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A wavelet analysis of the Rosenblatt process: chaos expansion and estimation of the self-similarity parameter

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Abstract

The purpose of this paper is to make a wavelet analysis of self-similar stochastic processes by using the techniques of the Malliavin calculus and the chaos expansion into multiple stochastic integrals. Our examples are the fractional Brownian motion and the Rosenblatt process. We study the asymptotic behavior of the statistics based on the wavelet coefficients of these processes. We find that, in the case when driven process is the Rosenblatt process, this statistics satisfy a non-central limit theorem although a part of it converges to a Gaussian limit. We also construct estimators for the self-similarity index and we illustrate our results by numerical simulations.

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Key words: Multiple Wiener integral, wavelet analysis, Rosenblatt process, fractional Brownian motion, non-central limit theorem, self-similarity, Malliavin calculus, parameter estimation.

1 Introduction

The self-similarity property for a stochastic process means that scaling of time is equivalent to an appropriate scaling of space. That is, a process $(Y_t)_{t \geq 0}$ is selfsimilar of order $H > 0$ if for all $c > 0$ the processes $(Y_{ct})_{t \geq 0}$ and $(c^H Y_t)_{t \geq 0}$ have the same finite dimensional distributions. This property is crucial in applications such as network traffic analysis, mathematical finance, astrophysics, hydrology or image processing. We refer to the monographs [8], [14] or [27] for complete expositions on theoretical and practical aspects of self-similar stochastic processes.

The most popular self-similar process is the fractional Brownian motion (fBm). Its practical applications are notorious. This process is defined as a centered Gaussian process $(B_t^H)_{t \geq 0}$ with covariance function

$$R^H(t, s) := \mathbb{E} (B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

It can be also defined as the only Gaussian self-similar process with stationary increments. Recently, this stochastic process has been widely studied from the stochastic calculus point of view as well as from the statistical analysis point of view. Various types of stochastic integrals with respect to it have been introduced and several types of stochastic differential equations driven by fBm have been considered. Another stochastic processes which are self-similar with stationary increments are the Hermite processes (see [9], [12], [28]); an Hermite process of order q is actually an iterated integral of a deterministic function with q variables with respect to the standard Brownian motion. These processes appears as limits in the so-called Non-Central Limit Theorem and they have the same covariance as the fBm. The fBm is obtained for $q = 1$ and it is the only Gaussian Hermite process. For $q = 2$ the corresponding process is known as the Rosenblatt process. Although it received a less important attention than the fractional Brownian

motion, this process is still of interest in practical applications because of its self-similarity, stationarity and long-range dependence of increments. Actually the numerous uses of the fractional Brownian motion in practice (hydrology, telecommunications) are due to these properties; one prefers in general fBm before other processes because it is a Gaussian process and the calculus for it is easier; but in concrete situations when the Gaussian hypothesis is not plausible for the model, Rosenblatt process may be an interesting alternative model.

When studying self-similar processes, a question of major interest is to estimate their self-similarity order. This is important because the self-similarity order characterizes in some sense the process: for example in the fBm case as well as for Hermite processes this order gives the long-range dependence property of its increments and it characterizes the regularity of the trajectories. Several statistics, applied directly to the process or to its increments, have been introduced to this end. Naturally, parametric statistics (exact or Whittle approached maximum likelihood) estimators were studied. But to enlarge the method to a more general class of models (that can be for instance locally or asymptotically self-similar), it can be interesting to apply semi-parametric methods as wavelets based, log-variogram or log-periodogram estimators. Informations and details on these various approaches can be found in the books of Beran [8] and Doukhan *et al.* [13].

Our purpose is to develop a wavelet-based analysis of the fBm and Rosenblatt process using multiple Wiener-Itô integrals and to apply asymptotic results for estimating the self-similar parameter. More precisely, let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with support included in the interval $[0, 1]$ (called "mother wavelet"). Assume that there exists an integer $Q \geq 1$ such that

$$\int_{\mathbb{R}} t^p \psi(t) dt = 0 \text{ for } p = 0, 1, \dots, Q - 1 \quad (1)$$

and

$$\int_{\mathbb{R}} t^Q \psi(t) dt \neq 0.$$

We will call the integer $Q \geq 1$ the *number of vanishing moments*. For a stochastic process $(X_t)_{t \in [0, N]}$ and for a "scale" $a \in \mathbb{N}^*$ we define its wavelet coefficient by

$$d(a, i) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{t}{a} - i\right) X_t dt = \sqrt{a} \int_0^1 \psi(x) X_{a(x+i)} dx \quad (2)$$

for $i = 1, 2, \dots, N_a$ with $N_a := [N/a] - 1$. Let us set

$$\tilde{d}(a, i) = \frac{d(a, i)}{(\mathbb{E} d^2(a, i))^{\frac{1}{2}}}$$

and

$$V_N(a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{d}^2(a, i) - 1 \right). \quad (3)$$

The wavelet analysis consists in studying the behavior of the sequence $V_N(a)$ when $N \rightarrow \infty$. But if X is respectively a stationary long-memory or a self-similar second-order process, $\mathbb{E} d^2(a, i)$ is a power-law function of a with, respectively, an exponent $2H - 1$ (when $a \rightarrow \infty$) or $2H + 1$. Therefore, if $V_N(a)$ is proved to converge to 0, a log-log-regression of $\frac{1}{N_a} \sum_{i=1}^{N_a} d^2(a_j, i)$ onto a_j will provide an estimator of H (with an appropriate choice of $(a_j)_j$). Hence, the asymptotic behavior of $V_N(a)$ will completely give the behavior of the estimator (see the Section 5 for details). There are four main advantages to use such an estimator: firstly, it is a semi-parametric method that may be easily generalized. Secondly, it is based on the log-regression of the wavelet coefficient sample variances onto several scales and the graph of such a regression provides interesting information concerning the goodness-of-fit of the model (χ^2 goodness-of-fit were defined and studied in [4] or [6] from this log-regression). Thirdly, it is very low consuming time estimator (this is due to the Mallat's algorithm for computing the wavelet coefficients). Finally, it is a very robust method: it is not sensitive to possible polynomial trends as soon that the number of vanishing moments Q is large enough.

Such a method was introduced by Flandrin [15] in the case of fBm, enlarged to more general processes in [2] or [1]. The asymptotic behavior of such an estimator was specified in the case of long-memory Gaussian processes in [7] or [19], of long memory linear processes in [26] or of locally fractional Gaussian processes in [6]. However the case of Rosenblatt process was not already studied (note that the wavelet synthesis of Rosenblatt processes was treated in [3] and we will use this method in the section devoted to simulations). Since this process is not a Gaussian or a linear process, the standard techniques for obtaining limit theorems (based respectively on Hermite or Appell decomposition) do not in principle work in its case. We will use a recently developed theory based on Malliavin calculus and Wiener-Itô multiple stochastic integrals. Let us briefly recall these new results. In [24] the authors gave necessary and sufficient conditions for a sequence of random variables in a fixed Wiener chaos (that means in essence that these random variables are iterated integrals of a fixed order with respect to a given Brownian motion) to converge to a standard normal random variable (one of these conditions is that the sequence of fourth order moments converges to 3 which represents the moment of order 4 of a standard normal random variable). Another equivalent condition is given in the paper [23] in terms of the Malliavin derivative. These results created a powerful link between the Malliavin calculus and limit theorems and they have already been used in several papers (for example in [31] and [10] to study the variations of the Hermite processes).

Recall (see [15] and [4]) that if $X = B^H$, a fBm, in (3) then the following fact happens: for any $Q > 1$ and $H \in (0, 1)$ the statistics $V_N(a)$ renormalized by \sqrt{N} converges to a centered normal random variable. If $Q = 1$ then the barrier $H = 3/4$ appears: that is the behavior of $V_N(a)$ is normal (that is, it satisfies a central limit theorem) only if $H \in (1/2, 3/4)$ and we prove in Section 3 that the limit of $V_N(a)$ (normalized by N^{2-2H}) is a Rosenblatt random variable Z_1^H when $H \in (3/4, 1)$. In this case we also prove that the limit is in law and not in \mathbb{L}^2 in contrast with the case of quadratic variations studied in [31].

The study of $V_N(a)$ in the Rosenblatt case (see Section 3, formula (14) for the definition) with $H \in (1/2, 1/4)$ put in light interesting and somehow intriguing phenomena. The main fact is that the number of vanishing moments Q does not affect its convergence and the limit of $V_N(a)$ is always non-Gaussian (it is still Rosenblatt). Actually, the statistics V_N can be decomposed into two parts: a term in the fourth chaos (an iterated integral of order 4 with respect to a Wiener process) and a term in the second chaos. We analyze here both terms and we deduce that the term in the fourth Wiener chaos keeps some of the characteristics of the Gaussian case (it has to be renormalized by \sqrt{N} and it has a Gaussian limit for $H \in (1/2, 3/4)$). But the main term in $V_N(a)$ which gives the normalization is the second chaos term and its detailed analysis shows that the normalization depends on H (it is N^{1-H}) and its limit is (in law) a Rosenblatt random variable. In contrast with the case of the statistics based on the quadratic variations of the observed process (see [31] and [10]), we prove that the convergence in the non-central case holds in law and not in \mathbb{L}^2 .

The consequence of these results are also interesting for the wavelet based estimator of the self-similarity. Assume that (X_1, \dots, X_N) is known, where X is a fBm or a Rosenblatt process. First, we consider a statistic $\widehat{V}_N(a)$ computed from (X_1, \dots, X_N) where approximated wavelet coefficients (for instance computed from Mallat's cascade algorithm) replace formula (2) and we prove that limit theorems satisfied by $V_N(a)$ also hold with $\widehat{V}_N(a)$ as soon as a is large enough with respect to N . Secondly, we deduce convergence rates for the wavelet based estimator of H following the cases: $Q \geq 2$ and X is a fBm, $Q = 1$, $H \in (3/3, 1)$ and X is a fBm, X is a Rosenblatt process (the regularity of ψ also plays a role). For practical use, it is clear that if X is fBm it is required to chose $Q \geq 2$ and ψ twice continuously differentiable (for instance, when ψ is a Daubechies wavelet with order ≥ 8). But if X is a Rosenblatt process Q plays no role. Simulations illustrate the convergence of $\widehat{V}_N(a)$ and of the estimator of H in this last case. These results are also interesting because they open other relative questions: for a process constituted by the increments of a Rosenblatt process, which is a parametric stationary long-memory process, what are the convergence rates of usual long-memory parameter estimators such as Whittle's maximum likelihood or log-periodogram estimators? As for wavelet based estimator, it may be possible that these convergence rates are different from those obtained for Gaussian or linear processes...

We organized the paper as follows. Section 2 contains some preliminaries on multiple Wiener-Itô integrals with respect to the Brownian motion. In Section 3 we treat the situation when the driven process is the fBm. In this case our new result is the Non-Central Limit Theorem satisfied by the wavelet based statistics proved in Theorem 2. In Section 4 we enter into a non-Gaussian world: our observed process is the Rosenblatt process and using the techniques of the Malliavin calculus and recent interesting results for the convergence of sequence of multiple stochastic integrals, we study in details the sequence $V_N(a)$. In Section 5 we construct an observable estimator based on the approximated wavelet coefficient and we study its asymptotic behavior (under certain hypothesis, it has the same limit as $V_N(a)$) and its convergence as well as wavelet based estimator convergence are then illustrated by numerical simulations.

2 Preliminaries

2.1 Basic tool on multiple Wiener-Itô integrals

Let $(W_t)_{t \in [0, T]}$ be a classical Wiener process on a standard Wiener space (Ω, \mathcal{F}, P) . If $f \in \mathbb{L}^2([0, T]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W . We refer to [22] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.

Let $f \in \mathcal{S}_n$, that means that there exists $n \geq 1$ integer such that

$$f := \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} 1_{A_{i_1} \times \dots \times A_{i_n}}$$

where the coefficients satisfy $c_{i_1, \dots, i_n} = 0$ if two indices i_k and i_ℓ are equal and the sets $A_i \in \mathcal{B}([0, T])$ are disjoint. For a such step function f we define

$$I_n(f) := \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} W(A_{i_1}) \dots W(A_{i_n})$$

where we put $W([a, b]) = W_b - W_a$. It can be seen that the application I_n constructed above from \mathcal{S}_n equipped with the scaled norm $\frac{1}{\sqrt{n!}} \|\cdot\|_{\mathbb{L}^2([0, T]^n)}$ to $\mathbb{L}^2(\Omega)$ is an isometry on \mathcal{S}_n , i.e. for m, n positive integers,

$$\begin{aligned} E(I_n(f)I_m(g)) &= n! \langle f, g \rangle_{\mathbb{L}^2([0, T]^n)} \quad \text{if } m = n, \\ E(I_n(f)I_m(g)) &= 0 \quad \text{if } m \neq n. \end{aligned}$$

It also holds that

$$I_n(f) = I_n(\tilde{f}) \tag{4}$$

where \tilde{f} denotes the symmetrization of f defined by $\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Since the set \mathcal{S}_n is dense in $\mathbb{L}^2([0, T]^n)$ for every $n \geq 2$ the mapping I_n can be extended to an isometry from $\mathbb{L}^2([0, T]^n)$ to $\mathbb{L}^2(\Omega)$ and the above properties hold true for this extension. Note also that I_n can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}$$

(this follows e.g. by Itô's formula).

