Resource modalities in game semantics (journal version)

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Abstract

The description of resources in game semantics has never achieved the simplicity and precision of linear logic, because of a misleading conception: the belief that linear logic is more primitive than game semantics. We advocate instead the contrary: that game semantics is conceptually more primitive than linear logic. Starting from this revised point of view, we design a categorical model of resources in game semantics, and construct a Conway game model extended with a notion of payoff in order to capture various resource policies: linear, affine, relevant and exponential.

Keywords: Game semantics, linear logic, resources, continuation monads, categorical models.

1 Introduction

Game semantics and linear logic. Contemporary game semantics is the younger sibling of linear logic: born (or rather reborn after the work of Lorenzen's school [28, 29]) at the beginning of the 1990s, in the turmoil produced by the recent discovery of linear logic by Girard [13], it remained under its spiritual influence for a long time.

In the early days, this radiance of linear logic was extraordinarily healthy and profitable. Properly guided, game semantics developed steadily, following the idea that every *formula* of linear logic describes a *game*; and that every *proof* of the formula describes a *strategy* for playing on that game.

This correspondence between formulas of linear logic and games is supported by a series of elegant and striking analogies. One basic principle of linear logic is that negation

$$A \mapsto \neg A$$

is involutive. This means that every formula A is equal (or at least isomorphic) to the formula negated twice:

$$A \quad \cong \quad \neg\neg A. \tag{1}$$

This principle is nicely reflected in game semantics by the idea that negating a game A consists in permuting the roles of the two players. Hence, negating a game twice amounts

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to permuting the role of Proponent and Opponent twice, which is just like doing nothing. Typically, if A is a chess board where White starts, $\neg A$ is a is the chess board where Black starts, and $\neg\neg A$ is again the chess board where White starts. Another basic principle of linear logic is that every formula behaves as a resource, which disappears once consumed. In particular, a proof of the formula

$$A \multimap B$$

enables to deduce the conclusion B by using (we rather say : consuming) exactly once its hypothesis A – seen here as a resource. Again, this principle is nicely reflected in game semantics, by the idea that playing a game is just like consuming a resource, the game itself.

The connectives of linear logic are also convincingly reflected in game semantics. For instance, the tensor product

$$A \otimes B$$

of two formulas A and B is suitably interpreted as the game (or formula) A played in parallel with the game (or formula) B, where only Opponent may switch from a component to the other one. This amounts to place two boards on the same table and to say that Black must respond on the board where White has just played. Similarly, the sum

$$A \oplus B$$

of two formulas A and B is interpreted as the game where Proponent plays the first move, which consists in choosing between the game A and the game B, before carrying on in the selected component. This amounts to place two boards on the same table and to let Black decides if he wants to play on the left or right board. This choice is then irreversible. Finally, the exponential modality of linear logic

$$!A$$

applied to the formula A is interpreted as the game where several copies of the game A are played in parallel, and only Opponent is allowed (a) to switch from a copy to another and (b) to open a fresh copy of the game A. This amounts to play on parallel chess boards as for tensor but with the ability for White to add a new chess board to those already there.

What we describe here is in essence the game semantics of linear logic defined by Blass in [9]. Simple and elegant, the model reflects the full flavor of the resource policy of linear logic. Interestingly, this game semantics is an early predecessor to linear logic [8]. It has first been introduced indirectly by Berry and Curien [7] through their model of concrete data structures which constitutes the first interactive semantics for sequential algorithms. Game semantics has then been used directly to obtain complete models of multiplicative linear logic [2, 18]. These works have then been pursued on more extended fragment of linear logic [25, 5]. Game semantics has also been related to the calculus of linear head-reduction computed by some abstract machines [11].

A schism with linear logic. The destiny of game semantics has been to emancipate itself from linear logic in the mid–1990s, in order to comply with its own designs, inherited from denotational semantics:

1. the desire to interpret *programs* written in programming languages with effects (recursion, states, etc.) and to characterize exactly their interactive behavior inside *fully abstract* models;

2. the desire to understand the algebraic principles of programming languages and effects, using the language of category theory.

A new generation of game semantics arose, propelled by (at least) two different lines of research:

1. Abramsky and Jagadeesan [2] noticed that the (alternating variant of the) Blass model does not define a categorical model of linear logic. Worse: it does not even define a category, for lack of associativity. Abramsky dubs this phenomenon the *Blass problem* and describes it in [1].

2. Hyland and Ong [19] introduced the notion of *arena game*, and characterized the interactive behavior of programs written in the functional language PCF — the simply-typed λ-calculus with conditional test, arithmetic and recursion. A similar result with a slightly different model has been obtained by Abramsky, Malacaria and Jagadeesan [3]. Note that despite their publication dates, those works have both been done during 1994.

So, the Blass problem indicates that it is difficult to construct a (sequential) game model of linear logic; and arena games became mainstream in the mid–1990s, although they do not define a model of linear logic. These two reasons (at least) opened a schism between game semantics and linear logic: it suddenly became accepted that categories of (sequential) games and strategies would only capture *fragments* of linear logic (intuitionistic or polarized) but not the whole thing.

A conciliation through tensorial logic. In order to understand in deep resource modalities of linear logic in game semantics, it appears necessary to reunify the two subjects. Since the disagreement started with category theory, this reunification should occur at the categorical level. We explain (in §2) how to achieve this by *relaxing* the involutive negation of linear logic into a less constrained tensorial negation. This negation induces in turn a *linear continuation* monad, whose unit

$$A \quad \longrightarrow \quad \neg\neg A \tag{2}$$

refines the isomorphism (1) of linear logic. Moving from an involutive to a tensorial negation means that we replace linear logic by a more general and primitive logic – which we call *tensorial logic*. As we will see, this shift to tensorial logic clarifies the Blass problem, and describes the structure of arena games. It also enables the expressions of resource modalities in game semantics, just as it is usually done in linear logic.

Tensorial logic provides a new insight on polarized logic introduced by Girard in his work on classical logic and system LC [14]. An unexpected phenomenon shows up in these polarized logic: resource modalities change the polarity of a formula. This peculiar fact is explained in tensorial logic by a decomposition of the lifting operator ${}^{\downarrow}_{\mathrm{pol}}$ into two constructors: the exponential modality ! which do not change the polarity of formulas, an the tensorial negation, noted $\downarrow$ in this framework (rather that $\neg$), whose role as a negation is to swap the point of view of Opponent and Proponent on a formula – which amounts to reverse its polarity:

$${}^{\downarrow}_{\mathrm{pol}}A \;=\; !\downarrow A.$$

We would like to promote a radical change of perspective on polarized logic. Indeed, according to us, tensorial logic is not reduced to a fragment of linear logic, as one used to think of polarized logic. On the contrary, we defend the thesis that tensorial logic is a more primitive logic that linear logic, closer to mechanisms of continuation described by game semantics. And in the same way that classical logic is interpreted in intuitionistic logic through the Gödel translation, we will see that linear logic is interpreted in tensorial logic through a similar translation, of a categorical nature (Kleisli construction). In a word, tensorial logic is to linear logic what intuitionistic logic it to classical logic: a formalism closer to computations and programs.

Plan of the paper. We describe (§2) a categorical semantics of resources in game semantics, and explain (§3) in what sense the resulting topography refines both linear logic and polarized logic. After that, we construct (§4) a compact-closed (that is, self-dual) category inspired by Conway games, where the resource policy is enforced by a notion of payoff. From this, we derive (§5) a model of our categorical semantics of resources, using a family construction, and conclude (§6).

