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Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions.

Stéphane Gerbi* and Belkacem Said-Houari†

Abstract

In this paper we consider a multi-dimensional wave equation with dynamic boundary conditions, related to the Kelvin-Voigt damping. Global existence and asymptotic stability of solutions starting in a stable set are proved. Blow up for solutions of the problem with linear dynamic boundary conditions with initial data in the unstable set is also obtained.

AMS Subject classification : 35L45, 35L70, 35B40.

Keywords : Damped wave equations, stable and unstable set, global solutions, blow up, Kelvin-Voigt damping, dynamic boundary conditions.

1 Introduction

In this paper we consider the following semilinear damped wave equation with dynamic boundary conditions:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2}u, & x \in \Omega, \ t > 0 \\ u(x, t) = 0, & x \in \Gamma_0, \ t > 0 \\ u_{tt}(x, t) = -a \left[\frac{\partial u}{\partial \nu}(x, t) + \frac{\alpha \partial u_t}{\partial \nu}(x, t) + r|u_t|^{m-2}u_t(x, t) \right] & x \in \Gamma_1, \ t > 0 \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) & x \in \Omega \quad , \end{cases} \quad (1.1)$$

where $u = u(x, t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of \mathbb{R}^N , ($N \geq 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, $m \geq 2$, a , α and r are positive constants, $p > 2$ and u_0 , u_1 are given functions.

From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called *dynamic boundary conditions*. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, the problem (1.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represents the Newton's law for the attached mass, (see [4, 2, 8] for more details). In the two dimension space, as showed in [28] and in the references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a

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region. Also some dynamic boundary conditions as in problem (1.1) appear when we assume that Ω is an exterior domain of \mathbb{R}^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [3] for more details). This type of dynamic boundary conditions are known as acoustic boundary conditions.

In one space dimension, in the case where $r \neq 0$ and $m = 2$, Pellicer and Solà-Morales [26] considered the one dimensional problem as an alternative model for the classical spring-mass damper system, and by using the dominant eigenvalues method, they proved that for small values of the parameter a the partial differential equations in the problem (1.1) has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t) = 0,$$

as a limit where the parameter m_1 , d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence. This result was followed by recent works [25, 27]. In particular in [27], the authors considered a one dimensional nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. In other word, they looked to the following problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0, t) = 0, \\ u_{tt}(1, t) = -\varepsilon [u_x + \alpha u_{tx} + r u_t](1, t) - \varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right), \end{cases} \quad (1.2)$$

with $x \in (0, 1)$, $t > 0$, $r, \alpha > 0$ and $\varepsilon \geq 0$. The above system models a spring-mass-damper system, where the term $\varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right)$ represents a control acceleration at $x = 1$. By using the invariant manifold theory, the authors proved that for small values of the parameter ε , the solutions of (1.2) are attracted to a two dimensional invariant manifold. See [27], for further details.

We recall that the presence of the strong damping term $-\Delta u_t$ in the problem (1.1) makes the problem different from that considered in [12] and widely studied in the literature [32, 29, 30, 11, 31] for instance. For this reason less results were known for the wave equation with a strong damping and many problems remained unsolved. Especially the blow-up of solutions in the presence of a strong damping and a nonlinear boundary damping at the same time is still an open problem. In [13], the present authors showed that the solution of (1.1) is unbounded and grows up exponentially when time goes to infinity if the initial data are large enough.

Recently, Gazzola and Squassina [11] studied the global solution and the finite time blow-up for a damped semilinear wave equations with Dirichlet boundary conditions by a careful study of the stationnary solutions and their stability using the Nehari manifold and a mountain pass energy level of the initial condition.

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1 . Very little attention has been paid to this type of boundary conditions. We mention only a few particular results in the one dimensional space and for a linear damping i.e. ($m = 2$) [15, 26, 9, 17].

A related problem to (1.1) is the following:

$$\begin{aligned} u_{tt} - \Delta u + g(u_t) &= f && \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial \nu} + K(u)u_{tt} + h(u_t) &= 0, && \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) && \text{in } \Omega \\ u_t(x, 0) &= u_1(x) && \text{in } \Omega \end{aligned}$$

where the boundary term $h(u_t) = |u_t|^\rho u_t$ arises when one studies flows of gas in a channel with porous walls. The term u_{tt} on the boundary appears from the internal forces, and the nonlinearity $K(u)u_{tt}$ on the boundary represents the internal forces when the density of the medium depends on the displacement. This problem has been studied in [9, 10]. By using the Faedo-Galerkin approximations and a compactness argument, they proved the global existence and the exponential decay of the solution of the problem.

We recall some results related to the interaction of an elastic medium with rigid mass. By using the classical semigroup theory, Littman and Markus [20] established a uniqueness result for a particular Euler-Bernoulli beam rigid body structure. They also proved the asymptotic stability of the structure by using the feedback boundary damping. In [21] the authors considered the Euler-Bernoulli beam equation which describes the dynamics of clamped elastic beam in which one segment of the beam is made with viscoelastic material and the other of elastic material. By combining the frequency domain method with the multiplier technique, they proved the exponential decay for the transversal motion but not for the longitudinal motion of the model, when the Kelvin-Voigt damping is distributed only on a subinterval of the domain. In relation with this point, see also the work by Chen *et al.* [7] concerning the Euler-Bernoulli beam equation with the global or local Kelvin-Voigt damping. Also models of vibrating strings with local viscoelasticity and Boltzmann damping, instead of the Kelvin-Voigt one, were considered in [22] and an exponential energy decay rate was established. Recently, Grobbelaar-Van Dalsen [14] considered an extensible thermo-elastic beam which is hanged at one end with rigid body attached to its free end, i.e. one dimensional hybrid thermoelastic structure, and showed that the method used in [23] is still valid to establish a uniform stabilization of the system. Concerning the controllability of the hybrid system we refer to the work by Castro and Zuazua [5], in which they considered flexible beams connected by point mass and the model takes account of the rotational inertia.