We recall the product for two multiple integrals (see [22]): if $f \in \mathbb{L}^2([0, T]^n)$ and $g \in \mathbb{L}^2([0, T]^m)$ are symmetric, then it holds that

$$I_n(f)I_m(g) = \sum_{\ell=0}^{m \wedge n} \ell! C_m^\ell C_n^\ell I_{m+n-2\ell}(f \otimes_\ell g) \tag{5}$$

where the contraction $f \otimes_\ell g$ belongs to $\mathbb{L}^2([0, T]^{m+n-2\ell})$ for $\ell = 0, 1, \dots, m \wedge n$ and is given by

$$(f \otimes_\ell g)(s_1, \dots, s_{n-\ell}, t_1, \dots, t_{m-\ell}) = \int_{[0, T]^\ell} f(s_1, \dots, s_{n-\ell}, u_1, \dots, u_\ell) g(t_1, \dots, t_{m-\ell}, u_1, \dots, u_\ell) du_1 \dots du_\ell.$$

3 The case of the fractional Brownian motion

3.1 A presentation using chaos expansion

We will assume in this part that $X = B^H$ a (normalized) fractional Brownian motion (fBm in the sequel) with Hurst parameter $H \in (0, 1)$. Recall that B^H is a centered Gaussian process with covariance function

$$R_H(t, s) := \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, N]. \quad (6)$$

It is the only normalized Gaussian H -self-similar process with stationary increments. Recall also the fBm $(B_t^H)_{t \in [0, N]}$ with Hurst parameter $H \in (0, 1)$ can be written

$$B_t^H = \int_0^t K^H(t, s) dW_s, \quad t \in [0, N]$$

where $(W_t, t \in [0, N])$ is a standard Wiener process and for $s < t$, and $H > \frac{1}{2}$

$$K^H(t, s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \quad (7)$$

with $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$ and $\beta(\cdot, \cdot)$ the beta function. For $t > s$, we have

$$\frac{\partial K^H}{\partial t}(t, s) := \partial_1 K^H(t, s) = c_H \left(\frac{s}{t} \right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}. \quad (8)$$

In this case it is trivial to decompose in chaos the wavelet coefficient $d(a, i)$. By a stochastic Fubini theorem we can write

$$\begin{aligned} d(a, i) &= \sqrt{a} \int_0^1 \psi(x) B_{a(x+i)}^H dx = \sqrt{a} \int_0^1 \psi(x) dx \left(\int_0^{a(x+i)} dB_u^H \right) \\ &= \sqrt{a} \int_0^1 \psi(x) dx \int_0^{a(x+i)} K^H(a(x+i), u) dW_u = I_1(f_{a,i}(\cdot)) \end{aligned}$$

where I_1 denote the multiple integral of order one (actually, the Wiener integral with respect to W) and we denoted by

$$f_{a,i}(u) := 1_{[0, a(i+1)]}(u) \sqrt{a} \int_{(\frac{u}{a}-i) \vee 0}^1 \psi(x) K^H(a(x+i), u) dx. \quad (9)$$

Thus, for all $a > 0$ and $i \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}(d^2(a, i)) &= \|f_{a,i}\|_{\mathcal{H}}^2 = a^{2H+1} C(H) \\ \text{with } C(H) &:= -\frac{1}{2} \int_0^1 \int_0^1 \psi(x) \psi(x') |x - x'|^{2H} dx dx' \end{aligned} \quad (10)$$

(see [4]). This formula will be essential for the estimation of H (see Section 5). Using the product formula

$$I_1(f) I_1(g) = I_2(f \otimes g) + \langle f, g \rangle_{\mathcal{H}}$$

(here and in the sequel \mathcal{H} denotes the space $\mathbb{L}^2([0, N])$) and we get

$$V_N(a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\frac{I_2(f_{a,i}^{\otimes 2}) + \|f_{a,i}\|_{\mathcal{H}}^2}{(\mathbb{E} d(a, i))^2} - 1 \right) = I_2(f_{N,a})$$

where

$$f_N^{(a)} := a^{-2H-1} C(H)^{-1} \frac{1}{N_a} \sum_{i=1}^{N_a} f_{a,i}^{\otimes 2}. \quad (11)$$

3.2 A multidimensional Central Limit Theorem satisfied by $(V_N(a_i))_{1 \leq i \leq m}$.

When the observed process is the fBm with $H < 3/4$, the statistics $V_N(a)$ satisfies a central limit theorem. This fact is known and we will not insist on this case. We just recall it to situate it in our context. Since $\mathbb{E} I_2(f) = 2! \|f\|_{\mathcal{H}}^2$ we have for $(a_i)_{1 \leq i \leq m}$ a family of integer numbers such that $a_i = i a$ for $i = 1, \dots, m$ and $a \in \mathbb{N}^*$,

$$\begin{aligned} \text{Cov}(V_N(a_p), V_N(a_q)) &= 2!(pq a^2)^{-2H-1} C(H)^{-2} \frac{1}{N_{a_p}} \frac{1}{N_{a_q}} \sum_{j=1}^{N_{a_p}} \sum_{j'=1}^{N_{a_q}} \langle f_{a_p, j}^{\otimes 2}, f_{a_q, j'}^{\otimes 2} \rangle_{\mathcal{H}^{\otimes 2}} \\ &= 2(pq a^2)^{-2H-1} C(H)^{-2} \frac{1}{N_{a_p}} \frac{1}{N_{a_q}} \sum_{j=1}^{N_{a_p}} \sum_{j'=1}^{N_{a_q}} \langle f_{a_p, j}, f_{a_q, j'} \rangle_{\mathcal{H}}^2. \end{aligned}$$

We know from [4] and [5] that:

$$\begin{aligned} \langle f_{a_p, j}, f_{a_q, j'} \rangle_{\mathcal{H}} &= \mathbb{E}(d(a_p, j)d(a_q, j')) \\ &= -\frac{1}{2}(pq a^2)^{1/2} a^{2H} \int_0^1 \int_0^1 \psi(x)\psi(x') |px - qx' + pj - qj'|^{2H} dx dx'. \end{aligned} \quad (12)$$

and from a Taylor expansion and using property (1) satisfied by ψ ,

$$\begin{aligned} \langle f_{a_p, j}, f_{a_q, j'} \rangle_{\mathcal{H}}^2 &= pq a^{4H+2} \mathcal{O}(1 + |pj - qj'|)^{4H-4Q} \\ \implies |\text{Cov}(V_N(a_p), V_N(a_q))| &\leq C \frac{1}{N_{a_q}^2} \sum_{j=1}^{N_{a_p}} \sum_{j'=1}^{N_{a_q}} \mathcal{O}(1 + |pj - qj'|)^{4H-4Q}. \end{aligned}$$

Consequently, if $Q > 1$ and $H \in (0, 1)$ or if $Q = 1$ and $H \in (0, 3/4)$,

$$\begin{aligned} \frac{N}{a} \text{Cov}(V_N(a_p), V_N(a_q)) &\xrightarrow{N \rightarrow \infty} \ell_1(p, q, H) \quad \text{with} \\ \ell_1(p, q, H) &= \frac{1}{2 d_{pq} (pq)^{2H-1}} \sum_{k=-\infty}^{\infty} \left(\frac{1}{C(H)} \int_0^1 \int_0^1 \psi(x)\psi(x') |px - qx' + kd_{pq}|^{2H} dx dx' \right)^2, \end{aligned} \quad (13)$$

where $d_{pq} = \text{GCD}(p, q)$. Moreover,

Theorem 1 *Let $V_N(a)$ be defined by (3) and $L_1(H) = (\ell_1(p, q, H))_{1 \leq p, q \leq m}$. Then if $Q > 1$ and $H \in (0, 1)$ or if $Q = 1$ and $H \in (0, 3/4)$, for all $a > 0$,*

$$\left(\sqrt{\frac{N}{a}} V_N(i a) \right)_{1 \leq i \leq m} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}_m(0, L_1(H)).$$

Proof: It is well-known in the literature (see e.g. [4]). ■

3.3 A non-Central Limit Theorem satisfied by $V_N(a)$.

We need at this point to define the Rosenblatt process. The Rosenblatt process, denoted in the sequel by $(Z^H(t))_{t \in [0, N]}$ appears as a limit in the so-called *Non Central Limit Theorem* (see [12] or [28]). It is not a Gaussian process and can be defined through its representation as double iterated integral with respect to a standard Wiener process (see [30]). More precisely, the Rosenblatt process with self-similarity order $H \in (\frac{1}{2}, 1)$ is defined by

$$Z_t^H := \int_0^t \int_0^t L_t^H(y_1, y_2) dW_{y_1} dW_{y_2} \quad (14)$$

where $(W_t, t \in [0, N])$ is a Brownian motion,

$$L_t^H(y_1, y_2) := d_H 1_{[0,t]}(y_1) 1_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2) du, \quad (15)$$

with K^H the standard kernel defined in (7) appearing in the Wiener integral representation of the fBm its derivatives defined in (8) and

$$H' := \frac{H+1}{2} \quad \text{and} \quad d_H := \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-\frac{1}{2}}.$$

Among the main properties of the Rosenblatt process, we recall

- it is H -self-similar in the sense that for any $c > 0$, $(Z^H(ct)) \stackrel{(d)}{=} (c^H Z^H(t))$, where " $\stackrel{(d)}{=}$ " means equivalence of all finite dimensional distributions;
- it has stationary increments, that is, the joint distribution of $(Z^H(t+h) - Z^H(h), t \in [0, T])$ is independent of $h > 0$.
- $\mathbb{E}(|Z_t^H|^p) < \infty$ for any $p > 0$, and $(Z^H(t))$ has the same variance and covariance than a standard fractional Brownian motion with parameter H .
- the Rosenblatt process is Hölder continuous of order $\delta < H$.

We obtain the following non central limit theorem for the wavelet coefficient of the fBm with $H > \frac{3}{4}$. Define

$$\ell_2(H) := \left(\frac{2H^2(2H-1)}{4H-3} \right)^{1/2} \frac{\left(\int_0^1 x \psi(x) dx \right)^2}{C(H)}. \quad (16)$$

Then,

Theorem 2 *If $Q = 1$ and $\frac{3}{4} < H < 1$ then there exists a Rosenblatt random variable R_1^{2H-1} with self-similarity order $2H-1$ such that*

$$\ell_2^{-1}(H) N_a^{2-2H} V_N(a) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} Z_1^{2H-1}.$$

where Z_1^{2H-1} is a Rosenblatt random variable given by (14).

Proof: With $f_N^{(a)}$ defined as in (11), we can write

$$N_a^{2-2H} V_N(a) = N_a^{2-2H} I_2(f_N^{(a)}).$$

But using the expression of $f_{a,i}$ provided in (9),

$$\begin{aligned} f_N^{(a)}(y_1, y_2) &:= \frac{1}{a^{2H} C_\psi(H)} \frac{1}{N_a} \sum_{i=1}^{N_a} 1_{[0, a(i+1)]}(y_1) 1_{[0, a(i+1)]}(y_2) \\ &\quad \times \int_{\frac{y_1}{a} - i}^1 \int_{\frac{y_2}{a} - i}^1 \psi(x) \psi(z) K^H(a(x+i), y_1) K^H(a(z+i), y_2) dx dz. \end{aligned}$$

To show that the sequence $\ell_2^{-1}(H) N_a^{2-2H} I_2(f_N^{(a)})$ converges in law to the Rosenblatt random variable Z_1^{2H-1} it suffices to show that its cumulants converges to the cumulants of Z_1^{2H-1} . We know (see [17], [24]) that the k -cumulant of a random variable $I_2(f)$ in the second Wiener chaos can be computed as follows

$$c_k(I_2(f)) = \int_{[0,1]^k} dy_1 \dots dy_k f(y_1, y_2) f(y_2, y_3) \dots f(y_{k-1}, y_k) f(y_k, y_1)$$

and thus

$$\begin{aligned}
& c_k \left(N_a^{2-2H} I_2 \left(f_N^{(a)} \right) \right) \\
&= N_a^{(2H-2)k} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} \int_{[0,1]^{2k}} dy_1 \dots dy_k \int_{[0,1]^{2k}} dx_1 dz_1 \dots dx_k dz_k \psi(x_1) \psi(z_1) \psi(x_2) \psi(z_2) \dots \psi(x_k) \psi(z_k) \\
&\quad \times K^H(a(x_1 + i_1), y_1) K^H(a(z_1 + i_1), y_2) K^H(a(x_2 + i_2), y_2) K^H(a(z_2 + i_2), y_3) \\
&\quad \dots K^H(a(x_{k-1} + i_{k-1}), y_{k-1}) K^H(a(z_k + i_k), y_k) K^H(a(x_k + i_k), y_k) K^H(a(z_k + i_k), y_1)
\end{aligned}$$

Using Fubini theorem and the fact that

$$\int_0^{a(x+i) \wedge a(x'+j)} K^H(a(x+i), y_1) K^H(a(x'+j), y_1) dy_1 = R^H((a(x+i), a(x'+j)))$$

we get

$$\begin{aligned}
& c_k \left(N_a^{2-2H} I_2 \left(f_N^{(a)} \right) \right) \\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} \int_{[0,1]^{2k}} dx_1 dz_1 \dots dx_k dz_k \psi(x_1) \psi(z_1) \psi(x_2) \psi(z_2) \dots \psi(x_k) \psi(z_k) \\
&\quad R^H(z_1 + i_1, x_2 + i_2) R^H(z_2 + i_2, x_3 + i_3) \dots R^H(z_{k-1} + i_{k-1}, x_k + i_k) R^H(z_k + i_k, x_1 + i_1) \\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} \int_{[0,1]^{2k}} dx_1 dz_1 \dots dx_k dz_k \psi(x_1) \psi(z_1) \psi(x_2) \psi(z_2) \dots \psi(x_k) \psi(z_k) \\
&\quad \times [|z_1 - x_2 + i_1 - i_2| \cdot |z_2 - x_3 + i_2 - i_3| \dots \cdot |z_{k-1} - x_k + i_{k-1} - i_k| \cdot |z_k - x_1 + i_k - i_1|]^{2H} \\
&= N_a^{(2H-2)k} a^{2Hk} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} (|i_1 - i_2| \dots \cdot |i_{k-1} - i_k| \cdot |i_k - i_1|)^{2H} \\
&\quad \times \int_{[0,1]^{2k}} dx_1 dz_1 \dots dx_k dz_k \psi(x_1) \psi(z_1) \psi(x_2) \psi(z_2) \dots \psi(x_k) \psi(z_k) \left| \left(1 + \frac{z_1 - x_2}{i_1 - i_2} \right)^{2H} \dots \left(1 + \frac{z_k - x_1}{i_k - i_1} \right)^{2H} \right| \\
&\sim N_a^{(2H-2)k} a^{2Hk} H^{2k} (2H-1)^{2k} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} (|i_1 - i_2| \dots \cdot |i_{k-1} - i_k| \cdot |i_k - i_1|)^{2H-2} \\
&\quad \int_{[0,1]^{2k}} dx_1 dz_1 \dots dx_k dz_k \psi(x_1) \psi(z_1) \psi(x_2) \psi(z_2) \dots \psi(x_k) \psi(z_k) x_1 z_1 \dots x_k z_k
\end{aligned}$$

and we used the fact that the integral of the mother wavelet vanishes and a Taylor expansion of second order. As a consequence, by a Riemann sum argument it is clear that the cumulant of $\ell_2^{-1}(H) N_a^{2-2H} I_2 \left(f_N^{(a)} \right)$ converges to

$$\int_{[0,1]^{2k}} [|x_1 - x_2| \dots |x_{k-1} - x_k| \cdot |x_k - x_1|]^{2H-2} dx_1 \dots dx_k$$

which represents the k cumulant of the Rosenblatt random variable Z_1^{2H-1} (see [28], [30]). \blacksquare

In the case of the statistics based of the variations of the fBm, in the case $H \in (3/4, 1)$ the renormalized statics converges in $\mathbb{L}^2(\Omega)$ to a Rosenblatt random variable at time 1. In the wavelet world, our above result gives only the convergence in law. The following question is then natural: can we get \mathbb{L}^2 convergence for the renormalized statistics $V_N(a)$? The answer is negative and it is proved below.