2 Categorical models of resources

Despite its apparent efficiency to model fragments of linear logic, game semantics has, for a long time, resist to multiple attempts to make it interpreting the whole linear logic. This is due in particular to the Blass problem , pointed out by Abramsky and Jagadeesan [2], which indicates that the game semantics introduced by Blass [9] does not give rise to a category. One had to wait the beginning of this century, and the work of the first author [33, 34] on asynchronous games, to get a first completeness result. Curiously, the solution is based on a quotient on strategies – quotient which, in the light of the work of Hasegawa, amounts to transform a strong continuation monad into a commutative continuation monad. It is then natural to wonder what would be a linear logic where the continuation monad is not commutative; which leads us to tensorial logic – a more primitive logic deeply related to game semantics.

It appears that most of the models of linear logic arise in this way, from model of tensorial logic where the continuation monad is commutative. This is the case for example for coherent spaces, phase spaces, finiteness spaces [12], etc. One of the interests of the approach is that we can define different negations on the same model – which harmoniously generate different models of linear logic (coherent spaces, finiteness spaces [38]). This insight of linear logic through tensorial logic looks trickily like the one brought by polarized logics [26]. The main difference lies in our desire to distinguish resource modalities, like the exponential, from the negation; and thus to get rid of the Cartesian paradigm which is inherent to polarized logics.

We introduce now the notion of *tensorial negation* on a symmetric monoidal category; and then explain how such a category with negation (called a *dialogue category*) may be equipped with additives and various resource modalities. The first author describes in [31] how to extract a *syntax* of proofs from a categorical semantics, using string diagrams and functorial boxes. The recipe may be applied here to extract the syntax of a logic, called *tensorial logic*. However, we provide in section 3 a sequent calculus for tensorial logic, in order to compare it to linear logic or Olivier Laurent's polarized linear logic [26].

Tensorial negation. A *tensorial negation* on a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is defined as a functor

$$\neg \quad : \quad \mathcal{C} \quad \longrightarrow \quad \mathcal{C}^{op}$$

together with a family of bijections

$$\varphi_{A,B,C} \quad : \quad \mathcal{C}(A \otimes B, \neg C) \quad \cong \quad \mathcal{C}(A, \neg(B \otimes C))$$

natural in A, B and C such that the diagram

$$\begin{array}{ccc}
\mathcal{C}(A \otimes (B \otimes C), \neg\, D) & \xrightarrow{\mathcal{C}(\alpha_{A,B,C}, \neg C)} & \mathcal{C}((A \otimes B) \otimes C, \neg\, D) \\
 & & \downarrow{\scriptstyle \varphi_{A\otimes B,C,D}} \\
\downarrow{\scriptstyle \varphi_{A,B\otimes C,D}} & & \mathcal{C}(A \otimes B, \neg\,(C \otimes D)) \\
 & & \downarrow{\scriptstyle \varphi_{A,B,C\otimes D}} \\
\mathcal{C}(A, \neg\,((B \otimes C) \otimes D)) & \xrightarrow[\mathcal{C}(A, \neg\alpha^{-1}_{B,C,D})]{} & \mathcal{C}(A, \neg\,(B \otimes (C \otimes D)))
\end{array}$$

commutes. A symmetric monoidal category equipped with a tensorial negation is said to be a *dialogue category*. Given a negation, it is customary to define the formula *false* as the object $\bot \stackrel{\text{def}}{=} \neg\, 1$ obtained by "negating" the unit object 1 of the monoidal category. Note that we use the notation 1 (instead of I or e) in order to remain consistent with the notations of linear logic. Note also that the bijection $\varphi_{A,B,1}$ provides then the category $\mathcal{C}$ with a one-to-one correspondence

$$\varphi_{A,B,1} \quad : \quad \mathcal{C}(A \otimes B, \bot) \quad \cong \quad \mathcal{C}(A, \neg B)$$

for all objects A and B. For that reason, the definition of a negation $\neg$ is often replaced by the – somewhat too informal – statement that "the object $\bot$ is exponentiable" in the symmetric monoidal category $\mathcal{C}$, with negation $\neg A$ noted $\bot^A$.

Self-adjunction. In his PhD thesis, Thielecke [39] observes for the first time a fundamental "self-adjunction" phenomenon, related to negation. This observation plays then a key role in an unpublished work by Selinger and the first author [32] on polar categories, a categorical semantics of polarized linear logic, continuations and games. The same idea reappears recently in a nice, comprehensive study on polarized categories (=distributors) by Cockett and Seely [10]. In our situation, the "self-adjunction" phenomenon amounts to the fact that every tensorial negation is left adjoint to the opposite functor

$$\neg \quad : \quad \mathcal{C}^{op} \quad \longrightarrow \quad \mathcal{C} \tag{3}$$

because of the natural bijection

$$\mathcal{C}^{op}(\neg A, B) \quad \cong \quad \mathcal{C}(A, \neg B).$$

Continuation monad. Every tensorial negation $\neg$ induces an adjunction, and thus a monad

$$\neg\neg \quad : \quad \mathcal{C} \longrightarrow \mathcal{C}$$

This monad is called the *continuation monad* of the negation. One fundamental fact observed by Moggi [35] is that the continuation monad is *strong* but not commutative in general. By strong monad, we mean that the monad $\neg\neg$ is equipped with a family of morphisms:

$$t_{A,B} \quad : \quad A \otimes \neg\neg B \quad \longrightarrow \quad \neg\neg\,(A \otimes B)$$

natural in A and B, and satisfying a series of coherence properties. By commutative monad, we mean a strong monad making the two canonical morphisms

$$\neg\neg A \otimes \neg\neg B \quad \rightrightarrows \quad \neg\neg\,(A \otimes B) \tag{4}$$

coincide. A tensorial negation $\neg$ is called *commutative* when the continuation monad induced in $\mathcal{C}$ is commutative — or equivalently, a monoidal monad in the lax sense.

Linear implication. A dialogue category $\mathcal{C}$, with negation $\neg$ is not very far from being monoidal *closed*. It is possible indeed to define a *linear implication* $\multimap$ when its target $\neg B$ is a negated object:

$$A \multimap \neg B \quad \stackrel{\text{def}}{=} \quad \neg\,(A \otimes B).$$

In this way, the functor (3) defines what we call an *exponential ideal* in the category $\mathcal{C}$. When the functor is faithful on objects and morphisms, we may identify this exponential ideal with the subcategory of *negated objects* in the category $\mathcal{C}$. The exponential ideal discussed in McCusker's PhD thesis [30] arises precisely in this way.

Continuation category. Every dialogue category $\mathcal{C}$, with negation $\neg$, induces a *category of continuations* $\mathcal{C}^{\neg}$ with the same objects as $\mathcal{C}$, and morphisms defined as

$$\mathcal{C}^{\neg}(A,B) \quad \stackrel{\text{def}}{=} \quad \mathcal{C}(\neg A, \neg B).$$

Note that the category $\mathcal{C}^{\neg}$ is the Kleisli category associated to the comonad in $\mathcal{C}^{op}$ induced by the adjunction; and that it is at the same time the opposite of the Kleisli category associated to the continuation monad in $\mathcal{C}$. Because the continuation monad is strong, the category $\mathcal{C}^{\neg}$ is *premonoidal* in the sense of Power and Robinson [36]. Note that string diagrams in premonoidal categories are inherently related to control flow charts in software engineering, as explained by Jeffrey [22].