In this paper we consider the problem (1.1) where we have set for the sake of simplicity $a = 1$. We will show that if the initial data are in the “stable set”, the solution continues to live there forever. In addition, we will prove that the presence of the strong damping forces the solution to go to zero uniformly and with an exponential decay rate. To obtain our results we combine the potential well method with the energy method. We will also prove that in the absence of a nonlinear boundary damping, in the case where $m = 2$, the solution blows up in finite time.

This paper is organized as follows. In the next section, we briefly recall the local existence result obtained by the authors in [13] and the main ingredients to prove it. In the section 3, we will prove that if the initial data are in the stable manifold, the solution continues to live there and so we will prove the global existence and the exponential decay of the solution. In section 4, we prove the blow up result of the problem (1.1), in the case of a linear boundary damping, in spite of the presence of the strong damping term Δu_t . The technique we use follows closely the method used in [11], which is based on the concavity argument due to Levine [18].

2 Local existence

In this section we will recall how we obtained in the article [13] the local existence and the uniqueness of the solution of the problem (1.1). We have adapted the ideas used by Georgiev and Todorova in [12], which consists in constructing approximations by the Faedo-Galerkin procedure in order to use the contraction mapping theorem.

We present here some material that we shall use in order to present the local

existence of the solution of problem (1.1). We denote

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) / u|_{\Gamma_0} = 0\}.$$

By (\cdot, \cdot) we denote the scalar product in $L^2(\Omega)$ i.e. $(u, v)(t) = \int_{\Omega} u(x, t)v(x, t)dx$. Also we mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and by $\|\cdot\|_{q, \Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Let $T > 0$ be a real number and X a Banach space endowed with norm $\|\cdot\|_X$. $L^p(0, T; X)$, $1 \leq p < \infty$ denotes the space of functions f which are L^p over $(0, T)$ with values in X , which are measurable and $\|f\|_X \in L^p(0, T)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0, T; X)} = \left(\int_0^T \|f\|_X^p dt \right)^{1/p}.$$

$L^\infty(0, T; X)$ denotes the space of functions $f :]0, T[\rightarrow X$ which are measurable and $\|f\|_X \in L^\infty(0, T)$. This space is a Banach space endowed with the norm:

$$\|f\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 < t < T} \|f\|_X.$$

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$ (continuous embedding), then

$$L^p(0, T; X) \hookrightarrow L^p(0, T; Y), \quad 1 \leq p \leq \infty.$$

We will also use the embedding (see [1, Theorem 5.8]).

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1), \quad 2 \leq q \leq \bar{q} \quad \text{where} \quad \bar{q} = \begin{cases} \frac{2(N-1)}{N-2}, & \text{if } N \geq 3 \\ +\infty, & \text{if } N = 1, 2 \end{cases}. \quad (2.1)$$

Let us denote $V = H_{\Gamma_0}^1(\Omega) \cap L^m(\Gamma_1)$.

In the article [13], we could not use “directly” the existence result of Georgiev and Todorova [12] nor the results of Vitillaro [33, 34] because of the presence of the strong linear damping $-\Delta u_t$ and the dynamic boundary conditions on Γ_1 . Therefore, we have the next local existence theorem.

Theorem 2.1 [13] *Let $2 \leq p \leq \bar{q}$ and $\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}$.*

Then given $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in L^2(\Omega)$, there exists $T > 0$ and a unique solution u of the problem (1.1) on $(0, T)$ such that

$$\begin{aligned} u &\in C([0, T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega)), \\ u_t &\in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0, T) \times \Gamma_1) \end{aligned}$$

We proved this theorem by using the Fadeo-Galerkin approximations and the well-known contraction mapping theorem. In order to understand the construction of the mapping, we briefly recall how we built the function for which a fixed point exists. For this sake, we considered first a related problem.

For $u \in C([0, T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ given, let us consider the following problem:

$$\begin{cases} v_{tt} - \Delta v - \alpha \Delta v_t = |u|^{p-2}u, & x \in \Omega, \quad t > 0 \\ v(x, t) = 0, & x \in \Gamma_0, \quad t > 0 \\ v_{tt}(x, t) = - \left[\frac{\partial v}{\partial \nu}(x, t) + \frac{\alpha \partial v_t}{\partial \nu}(x, t) + r|v_t|^{m-2}v_t(x, t) \right] & x \in \Gamma_1, \quad t > 0 \\ v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x) & x \in \Omega \end{cases} \quad (2.2)$$

We stated the following existence result :

Lemma 2.1 [13] *Let $2 \leq p \leq \bar{q}$ and $\max\left(2, \frac{\bar{q}}{\bar{q}+1-p}\right) \leq m \leq \bar{q}$. Then given $u_0 \in H_{\Gamma_0}^1(\Omega)$, $u_1 \in L^2(\Omega)$ there exist $T > 0$ and a unique solution v of the problem (2.2) on $(0, T)$ such that*

$$\begin{aligned} v &\in C([0, T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega)), \\ v_t &\in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0, T) \times \Gamma_1) \end{aligned}$$

and satisfies the energy identity:

$$\begin{aligned} \frac{1}{2} [\|\nabla v\|_2^2 + \|v_t\|_2^2 + \|v_t\|_{2, \Gamma_1}^2]_s^t &+ \alpha \int_s^t \|\nabla v_t(\tau)\|_2^2 d\tau + r \int_s^t \|v_t(\tau)\|_{m, \Gamma_1}^m d\tau \\ &= \int_s^t \int_{\Omega} |u(\tau)|^{p-2} u(\tau) v_t(\tau) d\tau dx \end{aligned}$$

for $0 \leq s \leq t \leq T$.