Proposition 1 *Under assumptions of Theorem 2, the statistics $N_a^{2-2H} V_N(a)$ does not converge in $\mathbb{L}^2(\Omega)$ as $N_a \rightarrow \infty$.*

Proof: It is equivalent to show that the sequence $N_a^{2-2H} f_N^{(a)}$ is not Cauchy in the space $\mathbb{L}^2([0, \infty)^2)$. We compute

$$\begin{aligned} & \|N_a^{2-2H} f_N^{(a)} - M_a^{2-2H} f_M^{(a)}\|_{\mathbb{L}^2([0, \infty)^2)}^2 \\ &= a^{-4H} C_\psi(H)^{-2} \left(N_a^{4-4H} \int_{[0,1]^2} dy_1 dy_2 (f_N^{(a)})^2(y_1, y_2) + M_a^{4-4H} \int_{[0,1]^2} dy_1 dy_2 (f_M^{(a)})^2(y_1, y_2) \right. \\ & \quad \left. - 2(M_a N_a)^{2-2H} f_M^{(a)}(y_1, y_2) f_N^{(a)}(y_1, y_2) dy_1 dy_2 \right) \end{aligned}$$

and

$$\begin{aligned} & N_a^{4-4H} \int_{[0,1]^2} dy_1 dy_2 (f_N^{(a)})^2(y_1, y_2) \\ &= N_a^{2-4H} \int_{[0,1]^2} dy_1 dy_2 \sum_{i,j=1}^{N_a} 1_{[0, a(1+i)]}(y_1) 1_{[0, a(1+i)]}(y_2) \left(\int_0^1 \int_0^1 dx dz \psi(x) \psi(z) \int_0^1 \int_0^1 dx' dz' \psi(x') \psi(z') \right) \\ & \quad \times K^H(a(x+i), y_1) K^H(a(z+i), y_2) K^H(a(x'+j), y_1) K^H(a(z'+j), y_2) \end{aligned}$$

and then by Fubini

$$\begin{aligned} & N_a^{4-4H} \int_{[0,1]^2} dy_1 dy_2 (f_N^{(a)})^2(y_1, y_2) \\ &= N_a^{2-4H} a^{4H} \frac{1}{4} \int_{[0,1]^4} \psi(x) \psi(z) \psi(x') \psi(z') dx dz dx' dz' \\ & \quad \times \sum_{i,j=1}^{N_a} (|x+i|^{2H} + |x'+j|^{2H} - |x-x'+i-j|^{2H}) (|z+i|^{2H} + |z'+j|^{2H} - |z-z'+i-j|^{2H}) \\ &= N_a^{2-4H} a^{4H} \frac{1}{4} \int_{[0,1]^4} \psi(x) \psi(z) \psi(x') \psi(z') dx dz dx' dz' \times \sum_{i,j=1}^{N_a} |x-x'+i-j|^{2H} |z-z'+i-j|^{2H} \\ &= N_a^{2-4H} a^{4H} \frac{1}{4} \int_{[0,1]^4} \psi(x) \psi(z) \psi(x') \psi(z') dx dz dx' dz' \times \sum_{i,j=1}^{N_a} |i-j|^{4H} \left(1 + \frac{x-x'}{|i-j|}\right)^{2H} \left(1 + \frac{z-z'}{|i-j|}\right)^{2H} \end{aligned}$$

Computing similarly the scalar product $M_a^{2-2H} N_a^{2-2H} \langle f_N^{(a)}, f_M^{(a)} \rangle_{\mathbb{L}^2([0, \infty)^2)}$ and repeating some of the above arguments, we can write

$$\begin{aligned} & \|N_a^{2-2H} f_N^{(a)} - M_a^{2-2H} f_M^{(a)}\|_{\mathbb{L}^2([0, \infty)^2)}^2 \\ & \sim a^{4H} H^2 (2H-1)^2 \int_{[0,1]^4} \psi(x) \psi(z) \psi(x') \psi(z') x x' z z' dx dz dx' dz' \\ & \quad \times \left[N_a^{2-4H} \sum_{i,j=1}^{N_a} |i-j|^{4H} ||i-j|^{-4} - 2M_a^{1-2H} N_a^{1-2H} \sum_{i=1}^{M_a} \sum_{j=1}^{N_a} |i-j|^{4H} |i-j|^{-4} + M_a^{2-4H} \sum_{i,j=1}^{M_a} |i-j|^{4H} ||i-j|^{-4} \right] \\ &= c_{a,H} \left[N_a^{2-4H} N_a^{4H-2} + M_a^{2-4H} M_a^{4H-2} - M_a^{1-2H} N_a^{1-2H} (M_a^{4H-2} + N_a^{4H-2} - |M_a - N_a|^{4H-2}) \right] \\ &= c_{a,H} \left[2 + \left(\frac{|M_a - N_a|^2}{N_a M_a} \right)^{2H-1} - \left(\frac{M_a}{N_a} \right)^{2H-1} - \left(\frac{N_a}{M_a} \right)^{2H-1} \right] \end{aligned}$$

where $c_{a,H}$ is a positive constant depending on a and H . The last expression does not tends to zero in general for $M_a, N_a \rightarrow \infty$ (take for instance $M_a = 2N_a$). \blacksquare

Let us further comment of the behavior of the sequence N_a^{2-2H} which is decisive for the behavior of our statistics $V_N(a)$ constructed from wavelet coefficients. The term $f_N^{(a)}$ can be written as

$$\begin{aligned}
f_N^{(a)}(y_1, y_2) &= \frac{1}{a^{2H}C_\psi(H)} \frac{1}{N_a} \sum_{i=1}^{N_a} \mathbf{1}_{[0, a(i+1)]}(y_1) \mathbf{1}_{[0, a(i+1)]}(y_2) \\
&\times \left(\mathbf{1}_{[0, ai]}(y_1) \mathbf{1}_{[0, ai]}(y_2) \int_0^1 \int_0^1 dx dz \psi(x) \psi(z) K^H(a(x+i), y_1) K^H(a(z+i), y_2) \right. \\
&\quad + \mathbf{1}_{[0, ai]}(y_1) \mathbf{1}_{[ai, a(1+i)]}(y_2) \int_0^1 \int_{\frac{y_2}{a}-i}^1 dx dz \psi(x) \psi(z) K^H(a(x+i), y_1) K^H(a(z+i), y_2) \\
&\quad + \mathbf{1}_{[0, ai]}(y_2) \mathbf{1}_{[ai, a(1+i)]}(y_1) \int_{\frac{y_1}{a}-i}^1 \int_0^1 dx dz \psi(x) \psi(z) K^H(a(x+i), y_1) K^H(a(z+i), y_2) \\
&\quad \left. + \mathbf{1}_{[ai, a(1+i)]}(y_1) \mathbf{1}_{[ai, a(1+i)]}(y_2) \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 dx dz \psi(x) \psi(z) K^H(a(x+i), y_1) K^H(a(z+i), y_2) \right) \\
&:= f_N^{(a,1)}(y_1, y_2) + f_N^{(a,2)}(y_1, y_2) + f_N^{(a,3)}(y_1, y_2) + f_N^{(a,4)}(y_1, y_2).
\end{aligned}$$

First we show that the terms $N_a^{2-2H} f_N^{(a,2)}$, $N_a^{2-2H} f_N^{(a,3)}$ and $N_a^{2-2H} f_N^{(a,4)}$ converge to zero in $L^2([0, \infty)^2)$ as $N_a \rightarrow \infty$. Similar techniques apply for all these three terms; we illustrate below the convergence of $f_N^{(a,4)}$:

$$\begin{aligned}
&\|f_N^{(a,4)}\|_{\mathbb{L}^2([0, \infty)^2)}^2 \\
&= \frac{1}{a^{4H}C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i=1}^{N_a} \int_{[0, \infty)^2} dy_1 dy_2 \mathbf{1}_{[ai, a(1+i)]}(y_1) \mathbf{1}_{[ai, a(1+i)]}(y_2) \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 \\
&\quad \psi(x) \psi(z) \psi(x') \psi(z') K^H(a(x+i), y_1) K^H(a(z+i), y_2) K^H(a(x'+i), y_1) K^H(a(z'+i), y_2) dx dz dx' dz' \\
&= \frac{1}{a^{4H}C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i=1}^{N_a} \int_{[0, 1]^4} dx dz dx' dz' \psi(x) \psi(z) \psi(x') \psi(z') \\
&\quad \times \left(\int_0^{(a(x+i)) \wedge (a(x'+i))} K^H(a(x+i), y_1) K^H(a(x'+i), y_1) dy_1 \right) \\
&\quad \times \left(\int_0^{(a(z+i)) \wedge (a(z'+i))} K^H(a(z+i), y_2) K^H(a(z'+i), y_2) dy_2 \right)
\end{aligned}$$

and since $\int_0^{t \wedge s} K^H(t, u) K^H(s, u) du = R^H(t, s)$ with $R^H(t, s)$ given in (6), we obtain

$$\begin{aligned}
&\|f_N^{(a,4)}\|_{\mathbb{L}^2([0, \infty)^2)}^2 \\
&= \frac{1}{a^{4H}C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i=1}^{N_a} \int_{[0, 1]^4} dx dz dx' dz' \psi(x) \psi(z) \psi(x') \psi(z') R^H(a(x+i), a(x'+i)) R^H(a(z+i), a(z'+i)) \\
&= \frac{1}{a^{4H}C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i=1}^{N_a} \int_{[0, 1]^4} dx dz dx' dz' \psi(x) \psi(z) \psi(x') \psi(z') \\
&\quad \times \frac{a^{4H}}{4} (|x|^{2H} + |x'|^{2H} - |x - x'|^{2H}) (|z|^{2H} + |z'|^{2H} - |z - z'|^{2H}) \\
&= \frac{1}{4C_\psi^2(H)} \frac{1}{N_a} \left(\int_{[0, 1]^2} dx dx' \psi(x) \psi(x') |x - x'|^{2H} \right)^2 = \frac{1}{N_a}
\end{aligned}$$

from property (1), and therefore $N_a^{4-4H} \|f_N^{(a,4)}\|_{\mathbb{L}^2([0, \infty)^2)}^2$ converges to zero as $N_a \rightarrow \infty$ since $H > \frac{3}{4}$.

It remains to study the convergence of the term $f_N^{(a,1)}$. Using again the property (1) of the mother wavelet

ψ we can write

$$f_N^{(a,1)}(y_1, y_2) = \frac{1}{a^{2H} C_\psi(H)} \frac{1}{N_a} \sum_{i=0}^{N_a} \int_0^1 \int_0^1 dx dz \psi(x) \psi(z) 1_{[0,ai]}(y_1) 1_{[0,ai]}(y_2) \\ \times \left(K^H(a(x+i), y_1) - K^H(ai, y_1) \right) \left(K^H(a(z+i), y_2) - K^H(ai, y_2) \right).$$

Therefore, with $\alpha(a, i, x)$ and $\beta(a, i, z)$ respectively located in $[ai, ax + ai]$ and $[ai, az + ai]$,

$$I_2(f_N^{(a,1)}) = \frac{1}{a^{2H} C_\psi(H)} \frac{1}{N_a} I_2 \left(\sum_{i=0}^{N_a} \int_0^1 \int_0^1 dx dz \psi(x) \psi(z) 1_{[0,ai]}(y_1) 1_{[0,ai]}(y_2) \right. \\ \left. \times ax \partial_1 K^H(\alpha(a, i, x), y_1) \times az \partial_1 K^H(\beta(a, i, z), y_2) \right)$$

and by approximating the points the points $\alpha(a, i, x)$ and $\beta(a, i, z)$ by ai and since from a usual approximation of a sum by a Riemann integral, when $N_a \rightarrow \infty$, with $y_1, y_2 \in [0, N]$,

$$\sum_{i=0}^{N_a} 1_{[0,ai]}(y_1) 1_{[0,ai]}(y_2) \partial_1 K^H(ai, y_1) \partial_1 K^H(ai, y_2) \sim \int_{(y_1 \vee y_2)/a}^{N_a} du \partial_1 K^H(au, y_1) \partial_1 K^H(au, y_2) \\ \sim \frac{1}{d_{2H-1} a} L_N^{2H-1}(y_1, y_2)$$

where L_N^{2H-1} is the kernel (see its definition in (15)) of the Rosenblatt process with self-similarity index $2H-1$. As a consequence $\ell_2^{-1}(H) N_a^{2-2H} f_N^{(a,1)}$ is equivalent (in the sense that it has the same limit pointwise) to $N^{1-2H} L_N^{2H-1}$. In some sense $\ell_2^{-1}(H) N_a^{2-2H} V_N(a)$ is equivalent to $N^{1-2H} I_2(L_N^{2H-1}) = N^{1-2H} Z_N^{2H-1} = Z_1^{2H-1}$ but this equivalence is only in law. The fact that the sequence $N_a^{2-2H} V_N(a)$ is not Cauchy in \mathbb{L}^2 comes from the fact that the sequence $N^{1-2H} Z_N^{2H-1}$ is not Cauchy in \mathbb{L}^2 as it can be easily seen.