Semantics of resources. A *resource modality* on a symmetric monoidal category $(\mathcal{C}, \otimes, e)$ is defined as an adjunction:

$$\mathcal{M} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{C} \qquad (\perp) \tag{5}$$

where

- $(\mathcal{M}, \bullet, u)$ is a symmetric monoidal category,

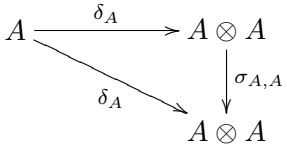
- U is a symmetric monoidal functor.

Recall that a *symmetric monoidal* functor U is a functor which transports the symmetric monoidal structure of $(\mathcal{M}, \bullet, u)$ to the symmetric monoidal structure of $(\mathcal{C}, \otimes, e)$, up to isomorphisms satisfying suitable coherence properties. Another more conceptual definition of a resource modality is possible: it is an adjunction defined in the 2-category of symmetric monoidal categories, *lax* symmetric monoidal functors, and monoidal transformations. Now, the resource modality is called

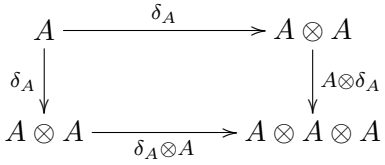
- *affine* when the unit u is the terminal object of the category $\mathcal{M}$,
- *relevant* when every object of $\mathcal{M}$ is duplicable, that is when there exists a diagonal

$$\delta_A \quad : \quad A \longrightarrow A \otimes A$$

  natural in A, compatible with the symmetry and satisfying the associativity diagram



(6)



(7)

- *exponential* when the tensor product $\bullet$ is a cartesian product, and the unit u is the terminal object of the category $\mathcal{M}$.

This definition of resource modality is inspired by the categorical semantics of linear logic, and more specifically by Benton's notion of Linear-Non-Linear model [6] — which may be now reformulated as a symmetric monoidal *closed* category $\mathcal{C}$ equipped with an exponential modality in our sense. Very often, we will identify the resource modality and the induced comonad $! = U \circ F$ on the category $\mathcal{C}$. We sum up the different resource modalities in the following table.

Modality	Category $(M, \otimes, e)$
Affine	the unit e is terminal
Relevant	every object is duplicable
Exponential	the structure is cartesian

The work of Jacobs on affine and relevant modalities [21] is based on a commutative monad on a cartesian closed category. He then considers the Eilenberg-Moore category induced by this (affine or relevant) monad in order to deduce models of intuitionistic linear logic (ILL) equipped with a (affine or relevant) modality. The weak point of this construction is its limitation to very special kind of model of ILL which are obtained as categories of algebras.

3 Tensorial logic

In our philosophy, tensorial logic is entirely described by its categorical semantics — which is defined in the following way. First, every dialogue category $\mathcal{C}$ defines a model of *multiplicative* tensorial logic. Such a category defines a model of *multiplicative additive*

tensorial logic when the category $\mathcal{C}$ has finite coproducts (noted $\oplus$) which *distribute* over the tensor product: this means that the canonical morphisms

$$(A \otimes B) \oplus (A \otimes C) \quad \longrightarrow \quad A \otimes (B \oplus C)$$

$$0 \quad \longrightarrow \quad A \otimes 0$$

are isomorphisms. Then, a model of (full) tensorial logic is defined as a model of multiplicative additive tensorial logic, equipped with an affine resource modality (with comonad noted $\underset{w}{!}$), a relevant resource modality (with comonad noted $\underset{c}{!}$) as well as an exponential resource modality (with comonad noted $\underset{e}{!}$).

The diagrammatic syntax of tensorial logic will be extracted from its categorical definition, using the recipe explained in [31]. To give a more familiar presentation of tensorial logic, we now formulate its sequent calculus in two different but equivalent ways: bilateral and monolateral.

Bilateral presentation. The formulas $A, B, \ldots$ of tensorial logic (in its bilateral presentation) are constructed as follows:

multiplicatives	$1 \mid \neg A \mid A \otimes B$
additives	$0 \mid A \oplus B$
resource modalities	$\underset{w}{!}A \mid \underset{c}{!}A \mid \underset{e}{!}A$

The sequents are of two forms: $\Gamma \vdash A$ where Γ is a context, and A is a formula; $\Gamma \vdash$ where Γ is a context. The sequent calculus of the multiplicative fragment appears in Figure 1. The first four rules express the monoidal structure on $\mathcal{C}$, the two below define a tensorial negation and the two last just represent identity and composition of our category $\mathcal{C}$. Figure 2 describes the rules managing finite coproducts. Figure 3 depicts the expected rules for the exponential modality (those are the rules of the ! of linear logic). The rules for the affine modality, given in Figure 4, are the same as for the exponential modality, but without contraction. The rules for the relevant modality, given in Figure 5, are the same as for the exponential modality, but without weakening.

Monolateral presentation. In order to switch to the monolateral formulation of tensorial logic, we need to introduce polarities. The formulas that were on the right in the bilateral presentation remain there, and are called *positive*. Dually, the formulas on the left move on the right, and are now called *negative*.

Bilateral presentation		Monolateral presentation
$\Gamma \vdash$	$\rightsquigarrow$	$\vdash \Gamma^*$
$\Gamma \vdash A$	$\rightsquigarrow$	$\vdash \Gamma^*, A$

So, there are two kinds of sequents in this formulation: the sequents $\vdash \Gamma$ where Γ contains only negative formulas, and the sequents $\vdash \Gamma, P$ containing exactly one positive formula P. To distinguish between positive and negative formulas, we have to clone each construct $0, 1, \oplus, \otimes, \underset{w}{!}, \underset{c}{!}, \underset{e}{!}$ into itself: $0, 1, \oplus, \otimes, \underset{w}{!}, \underset{c}{!}, \underset{e}{!}$ and its dual: $\top, \bot, \&, \parr, \underset{w}{?}, \underset{c}{?}, \underset{e}{?}$. The negation $\neg$ itself is cloned in two operations $\uparrow$ and $\downarrow$, each of them with a specific effect:

- $\uparrow$ transports the positive formulas into the negative formulas,

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \text{Tensor-Right} \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\ \text{Tensor-Left}$$

$$\frac{}{\vdash 1}\ \text{Unit-Right} \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\ \text{Unit-Left}$$

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \text{Negation-Right} \qquad \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}\ \text{Negation-Left}$$

$$\frac{}{A \vdash A}\ \text{Axiom} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \text{Cut}$$

Figure 1. Multiplicative tensorial logic: bilateral presentation

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \text{Sum-Right-1} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \text{Sum-Left}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \text{Sum-Right-2}$$

No right introduction rule for the zero $\qquad \frac{}{\Gamma, 0 \vdash A}\ \text{Zero-Left}$

Figure 2. Additive tensorial logic: bilateral presentation

$$\frac{!_e\Gamma \vdash A}{!_e\Gamma \vdash !_e A}\ \text{Strengthening} \qquad \frac{\Gamma, A \vdash}{\Gamma, !_e A \vdash}\ \text{Dereliction}$$

$$\frac{\Gamma \vdash}{\Gamma, !_e A \vdash}\ \text{Weakening} \qquad \frac{\Gamma, !_e A, !_e A \vdash}{\Gamma, !_e A \vdash}\ \text{Contraction}$$

Figure 3. Exponential modality: bilateral presentation

$$\frac{!_w\Gamma \vdash A}{!_w\Gamma \vdash !_w A}\ \text{Strengthening} \qquad \frac{\Gamma, A \vdash}{\Gamma, !_w A \vdash}\ \text{Dereliction}$$

$$\frac{\Gamma \vdash B}{\Gamma, !_w A \vdash}\ \text{Weakening}$$

Figure 4. Affine modality: bilateral presentation

$$\frac{!_c\Gamma \vdash A}{!_c\Gamma \vdash !_c A}\ \text{Strengthening} \qquad \frac{\Gamma, A \vdash}{\Gamma, !_c A \vdash}\ \text{Dereliction}$$

$$\frac{\Gamma, !_c A, !_c A \vdash}{\Gamma, !_c A \vdash}\ \text{Contraction}$$

Figure 5. Relevant modality: bilateral presentation

- $\downarrow$ transports the negative formulas into the positive formulas.