In order to prove lemma 2.1, we first studied for any $T > 0$ and $f \in H^1(0, T; L^2(\Omega))$ the following problem:

$$\begin{cases} v_{tt} - \Delta v - \alpha \Delta v_t = f(x, t), & x \in \Omega, \ t > 0 \\ v(x, t) = 0, & x \in \Gamma_0, \ t > 0 \\ v_{tt}(x, t) = - \left[\frac{\partial v}{\partial \nu}(x, t) + \frac{\alpha \partial v_t}{\partial \nu}(x, t) + r |v_t|^{m-2} v_t(x, t) \right] & x \in \Gamma_1, \ t > 0 \\ v(x, 0) = u_0(x), \ v_t(x, 0) = u_1(x) & x \in \Omega \end{cases} \quad (2.3)$$

At this point, as done by Doronin et al. [10], we had to precise exactly what type of solutions of the problem (2.3) we expected.

Definition 2.1 [13] *A function $v(x, t)$ such that*

$$\begin{aligned} v &\in L^\infty(0, T; H_{\Gamma_0}^1(\Omega)) \ , \\ v_t &\in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0, T) \times \Gamma_1) \ , \\ v_t &\in L^\infty(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^\infty(0, T; L^2(\Gamma_1)) \ , \\ v_{tt} &\in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; L^2(\Gamma_1)) \ , \\ v(x, 0) &= u_0(x) \ , \\ v_t(x, 0) &= u_1(x) \ , \end{aligned}$$

is a generalized solution to the problem (2.3) if for any function $\omega \in H_{\Gamma_0}^1(\Omega) \cap L^m(\Gamma_1)$ and $\varphi \in C^1(0, T)$ with $\varphi(T) = 0$, we have the following identity:

$$\begin{aligned} \int_0^T (f, w)(t) \varphi(t) dt &= \int_0^T \left[(v_{tt}, w)(t) + (\nabla v, \nabla w)(t) + \alpha (\nabla v_t, \nabla w)(t) \right] \varphi(t) dt \\ &+ \int_0^T \varphi(t) \int_{\Gamma_1} \left[v_{tt}(t) + r |v_t(t)|^{m-2} v_t(t) \right] w d\sigma dt. \end{aligned}$$

Lemma 2.2 [13] *Let $2 \leq p \leq \bar{q}$ and $2 \leq m \leq \bar{q}$.*

Let $u_0 \in H^2(\Omega) \cap V$, $u_1 \in H^2(\Omega)$ and $f \in H^1(0, T; L^2(\Omega))$, then for any $T > 0$, there exists a unique generalized solution (in the sense of definition 2.1), $v(t, x)$ of problem (2.3).

To prove the above lemma, we used the Faedo-Galerkin method, which consists in constructing approximations of the solution, then we obtain a priori estimates necessary to guarantee the convergence of these approximations. It appears some difficulties in order to derive a second order estimate of $v_{tt}(0)$. To get rid of them, and inspired by the ideas of Doronin and Larkin in [9] and Cavalcanti et al. [6], we introduced the following change of variables:

$$\tilde{v}(t, x) = v(t, x) - \phi(t, x) \text{ with } \phi(t, x) = u_0(x) + t u_1(x).$$

Consequently, we have the following problem with the unknown $\tilde{v}(t, x)$ and null initial conditions:

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} - \alpha \Delta \tilde{v}_t = f(x, t) + \Delta \phi + \alpha \Delta \phi_t, & x \in \Omega, t > 0 \\ \tilde{v}(x, t) = 0, & x \in \Gamma_0, t > 0 \\ \tilde{v}_{tt}(x, t) = - \left[\frac{\partial(\tilde{v} + \phi)}{\partial \nu}(x, t) + \frac{\alpha \partial(\tilde{v}_t + \phi_t)}{\partial \nu}(x, t) \right] - \\ \quad \left(r |(\tilde{v}_t + \phi_t)|^{m-2} (\tilde{v}_t + \phi_t)(x, t) \right) & x \in \Gamma_1, t > 0 \\ \tilde{v}(x, 0) = 0, \quad \tilde{v}_t(x, 0) = 0 & x \in \Omega \end{cases} \quad (2.4)$$

Now we constructed approximations of the solution \tilde{v} by the Faedo-Galerkin method as follows.

For every $n \geq 1$, let $W_n = \text{span}\{\omega_1, \dots, \omega_n\}$, where $\{\omega_j(x)\}_{1 \leq j \leq n}$ is a basis in the space V . By using the Gram-Schmidt orthogonalization process we can take $\omega = (\omega_1, \dots, \omega_n)$ to be orthonormal¹ in $L^2(\Omega) \cap L^2(\Gamma_1)$.

We define the approximations:

$$\tilde{v}_n(t) = \sum_{j=1}^n g_{jn}(t) w_j \quad (2.5)$$

where $\tilde{v}_n(t)$ are solutions to the finite dimensional Cauchy problem (written in normal form since ω is an orthonormal basis):

$$\begin{aligned} & \int_{\Omega} \tilde{v}_{ttn}(t) w_j dx + \int_{\Omega} \nabla(\tilde{v}_n + \phi) \nabla w_j + \alpha \int_{\Omega} \nabla(\tilde{v}_n + \phi)_t \nabla w_j dx \\ & + \int_{\Gamma_1} (\tilde{v}_{ttn}(t) + r |(\tilde{v}_n + \phi)_t|^{m-2} (\tilde{v}_n + \phi)_t) w_j d\sigma = \int_{\Omega} f w_j dx \quad . \end{aligned} \quad (2.6)$$

$$g_{jn}(0) = g'_{jn}(0) = 0, \quad j = 1, \dots, n$$

According to the Caratheodory theorem, the problem (2.6) has solution $(g_{jn}(t))_{j=1, n} \in H^3(0, t_n)$ defined on $[0, t_n]$.