4 The Rosenblatt case

We study in this section the wavelet-based statistics V_N given by (3) in the situation when the driving process is the Rosenblatt process. Although this section, assume that Z^H is a Rosenblatt process with self-similarity order H . In this case, the wavelet coefficient can be written as

$$d(a, i) = \sqrt{a} \int_0^1 \psi(x) Z_{a(x+i)}^H dx \\ = \sqrt{a} \int_0^1 \psi(x) dx \left(\int_0^{a(x+i)} \int_0^{a(x+i)} L_{a(x+i)}(y_1, y_2) dW_{y_1} dW_{y_2} \right) \\ = I_2(g_{a,i}(\cdot, \cdot))$$

with

$$g_{a,i}(y_1, y_2) := d_H \sqrt{a} 1_{[0, a(i+1)]}(y_1) 1_{[0, a(i+1)]}(y_2) \\ \times \int_{\frac{y_1 \vee y_2}{a} - i}^1 dx \psi(x) \left(\int_{y_1 \vee y_2}^{a(x+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2) du \right).$$

Recall the product formula for multiple stochastic integrals

$$I_2(f) I_2(g) = I_4(f \otimes g) + 4I_2(f \otimes_1 g) + 2\langle f, g \rangle_{\mathbb{L}^2[0, N]^2}$$

if $f, g \in \mathbb{L}^2([0, N]^2)$ are two symmetric functions and the contraction $f \otimes_1 g$ is defined by

$$(f \otimes_1 g)(y_1, y_2) = \int_0^N f(y_1, x)g(y_2, x)dx.$$

Thus, we obtain

$$d^2(a, i) = I_4(g_{a,i}^{\otimes 2}) + 4I_2(g_{a,i} \otimes_1 g_{a,i}) + 2\|g_{a,i}\|_{\mathbb{L}^2[0,N]^2}^2$$

and noting that, since the covariance of the Rosenblatt process is the same as the covariance of the fractional Brownian motion, we will also have

$$\mathbb{E}(d^2(a, i)) = \mathbb{E}(I_2(g_{a,i}))^2 = 2\|g_{a,i}\|_{\mathbb{L}^2[0,N]^2}^2 = a^{2H+1}C(H).$$

Therefore, we obtain the following decomposition for the statistic $V_N(a)$:

$$\begin{aligned} V_N(a) &= a^{-2H-1}C_\psi(H)^{-1} \frac{1}{N_a} \left[\sum_{i=1}^{N_a} I_4(g_{a,i}^{\otimes 2}) + 4 \sum_{i=1}^{N_a} I_2(g_{a,i} \otimes_1 g_{a,i}) \right] = T_2 + T_4 \\ \text{with} \quad &\begin{cases} T_2 &:= a^{-2H-1}C_\psi(H)^{-1} \frac{4}{N_a} \sum_{i=1}^{N_a} I_2(g_{a,i} \otimes_1 g_{a,i}) \\ T_4 &:= a^{-2H-1}C_\psi(H)^{-1} \frac{1}{N_a} \sum_{i=1}^{N_a} I_4(g_{a,i}^{\otimes 2}) \end{cases}. \end{aligned} \quad (17)$$

To understand the limit of the sequence V_N we need to regard the two terms above (note that similar terms appear in the decomposition of the variation statistics of the Rosenblatt process, see [31]). In essence, the following will happen: the term T_4 which lives in the fourth Wiener chaos keeps some characteristics of the fBm case (since it has to be renormalized by $\sqrt{N_a}$ except in the case $Q = 1$ where the normalization is N_a^{2-2H} for $H > \frac{3}{4}$) and its limit will be Gaussian (except for $Q = 1$ and $H > \frac{3}{4}$). Unfortunately, these somehow nice behavior does not affect the limit of V_N which is non-normal.

Now, let us study the asymptotic behavior of the term T_4 . From (17), we have

$$T_4 = I_4(g_N^{(a)})$$

where

$$g_N^{(a)} := a^{-2H-1}C_\psi(H)^{-1} \frac{1}{N_a} \sum_{i=1}^{N_a} g_{a,i}^{\otimes 2}, \quad (18)$$

and thus, by the isometry of multiple stochastic integrals,

$$\begin{aligned} \mathbb{E}T_4^2 &= 4!C_\psi(H)^{-2}a^{-4H-2} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \langle g_{a,i}^{\otimes 2}, g_{a,j}^{\otimes 2} \rangle_{\mathbb{L}^2[0,N]^4} \\ &= 4!C_\psi(H)^{-2}a^{-4H-2} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2[0,N]^2}^2 \end{aligned}$$

But,

$$\langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2[0,N]^2} = \frac{1}{2} \mathbb{E}(d(a, i)d(a, j))$$

and since the second order are the same, we obtain the same behavior (up to a multiplicative constant) as in the case of the fractional Brownian motion. That is,

Proposition 2 *From (13), if $Q > 1$ and $H \in (\frac{1}{2}, 1)$ or if $Q = 1$ then for $H \in (\frac{1}{2}, \frac{3}{4})$, then*

$$\frac{N}{a} \mathbb{E}(T_4^2) \xrightarrow{N \rightarrow \infty} 3\ell_1(1, 1, H) \quad (19)$$

and, from Theorem 2, if $Q = 1$ and $H \in (\frac{3}{4}, 1)$ then

$$\left(\frac{N}{a}\right)^{4-4H} \mathbb{E}(T_4^2) \xrightarrow{N \rightarrow \infty} 3\ell_2(H). \quad (20)$$

4.1 Asymptotic behavior of the term T_2

Evaluation of $\mathbb{E} T_2^2$

We evaluate in this part the L^2 -norm of the second chaos term and we will compare it with the corresponding norm of the term T_4 in order to determinate the normalization of V_N (note that $\mathbb{E}(T_4 T_2) = 0$ by the orthogonality of the multiple integrals). Recall that we have

$$T_2 = I_2(h_N^{(a)})$$

with

$$h_N^{(a)} := 4 \frac{1}{a^{2H+1} C(H)} \frac{1}{N_a} \sum_{i=1}^{N_a} g_{a,i} \otimes_1 g_{a,i}. \quad (21)$$

We compute the contraction $g_{a,i} \otimes_1 g_{a,i}$. We have

$$\begin{aligned} (g_{a,i} \otimes_1 g_{a,i})(y_1, y_2) &= \int_0^{N_a} g_{a,i}(y_1, z) g_{a,i}(y_2, z) dz \\ &= a d_H^2 1_{[0, a(i+1)]}(y_1) 1_{[0, a(i+1)]}(y_2) \int_0^{a(i+1)} dz \left[\int_{\frac{y_1 \vee z}{a} - i}^1 dx \psi(x) \left(\int_{y_1 \vee z}^{a(x+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, z) du \right) \right] \\ &\quad \times \left[\int_{\frac{y_2 \vee z}{a} - i}^1 dx' \psi(x') \left(\int_{y_2 \vee z}^{a(x'+i)} \partial_1 K^{H'}(u', y_2) \partial_1 K^{H'}(u', z) du' \right) \right] \\ &= a d_H^2 1_{[0, a(i+1)]}(y_1) 1_{[0, a(i+1)]}(y_2) \left(\left[\int_{\frac{y_1}{a} - i}^1 dx \psi(x) \int_{\frac{y_2}{a} - i}^1 dx' \psi(x') \right. \right. \\ &\quad \left. \left. \times \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} M(u, y_1, u', y_2) du du' \int_0^{u \wedge u'} M(u, z, u', z) dz \right] \right) \end{aligned}$$

where $M(u, y_1, u', y_2) := \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u', y_2)$ and $H' = (H + 1)/2$. Now, we have already seen that $\int_0^{t \wedge s} K^H(t, z) K^H(s, z) dz = R^H(t, s)$ with $R^H(t, s)$ given in (6) and therefore

$$\int_0^{u \wedge u'} M(u, z, u', z) dz = H'(2H' - 1) |u - u'|^{2H' - 2}. \quad (22)$$

Thus denoting $\alpha_H := H'(2H' - 1) = H(H + 1)/2$ and since ψ is $[0, 1]$ -supported, we obtain

$$\begin{aligned} (g_{a,i} \otimes_1 g_{a,i})(y_1, y_2) &= a d_H^2 \alpha_H 1_{[0, a(i+1)]}(y_1) 1_{[0, a(i+1)]}(y_2) \int_{\left(\frac{y_1}{a} - i\right) \vee 0}^1 \int_{\left(\frac{y_2}{a} - i\right) \vee 0}^1 dx dx' \psi(x) \psi(x') \\ &\quad \times \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} |u - u'|^{2H' - 2} M(u, y_1, u', y_2) du du'. \end{aligned}$$

We compute now the expectation of the square of the term T_2 . From (21) and by the isometry property of multiple Wiener-Itô integrals

$$\begin{aligned}
\mathbb{E} T_2^2 &= 16 a^{-4H-2} C(H)^{-2} 2! \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \langle g_{a,i} \otimes_1 g_{a,i}, g_{a,j} \otimes_1 g_{a,j} \rangle_{\mathbb{L}^2[0,N]^2} \\
&= 32 a^{-4H} \frac{\alpha_H^2 d_H^4}{C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \int_0^{a(i \wedge j+1)} \int_0^{a(i \vee j+1)} dy_1 dy_2 \\
&\quad \times \int_{(\frac{y_1}{a}-i) \vee 0}^1 dx \psi(x) \int_{(\frac{y_2}{a}-i) \vee 0}^1 dx' \psi(x') \int_{(\frac{y_1}{a}-j) \vee 0}^1 dz \psi(z) \int_{(\frac{y_2}{a}-j) \vee 0}^1 dz' \psi(z') \\
&\quad \times \int_{y_1}^{a(x+i)} du \int_{y_2}^{a(x'+i)} dv \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u-v|^{2H'-2} \\
&\quad \times \int_{y_1}^{a(z+j)} du' \int_{y_2}^{a(z'+j)} dv' \partial_1 K^{H'}(u', y_1) \partial_1 K^{H'}(v', y_2) |u'-v'|^{2H'-2}
\end{aligned}$$

and by calculating first the integrals dy_1 and dy_2

$$\begin{aligned}
\mathbb{E} T_2^2 &= 32 a^{-4H} \frac{\alpha_H^2 d_H^4}{C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \int_{[0,1]^4} \psi(x) \psi(x') \psi(z) \psi(z') dx dx' dz dz' \\
&\quad \times \int_0^{a(x+i)} du \int_0^{a(x'+i)} dv \int_0^{a(z+j)} du' \int_0^{a(z'+j)} dv' (|u-v| \cdot |u'-v'|)^{2H'-2} \\
&\quad \times \left(\int_0^{u \wedge u'} M(u, y_1, u', y_1) dy_1 \right) \left(\int_0^{v \wedge v'} M(v, y_2, v', y_2) dy_2 \right) \\
&= 32 \frac{\alpha_H^4 d_H^4}{C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \int_{[0,1]^4} \psi(x) \psi(x') \psi(z) \psi(z') dx dx' dz dz' \\
&\quad \times \int_0^{x+i} du \int_0^{x'+i} dv \int_0^{z+j} du' \int_0^{z'+j} dv' (|u-v| \cdot |u'-v'| \cdot |u-u'| \cdot |v-v'|)^{2H'-2} \\
&= 32 \frac{\alpha_H^4 d_H^4}{C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \int_{[0,1]^4} \psi(x) \psi(x') \psi(z) \psi(z') dx dx' dz dz' \\
&\quad \times \int_i^{x+i} du \int_i^{x'+i} dv \int_j^{z+j} du' \int_j^{z'+j} dv' (|u-v| \cdot |u'-v'| \cdot |u-u'| \cdot |v-v'|)^{2H'-2}
\end{aligned}$$

where the property (1) of ψ is applied four time to reduce the domain of integration. Using the change of variables $\bar{u} = \frac{1}{x}(u-i)$ (and similarly for the other variables) we obtain

$$\begin{aligned}
\mathbb{E} T_2^2 &= 32 \frac{\alpha_H^4 d_H^4}{C_\psi^2(H)} \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} \int_{[0,1]^4} \psi(x) \psi(x') \psi(z) \psi(z') x x' z z' dx dx' dz dz' \\
&\quad \times \int_{[0,1]^4} du du' dv dv' (|ux-vx'| \cdot |u'z-v'z'| \cdot |ux-u'z+i-j| \cdot |vx'-v'z'+i-j|)^{2H'-2}.
\end{aligned}$$

Denote by

$$S_{N_a} := \frac{1}{N_a^2} \sum_{i,j=1}^{N_a} (|ux-u'z+i-j| \cdot |vx'-v'z'+i-j|)^{2H'-2}.$$

We have

$$\begin{aligned} S_{N_a} &= \frac{1}{N_a^2} \sum_{k=-N_a}^{N_a} (N_a - |k|) |ux - u'z + k|^{2H'-2} |vx' - v'z' + k|^{2H'-2} \\ &= \frac{1}{N_a} \sum_{k=-N_a}^{N_a} N_a^{2(2H'-2)} \left(1 - \frac{|k|}{N_a}\right) \left| \frac{ux - v'z}{N_a} + \frac{k}{N_a} \right|^{2H'-2} \left| \frac{vx' - v'z'}{N_a} + \frac{k}{N_a} \right|^{2H'-2}. \end{aligned}$$

and clearly, since $2(2H' - 2) = 2H - 2$ and the terms $\frac{ux - v'z}{N_a}$ and $\frac{vx' - v'z'}{N_a}$ are negligible in front of $\frac{k}{N_a}$ for large k , we get by a Riemann sum argument that for every x, \dots, v' ,

$$N_a^{2-2H} S_{N_a} \xrightarrow{N \rightarrow \infty} 2 \int_0^1 (1-x)x^{2H-2} dx = \frac{1}{H(2H-1)}.$$

As a consequence, with $\int_{[0,1]^2} dudv |ux - vx'|^{2H'-2} = (2H'(2H' - 1))^{-1} (x^{2H'} + (x')^{2H'} - |x - x'|^{2H'})$ and the property (1) of ψ ,

$$\begin{aligned} N_a^{2-2H} \mathbb{E} T_2^2 &\xrightarrow{N \rightarrow \infty} 32 \frac{\alpha_H^4 d_H^4}{H(2H-1)C_\psi^2(H)} \left(\int_{[0,1]^4} \psi(x)\psi(x')xx'|ux - vx'|^{2H'-2} dx dx' dudv \right)^2 \\ &\xrightarrow{N \rightarrow \infty} 8 \frac{2H-1}{H(H+1)^2} \left(\frac{1}{C(H)} \int_{[0,1]^2} \psi(x)\psi(x')xx'|x - x'|^{H-1} dx dx' \right)^2 := C_{T_2}^2(H). \end{aligned} \quad (23)$$

In conclusion the asymptotic behavior of the term T_2 depends on H and, surprisingly and contrary to the Gaussian case, it is not influenced by the number Q of vanished moment of ψ . Thus for any $Q \geq 1$

$$\mathbb{E} \left[\left(\frac{N}{a} \right)^{1-H} T_2 \right]^2 \xrightarrow{N \rightarrow \infty} C_{T_2}^2(H).$$

Therefore, from this formula and (19) and (20), for any $Q \geq 1$ and $H \in (\frac{1}{2}, 1)$ the term T_2 is dominant with respect to T_4 for the behavior of $\mathbb{E} V_N^2(a)$. We are able to prove the following result:

Theorem 3 *Let $(Z_t^H)_{t \in [0, N]}$ be a Rosenblatt process with self-similarity index $H \in (\frac{1}{2}, 1)$ and the statistic $V_N(a)$ computed from $(Z_t^H)_{t \in [0, N]}$. Then for any $Q \geq 1$ and $H \in (\frac{1}{2}, 1)$,*

$$\mathbb{E} \left[N_a^{1-H} V_N(a) \right]^2 \xrightarrow{N \rightarrow \infty} C_{T_2}^2(H).$$

The limit of the term T_2

We prove in this paragraph that the sequence T_2 (and therefore the sequence $V_N(a)$) converges in $\mathbb{L}^2(\Omega)$ to a Rosenblatt random variable with self-similarity index H .

