dynamics we consider are of Glauber type. For any $\Lambda \Subset \mathbb{Z}^d$, any boundary condition $\tau \in \Omega$, let $(\mathbf{P}_t^{\Lambda, \tau})_{t \geq 0}$ be the Markov semi-group associated to the generator

$$(6) \quad \mathbf{L}_\Lambda^\tau = \sum_{x \in \Lambda} \Delta_x - \sum_{x \in \Lambda} \nabla_x H_\Lambda^\tau \cdot \nabla_x$$

where ∇_x and Δ_x stand respectively for the first and second partial derivative with respect to the variable σ_x . When there is no confusion, we shall drop the superscript τ in the definition and write simply \mathbf{P}_t^Λ and \mathbf{L}_Λ .

The generator \mathbf{L}_Λ^τ is symmetric in $\mathbb{L}^2(\mu_\Lambda^\tau)$. On the other hand, $\mathbf{P}_t^{\Lambda,\tau}$ is a contraction on $\mathbb{L}^p(\mu_\Lambda^\tau)$ for all $p \in [1, \infty]$.

For any $\Lambda \Subset \mathbb{Z}^d$ we denote by \mathcal{D}_Λ^τ (we shall also often drop the superscript τ) the *Dirichlet form* associated to \mathbf{L}_Λ^τ and defined by

$$(7) \quad \mathcal{D}_\Lambda^\tau(f) = \frac{1}{2} \sum_{x \in \Lambda} \mu_\Lambda^\tau (|\nabla_x f|^2)$$

for all sufficiently smooth f .

If (H1) and (H2) are satisfied, it is possible (see *e.g.* [24]) to construct the infinite volume semigroup by

$$(8) \quad \mathbf{P}_t = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \mathbf{P}_t^\Lambda$$

on the space of bounded smooth functions (in particular (H1) and (H2) guarantee that the limit above does not depend on the boundary condition). The associated infinite volume generator will be denoted by \mathbf{L} .

3. THE WEAK POINCARÉ INEQUALITIES.

The main tool we shall use is the so called weak Poincaré inequality. We now introduce this notion, first on \mathbb{R} , and then on Ω_Λ . We then collect some usefull results we shall use later.

3.1. Introduction. In this section we introduce the notion of weak Poincaré inequality on \mathbb{R} . Then we derive some (known) bounds on the decay to equilibrium. Finally we explain how to get weak Poincaré inequalities on product spaces.

Consider the probability measure $d\nu(u) = Z_V^{-1} e^{-V(u)} du$, on \mathbb{R} . We say that ν satisfies a *weak Poincaré inequality* with rate function $\beta : (0, \infty) \rightarrow [0, \infty)$, if for any bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ smooth enough, it holds

$$(9) \quad \mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s \text{Osc}(f)^2 \quad \forall s > 0.$$

where $\text{Osc}(f)$ is the *oscillation* of f : $\text{Osc}(f) := \sup f - \inf f$.

Note that if $\lim_{s \rightarrow 0} \beta(s) = \beta_0 < \infty$, then Inequality (9) reduces to the standard Poincaré inequality

$$\mathbf{Var}_\nu(f) \leq \beta_0 \int (f')^2 d\nu.$$

On the other hand, since $\mathbf{Var}_\nu(f) \leq \frac{1}{4} \text{Osc}(f)^2$, only the values $s \in (0, 1/4)$ are relevant. Most of the information is encoded in the behaviour of β near the origin. Weak Poincaré inequalities have been introduced by Röckner and Wang [33]. One interested feature of Inequality (9) is that it gives a control on the \mathbb{L}^2 decay to equilibrium of the Markov semi-group $(\mathbf{S}_t)_{t \geq 0}$ on \mathbb{R} with generator $L = \frac{d^2}{du^2} - V' \cdot \frac{d}{du}$.

Proposition 2 ([33]). *Let ν be a probability measure on \mathbb{R} with density $Z_V^{-1}e^{-V}$ with respect to the Lebesgue measure on \mathbb{R} . Let $(\mathbf{S}_t)_{t \geq 0}$ be the corresponding semi-group with generator $L := \frac{d^2}{du^2} - V' \cdot \frac{d}{du}$. If ν satisfies the weak Poincaré inequality (9) with rate function β , then, every smooth $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies*

$$(10) \quad \mathbf{Var}_\nu(\mathbf{S}_t f) \leq e^{-\frac{2t}{\beta(s)}} \mathbf{Var}_\nu(f) + 4s(1 - e^{-\frac{2t}{\beta(s)}}) \|f - \nu(f)\|_\infty^2 \quad \forall s, t > 0.$$

The result by Röckner and Wang holds in more general settings, see [33]. We sketch the proof for completeness.

Proof. Assume without loss of generality that $\nu(f) = 0$ (which implies $\nu(\mathbf{S}_t f) = 0$ for any t). If $u(t) = \mathbf{Var}_\nu(\mathbf{S}_t f) = \int (\mathbf{S}_t f)^2 d\nu$, the weak Poincaré inequality implies that

$$u'(t) = 2 \int \mathbf{S}_t f L \mathbf{P}_t f d\nu = -2 \int \left| \frac{d}{du} \mathbf{S}_t f \right|^2 d\nu \leq -\frac{2}{\beta(s)} [u(t) - 2s \|f\|_\infty]$$

since $\text{Osc}(\mathbf{S}_t f) \leq 2 \|\mathbf{S}_t f\|_\infty \leq 2 \|f\|_\infty$. The result follows by integration. \square

For the two classes of self-potentials V introduced above, the corresponding rate function β has been computed in [18] (see also [33, 3, 13, 11]). Given β , one can then optimise over $s > 0$ in (10) to get an explicit decay of the Markov semi-group $(\mathbf{S}_t)_{t \geq 0}$ in $\mathbb{L}^2(\nu)$. Let $U : \mathbb{R} \rightarrow (0, \infty)$ be a convex function.