Remark that the affine and exponential modalities do not change polarities themselves: this is a main difference with polarized logic. We use the letters P and Q for the positive formulas, the letters L and M for the negative formulas, and the letters Γ, Δ for the contexts of negative formulas. Formulas are constructed by the following grammar:

Positives	0	$\mid$	1	$\mid$	$\downarrow L$	$\mid$	$P \otimes Q$	$\mid$	$P \oplus Q$	$\mid$	$!_w P$	$\mid$	$!_c P$	$\mid$	$!_e P$
Negatives	$\bot$	$\mid$	$\top$	$\mid$	$\uparrow P$	$\mid$	$L \parr M$	$\mid$	$L \,\&\, M$	$\mid$	$?_w L$	$\mid$	$?_c L$	$\mid$	$?_e L$

Every positive formula P has a dual negative formula $P^\bot$, obtained by dualizing every logical construct appearing in the formula P. The sequent calculus in Figure 6 for the multiplicatives adapts Figure 1; Figure 7 for the additives adapts Figure 2. Figures 8, 9 and 10 for the resource modalities adapt Figures 3, 4 and 5.

Arena games and classical logic. Starting from Thielecke's work, Selinger [37] designs the notion of *control category* in order to axiomatize neatly the semantics of classical logic. Then, prompted by a completeness result established by Hofmann and Streicher in [17], he proves a beautiful structure theorem, stating that every control category $\mathbf{P}$ is the continuation category $\mathcal{C}^\neg$ of a *response category* $\mathcal{C}$. Now, a response category $\mathcal{C}$ — where the monic requirement on the units (2) is relaxed — is exactly the same thing as a model of multiplicative additive tensorial logic, where the tensor $\otimes$ is *cartesian* and the tensor unit 1 is *terminal*.

A purely proof-theoretic analysis of classical logic leads exactly to the same conclusion. Starting from Girard's work on polarities in LC [14], Laurent developed a comprehensive and perspicuous analysis of polarities in logic, incorporating classical logic [26], (non-well-bracketed) arena games [27] and control categories [27]. Now, it appears that Laurent's polarized logic coincides with multiplicative additive tensorial logic — where the monoidal structure is *cartesian*. This appears clearly in the monolateral formulation of tensorial logic. We sum up the difference between tensorial logic and classical logic in a very schematic table:

Tensorial logic	$\otimes$	is monoidal
	$\neg$	is tensorial
Classical logic	$\otimes$	is cartesian
	$\neg$	is tensorial

Game semantics and linear logic. The continuation monad $A \mapsto \overset{OP}{\neg\neg} A$ of game semantics lifts an Opponent-starting game A with an Opponent move $\neg_O$ followed by a Player move $\neg_P$. Now, it appears that the Blass problem mentioned in §1 arises precisely because the monad is strong, but not commutative [32, 33]. Indeed, one obtains a game model of (full) propositional linear logic by *identifying* the two canonical strategies (4) — this leading to a fully complete model of linear logic, in the way described in [34].

This construction in game semantics has a nice categorical counterpart. We already mentioned that the continuation category $\mathcal{C}^\neg$ inherits a premonoidal structure from the symmetric monoidal structure of $\mathcal{C}$. Now, Hasegawa Masahito shows [16] that the continuation category $\mathcal{C}^\neg$ equipped with this premonoidal structure is $*$-autonomous if and

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ \text{Tensor} \qquad \frac{\vdash \Gamma_1, L, M, \Gamma_2, P}{\vdash \Gamma_1, L \parr M, \Gamma_2, P}\ \text{Par}$$

$$\frac{}{\vdash 1}\ \text{One} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \text{Bottom}$$

$$\frac{\vdash \Gamma, L}{\vdash \Gamma, \downarrow L}\ \text{Linear strengthening} \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, \uparrow P}\ \text{Linear dereliction}$$

$$\frac{}{\vdash P^\bot, P}\ \text{Axiom} \qquad \frac{\vdash \Gamma, P \qquad \vdash P^\bot, \Delta, Q}{\vdash \Gamma, \Delta, Q}\ \text{Cut}$$

Figure 6. Multiplicative tensorial logic: monolateral presentation

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\ \text{Sum-Left} \qquad \frac{\vdash \Gamma, L \qquad \vdash \Gamma, M}{\vdash \Gamma, L \,\&\, M}\ \text{With}$$

$$\frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\ \text{Sum-Right}$$

No introduction rule for the zero $\qquad \frac{}{\vdash \Gamma, \top}\ \text{Top}$

Figure 7. Additive tensorial logic: monolateral presentation

$$\frac{\vdash ?_e\Gamma, P}{\vdash ?_e\Gamma, !_e P}\ \text{Strengthening} \qquad \frac{\vdash \Gamma, L}{\vdash \Gamma, ?_e L}\ \text{Dereliction}$$

$$\frac{\vdash \Gamma}{\vdash !_e\Gamma, ?_e L}\ \text{Weakening} \qquad \frac{\vdash \Gamma, ?_e L, ?_e L}{\vdash \Gamma, ?_e L}\ \text{Contraction}$$

Figure 8. Exponential modality: monolateral presentation

$$\frac{\vdash ?_w\Gamma, P}{\vdash ?_w\Gamma, !_w P}\ \text{Strengthening} \qquad \frac{\vdash \Gamma, L}{\vdash \Gamma, ?_w L}\ \text{Dereliction}$$

$$\frac{\vdash \Gamma, L}{\vdash \Gamma, ?_w L}\ \text{Weakening}$$

Figure 9. Affine modality: monolateral presentation

$$\frac{\vdash ?_c\Gamma, P}{\vdash ?_c\Gamma, !_c P}\ \text{Strengthening} \qquad \frac{\vdash \Gamma, L}{\vdash \Gamma, ?_c L}\ \text{Dereliction}$$

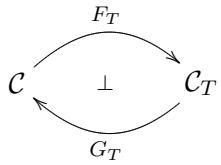
$$\frac{\vdash \Gamma, ?_c L, ?_c L}{\vdash \Gamma, ?_c L}\ \text{Contraction}$$

Figure 10. Relevant modality: monolateral presentation

only if the continuation monad is commutative. The specialist will recognize here a categorification of Girard's phase space semantics [13]. Anyway, this shows that linear logic is essentially tensorial logic in which the tensorial negation is commutative.

Linear logic	$\otimes$	is monoidal
	$\neg$	is commutative

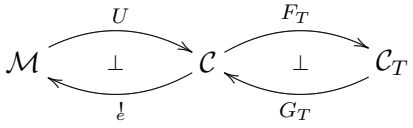
We now formalize this idea and show that any model of tensorial logic, for which the continuation monad is commutative, induces a model of linear logic. To show this, we will mainly work on the Kleisli category of the continuation monad



We first show that the exponential modality lift to the Kleisli category

Lemma 1 *The exponential modality* $\underset{e}{!}$ *lift to an exponential modality* ! *on the Kleisli category* $\mathcal{C}_T$.