Theorem 4 *Let $(Z_t^H)_{t \geq 0}$ be a Rosenblatt process and let T_2 be the sequence given by (17) and computed from $(Z_t^H)_{t \geq 0}$. Then, for any $Q \geq 1$ and $H \in (\frac{1}{2}, 1)$, there exists a Rosenblatt random variable R_1^H with self-similarity order H such as*

$$C_{T_2}^{-1}(H) \left(\frac{N}{a} \right)^{1-H} T_2 \xrightarrow{N \rightarrow \infty} R_1^H \implies C_{T_2}^{-1}(H) \left(\frac{N}{a} \right)^{1-H} V_N(a) \xrightarrow{N \rightarrow \infty} R_1^H,$$

where C_{T_2} is given by (23).

Proof: This proof follows the lines of the proof of Theorem 2. With $T_2 = I_2(h_N^{(a)})$ in mind, as in the proof of Theorem 2, a direct proof that the cumulants of the sequence $N_a^{1-H} I_2(h_N^{(a)})$ converge to those

of the Rosenblatt process can be given. Indeed, by combining the proof of theorem 2 and the estimation of the square mean of T_2 we will obtain

$$\begin{aligned}
c_k & \left(N_a^{1-H} I_2 \left(h_N^{(a)} \right) \right) \\
&= c_{a,H} N_a^{k(1-H)} N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} \int_{[0,1]^{4k}} \prod_{j=1}^k \psi(x_j) \psi(x'_j) \psi(z_j) \psi(z'_j) dx_j dx'_j dz_j dz'_j \\
&= \int_{[0,1]^{4k}} du_{k,j} du'_{k,j} dv_j dv'_j \prod_{j=1}^k (|u_j x_j - v_j x'_j| \cdot |u'_j z_j - v'_j z'_j|)^{2H'-2} \\
& \quad \prod_{j=1}^k (|u_j x_j - u'_j z_j + i_j - i_{j+1}| \cdot |v_j x'_j - v'_j z'_j + i_k - i_{j+1}|)^{2H'-2}
\end{aligned}$$

with the convention $i_{k+1} := i_1$. The key fact is that the sequence

$$S_{N_a}^k = N_a^{-k} \sum_{i_1, \dots, i_k=1}^{N_a} \prod_{j=1}^k \left(\frac{|u_j x_j - u'_j z_j + i_j - i_{j+1}| \cdot |v_j x'_j - v'_j z'_j + i_k - i_{j+1}|}{N_a} \right)^{2H'-2}$$

converges as a Riemman sum (for fixed $x_j, x'_j, z_j, z'_j, u_j, v_j, u'_j, v'_j$) to, modulo a constant, the integral

$$\int_{[0,1]^k} dx_1 \dots dx_k (|x_1 - x_2| \cdot |x_2 - x_3| \cdot \dots \cdot |x_k - x_1|)^{2H'-2}$$

which is the cumulant of Z_1^H .

But we prefer to do here a more detailed study of the sequence $N_a^{1-H} h_N^{(a)}$ to understand why we have only the convergence in law and the convergence \mathbb{L}^2 does not hold. Actually, as in the proof of Theorem 2, we prove first that the sequence $N_a^{1-H} h_N^{(a)}$ is NOT Cauchy in $L^2([0, \infty)^2)$ and by the isometry of multiple stochastic integrals we will obtain that $N_a^{1-H} I_2(h_N^{(a)})$ is NOT Cauchy in $\mathbb{L}^2(\Omega)$. But here we want to understand the limit and the behavior of this sequence. We can decompose $h_N^{(a)}$ (defined in (21)) as follows:

$$\begin{aligned}
h_N^{(a)}(y_1, y_2) &= 4 \frac{d_H^2 \alpha_H}{a^{2H} C(H)} \frac{1}{N_a} \times \\
& \sum_{i=1}^{N_a} \mathbb{1}_{[0, ai]}(y_1) \mathbb{1}_{[0, ai]}(y_2) \int_{[0,1]^2} dx dx' \psi(x) \psi(x') \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} dudv \\
& + \mathbb{1}_{[0, ai]}(y_1) \mathbb{1}_{[ai, a(i+1)]}(y_2) \int_0^1 \int_{\frac{y_2}{a}-i}^1 \psi(x) \psi(x') dx dx' \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} \partial_1 K^{H'}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) |u - v|^{2H'-2} dudv \\
& + \mathbb{1}_{[0, ai]}(y_2) \mathbb{1}_{[ai, a(i+1)]}(y_1) \int_{\frac{y_1}{a}-i}^1 \int_0^1 \psi(x) \psi(x') dx dx' \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} dudv \\
& + \mathbb{1}_{[ai, a(i+1)]}(y_1) \mathbb{1}_{[ai, a(i+1)]}(y_2) \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 \psi(x) \psi(x') dx dx' \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u - v|^{2H'-2} dudv \\
& := h_N^{(a,1)}(y_1, y_2) + h_N^{(a,2)}(y_1, y_2) + h_N^{(a,3)}(y_1, y_2) + h_N^{(a,4)}(y_1, y_2).
\end{aligned}$$

Following a similar proof as for Theorem 2, we first show that the terms $N_a^{1-H} h_N^{(a,2)}$, $N_a^{1-H} h_N^{(a,3)}$ and $N_a^{1-H} h_N^{(a,4)}$ converge to zero in $\mathbb{L}^2([0, \infty)^2)$ as $N \rightarrow \infty$. We treat the case $N_a^{1-H} h_N^{(a,4)}$ goes to zero as N goes to infinity; the convergence of the other sequences will follow similarly (the key fact is that the presence of the interval $[ai, a(i+1)]$ allows only diagonal terms in their \mathbb{L}^2 norm). Following the previous

computations in the proof of Theorem 2, there exists $k_H, k'_H > 0$ not depending on N and a such that

$$\begin{aligned} \|h_N^{(a,4)}\|_{\mathbb{L}^2([0,\infty)^2)}^2 &= k_H \frac{1}{N_a^2} \sum_{i=1}^{N_a} \int_{[ai, a(i+1)]^2} dy_1 dy_2 \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 \int_{\frac{y_1}{a}-i}^1 \int_{\frac{y_2}{a}-i}^1 \psi(x)\psi(x')\psi(z)\psi(z') dx dx' dz dz' \\ &\times \int_{y_1}^{a(x+i)} \int_{y_2}^{a(x'+i)} \int_{y_1}^{a(z+i)} \int_{y_2}^{a(z'+i)} |u-v|^{2H'-2} |u'-v'|^{2H'-2} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) \partial_1 K^{H'}(u', y_1) \partial_1 K^{H'}(v', y_2) dudvdu'dv' \\ &= \frac{k'_H}{N_a^2} \sum_{i=1}^{N_a} \int_{[0,1]^4} \psi(x)\psi(x')\psi(z)\psi(z') dx dx' dz dz' \int_{ai}^{a(x+i)} \int_{ai}^{a(x'+i)} \int_{ai}^{a(z+i)} \int_{ai}^{a(z'+i)} (|u-v||u'-v'||u-u'||v-v'|)^{2H'-2} dudvdu'dv' \end{aligned}$$

using once again the relation (22). With changes of variables,

$$\begin{aligned} &\int_{ai}^{a(x+i)} \int_{ai}^{a(x'+i)} \int_{ai}^{a(z+i)} \int_{ai}^{a(z'+i)} (|u-v||u'-v'||u-u'||v-v'|)^{2H'-2} dudvdu'dv' \\ &= a^{8H'-4} x x' z z' \int_{[0,1]^4} (|ux-vx'||u'z-v'z'||ux-zu'||vx'-v'z'|)^{2H'-2} dudvdu'dv' \end{aligned}$$

and therefore with $k''_H > 0$ not depending on N and a ,

$$\|h_N^{(a,4)}\|_{\mathbb{L}^2([0,\infty)^2)}^2 = k''_H a^{8H'-4} \frac{1}{N_a} \implies \|N_a^{1-H} h_N^{(a,4)}\|_{\mathbb{L}^2([0,\infty)^2)}^2 \xrightarrow{N \rightarrow \infty} 0.$$

Let us now study the term $h_N^{(a,1)}$. This term will give actually the limit of the sequence $V_N(a)$. Note that we can replace the integration interval $[y_1, a(x+i)]$ by $[ai, a(x+i)]$ since ψ satisfies the property (1). So we can write

$$\begin{aligned} h_N^{(a,1)}(y_1, y_2) &= \frac{4 d_H^2 \alpha_H}{a^{2H} C(H)} \frac{1}{N_a} \sum_{i=1}^{N_a} 1_{[0, ai]}(y_1) 1_{[0, ai]}(y_2) \int_0^1 \int_0^1 dx dx' \psi(x) \psi(x') \\ &\times \int_{ai}^{a(x+i)} \int_{ai}^{a(x'+i)} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(v, y_2) |u-v|^{2H'-2} dudv \end{aligned}$$

and by making the change of variable $\bar{u} = \frac{1}{x}(\frac{u}{a} - i)$ and $\bar{v} = \frac{1}{x'}(\frac{v}{a} - i)$, we get

$$\begin{aligned} N_a^{1-H} h_N^{(a,1)}(y_1, y_2) &= \frac{4 d_H^2 \alpha_H}{a^{2H-2} C(H)} \frac{1}{N_a^H} \sum_{i=1}^{N_a} 1_{[0, ai]}(y_1) 1_{[0, ai]}(y_2) \int_0^1 \int_0^1 x x' \psi(x) \psi(x') dx dx' \\ &\times \int_0^1 \int_0^1 \partial_1 K^{H'}(aux + ai, y_1) \partial_1 K^{H'}(avx' + ai, y_2) |aux - avx'|^{2H'-2} du dv \\ &\sim \frac{4 d_H^2 \alpha_H}{a^{H-1} C(H)} \frac{1}{N_a^H} \sum_{i=1}^{N_a} 1_{[0, ai]}(y_1) 1_{[0, ai]}(y_2) \int_0^1 \int_0^1 x x' \psi(x) \psi(x') dx dx' \\ &\times \int_0^1 \int_0^1 \partial_1 K^{H'}(ai, y_1) \partial_1 K^{H'}(ai, y_2) |ux - vx'|^{2H'-2} \end{aligned}$$

where $a_n \sim b_n$ means that the sequence a_n and b_n have the same limit as $n \rightarrow \infty$. We proceed to a first approximation. For $y_1, y_2 \in [0, N]$,

$$\begin{aligned} N_a^{1-H} h_N^{(a,1)}(y_1, y_2) &= \frac{4 d_H^2 \alpha_H}{a^{H-1} C(H)} \frac{1}{N_a^H} \sum_{i=\lfloor \frac{y_1 \vee y_2}{a} \rfloor}^{N_a} \int_0^1 \int_0^1 x x' \psi(x) \psi(x') dx dx' \\ &\times \int_0^1 \int_0^1 \partial_1 K^{H'}(aux + ai, y_1) \partial_1 K^{H'}(avx' + ai, y_2) |ux - vx'|^{2H'-2} dudv. \end{aligned}$$

But, for $i > y_1/a$, from the formula of $\partial_1 K^H(t, s)$ given in (8),

$$\left| \partial_1 K^{H'}(aux + ai, y_1) - \partial_1 K^{H'}(ai, y_1) \right| \leq \left(c_H \left(\frac{3}{2} - H \right) \left(\frac{ai}{y_1} \right)^{H-1/2} \frac{1}{(ai - y_1)^{5/2-H}} \right) aux.$$

Now, since for $ai - y_1$ large enough,

$$\left(\frac{ai}{y_1} \right)^{H-1/2} \frac{1}{(ai - y_1)^{5/2-H}} \sim \frac{1}{(ai - y_1)^{3-2H}} \implies \sum_{i > y_1/a} \left(\frac{ai}{y_1} \right)^{H-1/2} \frac{1}{(ai - y_1)^{5/2-H}} < \infty,$$

one deduces that

$$N_a^{1-H} h_N^{(a,1)}(y_1, y_2) \sim \frac{4 d_H^2 \alpha_H}{a^{H-1} C(H)} \frac{1}{N_a^H} C_0 \sum_{i=\lceil \frac{y_1 \vee y_2}{a} \rceil}^{N_a} \partial_1 K^{H'}(ai, y_1) \partial_1 K^{H'}(ai, y_2)$$

where

$$\begin{aligned} C_0 &= \int_0^1 \int_0^1 x x' \psi(x) \psi(x') dx dx' \int_0^1 \int_0^1 |ux - vx'|^{2H'-2} du dv \\ &= \int_0^1 \int_0^1 x x' \psi(x) \psi(x') dx dx' \times \frac{1}{2H'(2H'-1)x x'} (|x|^{2H'} + |x'|^{2H'} - |x - x'|^{2H'}) \\ &= -\frac{1}{H(H+1)} \int_0^1 \int_0^1 \psi(x) \psi(x') |x - x'|^{2H'} dx dx' = \frac{2}{H(H+1)} C(H'). \end{aligned}$$

As a consequence, from a usual comparison between a sum and an integral of positive terms,

$$\begin{aligned} N_a^{1-H} h_N^{(a,1)}(y_1, y_2) &\sim \frac{8 d_H^2 \alpha_H C(H')}{H(H+1) a^{H-1} C(H)} \frac{1}{N_a^H} 1_{[0, N]}(y_1) 1_{[0, N]}(y_2) \int_{\frac{y_1 \vee y_2}{a}}^{N_a} \partial_1 K^{H'}(au, y_1) \partial_1 K^{H'}(au, y_2) du \\ &\sim \left(\frac{8 d_H \alpha_H C(H')}{H(H+1) a^H C(H)} \right) \frac{1}{N_a^H} \times L_N^H(y_1, y_2) \\ &\sim C_{T_2}(H) N^{-H} L_N^H(y_1, y_2), \end{aligned}$$

where L_N is the kernel of the Rosenblatt process (see its definition in (15)). But from the self similarity property, $N^{-H} I_2(L_N^H)$ has the same law as $I_2(L_1^H)$. Using the equivalence $N_a^{1-H} h_N^{(a,1)}(y_1, y_2) \sim C_{T_2}(H) N^{-H} L_N^H(y_1, y_2)$ it can be easily deduced in the proof of Theorem 2 that the sequence $I_2(N_a^{1-H} h_N^{(a,1)})$ (and therefore $I_2(N_a^{1-H} h_N^{(a)})$) and thus $N_a^{1-H} C_{T_2}^{-1}(H) T_2$ has the cumulants convergent to those of the Rosenblatt random variable Z_1^H (but the direct argument proposed at the beginning of this proof also holds). In essence, the renormalized statistics $V_N(a)$ does not converge in \mathbb{L}^2 because $N^{-H} I_2(L_N^H) = N^{-H} Z_N^H$ does not converge in \mathbb{L}^2 as it can be easily seen. \blacksquare

4.2 Asymptotic analysis of the term T_4

We try here to understand the behavior of the term T_4 in the cases $Q > 1$ and $H \in (\frac{1}{2}, 1)$ or $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$. It can be already seen from its asymptotic variance that it is very close to the Gaussian case. We will show below that this term converges in law to a Gaussian random variable. This of course does not influence in principle the limit of V_N but we find that it is interesting from a theoretical point of view.