- If $V = (1 + \alpha) \log U$ with $\alpha > 0$ (the κ -concave case), then ν satisfies a weak Poincaré inequality with rate function $\beta(s) = c_\alpha s^{-2/\alpha}$ for some constant $c_\alpha > 0$ (see [18, Proposition 5.4]). Optimising (10) over s (together with some computations given in [33, Corollary 2.4], see also the proof of Corollary 6 below) leads to

$$(11) \quad \mathbf{Var}_\nu(\mathbf{S}_t f) \leq \frac{C}{t^{\alpha/2}} \|f - \nu(f)\|_\infty^2$$

for some constant $C = C(\alpha) > 0$.

- If $V = U^p$, $p \in (0, 1)$ (the sub-exponential case), then ν satisfies a weak Poincaré inequality with rate function $\beta(s) = c_p \left(\log \frac{2}{s \wedge 1}\right)^{2(\frac{1}{p}-1)}$ for some constant $c_p > 0$ (see [18, Proposition 5.6]). Optimising (10) over s (take $s = e^{-ct^{p/(2-p)}}$) leads to

$$(12) \quad \mathbf{Var}_\nu(\mathbf{S}_t f) \leq \frac{1}{C} e^{-Ct^{p/(2-p)}} \|f - \nu(f)\|_\infty^2$$

for some constants $C = C(p) > 0$. Note that $p/(2-p) \in (0, 1)$.

The previous results are optimal, in the sense that for $U(u) = 1 + |u|$, respectively $U(u) = |u|$, neither the rate function β nor the \mathbb{L}^2 decay can be improved. In particular there is no hope for a Poincaré inequality to hold, or equivalently, for an exponential decay to equilibrium in \mathbb{L}^2 .

Note that the limiting case $p = 1$ corresponds to the exponential measure for which it is known that a Poincaré inequality holds, and thus an

exponential decay of the semi-group. This fact is encoded in the rate function β (which becomes a constant) and on the decay (12) (which becomes exponential).

Contrary to the Poincaré inequality, the weak Poincaré inequalities do not tensorise in general. If the probability measure $\nu_n = \otimes \nu^{(i)}$ on \mathbb{R}^n is the tensor product of n copies of ν , it is possible (and actually very easy, see [3, Section 3]) to prove that

$$(13) \quad \mathbf{Var}_{\nu_n}(f) \leq \beta(s/n) \int \sum_{i=1}^n |\nabla_i f|^2 d\nu_n + s \text{Osc}(f)^2 \quad \forall s > 0$$

for all functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth enough. The rate function $\beta(\cdot/n)$ is best possible for the product of Cauchy measures and sub-exponential laws introduced in (3). In particular, there is no hope for those measures with heavy tails to get a weak Poincaré inequality in infinite dimension. A deep explanation of this phenomenon can be found in Talagrand's paper [35] (see also the introduction of [3]). It relies on the concentration of measure phenomenon.

However, quite remarkable is the fact that the decays (11) and (12) still hold in the infinite system Ω with infinite volume Gibbs measure μ and Markov semi-group $(\mathbf{P}_t)_{t \geq 0}$ introduced in the previous section. The aim of this paper is to prove such results.

3.2. The weak Poincaré inequalities for Gibbs measures. Now we turn to the Gibbs measure setting of the previous section. Let $\Lambda \Subset \mathbb{Z}^d$. For any $s > 0$, let $\beta_\Lambda(s)$ be the best non-negative number such that for any boundary condition τ and any smooth function $f : \Omega_\Lambda \rightarrow \mathbb{R}$,

$$(14) \quad \mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq \beta_\Lambda(s) \mathcal{D}_\Lambda^\tau(f) + s \text{Osc}(f)^2.$$

By this procedure we have defined a non-increasing function $\beta_\Lambda : (0, \infty) \rightarrow [0, \infty)$. Note that the system is invariant under translation and rotation. Hence, two finite subsets of \mathbb{Z}^d that are equal under translation and rotation lead to the same rate function β .

Our aim is to get the best possible rate function for (14) to hold. In view of (13), the best one can hope is $\beta(s/|\Lambda|)$ if β denotes the rate function associated to the one dimensional measure $d\nu = Z_V^{-1} e^{-V}$. This will actually be almost true, see Proposition 5 below. The difficulty here comes from the interacting part which can be of order $e^{|\Lambda|}$. The following perturbation result goes in this direction. Even if it is far from being optimal, it will be useful in the proof of our main result.

Proposition 3 (Perturbation). *Assume (H1). Also, assume that the self-potential V is such that $d\nu = Z_V^{-1} e^{-V}$ satisfies the following weak Poincaré inequality on \mathbb{R} for some non-increasing rate function β :*

$$\mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s \text{Osc}(f)^2 \quad \forall f, \forall s > 0.$$

Then, there exists a constant $C = C(r, T, d, \|W\|_\infty)$ such that for any $\Lambda \in \mathbb{Z}^d$, any boundary condition $\tau \in \Omega$, any smooth $f : \Omega_\Lambda \rightarrow \mathbb{R}$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq C e^{C|\Lambda|} \beta \left(\frac{s}{C|\Lambda|e^{C|\Lambda|}} \right) \mathcal{D}_\Lambda^\tau(f) + s \text{Osc}(f)^2 \quad \forall s > 0.$$

Remark 4. A somehow similar statement can be found in [13, Lemma 12.1].

Proof. Fix $\Lambda \in \mathbb{Z}^d$ and $\tau \in \Omega$. Let $d\nu_\Lambda(\sigma) = Z_V^{-|\Lambda|} \exp\{-\sum_{x \in \Lambda} V(\sigma_x)\} d\sigma_\Lambda$ be the probability measure corresponding to the product part of μ_Λ^τ . By the product property (13) we have

$$\mathbf{Var}_{\nu_\Lambda}(f) \leq \beta \left(\frac{s}{|\Lambda|} \right) \sum_{x \in \Lambda} \nu_\Lambda(|\nabla_x f|^2) + s \text{Osc}(f)^2 \quad \forall f : \Omega_\Lambda \rightarrow \mathbb{R}, \forall s > 0.$$