After having derived two a priori estimates, we showed that:

- firstly that $\forall n \in \mathbb{N}, t_n = T$,
- secondly that these approximations converge to a solution of the problem (2.4).

At this point, the proofs of lemma 2.1 and lemma 2.2 follow the works of Vitillaro [33, 34]. Finally the theorem 2.1 is proved using the contraction mapping theorem in a suitable Banach space. The complete proof has been carefully written in the article by the same authors [13].

¹Unfortunately, the presence of the nonlinear boundary conditions excludes us to use the spatial basis of eigenfunctions of $-\Delta$ in $H_{\Gamma_0}^1(\Omega)$ as done in [11]

3 Asymptotic stability

In this section we state and prove the global existence and exponential decay of the solution of problem (1.1). In order to do this, a suitable choice of the Lyapunov functional will be made.

Let us recall that the solution u of (1.1) belongs to the space:

$$Y_T = \left\{ \begin{array}{l} v \in C([0, T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega)), \\ v_t \in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0, T) \times \Gamma_1) \end{array} \right\}$$

endowed with the norm:

$$\|u\|_{Y_T}^2 = \max_{0 \leq t \leq T} [\|v_t\|_2^2 + \|\nabla v\|_2^2] + \|v_t\|_{L^m((0, T) \times \Gamma_1)}^2 + \int_0^T \|\nabla v_t(s)\|_2^2 ds.$$

Definition 3.1 Let $2 \leq p \leq \bar{q}$, $\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}$, $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We denote u the solution of (1.1). We define:

$$T_{max} = \sup\{T > 0, u = u(t) \text{ exists on } [0, T]\}$$

Since the solution $u \in Y_T$ (the solution is “enough regular”), let us recall that if $T_{max} < \infty$, then

$$\lim_{\substack{t \rightarrow T_{max} \\ t < T_{max}}} \|\nabla u\|_2 + \|u_t\|_2 = +\infty.$$

If $T_{max} < \infty$, we say that the solution of (1.1) blows up and that T_{max} is the blow up time.

If $T_{max} = \infty$, we say that the solution of (1.1) is global.

In order to study the blow up phenomenon or the global existence of the solution of (1.1), we define the following functions:

$$I(t) = I(u(t)) = \|\nabla u\|_2^2 - \|u\|_p^p, \quad (3.1)$$

$$J(t) = J(u(t)) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p} \|u\|_p^p, \quad (3.2)$$

and

$$E(u(t)) = E(t) = J(t) + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|u_t\|_{2, \Gamma_1}^2. \quad (3.3)$$

Let us remark that multiplying (1.1) by u_t , integrating over Ω and using integration by parts we obtain:

$$\frac{dE(t)}{dt} = -\alpha \|\nabla u_t\|_2^2 - r \|u_t\|_{m, \Gamma_1}^m \leq 0, \quad \forall t \geq 0. \quad (3.4)$$

Thus the function E is decreasing along the trajectories. As in [24], the potential well depth is defined as:

$$d = \inf_{u \in H_{\Gamma_0}^1(\Omega) \setminus \{0\}} \max_{\lambda \geq 0} J(\lambda u). \quad (3.5)$$

We can now define the so called “Nehari manifold” as follows:

$$\mathcal{N} = \{u \in H_{\Gamma_0}^1(\Omega) \setminus \{0\}; I(t) = 0\}.$$

\mathcal{N} separates the two unbounded sets:

$$\mathcal{N}^+ = \{u \in H_{\Gamma_0}^1(\Omega); I(t) > 0\} \cup \{0\} \quad \text{and} \quad \mathcal{N}^- = \{u \in H_{\Gamma_0}^1(\Omega); I(t) < 0\}.$$

The *stable* set \mathcal{W} and *unstable* set \mathcal{U} are defined respectively as:

$$\mathcal{W} = \{u \in H_{\Gamma_0}^1(\Omega); J(t) \leq d\} \cap \mathcal{N}^+ \quad \text{and} \quad \mathcal{U} = \{u \in H_{\Gamma_0}^1(\Omega); J(t) \leq d\} \cap \mathcal{N}^-.$$

It is readily seen that the potential depth d is also characterized by

$$d = \min_{u \in \mathcal{N}} J(u).$$

As it was remarked by Gazzola and Squassina in [11], this alternative characterization of d shows that

$$\beta = \text{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|\nabla u\|_2 = \sqrt{\frac{2dp}{p-2}} > 0. \quad (3.6)$$

In the lemma 3.1, we would like to prove the invariance of the set \mathcal{N}^+ : if the initial data u_0 is in the set \mathcal{N}^+ and if the initial energy $E(0)$ is not large (we will precise exactly how large may be the initial energy), then $u(t)$ stays in \mathcal{N}^+ forever.

For this purpose, as in [11, 32], we denote by C_* the best constant in the Poincaré-Sobolev embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^p(\Omega)$ defined by:

$$C_*^{-1} = \inf \{ \|\nabla u\|_2 : u \in H_{\Gamma_0}^1(\Omega), \|u\|_p = 1 \}. \quad (3.7)$$

Let us denote the Sobolev critical exponent:

$$\bar{p} = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3 \\ +\infty, & \text{if } N = 1, 2 \end{cases}.$$

Let us remark (as in [11, 32]) that if $p < \bar{p}$ the embedding is compact and the infimum in (3.7) (as well as in (3.5)) is attained. In such case (see, e.g. [24, Section 3]), any mountain pass solution of the stationary problem is a minimizer for (3.7) and C_* is related to its energy:

$$d = \frac{p-2}{2p} C_*^{-2p/(p-2)}. \quad (3.8)$$

Let us remark also that in the theorem 2.1, we have supposed that $p < \bar{q}$ where \bar{q} is defined by (2.1). As $\bar{q} < \bar{p}$, we may use the above characterisation of the potential well depth d .