We need now to introduce the Malliavin derivative. We will use only this derivative for random variables in a finite chaos. If $f \in \mathbb{L}^2([0, T]^n)$ is a symmetric function, we will use the following rule to differentiate in the Malliavin sense

$$D_t I_n(f) = n I_{n-1}(f(\cdot, t)), \quad t \in [0, 1].$$

Our strategy is based on the following result (see Theorem 4 in [23], see also [24]).

Proposition 3 Let $F_N = I_n(f_N)$ be a sequence of square integrable random variables in the n th Wiener chaos such that $\mathbb{E} [F_N^2] \xrightarrow{N \rightarrow \infty} 1$. Then the following are equivalent:

i) The sequence $(F_N)_{N \geq 0}$ converges to the normal law $\mathcal{N}(0, 1)$.

ii) $\|DF_N\|_{\mathbb{L}^2[0, T]}^2 = \int_0^T D_t I_n(f) dt \xrightarrow{N \rightarrow \infty} n$, where D is the Malliavin derivative with respect to the underlying Wiener process W .

Our main result concerning the limit of the renormalized statistic V_N is the following. Following the limit theorem (19), denote by $C_{T_4}(H)$ the positive constant such that

$$C_{T_4}^2(H) := 3 \ell(1, 1, H),$$

where $\ell(p, q, H)$ is defined in (13).

Theorem 5 Suppose that Z^H is a Rosenblatt process with self-similarity order H . Suppose that $Q > 1$ or $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$. Then

$$\sqrt{\frac{N}{a}} T_4 \xrightarrow{N \rightarrow \infty} \mathcal{N}(0, C_{T_4}^2(H)).$$

Remark 1 In the other case ($Q = 1$ and $H \in (\frac{3}{4}, 1)$) the limit in law of the renormalized V_N is a Rosenblatt random variable.

Proof of Theorem 5: We have seen in (17) that $T_4 = a^{-2H-1} C_\psi(H)^{-1} \frac{1}{N_a} \sum_{i=1}^{N_a} I_4(g_{a,i}^{\otimes 2})$ and $\mathbb{E} T_4^2 \xrightarrow{N \rightarrow \infty} 0$ from (19). Then by the criterium in Proposition 3 it suffices to show that

$$\int_0^N dr (D_r T_4)^2 \xrightarrow{N \rightarrow \infty} 4 C_{T_4}^2(H).$$

But we can write that

$$D_r T_4 = 4 a^{-2H-1} C_\psi(H)^{-1} \frac{1}{\sqrt{N_a}} \sum_{i=1}^{N_a} I_3((g_{a,i}^{\otimes 2})(\cdot, r))$$

and therefore

$$\int_0^N dr (D_r T_4)^2 = \frac{16}{a^{2H+1} C_\psi(H)} \frac{1}{N_a} \sum_{i,j=1}^{N_a} \int_0^N I_3((g_{a,i} \otimes g_{a,i})(\cdot, r)) I_3((g_{a,j} \otimes g_{a,j})(\cdot, r)) dr.$$

The product formula for multiple stochastic integrals (5) gives

$$\begin{aligned} & \int_0^N dr (D_r T_4)^2 \\ &= \frac{16}{a^{2H+1} C_\psi(H)} \frac{1}{N_a} \sum_{i,j=1}^{N_a} \int_0^N dr \left(I_6((g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes (g_{a,j} \otimes g_{a,j})(\cdot, r)) + 9 I_4((g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes_1 (g_{a,j} \otimes g_{a,j})(\cdot, r)) \right. \\ & \quad \left. + 9 I_2((g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes_2 (g_{a,j} \otimes g_{a,j})(\cdot, r)) + I_0((g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes_3 (g_{a,j} \otimes g_{a,j})(\cdot, r)) \right) \\ & := T_{46} + T_{44} + T_{42} + T_{4,0}. \end{aligned}$$

First the limit theorem (19) we note that:

$$\mathbb{E} \left[\int_0^N (D_r T_4)^2 dr \right] \xrightarrow{N \rightarrow \infty} 4 C_{T_4}^2(H).$$

Indeed, this follows easily because for a random variable G in the n th Wiener chaos, we have

$$n \mathbb{E} [G^2] = \int_0^N (D_r G)^2 dr.$$

This limit is provided by the term $T_{4,0}$ as for the fBm case. Let us show that the other terms, *i.e.* T_{46} , T_{44} and T_{42} , converge to zero. First we regard the term T_{46} .

$$\begin{aligned} T_{46} &= \frac{16}{a^{2H+1} C_\psi(H)} \frac{1}{N_a} I_6 \left(\sum_{i,j=1}^{N_a} \int_0^N dr (g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes (g_{a,j} \otimes g_{a,j})(\cdot, r) \right) \\ &= \frac{16}{a^{2H+1} C_\psi(H)} \frac{1}{N_a} I_6 \left(\sum_{i,j=1}^{N_a} (g_{a,i} \otimes g_{a,j}) \otimes (g_{a,i} \otimes_1 g_{a,j}) \right). \end{aligned}$$

Since for any function $f \in \mathbb{L}^2([0, \infty)^2)$ one has $\|f\|_{\mathbb{L}^2} \leq \|\tilde{f}\|_{\mathbb{L}^2}$ (see (4)), we obtain, with $C > 0$ not depending on N ,

$$\begin{aligned} \mathbb{E} T_{46}^2 &\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \langle g_{a,i} \otimes g_{a,j}, g_{a,k} \otimes g_{a,l} \rangle_{\mathbb{L}^2([0,N]^4)} \langle g_{a,i} \otimes_1 g_{a,j}, g_{a,l} \otimes_1 g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)} \\ &\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \langle g_{a,i}, g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,j}, g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,i} \otimes_1 g_{a,j}, g_{a,l} \otimes_1 g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)}. \end{aligned}$$

It is not difficult to see that the dominant term of the above expression appears when $|i-j|, |i-k|, |k-l|, |j-l| \geq 2$. In this case, the scalar product $\langle g_{a,i}, g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)}$ has been already evaluated before: it behaves for $|i-k| \geq 1$ as $|i-k|^{2H-2Q}$. Let us regard the factor $\langle g_{a,i} \otimes_1 g_{a,j}, g_{a,l} \otimes_1 g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)}$. It can be computed as follows

$$\begin{aligned} &\langle g_{a,i} \otimes_1 g_{a,j}, g_{a,l} \otimes_1 g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)} \\ &= \int_{[0,1]^4} dx dx' dz dx' \psi(x) \psi(x') \psi(z) \psi(z') \\ &\quad \times \int_0^{a(x+i)} du \int_0^{a(x'+j)} dv \int_0^{a(z+k)} \int_0^{a(z'+l)} dv' (|u-v||u'-v'||u-u'||v-v'|)^{2H'-2} \\ &= \int_{[0,1]^4} dx dx' dz dx' \psi(x) \psi(x') \psi(z) \psi(z') x x' z z' \\ &\quad \times \int_{[0,1]^4} du dv du' dv' (|ux - vx' + i - j||u'z - v'z' + j - k||ux - zu' + i - k||vx' - v'z'|)^{2H'-2} \end{aligned}$$

and therefore there exists $C > 0$ not depending on N such that

$$\left| \langle g_{a,i} \otimes_1 g_{a,j}, g_{a,l} \otimes_1 g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)} \right| \leq C ((1 + |i-j|)(1 + |k-l|)(1 + |i-k|)(1 + |j-l|))^{2H'-2}.$$

Consequently since for $N_a \rightarrow \infty$

$$\begin{aligned} &\frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} ((1 + |i-j|)(1 + |k-l|)(1 + |i-k|)(1 + |j-l|))^{H-1} ((1 + |i-k|)(1 + |j-l|))^{2H-2Q} \\ &\leq \frac{1}{N_a} \int_{[1, N_a]^3} (xzt(x+y+z))^{2H'-2} (zt)^{2H-2Q} dx dy dz dt = \frac{C}{N_a} N_a^{2H-1} \times (N_a^{6H-4} \mathbf{1}_{Q=1} + C \mathbf{1}_{Q \geq 2}), \end{aligned}$$

then

$$\mathbb{E} T_{46}^2 \xrightarrow{N \rightarrow \infty} 0 \tag{24}$$

for $Q > 1$ or $Q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$.

We prove now the convergence of T_{44} to zero. It holds that (here $(f)^s$ means \tilde{f})

$$\begin{aligned}
T_{44} &= C \frac{1}{N_a} I_4 \left(\sum_{i,j=1}^{N_a} \int_0^N dr (g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes_1 (g_{a,j} \otimes g_{a,j})(\cdot, r) \right)^s \\
&= C \frac{1}{N_a} I_4 \left(\sum_{i,j=1}^{N_a} \langle g_{a,i} \otimes g_{a,j} \rangle_{\mathbb{L}^2([0,N]^2)} \right) + C \frac{1}{N_a} I_4 \left(\sum_{i,j=1}^{N_a} (g_{a,i} \otimes_1 g_{a,j}) \otimes (g_{a,j} \otimes_1 g_{a,i}) \right) \\
&:= T_{441} + T_{442}.
\end{aligned}$$

The expectation of square T_{441} can be handled as follows

$$\begin{aligned}
\mathbb{E} T_{441}^2 &\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \langle g_{a,i} \otimes g_{a,j}, g_{a,k} \otimes g_{a,l} \rangle_{\mathbb{L}^2([0,N]^4)} \langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,k}, g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \\
&\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \langle g_{a,i}, g_{a,k} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,j}, g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,k}, g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \\
&\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} ((1 + |i - k|)(1 + |j - l|)(1 + |i - j|)(1 + |k - l|))^{2H-2Q} \\
&\leq C \frac{1}{N_a} \int_1^{N_a} \int_1^{N_a} \int_1^{N_a} (x y z (x + y + z))^{2H-2Q} dx dy dz \\
&\leq C \frac{1}{N_a} \int_1^{N_a} \int_1^{N_a} \int_1^{N_a} (x^2 y z)^{2H-2Q} dx dy dz \leq C N_a^{2+8H-8Q} \xrightarrow{N \rightarrow \infty} 0
\end{aligned} \tag{25}$$

since $Q > 1$ or when $Q = 1$ then $\frac{1}{2} < H < \frac{3}{4}$. Concerning the term T_{442} , since

$$\langle g_{a,i} \otimes_1 g_{a,j}, g_{a,k} \otimes_1 g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \sim C ((1 + |i - j|)(1 + |k - l|)(1 + |i - k|)(1 + |j - l|))^{2H'-2},$$

from a similar bound as for $\mathbb{E} T_{441}^2$, one obtains that

$$\begin{aligned}
\mathbb{E} T_{442}^2 &\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} (\langle g_{a,i} \otimes_1 g_{a,j}, g_{a,k} \otimes_1 g_{a,l} \rangle_{\mathbb{L}^2([0,N]^4)})^2 \\
&\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} ((1 + |i - j|)(1 + |k - l|)(1 + |i - k|)(1 + |j - l|))^{4H'-4} \\
&\leq C N_a^{8H-6} \xrightarrow{N \rightarrow \infty} 0.
\end{aligned} \tag{26}$$

Therefore, from (25) and (26), one deduces that $\mathbb{E} T_{44}^2 \xrightarrow{N \rightarrow \infty} 0$ when $Q > 1$ or when $Q = 1$ and $\frac{1}{2} < H < \frac{3}{4}$.