Now Hypothesis (H1) guarantees that there exists a constant C (depending on r, T, d and $\|W\|_\infty$ but independent of the boundary condition τ and Λ) such that

$$C^{-1} e^{-C|\Lambda|} \leq \frac{\nu_\Lambda(\sigma)}{\mu_\Lambda^\tau(\sigma)} \leq C e^{C|\Lambda|} \quad \forall \sigma \in \Omega_\Lambda.$$

Hence, since $\mathbf{Var}_{\mu_\Lambda^\tau}(f) = \inf_a \mu_\Lambda^\tau((f - a)^2)$, we get for any $s > 0$,

$$\begin{aligned} \mathbf{Var}_{\mu_\Lambda^\tau}(f) &\leq C e^{C|\Lambda|} \mathbf{Var}_{\nu_\Lambda}(f) \\ &\leq C e^{C|\Lambda|} \beta \left(\frac{s}{|\Lambda|} \right) \sum_{x \in \Lambda} \nu_\Lambda(|\nabla_x f|^2) + s C e^{C|\Lambda|} \text{Osc}(f)^2 \\ &\leq 2C^2 e^{2C|\Lambda|} \beta \left(\frac{s}{|\Lambda|} \right) \mathcal{D}_\Lambda^\tau(f) + s C e^{C|\Lambda|} \text{Osc}(f)^2. \end{aligned}$$

The result follows. \square

As for the tensorisation property, the previous result is of no help in order to get directly infinite volume estimates, since when $|\Lambda| \rightarrow \infty$ the weak Poincaré inequality becomes trivial.

Using the bisection technique [29], the result of Proposition 3 can be improved for volumes Λ that are cubes. More precisely we have the following proposition.

Proposition 5 (Perturbation improved). *Assume (H1) and (H2). Also, assume that the self-potential V is such that $d\nu = Z_V^{-1} e^{-V}$ satisfies the following weak Poincaré inequality on \mathbb{R} for some non-increasing rate function β :*

$$\mathbf{Var}_\nu(f) \leq \beta(s) \int (f')^2 d\nu + s \text{Osc}(f)^2 \quad \forall f, \forall s > 0.$$

Then, for any $\varepsilon \in (0, 1)$, there exists a constant $C = C(\varepsilon, r, T, d, \|W\|_\infty)$ such that for any integer L ,

(15)

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq C \beta \left(\frac{s}{C|\Lambda|^{1+\varepsilon}} \right) \mathcal{D}_\Lambda^\tau(f) + s \text{Osc}(f) \quad \forall \tau \in \Omega, \forall f, \forall s > 0,$$

where $\Lambda = [-L, L]^d$.

The proof is postponed to Section 5. Note that we obtained a quasi optimal inequality, up to the power ε . Indeed Inequality (15) is closed to the non-interacting case (13). Proposition 5 is at the heart of the proof of the main theorems through the following two Lemmas. In fact, using the semi-group strategy [33] explained in Proposition 2, Inequality (15) already leads to some finite volume decay of the semi-group $(\mathbf{P}_t^{\Lambda, \tau})_{t \geq 0}$, for cubes.

Corollary 6 (κ -concave case). *Let $U : \mathbb{R} \rightarrow (0, \infty)$ be a convex function and $V = (1 + \alpha) \log U$ with $\alpha > 0$. Assume (H1) and (H2). Then, for any ε , there exists a constant $C = C(\varepsilon, \alpha, r, T, d, \|W\|_\infty)$ such that for any integer L , any local function f satisfies*

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f) \leq C \frac{|\Lambda|^{1+\varepsilon}}{t^{\alpha/2}} \|f - \mu_\Lambda^\tau(f)\|_\infty^2 \quad \forall t > 0, \forall \tau \in \Omega,$$

where $\Lambda = [-L, L]^d$.

Proof. Fix an integer L , $\tau \in \Omega$, $\varepsilon > 0$ and a local function f . Set $\Lambda = [-L, L]^d$. Assume without loss of generality that $\mu_\Lambda^\tau(f) = 0$.

As mentioned before Inequality (11), the measure $d\nu = Z_V^{-1} e^{-V}$ on \mathbb{R} satisfies a weak Poincaré inequality with rate function $\beta(s) = c_\alpha s^{-2/\alpha}$ for some constant $c_\alpha > 0$. Hence, using Proposition 5, μ_Λ^τ satisfies a weak Poincaré inequality with rate function $\gamma(s) = C s^{-2/\alpha} |\Lambda|^{2(1+\varepsilon)/\alpha}$, for some constant $C = C(\varepsilon, \alpha, r, T, d, \|W\|_\infty)$. In turn, using the strategy of the proof of Proposition 2 (we omit the details), we get that

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f) \leq e^{-\frac{2t}{\gamma(s)}} \mathbf{Var}_\nu(f) + 4s(1 - e^{-\frac{2t}{\gamma(s)}}) \|f\|_\infty^2 \quad \forall s, t > 0.$$

Following [33], we take $s = (\lambda/t)^{\alpha/2}$ with $\lambda > 0$ chosen in such a way that

$$e^{-\frac{2t}{\gamma(s)}} = e^{-\frac{2ts^{2/\alpha}}{C|\Lambda|^{2(1+\varepsilon)/\alpha}}} = e^{-\frac{2\lambda}{C|\Lambda|^{2(1+\varepsilon)/\alpha}}} = \left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1}.$$

It follows that

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f) \leq \left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1} \mathbf{Var}_\nu(f) + 4 \left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}} \|f\|_\infty^2 \quad \forall t > 0.$$

We omit the superscript τ . Applying this inequality repeatedly, we obtain (using also the fact that \mathbf{P}_t^Λ is a contaction in the sup-norm)

$$\begin{aligned} \mathbf{Var}_{\mu_\Lambda}(\mathbf{P}_t^\Lambda f) &= \mathbf{Var}_{\mu_\Lambda}\left(\mathbf{P}_{t/2}^\Lambda\left[\mathbf{P}_{t/2}^\Lambda f\right]\right) \\ &\leq \left(\frac{1}{2}\right)^{\frac{\alpha}{2}+1} \mathbf{Var}_\nu\left(\mathbf{P}_{t/2}^\Lambda f\right) + 4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \|f\|_\infty^2 \\ &\leq \left(\frac{1}{4}\right)^{\frac{\alpha}{2}+1} \mathbf{Var}_\nu\left(\mathbf{P}_{t/4}^\Lambda f\right) + 4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \|f\|_\infty^2 \left(1 + \frac{1}{2}\right) \\ &\leq \dots \\ &\leq 4\left(\frac{\lambda}{t}\right)^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \|f\|_\infty^2 \sum_{n \geq 0} 2^{-n}. \end{aligned}$$

The result follows by our choice of λ . \square

The next result concerns sub-exponential type laws.