Proof: According to Power and Robinson [36], the left adjoint F_T between $\mathcal{C}$ and $\mathcal{C}_T$ is a symmetric monoidal strong functor. We then deduce that the adjunction is symmetric monoidal. As symmetric monoidal adjunctions do compose, the resulting adjunction



is symmetric monoidal. Now, $\mathcal{M}$ is cartesian, so this adjunction builds a commutative $\otimes$-comonoid

$$!A \stackrel{\text{def}}{=} \underset{e}{!}\neg\neg A$$

on $\mathcal{C}_T$. $\square$

Now, we recall a folklore result among category theorist: the Kleisli catégory preserves finite coproducts. More precisely

Proposition 1 *Let* T *be a monad on a category* $\mathcal{C}$ *equipped with finite coproducts. The Kleisli category* $\mathcal{C}_T$ *is also equipped with finite coproducts. Furthermore, if finite coproducts commute with the tensor product in* $\mathcal{C}$*, they still do in* $\mathcal{C}_T$.

Proof: Note $A+B$ the coproduct of A and B. Injections are defined as

$$A \xrightarrow{\text{inj}_1} A+B \xrightarrow{\eta_{A+B}} T(A+B) \qquad\qquad B \xrightarrow{\text{inj}_2} A+B \xrightarrow{\eta_{A+B}} T(A+B)$$

Let X be an object of $\mathcal{C}$ and $f : A \to TX$ et $g : B \to TX$ be two morphisms, one defines the sum $f+g$ by

$$A+B \xrightarrow{f+g} TX$$

One easily check that this arrow gives rise to a coproduct.

For the commutation with the tensor product, it suffices to notice that the definition of the tensor product and of finite coproducts are given by the image of F_T of those in $\mathcal{C}$. Then, the canonical distributivity morphism

$$(A \otimes B) \oplus (A \otimes C) \;\cong\; A \otimes (B \oplus C)$$

in $\mathcal{C}_T$ is the image by F_T of the distributivity morphism in $\mathcal{C}$. It is an isomorphism in $\mathcal{C}$ so it is an isomorphism in $\mathcal{C}_T$ as any functor preserves isomorphisms. □

By duality, we deduce that the Kleisli category $\mathcal{C}_T$ on the continuation monad is quipped with finite products

$$A \,\&\, B \;\stackrel{\text{def}}{=}\; \neg(\neg A \oplus \neg B).$$

We can now express a last lemma.

Lemma 2 *The isomorphism*

$$!(A \,\&\, B) \;\cong\; !A \;\otimes\; !B$$

holds in the category $\mathcal{C}_T$.

Proof: The isomorphism comes from the fact that right adjoints (here $\stackrel{!}{e}$ and G_T) preserve finite products and that the left adjoints F_T and U are strong monoidal functors. □

The Kleisli category $\mathcal{C}_T$ thus defines a model of linear logic. It is also the case for its opposite category $\mathcal{C}^\neg$, the only difference is that the exponential modality is, in that case, given by

$$!A \;\stackrel{\text{def}}{=}\; \neg\neg\stackrel{!}{e}A.$$

Theorem 1 *Let $\mathcal{C}$ be a model of tensorial logic (multiplicative, additive, exponential) for which the continuation monad $\neg\neg$ is commutative. The category of continuation $\mathcal{C}^\neg$ gives rise to a model of linear logic (multiplicative, additive, exponential).*

4 Payoff Conway games

We define here and in §5 a simple game semantics with resource modalities in order to give a model of tensorial logic. We achieve this by enriching the model of Conway games, introduced by Joyal in his pioneer work [23], with a notion of *payoff*.

Conway games. Conway games give a very intuitive formalism which connects directly game semantics to algorithmic notions using the structure of graphs explicitly.

A *Conway game* A is an oriented rooted graph (V_A, E_A, λ_A) consisting of

- a set V_A of vertices called the *positions* of the game,
- a set $E_A \subset V_A \times V_A$ of edges called the *moves* of the game,
- a function $\lambda_A : E_A \to \{-1, +1\}$ indicating whether a move belongs to Opponent (-1) or Proponent $(+1)$,

We note $\star_A$ the root of the underlying graph. A Conway game is said to be *negative* (resp. *positive*) when all its move starting from the root belongs to Opponent (resp. Proponent).

Path and play. A *play* $m_1 \cdot m_2 \cdot \ldots \cdot m_{k-1} \cdot m_k$ of a Conway game A is a path starting from the root $\star_A$:

$$\star_A \xrightarrow{m_1} x_1 \xrightarrow{m_2} \ldots \xrightarrow{m_{k-1}} x_{k-1} \xrightarrow{m_k} x_k \tag{8}$$

Two paths are parallel when they have the same initial and final positions. A play (8) is *alternating* when:

$$\forall i \in \{1, \ldots, k-1\}, \qquad \lambda_A(m_{i+1}) = -\lambda_A(m_i).$$

We note Play_A the set of plays of a game A.

Strategies. Remark that the definition of a Conway game does not imply that all the plays are alternating. The notion of alternation between Opponent and Proponent only appears at the level of strategies (i.e. programs) and not at the level of games (i.e. types). This corresponds to the intuition that a game describes a fairly liberal space of computation whereas a strategy describes regulated executions.

A *strategy* σ of a Conway game A is defined as a non empty set of *alternating plays* of even length such that

- every non empty play starts with an Opponent move,
- σ is closed by even length prefix: for all plays s and for all moves m, n,

$$s \cdot m \cdot n \in \sigma \quad \text{implies} \quad s \in \sigma;$$

- σ is determinism: for all plays s, and for all moves m, n, n',

$$s \cdot m \cdot n \in \sigma \quad \text{and} \quad s \cdot m \cdot n' \in \sigma \quad \text{implies} \quad n = n'.$$

Note that our notion of strategy is *partial* because a strategy does not necessarily have to answer to an Opponent move. We write $\sigma : A$ to indicate that σ is strategy over the game A.

Dual. Every Conway game A induces a *dual* game A^* obtained simply by reversing the polarity of moves. More formally, $A^* = (V_{A^*}, E_{A^*}, \lambda_{A^*})$ is defined by

- $V_{A^*} = V_A$;
- $E_{A^*} = E_A$;
- $\lambda_{A^*} = -\lambda_A$.

Tensor product. The tensor product $A \otimes B$ of two Conway games A and B is essentially the asynchronous product of the two underlying graphs. More formally, it is defined as:

- its positions are the pairs (x, y) noted $x \otimes y$ with $\star_{A \otimes B} = \star_A \otimes \star_B$, that is

$$V_{A \otimes B} = V_A \times V_B,$$

- its moves are of two kinds :

$$x \otimes y \to \begin{cases} z \otimes y \text{ if } x \to z \text{ in the game } A \\ x \otimes z \text{ if } y \to z \text{ in the game } B, \end{cases}$$

- the polarity of moves in the game $A \otimes B$ is inherited from games A and B.

The Conway game 1 with a unique position $\star$ and no move is the neutral element of the tensor product. As usual in game semantics, every play s of the game $A \otimes B$ can be seen as the interleaving of a play $s_{|A}$ of the game A and a play $s_{|B}$ of the game B.

Composition. We now define the composition of two strategies using the concept of "parallel and hiding" which works for game semantics as well as for more abstract settings such as traced monoidal category or compact closed category.