Finally let us deal with the term T_{42} . It can be also decomposed into two parts as follows

$$\begin{aligned}
T_{42} &= C \frac{1}{N_a} I_2 \left(\sum_{i,j=1}^{N_a} \int_0^N dr (g_{a,i} \otimes g_{a,i})(\cdot, r) \otimes_2 (g_{a,j} \otimes g_{a,j})(\cdot, r) \right)^s \\
&= C \frac{1}{N_a} I_2 \left(\sum_{i,j}^{N_a} \langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2([0,N]^2)} (g_{a,i} \otimes_1 g_{a,j}) \right) + C \frac{1}{N_a} I_2 \left(\sum_{i,j=1}^{N_a} (g_{a,i} \otimes_1 g_{a,j}) \otimes_1 (g_{a,j} \otimes_1 g_{a,i}) \right) \\
&:= T_{421} + T_{422}
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E} T_{421}^2 &\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \langle g_{a,i}, g_{a,j} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,k}, g_{a,l} \rangle_{\mathbb{L}^2([0,N]^2)} \langle g_{a,i} \otimes_1 g_{a,j}, g_{a,k} \otimes_1 g_{a,l} \rangle_{\mathbb{L}^4([0,N]^2)} \\
&\leq C \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} ((1+|i-j|)(1+|k-l|))^{3H-2Q-1} ((1+|i-k|)(1+|j-l|))^{2H'-2} \\
&\leq C N_a^{8H-4Q-2} \xrightarrow{N \rightarrow \infty} 0,
\end{aligned} \tag{27}$$

since $Q > 1$ or $Q = 1$ and $\frac{1}{2} < H < \frac{3}{4}$. Concerning T_{422} , we have

$$\begin{aligned}
(g_{a,i} \otimes_1 g_{a,j}) \otimes_1 (g_{a,j} \otimes_1 g_{a,i})(y_2, y'_2) &= \int_0^1 \int_{\frac{y_2}{a}-j} \int_0^1 \int_{\frac{y'_2}{a}-i} dx dx' dz dz' \psi(x) \psi(x') \psi(z) \psi(z') \\
&\times \int_0^{a(x+i)} \int_{y_2}^{a(x'+j)} \int_0^{a(z+j)} \int_{y'_2}^{a(z'+i)} dudvdu'dv' (|u-v||u'-v'||u-u'|)^{2H'-2} \partial_1 K^{H'}(v, y_2) \partial_1 K^{H'}(v', y_2)
\end{aligned}$$

and therefore with the same changes of variables than in the previous proofs and the property satisfied by ψ ,

$$\begin{aligned}
\mathbb{E} T_{422}^2 &= \frac{C}{N_a^2} \sum_{i,j,k,l=1}^{N_a} \int_{[0,1]^8} \prod_{i=1}^8 \psi(x_i) dx_i \int_0^{a(x_1+i)} \int_0^{a(x_2+j)} \int_0^{a(x_3+j)} \int_0^{a(x_4+i)} \int_0^{a(x_5+k)} \int_0^{a(x_6+l)} \int_0^{a(x_7+l)} \int_0^{a(x_8+k)} \prod_{i=1}^8 du_i \\
&\times (|u_1 - u_2||u_3 - u_4||u_2 - u_4|)^{2H'-2} (|u_5 - u_6||u_7 - u_8||u_6 - u_8|)^{2H'-2} (|u_1 - u_5||u_2 - u_6|)^{2H'-2} \\
&= C \int_{[0,1]^8} \prod_{i=1}^8 x_i \psi(x_i) dx_i \int_{[0,1]^8} \prod_{i=1}^8 du_i (S(i, j, k, l, (u_p x_p)_p))^{2H'-2}
\end{aligned}$$

with

$$\begin{aligned}
S(i, j, k, l, (u_p x_p)_p) &:= \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} (|x_1 u_1 - x_2 u_2 + i - j| |x_3 u_3 - x_4 u_4 + j - i| |x_2 u_2 - x_4 u_4 + j - i| |x_5 u_5 - x_6 u_6 + k - l| \\
&|x_7 u_7 - x_8 u_8 + l - k| |x_5 u_5 - x_7 u_7 + k - l| |x_1 u_1 - x_5 u_5 + i - k| |x_2 u_2 - x_6 u_6 + j - l|)^{2H'-2}.
\end{aligned}$$

But, with a first order approximation, it is clear that $S(i, j, k, l, (u_p x_p)_p)$ behaves as $S(i, j, k, l, 0)$ since $u_p x_p \in [0, 1]$. Thus,

$$\begin{aligned}
S(i, j, k, l, 0) &\sim \frac{1}{N_a^2} \sum_{i,j,k,l=1}^{N_a} (|i-j|^3 |k-l|^3 |i-k||j-l|)^{H-1} \\
&\sim \frac{C}{N_a} \int_0^{N_a} \int_0^{N_a} \int_0^{N_a} x^{3H-3} y^{3H-3} z^{H-1} (x+y+z)^{H-1} dx dy dz \\
&\sim C_1 N_a^{8H-6} \mathbf{1}_{H>3/4} + C_2 \log(N_a) \mathbf{1}_{H=3/4} + C_3 N_a^{4H-3} \mathbf{1}_{2/3 < H < 3/4} \\
&\quad + C_4 N_a^{-1/3} \log(N_a) \mathbf{1}_{H=3/4} + C_5 N_a^{2H-2} \mathbf{1}_{1/2 < H < 2/3}.
\end{aligned}$$

Thus if $Q = 1$ then $\int_{[0,1]^8} \prod_{i=1}^8 x_i \psi(x_i) dx_i \neq 0$ and it implies that $\mathbb{E} T_{422}^2 \xrightarrow{N \rightarrow \infty} 0$ when $1/2 < H < 3/4$.

But if $Q \geq 2$ then then $\int_{[0,1]^8} \prod_{i=1}^8 x_i \psi(x_i) dx_i = 0$ and a second order approximation of $S(i, j, k, l, (u_p x_p)_p)$ has to be considered. For instance, from a Taylor expansion,

$$|x_1 u_1 - x_2 u_2 + i - j|^{H-1} \simeq |i - j|^{H-1} \left(1 + (H-1) \frac{x_1 u_1 - x_2 u_2}{i - j} \right).$$

Therefore in a second order approximation, and using $\int_{[0,1]^s} \prod_{i=1}^s x_i \psi(x_i) dx_i = 0$ then

$$\begin{aligned} \int_{[0,1]^s} \prod_{i=1}^s x_i \psi(x_i) dx_i S(i, j, k, l, (u_p x_p)_p) &\sim \frac{C}{N_a^2} \int_{[0,1]^s} \prod_{i=1}^s u_i x_i^2 \psi(x_i) dx_i \sum_{i,j,k,l=1}^{N_a} |i-j|^{3H-6} |k-l|^{3H-6} |i-k|^{H-2} |j-l|^{H-2} \\ &\sim \frac{C}{N_a} \prod_{i=1}^s u_i. \end{aligned}$$

Hence, when $Q \geq 2$, $\mathbb{E} T_{422}^2 \xrightarrow{N \rightarrow \infty} 0$ for all $1/2 < H < 1$. As a consequence, for $Q \geq 2$ or $Q = 1$ and $1/2 < H < 3/4$, with (24), (25), (26) and (27), we have proved that T_{46} , T_{44} and T_{42} converge to zero in $\mathbb{L}^2(\Omega)$. Therefore T_4 behaves as T_{40} , and the behavior of this term is the same as in the case of a fBm with parameter H . \blacksquare

5 Applications and simulations

Here, we will denote X^H as well a fBm with $H \in (0, 1)$ or a Rosenblatt process with $H \in (1/2, 1)$.

5.1 Asymptotic normality of the sample variance of approximated wavelet coefficients

Here a sample $(X_0^H, X_1^H, \dots, X_N^H)$ of X^H is supposed to be observed. For (a, b) , define the approximated wavelet coefficients of $d(a, b)$ and $\tilde{d}(a, b)$,

$$e(a, b) = \frac{1}{\sqrt{a}} \sum_{k=1}^N X_k^H \psi\left(\frac{k}{a} - b\right) \quad \text{and} \quad \tilde{e}(a, b) = \frac{e(a, b)}{a^{H+1/2} C^{1/2}(H)}, \quad (28)$$

that are the usual Riemann approximations. Define also for $a > 0$,

$$\widehat{V}_N(a) = \frac{1}{N_a} \sum_{i=1}^{N_a} (\tilde{e}^2(a, i) - 1). \quad (29)$$

Remark 2 *These approximations of wavelet coefficients and their sample variance can be directly computed from data for all mother wavelet ψ . In the case of a multiresolution analysis with orthogonal discrete wavelet transform, the very fast Mallat's algorithm can be applied to obtain such approximations. It provides a clear advantage to the wavelet based estimator of the parameter H with regard to the estimators based on a minimisation of a criterium (such as maximum likelihood estimators).*

Now, it can be proved:

Proposition 4 *Let $a(N)$ a sequence of integer numbers satisfying $N a(N)^{-1} \xrightarrow{N \rightarrow \infty} \infty$, and $a(N) \xrightarrow{N \rightarrow \infty} \infty$. Assume also that $\psi \in \mathcal{C}^m(\mathbb{R})$ with $m \geq 1$ and ψ is $[0, 1]$ -supported. Then,*

1. *if X^H is a fBm and $Q \geq 2$ or $Q = 1$ and $0 < H < 3/4$, and if $N a(N)^{-2} \xrightarrow{N \rightarrow \infty} 0$ and $m \geq 2$, then*

Theorem 1 holds when $V_N(a)$ is replaced by $\widehat{V}_N(a)$.

2. *if X^H is a fBm and $Q = 1$ and $3/4 < H < 1$, and if*

$$N a(N)^{-(5-4H)/(4-4H)} \xrightarrow{N \rightarrow \infty} 0 \quad \text{and} \quad N a(N)^{-(3-2H+m)/(3-2H)} \xrightarrow{N \rightarrow \infty} 0$$

then $\ell_2^{-1}(H) \left(\frac{N}{a}\right)^{2-2H} \widehat{V}_N(a) \xrightarrow{N \rightarrow \infty} R_1^{2H-1}$, where R_1^{2H-1} is a $(2H-1)$ -Rosenblatt random variable.

3. if X^H is a Rosenblatt process, if $N a(N)^{-(3-2H)/(2-2H)} \xrightarrow{N \rightarrow \infty} 0$ and $N a(N)^{-(1+m)} \xrightarrow{N \rightarrow \infty} 0$, then for all $Q \geq 1$, $C_{T_2}^{-1}(H) \left(\frac{N}{a}\right)^{1-H} \widehat{V}_N(a) \xrightarrow{N \rightarrow \infty} R_1^H$ where R_1^H is a H -Rosenblatt random variable.

Proof of Proposition 4: Let $Q \geq 1$ and X^H be as well a fBm with $H \in (0, 1)$ or a Rosenblatt process with $H \in (1/2, 3/4)$ (in the sequel we will only use the second-order properties of the process which are the same for a H -fBm and a H -Rosenblatt process). First,

$$\begin{aligned} \mathbb{E} [(\tilde{e}(a, i) - \tilde{d}(a, i))^2] &= \frac{1}{2C(H)} \left(- \int_0^1 \int_0^1 dt dt' \psi(t) \psi(t') |t - t'|^{2H} - \frac{2}{a} \int_0^1 dt \psi(t) \sum_{k'=0}^{a-1} \psi\left(\frac{k'}{a}\right) \left(|t + i|^{2H} - \left|t - \frac{k'}{a}\right|^{2H} \right) \right. \\ &\quad \left. + \frac{1}{a^2} \sum_{k, k'=0}^{a-1} \psi\left(\frac{k}{a}\right) \psi\left(\frac{k'}{a}\right) \left(\left| i + \frac{k}{a} \right|^{2H} + \left| i + \frac{k'}{a} \right|^{2H} - \left| \frac{k}{a} - \frac{k'}{a} \right|^{2H} \right) \right). \end{aligned}$$

Now, from usual Taylor expansions, if g is supposed to be a m times continuously differentiable $[0, 1]$ -supported function for all $a > 0$,

$$\left| \frac{1}{a} \sum_{k=0}^{a-1} g\left(\frac{k}{a}\right) - \int_0^1 g(t) dt \right| \leq \sup_{t \in [0, 1]} |g^{(m)}(t)| \frac{1}{a^m}.$$

From a Taylor expansion, there exists C depending only on H , Q and ψ such that $\left| \int_0^1 \psi(t) (i+t)^{2H} dt \right| \leq C((1+i)^{2H-Q})$. Therefore, there exists C depending only on H , Q , m and ψ such that for all $a > 0$

$$\left| \frac{1}{a} \sum_{k=0}^{a-1} \psi\left(\frac{k}{a}\right) \right| \leq \frac{C}{a^m} \quad \text{and} \quad \left| \frac{1}{a} \sum_{k=0}^{a-1} \psi\left(\frac{k}{a}\right) \left| i + \frac{k}{a} \right|^{2H} \right| = C \left((1+i)^{2H-Q} + \frac{(1+i)^{2H}}{a^m} \right).$$

Finally, as it was already proved in [4], there exists C depending only on H and ψ such that for all $m \geq 1$ and $a > 0$,

$$\begin{aligned} \left| \frac{1}{a^2} \sum_{k, k'=0}^{a-1} \psi\left(\frac{k}{a}\right) \psi\left(\frac{k'}{a}\right) \left| \frac{k}{a} - \frac{k'}{a} \right|^{2H} - \int_0^1 \int_0^1 dt dt' \psi(t) \psi(t') |t - t'|^{2H} \right| &\leq \frac{C}{a} \\ \left| \frac{1}{a} \int_0^1 dt \psi(t) \sum_{k'=0}^{a-1} \psi\left(\frac{k'}{a}\right) \left| t - \frac{k'}{a} \right|^{2H} - \int_0^1 \int_0^1 dt dt' \psi(t) \psi(t') |t - t'|^{2H} \right| &\leq \frac{C}{a}. \end{aligned}$$

All those inequalities imply that there exists C depending only on H , Q , m and ψ such that for all $a > 0$,

$$\mathbb{E} [(\tilde{e}(a, i) - \tilde{d}(a, i))^2] \leq C \left(\frac{1}{a} + \frac{(1+i)^{2H-Q}}{a^m} + \frac{(1+i)^{2H}}{a^{2m}} \right). \quad (30)$$

But, with Cauchy-Schwarz Inequality,

$$\begin{aligned} \mathbb{E} |\widehat{V}_N(a) - V_N(a)| &\leq \frac{1}{N_a} \sum_{i=1}^{N_a} \mathbb{E} |\tilde{e}^2(a, i) - \tilde{d}^2(a, i)| \\ &\leq \frac{1}{N_a} \sum_{i=1}^{N_a} \sqrt{\mathbb{E} [(\tilde{e}(a, i) - \tilde{d}(a, i))^2]} \sqrt{\mathbb{E} [(\tilde{e}(a, i) + \tilde{d}(a, i))^2]} \\ &\leq \frac{1}{N_a} \sum_{i=1}^{N_a} \sqrt{\mathbb{E} [(\tilde{e}(a, i) - \tilde{d}(a, i))^2]} \sqrt{\mathbb{E} [8\tilde{d}^2(a, i) + 2(\tilde{e}(a, i) - \tilde{d}(a, i))^2]} \\ &\leq \sqrt{2} \left(\frac{1}{N_a} \sum_{i=1}^{N_a} \mathbb{E} [(\tilde{e}(a, i) - \tilde{d}(a, i))^2] \right)^{1/2} \left(\frac{1}{N_a} \sum_{i=1}^{N_a} \mathbb{E} [4\tilde{d}^2(a, i) + 2(\tilde{e}(a, i) - \tilde{d}(a, i))^2] \right)^{1/2}, \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz Inequality. Now since $a = a(N)$ is supposed to be such that $a(N)N^{-H/(H+m)} \xrightarrow{N \rightarrow \infty} 0$, from (30), there exists $C > 0$ not depending on a and N such that,

$$\mathbb{E} |\widehat{V}_N(a) - V_N(a)| \leq C \left(\frac{1}{\sqrt{a}} + \frac{N_a^H}{a^m} + \frac{N_a^{H-1/2}}{a^{m/2}} \mathbf{1}_{Q=1, H>3/4} \right). \quad (31)$$