Remark 3.1 [24, 16] For every solution of (1.1), given by Theorem 2.1, only one of the following assumption holds:

- i) if there exists some $t_0 \geq 0$ such that $u(t_0) \in \mathcal{W}$ and $E(t_0) < d$, then $\forall t \geq t_0$, $u(t) \in \mathcal{W}$ and $E(t) < d$.
- ii) if there exists some $t_0 \geq 0$ such that $u(t_0) \in \mathcal{U}$ and $E(t_0) < d$, then $\forall t \geq t_0$, $u(t) \in \mathcal{U}$ and $E(t) < d$.
- iii) $\forall t \geq 0$, $E(t) \geq d$.

We can now proceed in the global existence result investigation. For this sake, let us state two lemmas.

Lemma 3.1 Assume $2 \leq p \leq \bar{q}$ and $\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then $u(t, \cdot) \in \mathcal{N}^+$ for each $t \in [0, T)$.

Remark 3.2 Let us remark, that if there exists $t_0 \in [0, T)$ such that

$$E(t_0) < d$$

the same result stays true. It is the reason why we choose $t_0 = 0$.

Moreover, one can easily see that, from (3.8), the condition $E(0) < d$ is equivalent to the inequality:

$$C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1 \quad (3.9)$$

This last inequality will be used in the remaining proofs.

Proof of lemma 3.1: Since $I(u_0) > 0$, then by continuity, there exists $T_* \leq T$ such that $I(u(t, \cdot)) \geq 0$, for all $t \in [0, T_*)$. Since we have the relation:

$$J(t) = \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{1}{p} I(t)$$

we easily obtain :

$$J(t) \geq \frac{p-2}{2p} \|\nabla u\|_2^2, \quad \forall t \in [0, T_*) \quad .$$

Hence we have:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} J(t) \quad .$$

From (3.2) and (3.3), we obviously have $\forall t \in [0, T_*)$, $J(t) \leq E(t)$. Thus we obtain:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} E(t)$$

Since E is a decreasing function of t , we finally have:

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2} E(0), \quad \forall t \in [0, T_*) \quad . \quad (3.10)$$

By definition of C_* , we have:

$$\|u\|_p^p \leq C_*^p \|\nabla u\|_2^p = C_*^p \|\nabla u\|_2^{p-2} \|\nabla u\|_2^2$$

Using the inequality (3.10), we deduce:

$$\|u\|_p^p \leq C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \|\nabla u\|_2^2 \quad .$$

Now exploiting the inequality on the initial condition (3.9) we obtain:

$$\|u\|_p^p < \|\nabla u\|_2^2 \quad .$$

Hence $\|\nabla u\|_2^2 - \|u\|_p^p > 0$, $\forall t \in [0, T_*)$, this shows that $u(t, \cdot) \in \mathcal{N}^+$, $\forall t \in [0, T_*)$. By repeating this procedure, T_* is extended to T . ■

Lemma 3.2 Assume $2 \leq p \leq \bar{q}$ and $\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then the solution of the problem (1.1) is global in time.

Proof of lemma 3.2: Since the map $t \mapsto E(t)$ is a decreasing function of time t , we have:

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{(p-2)}{2p}\|\nabla u\|_2^2 + \frac{1}{p}I(t) \quad ,$$

which gives us:

$$E(0) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{(p-2)}{2p}\|\nabla u\|_2^2 \quad .$$

Thus, $\forall t \in [0, T)$, $\|\nabla u\|_2 + \|u_t\|_2$ is uniformly bounded by a constant depending only on $E(0)$ and p . Then by definition 3.1, the solution is global, so $T_{max} = \infty$. ■

We can now state the asymptotic behavior of the solution of (1.1).

Theorem 3.1 Assume $2 \leq p \leq \bar{q}$ and $\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then there exist two positive constants \hat{C} and ξ independent of t such that:

$$0 < E(t) \leq \hat{C}e^{-\xi t}, \quad \forall t \geq 0.$$

Remark 3.3 Let us remark that these inequalities imply that there exist positive constants K and ζ independent of t such that:

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq Ke^{-\zeta t}, \quad \forall t \geq 0.$$

Thus, this result improves the decay rate of Gazzola and Squassina [11, Theorem 3.8], in which the authors showed only the polynomial decay. Here we show that we can always find initial data satisfying $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$ which verify the inequality (3.9), such that the solution can decay faster than $1/t$, in fact with an exponential rate, even in the case $m > 2$. Also, the same situation happens in absence of strong damping ($\alpha = 0$) and $m = 2$.

Proof of theorem 3.1: Since we have proved that $\forall t \geq 0$, $u(t) \in \mathcal{N}^+$, we already have:

$$0 < E(t) \quad \forall t \geq 0.$$

The proof of the other inequality relies on the construction of a Lyapunov functional by performing a suitable modification of the energy. To this end, for $\varepsilon > 0$, to be chosen later, we define

$$L(t) = E(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \int_{\Gamma_1} u u_t d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u\|_2^2 \quad . \quad (3.11)$$

It is straightforward to see that $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and $\beta_2 > 0$ depending on ε such that for $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \quad (3.12)$$

By taking the time derivative of the function L defined above in equation (3.11), using problem (1.1), and performing several integration by parts, we get:

$$\begin{aligned} \frac{dL(t)}{dt} &= -\alpha \|\nabla u_t\|_2^2 - r \|u_t\|_{m,\Gamma_1}^m + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ &\quad + \varepsilon \|u\|_p^p + \varepsilon \|u_t\|_{2,\Gamma_1}^2 - \varepsilon r \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \quad . \end{aligned} \quad (3.13)$$

Now, we estimate the last term in the right hand side of (3.13) as follows.
By using Young's inequality, we obtain, for any $\delta > 0$

$$\left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right| \leq \frac{\delta^{-m}}{m} \|u\|_{m,\Gamma_1}^m + \frac{m-1}{m} \delta^{m/(m-2)} \|u_t\|_{m,\Gamma_1}^m. \quad (3.14)$$