We proceed as in [30, 15], and say that u is an interaction on three games A, B, C, this noted $u \in int_{ABC}$, when the projection of u on each game $A^* \otimes B$, $B^* \otimes C$ and $A^* \otimes C$ is a play. Given two strategies $\sigma : A^* \otimes B, \tau : B^* \otimes C$, we define the composition of these strategies as follows:

$$\sigma; \tau = \{u_{|A^* \otimes C} \mid u \in int_{ABC}, u_{|A^* \otimes B} \in \sigma, u_{|B^* \otimes C} \in \tau\}$$

The composition of two strategies is a strategy.

Identity morphism. We define the identity morphism id_A on a game A as a variation on the copycat strategy on the game $A^* \otimes A$ described by André Joyal in [23]. Coarsely, for every Opponent move in one of the component A^* or A, the copycat strategy responses with the dual move in the other component. More formally, the identity is defined as

$$id_A \quad \stackrel{\text{def}}{=} \quad \{s \in \text{Play}^{even}_{A_1^* \otimes A_2} \mid \forall t \prec^{even} s \ , \ t_{|A_1} = t_{|A_2}\}$$

where we use the tags 1 and 2 to distinguish between the two occurrences of A and where the exponent *even* restricts the prefix relation on paths to the prefix relation on even paths.

The category Conway of Conway games. The category **Conway** has Conway games as objects, and strategies σ of $A^* \otimes B$ as morphisms $\sigma : A \to B$. The resulting category **Conway** is compact-closed in the sense of [24] with units $\eta_A : 1 \to A \otimes A^*$ and counits $\varepsilon_A : A^* \otimes A \to 1$ defined as copycat strategies. Compact closure means that $\otimes = \parr$ from a logical point of view, or that there exist an isomorphism $(A \otimes B)^* \cong A^* \otimes B^*$ natural in A and B. All we need here is that **Conway** is automatically monoidal closed, with closure defined as

$$A^* \otimes B.$$

Before introducing a notion of resources through a payoff function, we have to restrict ourselves to negative Conway games. Indeed, if we stay in an entirely self dual setting, having a notion of winning strategies that compose is quite tricky. In particular, the payoff function can not be positional and the constraints imposed on strategies have to deal with every subpath of a play rather than every reached positions. Such a notion has been developed in [38] in order to extends the notion of bracketing to a self dual

framework, leading to the notion of multibracketing. Hopefully, it is possible to give a model of tensorial logic by much simpler conditions.

The full subcategory $\mathcal{N}$ of negative Conway games is no longer compact closed but inherits the closure of **Conway**. Given a Conway game A, we note A^- the negative Conway game obtained by removing all Proponent moves starting from the root.

This construction extends to a fully faithful functor from **Conway** to $\mathcal{N}$, which makes $\mathcal{N}$ a coreflexive category of **Conway**. This is enough to export the closure of the category **Conway** to the category $\mathcal{N}$ by

$$A \multimap B \quad \stackrel{\text{def}}{=} \quad (A^* \otimes B)^-$$

as shown by the following proposition.

Proposition 2 (transport of closure) *Let $(\mathcal{C}, \otimes, \twoheadrightarrow)$ be a symmetric monoidal closed category and $(\mathcal{D}, \bullet)$ be a symmetric monoidal category. Suppose that there exists a monoidal adjunction $U \dashv F : \mathcal{D} \to \mathcal{C}$ where U is fully faithful. We can export the closure on $\mathcal{C}$ to a closure on $\mathcal{D}$ which is defined, for every A, B in $\mathcal{D}$, by:*

$$A \multimap B = F(U(A) \twoheadrightarrow U(B))$$

Proof: As the adjunction is monoidal, the functor U is strong monoidal. The closure is deduced from the following cascade of bijections:

$$\begin{array}{lcll} \mathcal{D}(B, A \multimap C) & \cong & \mathcal{D}(B, F(U(A) \twoheadrightarrow U(C))) & \\ & \cong & \mathcal{C}(U(B), U(A) \twoheadrightarrow U(C)) & \text{adjunction } U \dashv F \\ & \cong & \mathcal{C}(U(A) \otimes U(B), U(C)) & \text{closure in } \mathcal{C} \\ & \cong & \mathcal{C}(U(A \otimes B), U(C)) & U \text{ strong monoidal} \\ & \cong & \mathcal{D}(A \bullet B, C) & U \text{ fully faithful} \end{array}$$

□

Proposition 3 *The category $\mathcal{N}$ is symmetric monoidal closed.*

This category will be the new base on which we will define our notion of gain.

Payoff Connway games. A *payoff Conway game* is a negative Conway game $A = (V_A, E_A, \lambda_A)$ equipped with a payoff function (defined on positions)

$$\kappa_A : V_A \to \{-1, 0, +1\}.$$

A position is said to be *winning* if $\kappa_A(x) \in \{0, +1\}$. Intuitively, the value -1 denotes a winning position for Opponent, the value $+1$ denotes a winning position for Proponent, and the value 0 denotes a "neutral" position.

$\otimes$	-1	0	$+1$
-1	-1	-1	-1
0	-1	0	$+1$
$+1$	-1	$+1$	$+1$

$\multimap$	-1	0	$+1$
-1	$+1$	$+1$	$+1$
0	-1	0	$+1$
$+1$	-1	-1	$+1$

Table 1. "Truth tables" of the tensor product and the closure.

Tensor product and closure of payoff games. We now extend the payoff function to the tensor product and the closure operator. As the payoff is positional, it is sufficient to provide a "truth table" (cf. Table 1). Those "Truth tables" are guided by the intuition that $\otimes$ corresponds to the boolean conjunction $\wedge$, $\multimap$ corresponds to the boolean implication $\Rightarrow$, -1 means false, $+1$ means true and 0 never changes the polarity.

Thus, the payoff Conway game $A \otimes B$ is defined as the underlying Conway game $A \otimes B$, equipped with the payoff function

$$\kappa_{A\otimes B}(x \otimes y) = \kappa_A(x) \otimes \kappa_B(y);$$

and the payoff Conway game $A \multimap B$ is defined as the underlying Conway game $A \multimap B$, equipped with the payoff function

$$\kappa_{A\multimap B}(x \multimap y) = \kappa_A(x) \multimap \kappa_B(y).$$

We set to 0 the unique position of the game 1.

Winning strategies. With the general notion of strategy, every negative game has a unique morphism to 1. We will use the payoff function to define *winning strategies*, which will help us to distinguish between *affine games* (whose polarity of the root is 0) and *linear games* (whose polarity of the root is $+1$).

A strategy σ on a payoff Conway game A is *winning* when all the plays in the strategy lead to winning positions, that is position of payoff 0 or $+1$:

$$\text{for all } s : x \twoheadrightarrow y, \quad s \in \sigma \quad \text{implies} \quad \kappa_A(s) \in \{0, +1\}$$

We will define a category of payoff Conway games whose morphisms from a game A to a game B are winning strategies on $A \multimap B$. Remark that our definition of payoff on $A \multimap B$ implies that there is no wining strategies from A to B when $\kappa_A(\star_A) \multimap \kappa_B(\star_B) = -1$. This is the case when the game A is linear and the game B is affine. To define a category, we need to show that winning strategies do compose.