Corollary 7 (Sub-exponential case). *Let $U : \mathbb{R} \rightarrow (0, \infty)$ be a convex function and $V = |U|^p$ with $p \in (0, 1)$. Assume (H1) and (H2). Fix $A > 0$. Then, there exists a constant $C = C(p, A, r, T, d, \|W\|_\infty)$ such that for any integer L , any local function f satisfies*

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f) \leq \frac{1}{C} e^{-Ct^{p/(2-p)}} \|f - \mu_\Lambda^\tau(f)\|_\infty^2 \quad \forall \tau \in \Omega,$$

provided $t^{p/(2-p)} \geq 2A \log(|\Lambda|)$, where $\Lambda = [-L, L]^d$.

Proof. Fix an integer L , $\tau \in \Omega$ and a local function f with $\mu_\Lambda^\tau(f) = 0$. We start as in the proof of Corollary 6, using instead that the one dimensional measure $d\nu = Z_V^{-1} e^{-V}$ satisfies a weak Poincaré inequality with rate function $\beta(s) = c_p \left(\log \frac{2}{s \wedge 1}\right)^{2(\frac{1}{p}-1)}$ for some constant $c_p > 0$ (see before Inequality (12)), to get that

$$\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f) \leq e^{-\frac{2t}{\gamma(s)}} \mathbf{Var}_\nu(f) + 4s(1 - e^{-\frac{2t}{\gamma(s)}}) \|f\|_\infty^2 \quad \forall s, t > 0$$

with $\gamma(s) = C \left(\log \frac{2|\Lambda|^{3/2}}{s \wedge |\Lambda|^{3/2}}\right)^{2(\frac{1}{p}-1)}$ for some $C = C(p, r, T, d, \|W\|_\infty)$ (we have chosen $\varepsilon = 3/2$ in Proposition 5).

Choose $s = e^{-t^{p/(2-p)}}$. Under the assumption $t^{p/(2-p)} \geq A \log(|\Lambda|^{3/2})$, the expected result follows after few rearrangements. \square

4. THE RESULTS.

In this section we shall deal with the two classes of examples of self potential V introduced in Section 2.2. We assume that Hypothesis (H1) and (H2) are satisfied in such a way that the infinite volume Gibbs measure μ exists. Recall that the Markov semi-group $(\mathbf{P}_t)_{t \geq 0}$ has been defined in (8). Also, we set $\|f\| = |||f||| + \|f\|_\infty$.

Theorem 8 (κ -concave case). *Let $U : \mathbb{R} \rightarrow (0, \infty)$ be a convex function and $\alpha > 0$. Set $V = (1 + \alpha) \log U$. Let $W : \mathbb{R} \rightarrow \mathbb{R}$. Assume (H1) and (H2). Fix and integer $\ell \geq 1$. Then, for any $\varepsilon \in (0, 1)$, there exists a constant C depending on $\varepsilon, \ell, \alpha, T, r, d, \|W\|_\infty, \|W'\|_\infty$ and $\|W''\|_\infty$ such that for all bounded local functions $f : \Omega \rightarrow \mathbb{R}$ with $|\Delta_f| \leq \ell^d$,*

$$(16) \quad \mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{C}{t^{\frac{\alpha}{2} - d(1+\varepsilon)}} \|f - \mu(f)\|^2 \quad \forall t > 0.$$

Remark 9. The spurious term $d(1 + \varepsilon)$ is, a priori, technical and we believe that the correct decay should be with the exponent $\alpha/2$ as in the one dimensional case. But on the other hand, it could be that the very heavy tails of the Cauchy type distributions slow down the dynamics with some strange unattended phenomenon (that we have not been able to catch).

Observe also that we obtain a polynomial decay only for $\alpha > 2d$. For $\alpha \leq 2d$, the previous bound is useless since \mathbf{P}_t is a contraction: we already know that $\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \|f - \mu(f)\|_\infty^2$.

Similarly, we have for sub-exponential self-potentials:

Theorem 10 (Sub-exponential case). *Let $U : \mathbb{R} \rightarrow (0, \infty)$ be a convex function and $p \in (0, 1)$. Set $V = |U|^p$. Let $W : \mathbb{R} \rightarrow \mathbb{R}$. Assume (H1) and (H2). Fix an integer $\ell \geq 1$. Then, there exists a constant C depending on $p, \ell, T, r, d, \|W\|_\infty, \|W'\|_\infty$ and $\|W''\|_\infty$ such that for all bounded local functions $f : \Omega \rightarrow \mathbb{R}$ with $|\Delta_f| \leq \ell^d$,*

$$(17) \quad \mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{1}{C} e^{-Ct^{p/(2-p)}} \|f - \mu(f)\|^2 \quad \forall t > 0.$$

The proof of Theorem 8 and Theorem 10 relies on two main ingredients: the bisection technique [29] through Corollary 6 and Corollary 7, and the following property known as finite speed of propagation.

Proposition 11 (Finite speed of propagation). *Fix and integer $\ell \geq 1$ and assume (H1) and (H2). Then, for any local function f with support $\Delta_f \subset [-\ell, \ell]^d$, any L multiple of r , any boundary condition $\tau \in \Omega$,*

$$\|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda, \tau} f\|_\infty \leq C \|f\| \left(\frac{C't}{L} \right)^{C''L} e^{Ct} \quad \forall t > 0$$

with $\Lambda = [-L, L]^d$, for some constant $C, C', C'' > 0$ depending only on $r, d, \|W'\|_\infty, \|W''\|_\infty$ and ℓ .