The trace inequality implies that:

$$\|u\|_{m,\Gamma_1}^m \leq C \|\nabla u\|_2^m \quad .$$

where C here and in the sequel denotes a generic positive constant which may change from line into line. Since the inequality (3.10) holds, we have

$$\|u\|_{m,\Gamma_1}^m \leq C \left(\frac{2p E(0)}{p-2} \right)^{\frac{m-2}{2}} \|\nabla u\|_2^2 \quad . \quad (3.15)$$

Inserting the two inequalities (3.14) and (3.15) in (3.13), we have:

$$\begin{aligned} \frac{dL(t)}{dt} &\leq -\alpha \|\nabla u_t\|_2^2 + r \left(\varepsilon \frac{m-1}{m} \delta^{m/(m-2)} - 1 \right) \|u_t\|_{m,\Gamma_1}^m \\ &\quad + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2,\Gamma_1}^2 \\ &\quad + \varepsilon \left(\frac{r\delta^{-m}}{m} C \left(\frac{2p E(0)}{p-2} \right)^{\frac{m-2}{2}} + \underbrace{C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}}}_{<0} - 1 \right) \|\nabla u\|_2^2. \end{aligned} \quad (3.16)$$

From (3.9), we have

$$C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} - 1 < 0 \quad .$$

Now, let us choose δ large enough such that:

$$\left(\frac{r\delta^{-m}}{m} C \left(\frac{2p E(0)}{p-2} \right)^{\frac{m-2}{2}} + C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} - 1 \right) < 0 \quad .$$

Once δ is fixed, we choose ε small enough such that:

$$\left(\varepsilon \frac{m-1}{m} \delta^{m/(m-2)} - 1 \right) < 0 \quad .$$

From (3.16), we may find $\eta > 0$, which depends only on δ , such that:

$$\frac{dL(t)}{dt} \leq -\alpha \|\nabla u_t\|_2^2 + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2,\Gamma_1}^2 - \varepsilon \eta \|\nabla u\|_2^2.$$

Consequently, using the definition of the energy (3.3), for any positive constant M , we obtain:

$$\begin{aligned} \frac{dL(t)}{dt} &\leq -M\varepsilon E(t) + \varepsilon \left(1 + \frac{M}{2} \right) \|u_t\|_2^2 - \alpha \|\nabla u_t\|_2^2 \\ &\quad + \left(\frac{M\varepsilon}{2} + \varepsilon \right) \|u_t\|_{2,\Gamma_1}^2 + \varepsilon \left(\frac{M}{2} - \eta \right) \|\nabla u\|_2^2 \quad . \end{aligned} \quad (3.17)$$

By using the Poincaré inequality and the trace inequality

$$\begin{aligned} \|u_t\|_2^2 &\leq C \|\nabla u_t\|_2^2 \\ \|u_t\|_{2,\Gamma_1}^2 &\leq C \|\nabla u_t\|_2^2, \end{aligned}$$

choosing again ε small enough and $M \leq 2\eta$, from (3.17), we have:

$$\frac{dL(t)}{dt} \leq -M\varepsilon E(t), \quad \forall t \geq 0.$$

On the other hand, by virtue of (3.12), setting $\xi = -M\varepsilon/\beta_2$, the last inequality becomes:

$$\frac{dL(t)}{dt} \leq -\xi L(t), \quad \forall t \geq 0. \quad (3.18)$$

Integrating the previous differential inequality (3.18) between 0 and t gives the following estimate for the function L :

$$L(t) \leq Ce^{-\xi t}, \quad \forall t \geq 0.$$

Consequently, by using (3.12) once again, we conclude

$$E(t) \leq \widehat{C}e^{-\xi t}, \quad \forall t \geq 0.$$

This completes the proof. ■

Remark 3.4 In [13], we have proved the following result:

Theorem 3.2 *Assume $2 \leq p \leq \bar{q}$ and $m < p$. Let $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Suppose that*

$$E(0) < d \text{ and } \|\nabla u_0\|_2 > C_*^{-p/p-2}.$$

Then the solution of problem (1.1) grows exponentially in the L^p norm.

The present result on the asymptotic stability completes the above result on the exponential growth since when $u_0 \in \mathcal{N}^+$, we have: $\|\nabla u_0\|_2 \leq C_*^{-p/p-2}$.

Indeed, since d is the mountain pass level of the function J , we have $J(u_0) \leq d$. This writes:

$$\frac{p-2}{2p} \|\nabla u_0\|_2^2 + \frac{1}{p} I(0) \leq d$$

Since $u_0 \in \mathcal{N}^+$, we have:

$$\frac{p-2}{2p} \|\nabla u_0\|_2^2 \leq d.$$

Using identity (3.8), we get finally $\|\nabla u_0\|_2 \leq C_*^{-p/p-2}$.

4 Blow up

In this section we consider the problem (1.1) in the linear boundary damping case (i.e. $m = 2$) and we show that if $u_0 \in \mathcal{U}$ and $E(0) \leq d$ then any solution of (1.1) blows up in finite time. Our result reads as follows:

Theorem 4.1 *Assume $2 \leq p \leq \bar{q}$ and $m = 2$. Let u be the solution of (1.1) on $[0, T_{\max})$. Then $T_{\max} < \infty$ if and only if there exists $\bar{t} \in [0, T_{\max})$ such that*

$$u(\bar{t}) \in \mathcal{U} \text{ and } E(\bar{t}) \leq d. \quad (4.1)$$

Proof of theorem 4.1: Let us firstly suppose that there exists $\bar{t} \in [0, T_{\max})$ satisfying the conditions (4.1). Without loss of generality, we may assume that $\bar{t} = 0$.