Now, the 3 different cases of the Proposition can be deduced:

- Using the Markov Inequality, and if $Na(N)^{-2} \xrightarrow{N \rightarrow \infty} 0$ and $m \geq 2$, then $\mathbb{P}(N_a^{1/2} |\widehat{V}_N(a) - V_N(a)|) \xrightarrow{N \rightarrow \infty} 0$. Therefore, if X^H is a fBm and $Q \geq 2$ or $Q = 1$ and $0 < H < 3/4$, Theorem 1 is still valid when $V_N(a)$ is replaced by $\widehat{V}_N(a)$.
- if X^H is a fBm, $Q = 1$ and $3/4 < H$, then if $Na(N)^{-\frac{5-4H}{4-4H}} \xrightarrow{N \rightarrow \infty} 0$ and $Na(N)^{-\frac{3-2H+m}{3-2H}} \xrightarrow{N \rightarrow \infty} 0$, then $\mathbb{P}(N_a^{2-2H} |\widehat{V}_N(a) - V_N(a)|) \xrightarrow{N \rightarrow \infty} 0$ and therefore $\ell_2^{-1}(H) \left(\frac{N}{a}\right)^{2-2H} \widehat{V}_N(a) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} R_1^{2H-1}$ from Theorem 2, where R_1^{2H-1} is a $(2H-1)$ -Rosenblatt random variable.
- if X^H is a Rosenblatt process, if $Na(N)^{-\frac{3-2H}{2-2H}} \xrightarrow{N \rightarrow \infty} 0$ and $Na(N)^{-(1+m)} \xrightarrow{N \rightarrow \infty} 0$, then for all $Q \geq 1$, from Theorem 4, $C_{T_2}^{-1}(H) \left(\frac{N}{a}\right)^{1-H} \widehat{V}_N(a) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} R_1^H$ where R_1^H is a H -Rosenblatt random variable. ■

Remark 3 *We do not think that the conditions provided on $a(N)$ in Proposition 4 are optimal. They could be improved by computing of $\mathbb{E} [(\widehat{V}_N(a) - V_N(a))^2]$ instead of $\mathbb{E} |\widehat{V}_N(a) - V_N(a)|$ in its proof. However, such computation should be very long and technical in the case of the Rosenblatt process and we have preferred not to present them.*

Since the case of fBm was already studied (see for instance [4]) we only provide below the results of simulations when X^H is a Rosenblatt process. Thus, we first exhibit the main result of this paper, *i.e.* the limit theorem $C_{T_2}^{-1}(H) \left(\frac{N}{a(N)}\right)^{1-H} \widehat{V}_N(a(N)) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} R_1^H$ following the following procedure.

Concrete procedure of simulations:

- The samples of Rosenblatt processes are obtained following a similar procedure as the one presented in [3]. It is a wavelet based method introduced by Sellan in the case of the fBm (see for instance [18]). The Matlab procedures of generation of fBm or Rosenblatt processes can be downloaded from <http://samoss.univ-paris1.fr/~Jean-Marc-Bardet>.
- The chosen mother wavelet ψ is a Daubechies wavelet of order 10 (which is such that $Q \geq 2$ and $\psi \in \mathcal{C}^2(\mathbb{R})$).
- The values of both the constants $C(H)$ and $C_{T_2}(H)$ are obtained from usual approximations of integrals by Riemann sums.

Montecarlo experiments using 100 independent replications of trajectories are realized for each $H = 0.6, 0.7, 0.8$ and 0.9 and for $N = 10^3$ and 10^4 . The sequence of scales $(a(N))_N$ is selected to be such that $a(N) = N^{0.4}$ and $a(N) = N^{0.6}$. The following Table 5.1 provides the results of simulations. It appears that $(C_{T_2}^{-1}(H) \left(\frac{N}{a(N)}\right)^{1-H} \widehat{V}_N(a(N)))_N$ converges in distribution to a centered distribution with

		0.6	0.7	0.8	0.9
$N = 10^3$	$\sqrt{\text{MSE}} \ a(N) = N^{0.4}, \ a(N) = N^{0.6}$	1.44, 1.21	1.23, 1.06	0.82, 0.91	0.72, 0.97
$N = 10^4$	$\sqrt{\text{MSE}} \ a(N) = N^{0.4}, \ a(N) = N^{0.6}$	1.18, 1.13	0.89, 1.19	0.66, 0.93	0.04, 0.04

Table 1: Convergence of $(C_{T_2}^{-1}(H) \left(\frac{N}{a(N)}\right)^{1-H} \widehat{V}_N(a(N)))_N$ for different choices of H , N and $a(N)$, from 100 independent replications.

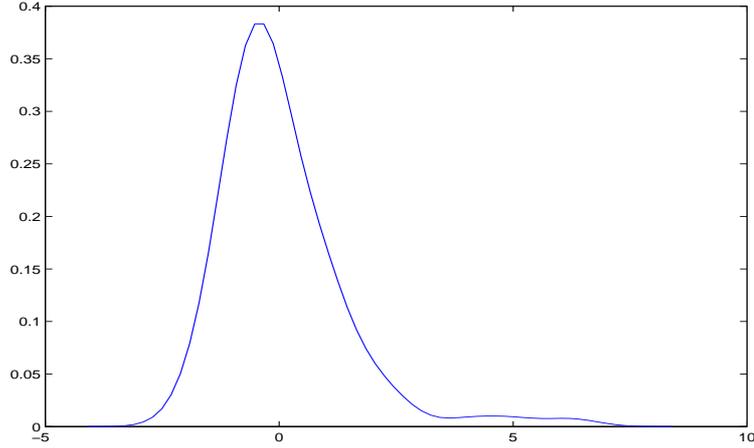


Figure 1: FFT estimation (Silverman's method) of the density of the limit of $(C_{T_2}^{-1}(H) \left(\frac{N}{a(N)}\right)^{1-H} \widehat{V}_N(a(N)))_N$ for $H = 0.7$, $N = 10000$ and $a(N) = N^{0.6}$ from 100 independent replications.

a variance close to 1. However, the case $H = 0.9$ is not as convincing as the other ones because a bias appears in the limit distribution (the procedure of the generation of a Rosenblatt process for large H and the very slow convergence rate $(N/a(N))^{0.1}$ of the limit theorem could explain this weak result).

An example of the estimation of the limit density is also presented in Figure 5.1 in the case $H = 0.7$, $N = 10000$ and $a(N) = N^{0.6}$. Such a density is quite similar to a standard Gaussian density but a Kolmogorov-Smirnov test invalides the hypothesis that this distribution is a $\mathcal{N}(0, 1)$ law. This result should be compared with the numerical simulation of the Rosenblatt density given in [29].

Figure 5.1 shows the convergence of the sequence $(C_{T_2}^{-1}(H) \left(\frac{N}{a(N)}\right)^{1-H} \widehat{V}_N(a(N)))_N$ when N increases. It can be noted that this sequence seems not converge in $\mathbb{L}^2(\Omega)$.

5.2 Estimation of H

Here we consider that a sample (X_1, \dots, X_N) of $X = \sigma^2 X^H$ is known, but H and σ^2 are unknown. For any 3 cases of Proposition 4, one deduces that

$$\frac{1}{a(N)^{2H+1} \sigma^2 C(H)} \frac{1}{N_{a(N)}} \sum_{j=1}^{N_{a(N)}} e^2(a(N), j) = 1 + \varepsilon_N,$$

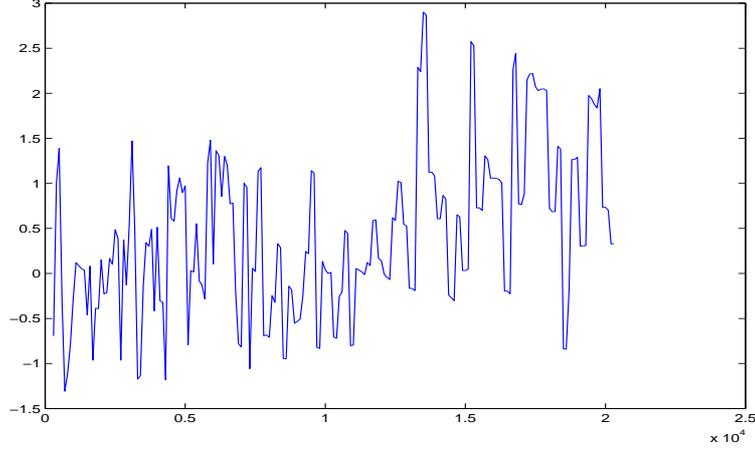


Figure 2: Convergence of the sequence $(C_{T_2}^{-1}(H) (\frac{N}{a(N)})^{1-H} \widehat{V}_N(a(N)))_N$ for $H = 0.7$ and $a(N) = N^{0.5}$.

where $\varepsilon_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} 0$. Therefore,

$$\log \left(\frac{1}{N_{a(N)}} \sum_{j=1}^{N_{a(N)}} e^2(a(N), j) \right) - (2H + 1) \log(a(N)) - \log(\sigma^2 C(H)) = \varepsilon'_N,$$

where $\varepsilon'_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} 0$ and the asymptotic distribution of $N_{a(N)}^\alpha \varepsilon_N$ and $N_{a(N)}^\alpha \varepsilon'_N$ are the same (a Gaussian or a Rosenblatt distribution) with $\alpha = 1/2, 2 - 2H$ or $1 - H$ following the case. Therefore, if $\ell \in \mathbf{N}^* \setminus \{1\}$ and $a_i(N) = ia(N)$ for $i = 1, \dots, \ell$, a log-log-regression of $\left(\frac{1}{N_{a_i(N)}} \sum_{j=1}^{N_{a_i(N)}} e^2(a_i(N), j) \right)_{1 \leq i \leq \ell}$ by $(ia(N))_{1 \leq i \leq \ell}$ (or $(i)_{1 \leq i \leq \ell}$) provides an estimator of H . Such an estimator is defined by

$$\widehat{H}_N := \left(\frac{1}{2}, 0 \right)' \cdot (Z'_\ell Z_\ell)^{-1} Z'_\ell \left(\frac{1}{N_{a_i(N)}} \sum_{i=1}^{N_{a_i(N)}} e^2(a, i) \right)_{1 \leq i \leq \ell} - \frac{1}{2},$$

where $Z_\ell(i, 1) = 1$ and $Z_\ell(i, 2) = \log i$ for all $i = 1, \dots, \ell$. Then Proposition 4 implies

Proposition 5 *Using the same assumptions as in Proposition 4, Then,*

1. if X^H is a fBm and $Q \geq 2$ or $Q = 1$ and $0 < H < 3/4$, and if $N a(N)^{-2} \xrightarrow[N \rightarrow \infty]{} 0$ and $m \geq 2$, then there exists $\gamma^2(H, \ell, \psi) > 0$ depending only on H, ℓ and ψ such that

$$\sqrt{\frac{N}{a(N)}} (\widehat{H}_N - H) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \gamma^2(H, \ell, \psi))$$

2. if X^H is a fBm and $Q = 1$ and $3/4 < H < 1$, and if $N a(N)^{-\frac{5-4H}{4-4H}} \xrightarrow[N \rightarrow \infty]{} 0$ and $N a(N)^{-\frac{3-2H+m}{3-2H}} \xrightarrow[N \rightarrow \infty]{} 0$, then there exists $C > 0$ depending only on H, ℓ and ψ such that $\left(\frac{N}{a(N)} \right)^{2-2H} (\widehat{H}_N - H) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} L$ where L is a distribution depending only on H, ℓ and ψ .
3. if X^H is a Rosenblatt process, if $N a(N)^{-\frac{3-2H}{2-2H}} \xrightarrow[N \rightarrow \infty]{} 0$ and $N a(N)^{-(1+m)} \xrightarrow[N \rightarrow \infty]{} 0$, then for all $Q \geq 1$, there exists $C > 0$ depending only on H, ℓ and ψ such that $\left(\frac{N}{a(N)} \right)^{1-H} (\widehat{H}_N - H) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} L$ where L is a distribution depending only on H, ℓ and ψ .

	H	0.6	0.75	0.9
$N = 500$	Mean, STD, $\sqrt{\text{MSE}}$	0.59, 0.15, 0.15	0.67, 0.17, 0.18	0.79, 0.17, 0.19
$N = 2000$	Mean, STD, $\sqrt{\text{MSE}}$	0.61, 0.058, 0.059	0.72, 0.067, 0.075	0.83, 0.075, 0.10
$N = 10000$	Mean, STD, $\sqrt{\text{MSE}}$	0.61, 0.032, 0.033	0.73, 0.035, 0.038	0.86, 0.045, 0.052

Table 2: Convergence of \hat{H}_N for different choices of H and N , from 100 independent replications.

Remark 4 *i.* An estimator of $\log(\sigma^2 C(H))$ (and therefore of σ^2) can also be provided by this method, with the same convergence rate.

ii. Concerning the convergence rates, it is obvious that the choice of the sequence $a(N)$ can not depend on H . The following table summarizes the optimal choice of $a(N)$ and the corresponding convergence rate for \hat{H}_N (with $\delta > 0$ arbitrary small and $m \geq 2$):

	Choice of $a(N)$	Convergence rate
fBm and $Q \geq 2$ or $Q = 1$ and $0 < H < 3/4$	$N^{1/2+\delta}$	$N^{1/4-\delta/2}$
fBm and $Q = 1$ and $3/4 < H < 1$	$N^{1/2+\delta}$	$N^{1-H-\delta(2-2H)}$
Rosenblatt process	$N^{1/2+\delta}$	$N^{(1-H)/2-\delta(1-H)}$

It is clear that such convergence rate are weak in a parametric frame. For instance, applied to the increments of a fBm , the convergence rate of the maximum likelihood or the approximated Whittle maximum likelihood estimator is $N^{1/2}$ (see [11] and [16]; we do not know such result concerning the Rosenblatt process). However, as it was previously recalled in the introduction, the wavelet based estimator can be also applied to semi-parametric frames.

The results of simulations are given in Table 5.2. It appears that the convergence rate of the estimator depends on H as it was specified in Proposition 5. However, as m is chosen to increase with N the convergence rate are better than in Proposition 5.

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