The proof of Proposition 11 is postponed to Section 6 for the clarity of the exposition.

Proof of Theorem 8. Fix $t > 0$ and $\varepsilon \in (0, 1)$. Let f be a local function. Since the system is invariant under translation and rotation we may assume as we shall that the support Δ_f of f contains the origin $0 \in \mathbb{Z}^d$. Furthermore, we can assume that $\mu(f) = 0$.

Let $\Lambda = [-L, L]^d \Subset \mathbb{Z}^d$ with $L = \lambda t + \lambda'$, where $\lambda, \lambda' > 0$ are parameters that will be chosen later. We assume that λ' is large enough in such a way that $\Delta_f \subset \Lambda$. Let $\tau \in \Omega$ be a boundary condition. Our starting point is the following bound

$$(18) \quad \mathbf{Var}_\mu(\mathbf{P}_t f) \leq 2\|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda, \tau} f\|_\infty^2 + 2\mathbf{Var}_{\mu_\Lambda^\tau}(\mathbf{P}_t^{\Lambda, \tau} f).$$

The first term of (18) is controlled by the finite speed of propagation result above. Indeed, we can choose λ and λ' large enough in such a way that L is a multiple of r and

$$\left(\frac{C't}{L}\right)^{C''L} e^{Ct} \leq e^{-ct}$$

for some constant c depending on C, C', C'', λ and λ' , where C, C' and C'' are defined in Proposition 11.

Going back to (18) and thanks to Corollary 6, we get that

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{\|f\|_\infty}{c} e^{-ct} + \frac{1}{c} \frac{|\Lambda|^{1+\varepsilon}}{t^{\alpha/2}} \|f\|_\infty^2$$

for some constant c depending only on $\varepsilon, \alpha, T, r, d, \|W\|_\infty, \|W'\|_\infty, \|W''\|_\infty$ and ℓ . The expected result follows. \square

Proof of Theorem 10. The proof of Theorem 10 is identical to the one of Theorem 8. We use the same notations.

Note that, for $L = \lambda t + \lambda'$ and $\Lambda = [-L, L]^d$, there exists $A = A(\lambda, \lambda', p)$ such that for any $t \geq 1$, $t^{p/(2-p)} \geq 2A \log(|\Lambda|)$. Hence, using the finite speed of propagation result together with Corollary 7, we get (details are left to the reader) that

$$\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \frac{\|f\|_\infty}{c} e^{-ct} + \frac{1}{c} e^{-ct^{p/(2-p)}} \|f\|_\infty^2$$

for any $t \geq 1$ and for some constant c depending only on $p, T, r, d, \|W\|_\infty, \|W'\|_\infty, \|W''\|_\infty$ and ℓ . The expected result follows for $t \geq 1$. Since trivially $\mathbf{Var}_\mu(\mathbf{P}_t f) \leq \|f - \mu(f)\|_\infty^2$, the expected result follows for any $t > 0$. \square

5. THE PERTURBATION PROPERTY IMPROVED.

In this section we prove Proposition 5 that improves for cubes the result of Proposition 3. We need to introduce a family of rectangles that will be useful for our purposes.

Fix $\varepsilon \in (0, 1)$. Let $l_k := (2 - \varepsilon)^{k/d}$, and let \mathbb{F}_k be the set of all rectangles $V \Subset \mathbb{Z}^d$ which, modulo translations and permutations of the coordinates, are contained in

$$[0, l_{k+1}] \times \cdots \times [0, l_{k+d}]$$

The main property of \mathbb{F}_k is that each rectangle in $\mathbb{F}_k \setminus \mathbb{F}_{k-1}$ can be obtained as a ‘‘slightly overlapping union’’ of two rectangles in \mathbb{F}_{k-1} . More precisely we have:

Lemma 12 ([8]). *For all $k \in \mathbb{Z}_+$, for all $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ there exists a finite sequence $\{\Lambda_1^{(i)}, \Lambda_2^{(i)}\}_{i=1}^{s_k}$ in \mathbb{F}_{k-1} , where $s_k := \lfloor l_k^{1/3} \rfloor$, such that, letting $\delta_k := \frac{\varepsilon}{4} \sqrt{l_k}$,*

- (i) $\Lambda = \Lambda_1^{(i)} \cup \Lambda_2^{(i)}$,
- (ii) $d(\Lambda \setminus \Lambda_1^{(i)}, \Lambda \setminus \Lambda_2^{(i)}) \geq \delta_k$,
- (iii) $(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}) \cap (\Lambda_1^{(j)} \cap \Lambda_2^{(j)}) = \emptyset$, if $i \neq j$

Proof. The proof is given in [8, Proposition 3.2] for $\varepsilon = 1/2$. The general case given here follows exactly the same line (details are left to the reader). \square

Proof of Proposition 5. The proof of Proposition 5 relies on the bisection technique together with the quasi factorisation of the variance.

The bisection method establishes a simple recursive inequality between the quantity $\gamma_k(s) := \sup_{\Lambda \in \mathbb{F}_k} \beta_\Lambda(s)$ (recall (14)) on scale k and the same quantity on scale $k - 1$. Note that, by construction γ_k is non-increasing.

Fix $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ and write it as $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ satisfying the properties described in Lemma 12 above. Without loss of generality we can assume that all the faces of Λ_1 and of Λ_2 lay on the faces of Λ except for one face orthogonal to the first direction $\vec{e}_1 := (1, 0, \dots, 0)$ and that, along that direction, Λ_1 comes before Λ_2 , see Figure 1.

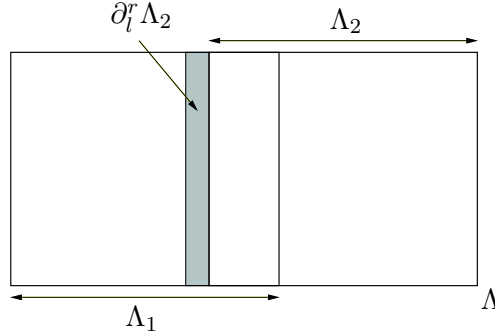


FIGURE 1. The set $\Lambda = \Lambda_1 \cup \Lambda_2$. The grey region is $\partial_f^r \Lambda_2$.