Proposition 4 *When $\sigma : A \multimap B$ and $\tau : B \multimap C$ are winning strategies, $\tau \circ \sigma : A \multimap C$ is also a winning strategy.*

Proof: We already know that strategies do compose, it just remains to check the winning condition. As it is defined positionally, it suffices to observe by a case analysis that the composite of two winning positions on $\multimap$ is winning, in the sense that

$$\frac{\begin{array}{ll}\kappa_A(x) \multimap \kappa_B(y) \in \{0, +1\} & (x \multimap y : \text{ winning}) \\ \kappa_B(y) \multimap \kappa_C(z) \in \{0, +1\} & (y \multimap z : \text{ winning})\end{array}}{\kappa_A(x) \multimap \kappa_C(z) \in \{0, +1\} \qquad (x \multimap z : \text{ winning})}$$

This works because the definition of the payoff function on $\multimap$ comes from the boolean implication $\Rightarrow$, which is itself stable by composition. $\square$

Proposition 5 (category of payoff Conway games) *The category – whose objects are payoff Conway games and whose morphisms from A to B are winning strategies on $A \multimap B$ – is symmetric monoidal closed.*

Proof: We already know that $\mathcal{N}$ is a symmetric monoidal closed category. It remains to check that

$$(\kappa_A(x) \otimes \kappa_B(y)) \multimap \kappa_C(z) = \kappa_A(x) \multimap (\kappa_B(y) \multimap \kappa_C(z))$$

for all positions $x \in V_A$, $y \in V_B$ and $z \in V_C$. This equation can be deduced from the validity of the boolean formula

$$(A \wedge B) \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C).$$

$\square$

Technically, in order to define an exponential modality on our category, we need to restrict ourselves to games whose root is winning. From now on, we will work with the full subcategory $\mathcal{P}$ whose objects are restricted to payoff Conway games with a winning root. For short, we will denote such games as *payoff games*. Remark that $\mathcal{P}$ is still symmetric monoidal but no longer closed as $\multimap$ does not preserve winning root. It is nevertheless possible to define a tensorial negation on $\mathcal{P}$.

Tensorial negation. Let $\bot$ the linear game as follows

- two position $\star_\bot$ et x,
- a unique Opponent move $m_\bot : \star_\bot \twoheadrightarrow x$,
- $\kappa_\bot(\star_\bot) = +1$ and $\kappa_\bot(x) = 0$.

As presented in Section 2, the closure operator gives rise to a tensorial negation in the following way: the tensorial negation of a payoff game A is defined as

$$\neg A \stackrel{\text{def}}{=} A \multimap \bot.$$

Proposition 6 *The category $\mathcal{P}$ is a dialogue category.*

5 A game model with resources

We now give an explicit description of the three resource modalities of tensorial logic in our game model. We prove the additional properties that the affine and exponential modalities are free, but not the relevant modality.

Affine modality. We have already mentioned that a payoff games A is said to be *affine* when its root is of payoff 0. The unique strategy from A to 1, noted t_A, is then winning. We note $\mathcal{P}_w$ the subcategory of affine payoff games.

The affine game $!_w A$ associated to a payoff game A is obtained by posing that the root of A is of payoff 0. This modality extends to a strategy $\sigma : A \multimap B$ by posing $!_w\sigma = \sigma$. This definition is valid because making the roots of A and B neutral only grows up the number of winning positions on $A \multimap B$.

Lemma 3 *Let A be an affine payoff game and B be a payoff game. Every position $x \multimap y$ of $A \multimap B$ (or indifferently of $A \multimap !_w B$) satisfies*

$$\kappa_A(x) \multimap \kappa_B(y) \in \{0, +1\} \quad \textit{iff} \quad \kappa_A(x) \multimap \kappa_{!B}(y) \in \{0, +1\}$$

Proof: If y is not the root of B, y has the same polarity in both games B and $!_w B$. Thus, it suffices to consider the case where $y = \star_B$. In that case, $x = \star_A$ because Opponent must play its first move in B (both game are negative). As A is affine, we deduce that

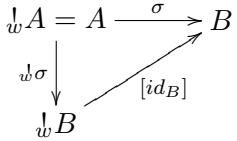
$$\begin{cases} \kappa_A(x) \multimap \kappa_B(y) = \kappa_B(\star_B) \in \{0, +1\} & \text{car la racine est gagnante} \\ \kappa_A(x) \multimap \kappa_{!B}(y) = \kappa_{!B}(\star_B) = 0 & \text{car } !_w B \text{ est affine.} \end{cases}$$

□

Proposition 7 *The functor $!_w : \mathcal{P} \to \mathcal{P}_w$, which associates $!_w A$ to a payoff game A, defines an* affine modalities *on $\mathcal{P}$.*

Proof: The category of affine payoff games $\mathcal{P}_w$ is a symmetric monoidal subcategory of $\mathcal{P}$, having 1 as terminal object. The backward functor from $\mathcal{P}_w$ to $\mathcal{P}$ is simply the inclusion functor. It remains to define a bijection between $\mathcal{P}_w(A, !_w B)$ and $\mathcal{P}(A, B)$ for any affine payoff game A and any payoff game B. But Lemma 3 points out that a strategy on $A^* \otimes B$ is winning if and only if it is winning on $A^* \otimes !_w B$. the bijection is thus given by the identity. □

Remark, by the way, that this modality is free. Indeed, noting $[id_B] : !_w B \to B$ the copycat strategy which has the same plays as the identity on B, we know that for every strategy $\sigma : A \to B$, where A is affine, the following diagram commutes uniquely for the strategy $!_w\sigma$:



Relevant modality. Every payoff game A induces a relevant game $!_c A$ – which can be seen as the infinite tensor of A – as follows:

- its positions are words $w = x_1 \cdots x_k$ whose letters are positions x_i of the game A distinct from the root; the intuition is that each letter x_i describes the current position in the i^{th} copy of A,

- its root $\star_{!A}$ is the empty word,
- its moves $m : w \to w'$ are either moves played in one copy :

$$w_1 \cdot x \cdot w_2 \xrightarrow{m} w_1 \cdot y \cdot w_2$$

where $\underline{m} : x \to y$ is a move of the game A; or moves where Opponent opens a new copy:

$$w \xrightarrow{m} w \cdot x$$

where $\underline{m} : \star_A \to x$ is an initial move of A. In particular, if m is a move of $!A$, we note $\underline{m}$ the underlying move in A,

- its payoff function on a position $w = x_1 \cdots x_k$ is obtained by

$$\kappa_{!A}(w) = \bigotimes_{1 \leq i \leq n} \kappa_A(x_i)$$

The polarity of a move in $!A$ is directly inherited from the polarity of its underlying move in A. We will now describes the diagonal strategy

$$\delta_A \quad : \quad !A \longrightarrow !A \otimes !A$$

which makes $!$ a relevant modality on $\mathcal{P}$. Coarsely, δ_A is a copycat strategy plus a management of indices. When Opponent opens a new copy in $!A \otimes !A$, δ_A responses by opening a new copy in $!A$. This copy is then linked to the former until the end of the interaction. Consequently, when Opponent plays a move in an already opened copy, δ_A responses by the same move in the related copy. More formally, we define by induction the interleaving of a play s of $!A \otimes !A$ as the play $\langle s \rangle$ of $!A$ that mimics the moves of s. The empty play in $!A \otimes !A$ is unsurprisingly translated into the empty play in $!A$ and a play $s \cdot m$ is translated into

$$\langle s \cdot m \rangle \quad \stackrel{\text{def}}{=} \quad \langle s \rangle \cdot m'$$

where m' has the same underlying move

$$\underline{m'} \quad = \quad \underline{m}$$

played in the corresponding copy if $\underline{m}$ is not initial in A, or in a new copy if $\underline{m}$ is initial in A. Using this interleaving function, the strategy δ_A can be expressed as

$$\delta_A \quad \stackrel{\text{def}}{=} \quad \{s \in \text{Play}^{even}_{!A_1 \multimap (!A_2 \otimes !A_3)} \mid \forall t \prec^{even} s \ , \ t_{|!A_1} = \langle t_{|!A_2 \otimes !A_3} \rangle\}$$

Again, the tag 1, 2 and 3 are here to distinguish between the different copies of the game A. This strategy is winning because every position it plays has the same polarity on the left and on the right of $\multimap$. It is not difficult to see that δ_A satisfies the diagrams (6) and (7) and thus defines a diagonal on $!A$. This modality extends to winning strategy $\sigma : A \multimap B$ by posing

$$! \, \sigma \stackrel{\text{def}}{=} \{(x_1 \cdots x_k) \multimap (y_1 \cdots y_k) \mid \forall 1 \leq i \leq k, x_i \multimap y_i \in \sigma\}$$

which is a morphism of $\mathcal{P}$ as pointed out by the following lemma.