We will prove that $T_{\max} < \infty$ by contradiction. We will suppose that the solution is global “in time” and we will use the concavity argument of Levine [18, 19]: the basic idea is to construct a positive functional $\theta(t)$ of the solution and show that

for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t . Thus it will exists T^* such that $\lim_{t \rightarrow T^*} \theta^{-\gamma}(t) = 0$. From the construction of the function θ , this will imply that:

$$\lim_{\substack{t \rightarrow T^* \\ t < T^*}} \|\nabla u\|_2 + \|u_t\|_2 = +\infty \quad .$$

In order to find such γ , we will verify that:

$$\frac{d^2 \theta^{-\gamma}(t)}{dt^2} = -\gamma \theta^{-\gamma-2}(t) \left[\theta \theta'' - (1 + \gamma) \theta'^2(t) \right] \leq 0, \quad \forall t \geq 0 \quad . \quad (4.2)$$

Thus it suffices to prove that $\theta(t)$ satisfies the differential inequality

$$\theta \theta'' - (1 + \gamma) \theta'^2(t) \geq 0, \quad \forall t \geq 0 \quad . \quad (4.3)$$

From the remark 3.1, we firstly have:

$$\forall t \in [0, T_{\max}), \quad E(t) \leq d \text{ and } u(t) \in \mathcal{U} \quad .$$

Hence by (3.6), we obtain:

$$\frac{2dp}{p-2} < \|\nabla u(t)\|_2^2, \quad \forall t \in [0, T_{\max}) \quad . \quad (4.4)$$

Assume by contradiction that the solution u is global “in time”. Then for any $T > 0$, let us define the functional θ as follows

$$\begin{aligned} \theta(t) &= \|u(t)\|_2^2 + \|u(t)\|_{2,\Gamma_1}^2 + \alpha \int_0^t \|\nabla u(s)\|_2^2 ds + r \int_0^t \|u(s)\|_{2,\Gamma_1}^2 ds \\ &\quad + (T-t) [\alpha \|\nabla u_0\|_2^2 + r \|u_0\|_{2,\Gamma_1}^2], \quad \forall t \in [0, T) \quad . \end{aligned} \quad (4.5)$$

Taking the time derivative of (4.5) we have:

$$\begin{aligned} \theta'(t) &= 2 \int_{\Omega} u_t u dx + 2 \int_{\Gamma_1} u_t u d\sigma + 2\alpha \int_0^t \int_{\Omega} \nabla u \nabla u_t dx ds \\ &\quad + 2r \int_0^t \int_{\Gamma_1} u_t u d\sigma ds \quad . \end{aligned} \quad (4.6)$$

Thus, as u is the solution of problem (1.1), differentiating equation (4.6) with respect to t gives us:

$$\theta''(t) = 2 [\|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 + \|u\|_p^p + \|u_t(t)\|_{2,\Gamma_1}^2] \quad .$$

Therefore, using the definition of θ given by (4.5), we can easily see that:

$$\begin{aligned} \theta(t) \theta''(t) &- \frac{p+2}{4} \theta'(t)^2 = 2\theta(t) [\|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 + \|u\|_p^p + \|u_t(t)\|_{2,\Gamma_1}^2] \\ &- (p+2) \left[\theta(t) - (T-t) [\alpha \|\nabla u_0\|_2^2 + r \|u_0\|_{2,\Gamma_1}^2] \right] \\ &\times \left[\|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 + \alpha \int_0^t \|\nabla u_t(t)\|_2^2 ds + r \int_0^t \|u_t(t)\|_{2,\Gamma_1}^2 ds \right] \\ &+ (p+2) \eta(t) \end{aligned} \quad (4.7)$$

where the function η is defined by:

$$\begin{aligned} \eta(t) &= \left[\|u(t)\|_2^2 + \|u(t)\|_{2,\Gamma_1}^2 + \alpha \int_0^t \|\nabla u(t)\|_2^2 ds + r \int_0^t \|u(t)\|_{2,\Gamma_1}^2 ds \right] \\ &\times \left[\|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 + \alpha \int_0^t \|\nabla u_t(t)\|_2^2 ds + r \int_0^t \|u_t(t)\|_{2,\Gamma_1}^2 ds \right] \\ &- \left[\int_{\Omega} u_t u dx + \int_{\Gamma_1} u_t u d\sigma + \alpha \int_0^t \int_{\Omega} \nabla u \nabla u_t dx ds + r \int_0^t \int_{\Gamma_1} u_t u d\sigma ds \right]^2 \quad . \end{aligned} \quad (4.8)$$

Our purpose now is to show that the right hand side of the equality (4.7) is non negative. Let us firstly show that $\eta(t) \geq 0$ for every $t \in [0, T]$. To do this, we estimate all the terms in the third line of (4.8) making use of Cauchy-Schwarz inequality, and compared the results with the terms in the first and second line in (4.8). For instance, when we develop the square term in the inequality (4.8), we estimate the terms as follows:

$$\begin{aligned} \left(\int_{\Omega} u_t u dx \right)^2 &\leq \|u(t)\|_2^2 \|u_t(t)\|_2^2 \text{ and} \\ 2 \int_{\Omega} u_t u dx \int_{\Gamma_1} u_t u d\sigma &\leq \|u(t)\|_{2,\Gamma_1}^2 \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 \|u(t)\|_2^2. \end{aligned}$$

Also, the following estimate holds:

$$\begin{aligned} 2\alpha \int_0^t \int_{\Omega} \nabla u \nabla u_t dx ds \int_{\Omega} u_t u dx &\leq \alpha \|u_t(t)\|_2^2 \int_0^t \|\nabla u(s)\|_2^2 ds \\ &\quad + \alpha \|u(t)\|_2^2 \int_0^t \|\nabla u_t(s)\|_2^2 ds. \end{aligned}$$