We claim that

Claim 13. *There exists k_0 such that for $k \geq k_0$,*

$$(19) \quad \mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \mu_\Lambda^\tau \left(\mathbf{Var}_{\mu_{\Lambda_1}}(f) + \mathbf{Var}_{\mu_{\Lambda_2}}(f) \right)$$

for some constant c_1 and c_2 depending on r, T, d and $\|W\|_\infty$.

This bound measures the weak dependence between μ_{Λ_1} and μ_{Λ_2} since it would hold with $c_1 = 0$ if μ_Λ^τ was the product $\mu_{\Lambda_1} \otimes \mu_{\Lambda_2}$. In other words it is a kind of weak factorisation of the variance.

Proof of the Claim. To prove the claim, let g be a measurable function with respect to $\mathcal{B}_{\Lambda_1^c \cap \Lambda}$. Then, by the DLR condition, we have

$$\|\mu_{\Lambda_2}(g) - \mu_{\Lambda}^{\tau}(g)\|_{\infty} = \|\mu_{\Lambda_2}(g) - \mu_{\Lambda}^{\tau}(\mu_{\Lambda_2}(g))\|_{\infty} \leq \sup_{\substack{\eta, \omega \in \Omega: \\ \eta_{\Lambda^c} = \omega_{\Lambda^c}}} |\mu_{\Lambda_2}^{\eta}(g) - \mu_{\Lambda_2}^{\omega}(g)|$$

Let $\partial_l^r \Lambda_2 := \{x \in \Lambda \setminus \Lambda_2 \text{ such that } x + i\vec{e}_1 \in \Lambda_2 \text{ for some } i = 1, \dots, r\}$ be the left boundary of width r of Λ_2 , see Figure 1. Note that $\sigma \mapsto \mu_{\Lambda_2}^{\sigma}(g)$ does not depend on any site x such that $d(x, \Lambda_2) > r$. Hence, if $\eta_{\Lambda^c} = \omega_{\Lambda^c}$, $\mu_{\Lambda_2}^{\eta}(g) - \mu_{\Lambda_2}^{\omega}(g)$ depends only on the sites in $\partial_l^r \Lambda_2$. In turn, using a telescopic sum over all $x \in \partial_l^r \Lambda_2$, one has for any $\eta, \omega \in \Omega$ such that $\eta_{\Lambda^c} = \omega_{\Lambda^c}$,

$$|\mu_{\Lambda_2}^{\eta}(g) - \mu_{\Lambda_2}^{\omega}(g)| \leq |\partial_l^r \Lambda_2| \sup_{\substack{x \in \partial_l^r \Lambda_2, \tau, \tau' \in \Omega: \\ \tau_{\Lambda_2^c \setminus \{x\}} = \tau'_{\Lambda_2^c \setminus \{x\}}}} |\mu_{\Lambda_2}^{\tau}(g) - \mu_{\Lambda_2}^{\tau'}(g)|.$$

Now set $h_x := \frac{Z_{\Lambda_2}^{\tau}}{Z_{\Lambda_2}^{\tau'}} e^{H_{\Lambda_2}^{\tau} - H_{\Lambda_2}^{\tau'}}$ and observe that h_x is a local function with support $\Delta_{h_x} = \{x\}$ and that $\|h_x\|_{\infty} \leq C$ for some constant $C = C(r, T, \|W\|_{\infty})$. Then, by a simple computation and using Hypothesis (H2), we have

$$|\mu_{\Lambda_2}^{\tau}(g) - \mu_{\Lambda_2}^{\tau'}(g)| = |\mu_{\Lambda_2}^{\tau}(g, h_x)| \leq C' |\Lambda_1^c \cap \Lambda| \|g\|_{\infty} e^{-md(\Lambda \setminus \Lambda_1, \Lambda \setminus \Lambda_2)}$$

for some constants C' and m (depending on r, d, T and $\|W\|_{\infty}$). All the previous computations together (recall the definition of δ_k in Lemma 12) lead to

$$\|\mu_{\Lambda_2}(g) - \mu_{\Lambda}^{\tau}(g)\|_{\infty} \leq C' \|g\|_{\infty} r \ell_{k+d}^{2d-1} e^{-m\delta_k} \leq C'' \|g\|_{\infty} e^{-c_2 \delta_k}$$

for some constants C'' and c_2 depending on r, T, d and $\|W\|_{\infty}$.

The same holds for $\|\mu_{\Lambda_1}(g) - \mu_{\Lambda}^{\tau}(g)\|_{\infty}$ with g measurable with respect to $\mathcal{B}_{\Lambda_2^c \cap \Lambda}$. The claim follows at once by the following quasi factorisation lemma of [8]. \square

Lemma 14 (Quasi factorisation of the Variance [8]). *Let $\Lambda, A, B \Subset \mathbb{Z}^d$ such that $\Lambda = A \cup B$. Assume that for some $\tau \in \Omega$ and $\varepsilon \in [0, \sqrt{2} - 1]$,*

$$\begin{aligned} \|\mu_B(g) - \mu_{\Lambda}^{\tau}(g)\|_{\infty} &\leq \varepsilon \|g\|_{\infty} \quad \forall g \in \mathbb{L}^{\infty}(\Omega, \mathcal{B}_{A^c \cap \Lambda}, \mu_{\Lambda}^{\tau}) \\ \|\mu_A(g) - \mu_{\Lambda}^{\tau}(g)\|_{\infty} &\leq \varepsilon \|g\|_{\infty} \quad \forall g \in \mathbb{L}^{\infty}(\Omega, \mathcal{B}_{B^c \cap \Lambda}, \mu_{\Lambda}^{\tau}). \end{aligned}$$

Then,

$$\mathbf{Var}_{\mu_{\Lambda}^{\tau}}(f) \leq \frac{1}{1 - 2\varepsilon - \varepsilon^2} \mu_{\Lambda}^{\tau}(\mathbf{Var}_{\mu_A}(f) + \mathbf{Var}_{\mu_B}(f)) \quad \forall f \in \mathbb{L}^2(\mu_{\Lambda}^{\tau}).$$

Proof. See [8, Lemma 3.1] \square

Remark 15. A similar result for the entropy can be found in [19].

Back to (19), we can use the definition of γ_{k-1} twice to get the following weak Poincaré inequality: for any $f : \Omega_\Lambda \rightarrow \mathbb{R}$, any $s > 0$, it holds

$$\begin{aligned} \mathbf{Var}_{\mu_\Lambda^\tau}(f) &\leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \gamma_{k-1}(s) \left[\mathcal{D}_\Lambda^\tau(f) + \sum_{x \in \Lambda_1 \cap \Lambda_2} \mu_\Lambda^\tau(|\nabla_x f|^2) \right] \\ &\quad + 2s \left(1 + c_1 e^{-c_2 \delta_k}\right) \text{Osc}(f). \end{aligned}$$

In order to get rid of the overlapping term $\sum_{x \in \Lambda_1 \cap \Lambda_2} \mu_\Lambda^\tau(|\nabla_x f|^2)$ in the latter, as observed in [29], one can average over the various positions of the pair $(\Lambda_1^{(i)}, \Lambda_2^{(i)})$ given in Lemma 12. In fact, by averaging the previous bound over the s_k possible choices of $(\Lambda_1^{(i)}, \Lambda_2^{(i)})$, we get

$$\begin{aligned} \mathbf{Var}_{\mu_\Lambda^\tau}(f) &\leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \gamma_{k-1}(s) \left[\mathcal{D}_\Lambda^\tau(f) + \frac{1}{s_k} \sum_{i=1}^{s_k} \sum_{x \in \Lambda_1^{(i)} \cap \Lambda_2^{(i)}} \mu_\Lambda^\tau(|\nabla_x f|^2) \right] \\ &\quad + 2s \left(1 + c_1 e^{-c_2 \delta_k}\right) \text{Osc}(f) \\ &\leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \left(1 + \frac{2}{s_k}\right) \gamma_{k-1}(s) \mathcal{D}_\Lambda^\tau(f) \\ &\quad + 2s \left(1 + c_1 e^{-c_2 \delta_k}\right) \text{Osc}(f). \end{aligned}$$

In the last line we used that $(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}) \cap (\Lambda_1^{(j)} \cap \Lambda_2^{(j)}) = \emptyset$ for $i \neq j$, *i.e.* Point (iii) of Lemma 12. It follows that

$$\gamma_k(s) \leq \left(1 + c_1 e^{-c_2 \delta_k}\right) \left(1 + \frac{2}{s_k}\right) \gamma_{k-1} \left(\frac{s}{2(1 + c_1 e^{-c_2 \delta_k})} \right) \quad \forall s > 0.$$

By iteration, we get for any $k \geq k_0$ and any $s > 0$,

$$\gamma_k(s) \leq \prod_{i=k_0+1}^k \left(1 + c_1 e^{-c_2 \delta_i}\right) \left(1 + \frac{2}{s_i}\right) \gamma_{k_0} \left(\frac{s}{2^{k-k_0} \prod_{i=k_0+1}^k (1 + c_1 e^{-c_2 \delta_i})} \right).$$

Note that for some $C = C(r, T, d, \|W\|_\infty)$,

$$1 \leq \prod_{i=k_0+1}^k \left(1 + c_1 e^{-c_2 \delta_i}\right) \leq \prod_{i=0}^{\infty} \left(1 + c_1 e^{-c_2 \delta_i}\right) \leq C$$

and similarly for $\prod_{i=k_0+1}^k \left(1 + \frac{2}{s_i}\right)$. Hence, since γ_{k_0} is non-increasing,

$$\gamma_k(s) \leq C^2 \gamma_{k_0} \left(\frac{s}{C 2^k} \right) \quad \forall k \geq k_0, \forall s > 0.$$

We are left with an estimate of γ_{k_0} . This is given by Proposition 3. Indeed, since k_0 is a constant depending only on r, T, d and $\|W\|_\infty$, Proposition 3 guarantees that $\gamma_{k_0}(s) \leq C' \beta(s/C')$ for some $C' = C'(r, T, d, \|W\|_\infty)$.

In conclusion, we have proved that for any $\Lambda \in \mathbb{F}_k$,

$$\mathbf{Var}_{\mu_\Lambda^\tau}(f) \leq C'' \beta \left(\frac{s}{2^k C''} \right) \mathcal{D}_\Lambda^\tau(f) + s \text{Osc}(f) \quad \forall \tau \in \Omega, \forall f, \forall s > 0$$

for some $C'' = C''(r, T, d, \|W\|_\infty)$.

Now consider a volume $\Lambda = [-L, L]^d$. Observe that $\Lambda \in \mathbb{F}_k$ as soon as $2L \leq l_{k+1}$. Take k to be the smallest integer satisfying such a property. After some computations, this leads to $2^k \leq c|\Lambda|^{\frac{\log 2}{\log(2-\varepsilon)}}$ for some universal constant $c > 0$. Since β_Λ is non-increasing, we get the expected result. This achieves the proof. \square

6. FINITE SPEED OF PROPAGATION

This section is dedicated to the proof of Proposition 11 (we recall below) on the finite speed of propagation. This result is somehow standard and would certainly not surprise the specialists. Nevertheless we give the proof for completeness.

Recall the definition of the finite volume and infinite volume Markov semi-groups $(\mathbf{P}_t^{\Lambda, \tau})_{t \geq 0}$ and $(\mathbf{P}_t)_{t \geq 0}$. Recall also the definition of $\|f\|$.

Proposition 16 (Finite speed of propagation). *Assume (H_1) and (H_2) . Fix and integer $\ell \geq 1$. Then, for any local function f with support $\Delta_f \subset [-\ell, \ell]^d$, any L multiple of r , any boundary condition $\tau \in \Omega$,*

$$\|\mathbf{P}_t f - \mathbf{P}_t^{\Lambda, \tau} f\|_\infty \leq C \|f\| \left(\frac{C' t}{L} \right)^{C'' L} e^{Ct} \quad \forall t > 0$$

with $\Lambda = [-L, L]^d$, for some constant $C, C', C'' > 0$ depending only on $r, d, \|W'\|_\infty, \|W''\|_\infty$ and ℓ .