By carrying ‘‘carefully’’ all computations based on the same estimates as above, we finally obtain

$$\forall t \in [0, T], \eta(t) \geq 0.$$

Consequently, the equality (4.7) becomes

$$\theta(t)\theta''(t) - \frac{p+2}{4}\theta'(t)^2 \geq \theta(t)\zeta(t), \quad \forall t \in [0, T].$$

where

$$\begin{aligned} \zeta(t) &= 2 [\|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 + \|u\|_p^p + \|u_t(t)\|_{2,\Gamma_1}^2] \\ &\quad - (p+2) \left\{ \|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 \right. \\ &\quad \left. + \alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds + r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds \right\} \end{aligned}$$

Let us remark that

$$\begin{aligned} \zeta(t) &= -2pE(t) + (p-2)\|\nabla u(t)\|_2^2 - (p+2)\alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds \\ &\quad - (p+2)r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds. \end{aligned}$$

From the equality (3.4), we have:

$$\forall t \in [0, T], E(t) + \alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds + r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds = E(0). \quad (4.9)$$

Thus we can write:

$$\begin{aligned} \zeta(t) &= -2pE(0) + (p-2)\|\nabla u(t)\|_2^2 - (p-2)\|\nabla u_0\|_2^2 \\ &\quad + (p-2)\alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds + (p-2)r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds. \end{aligned}$$

Therefore, by using (4.4) and since $E(0) \leq d$ we have:

$$\begin{aligned} \zeta(t) &> 2p(d - E(0)) + (p-2)\alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds + (p-2)r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds \\ &\geq (p-2)\alpha \int_0^t \|\nabla u_t(s)\|_2^2 ds + (p-2)r \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds. \end{aligned}$$

Hence, there exists $\delta > 0$ such that

$$\zeta(t) \geq \delta, \quad \forall t \in [t_0, T]$$

for some $t_0 > 0$. But, since $\theta(t)$ is continuous and positive, there exists $\rho > 0$ such that

$$\theta(t) \geq \rho, \quad \forall t \in [t_0, T]$$

Consequently,

$$\theta(t)\theta''(t) - \frac{p+2}{4}\theta'(t)^2 \geq \rho\delta, \quad \forall t \in [t_0, T]$$

Setting

$$\gamma = \frac{p-2}{4} > 0, \quad ,$$

the differential inequality (4.3) is verified on $[t_0, T]$. This proves that $\theta(t)^{-\gamma}$ reaches 0 in finite time, say as $t \rightarrow T^*$. Since T^* is independent of the initial choice of T , we may assume that $T^* < T$. This tells us that:

$$\lim_{t \rightarrow T^*} \theta(t) = +\infty.$$

In turn this implies that:

$$\lim_{\substack{t \rightarrow T^* \\ t < T^*}} \|\nabla u\|_2 + \|u_t\|_2 = +\infty.$$

Thus we cannot suppose that the solution of (1.1) with $m = 2$ is global “in time”, that is $T_{max} < \infty$.

Conversely, let us suppose that $T_{max} < \infty$.

By Holder’s and Young’s inequalities, we have:

$$\frac{1}{2t} \left(\int_0^t \sqrt{\alpha} \|u_t\|_{2,\Gamma_1} + \sqrt{r} \|\nabla u_t\|_2 ds \right)^2 \leq \int_0^t (\alpha \|u_t\|_{2,\Gamma_1}^2 + r \|\nabla u_t\|_2^2) ds.$$

This implies

$$\int_0^t (\alpha \|u(t)\|_{2,\Gamma_1}^2 + r \|\nabla u\|_2^2) ds \geq \frac{1}{2t} \left((\sqrt{\alpha} \|u\|_{2,\Gamma_1} + \sqrt{r} \|\nabla u\|_2)^2 - (\sqrt{\alpha} \|u(0)\|_{2,\Gamma_1} + \sqrt{r} \|\nabla u(0)\|_2)^2 \right).$$

Since $p \geq 2$, we get from (3.2) and (3.3)

$$\frac{1}{2} I(t) = \frac{1}{2} (\|\nabla u(t)\|_2^2 - \|u(t)\|_p^p) \leq \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{p} \|u(t)\|_p^p \leq E(t) \quad (4.10)$$

By the help of (4.9), we thus have:

$$E(t) \leq E(0) - \frac{1}{2t} \left((\sqrt{\alpha} \|u\|_{2,\Gamma_1} + \sqrt{r} \|\nabla u\|_2)^2 - (\sqrt{\alpha} \|u(0)\|_{2,\Gamma_1} + \sqrt{r} \|\nabla u(0)\|_2)^2 \right). \quad (4.11)$$

Since $T_{max} < \infty$, the solution is not global in time. Thus by the definition of T_{max} , (see definition 3.1), we have:

$$\lim_{t \rightarrow T_{max}} (\|u(t)\|_{2,\Gamma_1} + \|\nabla u(t)\|_2) = +\infty \quad (4.12)$$

From (4.10) and (4.11), we obtain

$$\lim_{t \rightarrow T_{max}} I(t) = \lim_{t \rightarrow T_{max}} E(t) = -\infty. \quad (4.13)$$

Since the functions $I(t)$ and $E(t)$ are continuous, there exists $t_1 \in [0, T_{max})$ such that

$$E(t_1) \leq d \text{ and } I(t_1) < 0 \quad .$$

Thus the time t_1 verifies the condition (4.1). ■

Remark 4.1 The term $f(u) = |u|^{p-2}u$ is clearly responsible for the blow up situation. It is often called the “blow up term”. Consequently when $f(u) = 0$, or $f(u) = -|u|^{p-2}u$ any solution with arbitrary initial data is global in time and the result of theorem 4.1 holds without condition (3.9).

Remark 4.2 It's early well known ([18, 19]) that this blow up result appears for solutions with large initial data i.e. $E(0) < 0$. We note here that if $E(0) < 0$, then the blow up conditions (4.1) hold.

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