Remark 17. Note that this bound is particularly interesting when $L \gg t$.

Proof. We follow [39]. Fix $t > 0$, $\Lambda = [-L, L]^d \in \mathbb{Z}^d$ with L a multiple of r , a boundary condition $\tau \in \Omega$ and a local function f with support Δ_f containing 0. Then,

(20)

$$(\mathbf{P}_t - \mathbf{P}_t^{\Lambda, \tau})f = - \int_0^t \left(\frac{d}{ds} (\mathbf{P}_{t-s} \mathbf{P}_s^{\Lambda, \tau}) f \right) ds = \int_0^t \mathbf{P}_{t-s} (\mathbf{L} - \mathbf{L}_\Lambda^\tau) \mathbf{P}_s^{\Lambda, \tau} f ds.$$

For simplicity let $f_s^\Lambda := \mathbf{P}_s^{\Lambda, \tau} f$ and note that its support $\Delta_{f_s^\Lambda} \subset \Lambda$. Therefore,

$$\begin{aligned} (\mathbf{L} - \mathbf{L}_\Lambda^\tau) f_s^\Lambda &= \sum_{x \in \Lambda} (\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau) \cdot \nabla_x f_s^\Lambda \\ (21) \quad &= \sum_{x \in \Lambda: d(x, \Lambda^c) \leq r} (\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau) \cdot \nabla_x f_s^\Lambda. \end{aligned}$$

Now our aim is to control $\nabla_x f_s^\Lambda$.

Take $y \in \Lambda$. By definition of L_Λ^τ , we have

$$[\nabla_y, L_\Lambda^\tau] := \nabla_y L_\Lambda^\tau - L_\Lambda^\tau \nabla_y = \sum_{x \in \Lambda} \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x = \sum_{x \in \Lambda: d(x,y) \leq r} \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x.$$

Thus, (we skip the superscript τ)

$$\begin{aligned} \nabla_y f_s^\Lambda &= \mathbf{P}_s^\Lambda \nabla_y f + \int_0^s \left(\frac{d}{du} \mathbf{P}_{s-u}^\Lambda \nabla_y f_u^\Lambda \right) du \\ &= \mathbf{P}_s^\Lambda \nabla_y f + \int_0^s \mathbf{P}_{s-u}^\Lambda [\nabla_y, L_\Lambda^\tau] f_u^\Lambda du \\ &= \mathbf{P}_s^\Lambda \nabla_y f + \sum_{x \in \Lambda: d(x,y) \leq r} \int_0^s \mathbf{P}_{s-u}^\Lambda \nabla_y \nabla_x H_\Lambda^\tau \cdot \nabla_x f_u^\Lambda du. \end{aligned}$$

Hence, thanks to Hypothesis (H_1) and the fact that \mathbf{P}_t^Λ is a contraction in the sup norm,

$$(22) \quad \|\nabla_y f_s^\Lambda\|_\infty \leq \|\nabla_y f\|_\infty + \|W''\|_\infty \sum_{x \in \Lambda: d(x,y) \leq r} \int_0^s \|\nabla_x f_u^\Lambda\|_\infty du.$$

Then, for any $n = 0, 1, \dots, L/r$, define

$$Y_n(u) := \sum_{x \in \Lambda: d(x, \Lambda^c) \leq L - nr} \|\nabla_x f_u^\Lambda\|_\infty.$$

Recall that $\Delta_f \subset [-\ell, \ell]^d$. Since $\nabla_x f = 0$ unless $x \in \Delta_f$, we get from (22) that for $n = \ell + 1, \dots, L/r$,

$$Y_n(s) \leq (2r)^d \|W''\|_\infty \int_0^s Y_{n-1}(u) du.$$

On the other hand, for $n = 0, 1, \dots, \ell$,

$$Y_n(s) \leq \|f\| + (2r)^d \|W''\|_\infty \int_0^s Y_{n-1}(u) du,$$

with the convention that $Y_{-1} := Y_0$.

It follows that $Y_n(t) \leq \|f\| \exp\{Ct\}$ for any $0 \leq n \leq \ell$, with $C := (2r)^d \|W''\|_\infty$. Moreover, an easy induction gives for any $\ell < n \leq L/r$

$$(23) \quad Y_n(t) \leq \|f\| R(n-\ell, t) \quad \text{with } R(m, t) := e^{Ct} - \sum_{k=0}^m \frac{(Ct)^k}{k!} \leq \left(\frac{Cte}{m} \right)^m e^{Ct}.$$

Finally, using the fact that \mathbf{P}_t is a contraction in the sup norm and Hypothesis (H1), we get from (20) and (23) that

$$\begin{aligned}
\|(\mathbf{P}_t - \mathbf{P}_t^{\Lambda, \tau})f\|_\infty &\leq \int_0^t \|(\mathbf{L} - \mathbf{L}_\Lambda^\tau)f_s^\Lambda\|_\infty ds \\
&\leq \sum_{x \in \Lambda: d(x, \Lambda^c) \leq r} \int_0^t \|\nabla_x H_\Lambda - \nabla_x H_\Lambda^\tau\|_\infty \|f_s^\Lambda\|_\infty ds \\
&\leq 4(2r)^d \|W'\|_\infty \int_0^t Y_{\frac{L}{r}-1}(s) ds \\
&\leq \frac{4(2r)^d \|W'\|_\infty}{C} \|f\| \|R\left(\frac{L}{r} - \ell, t\right)
\end{aligned}$$

The expected result follows from (23). This achieves the proof. \square

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Cyril ROBERTO, LABORATOIRE D'ANALYSE ET MATHÉMATIQUES APPLIQUÉES-UMR 8050, UNIVERSITÉS DE PARIS-EST, MARNE LA VALLÉE, 5 BOULEVARD DESCARTES, CITÉ DESCARTES, CHAMPS SUR MARNE, 77454 MARNE LA VALLÉE CEDEX 2, FRANCE.
E-mail address: `cyril.roborto@univ-mlv.fr`