Lemma 4 *If σ is a winning strategy, then the strategy $! \, \sigma$ is also winning.*

Proof: It suffices to observe that for all positions $x \in V_A$, $y \in V_B$, $z \in V_C$ and $t \in V_D$ such that

$$\kappa_{(A \multimap B) \otimes (C \multimap D)}((x \multimap y) \otimes (z \multimap t)) \in \{0, +1\},$$

we have

$$\kappa_{(A \otimes C) \multimap (B \otimes D)}((x \otimes z) \multimap (y \otimes t)) \in \{0, +1\}.$$

□

We now claim that $!_c$ defines a relevant modality. We note $\mathcal{P}_c$ the subcategory of $\mathcal{P}$ of objects equipped with a diagonal and winning strategies preserving this diagonal.

Proposition 8 *The functor $!_c : \mathcal{P} \to \mathcal{P}_c$, which associates the relevant game $!_c A$ to a game A, defines a* relevant modality *on $\mathcal{P}$.*

Proof: To define an adjunction between $\mathcal{P}_c$ and $\mathcal{P}$, it is sufficient to show that $!_c$ defines a comonad on $\mathcal{P}$ because its adjoint is the inclusion functor. The counit e_A and comultiplication d_A of the comonad

$$e_A : !_c A \longrightarrow A \qquad\qquad d_A : !_c A \longrightarrow !_c !_c A$$

are defined as for the diagonal – by first defining two interleaving functions from plays of $!_c A$ to plays of A and $!_c !_c A$ and then defining a copycat-like strategy. The empty play is send to the empty play. Given a play $s \cdot m$ of $!_c A$, we define

$$\langle s \cdot m \rangle_e = \langle s \rangle_e \cdot \underline{m} \qquad\qquad \langle s \cdot m \rangle_d = \langle s \rangle_d \cdot m'$$

where m' has the same underlying move $\underline{m}$ played in the corresponding copy if $\underline{m}$ is not initial in A, or in a new copy if $\underline{m}$ is initial in A. We then define

$$\begin{array}{lcll} e_A & \stackrel{\text{def}}{=} & \{s \in \mathrm{Play}^{even}_{!_c A_1 \multimap A_2} & | \quad \forall t \prec^{even} s \ , \ t_{|!_c A_1} = \langle t_{|A_2} \rangle_e\} \\ d_A & \stackrel{\text{def}}{=} & \{s \in \mathrm{Play}^{even}_{!_c A_1 \multimap !_c !_c A_2} & | \quad \forall t \prec^{even} s \ , \ t_{|!_c A_1} = \langle t_{|!_c !_c A_2} \rangle_d\} \end{array}$$

It is easy to check that the coherence diagrams for a comonad are satisfied. □

We can now wonder whether this comonad defines the free duplicable object on the category $\mathcal{P}$ in the sense that any arrow $D \xrightarrow{f} A$ from a duplicable game D to a game A can be uniquely factorized as $D \xrightarrow{f^\dagger} !_c A \xrightarrow{e_A} A$, where $f^\dagger$ preserves the diagonal from D to $!_c A$. This is not possible in general as we can see by considering the game

$$\mathtt{Unit} \stackrel{\text{def}}{=} !_w(\bot \multimap \bot).$$

This game is affine and we can define a strategy $com : 1 \to \mathtt{Unit}$ that answer to the unique Opponent move by the unique Proponent move. Then,

$$\mathtt{Unit} \xrightarrow{t_{\mathtt{Unit}}} 1 \cong 1 \otimes 1 \xrightarrow{com \otimes com} \mathtt{Unit} \otimes \mathtt{Unit}$$

makes the game $\mathtt{Unit}$ a duplicable object. Now, any strategy $\sigma : \mathtt{Unit} \to !_c A$ that preserves the duplication cannot play in $\mathtt{Unit}$ as expressed by the diagram

$$\begin{array}{ccccc} \mathtt{Unit} & \xrightarrow{t_A} & 1 \cong 1 \otimes 1 & \xrightarrow{com \otimes com} & \mathtt{Unit} \otimes \mathtt{Unit} \\ {\scriptstyle \sigma}\downarrow & & & & \downarrow{\scriptstyle \sigma \otimes \sigma} \\ !_c A & & \xrightarrow{\qquad\qquad \delta_A \qquad\qquad} & & !_c A \otimes !_c A \end{array}$$

It follows that it is only possible to factor arrows of the form

$$\mathtt{Unit} \xrightarrow{t_{\mathtt{Unit}}} 1 \xrightarrow{f} A$$

Thus, the relevant modality is not free in $\mathcal{P}$. But this is only due to the lack of coherence requirements between the duplication and the weakening. This enables any affine game to act as a duplicable game without implementing a *"real"* duplication. Nevertheless, this kind of fake duplication will be rejected by considering commutative comonoids instead of duplicable objects as we will see in the next paragraph.

Exponential modality. The comonoidal game $!_e A$ associated to a payoff game A is obtained by applying successively both affine and relevant modalities

$$!_e A \stackrel{\text{def}}{=} !_c !_w A \; = \; !_w !_c A.$$

The order of application does not matter as the two comonads commute. We easily check that the game $!_e A$ lives in the cartesian category $\mathcal{P}_e$ of commutative comonoids on $\mathcal{P}$.

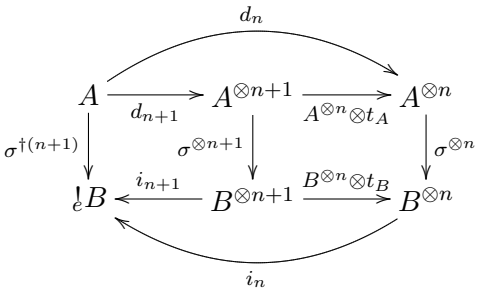
Proposition 9 *The functor $!_e : \mathcal{P} \to \mathcal{P}_e$ which associates the comonoidal game $!_e A$ to each game A defines an* exponential modality *on $\mathcal{P}$.*

Proof: The commutation of $!_w$ and $!_c$ induces a distributive law between those two comonads, equipping automatically $!_e$ of a comonadic structure on $\mathcal{P}$. □

We will now show that this comonad defines the free commutative comonoid. We suppose – without lost of generality because the affine modality is free – that B is an affine game. Let us introduce the strategy $i_n : B^{\otimes n} \to !_e B$ that plays a copycat on the first n copies of $!_e B$ in $B^{\otimes n}$ and does not answer to the opening of the $n+1^{\text{th}}$ copy. Given a strategy $\sigma : A \to B$ from a commutative comonoid A – with counit noted t_A and comultiplication noted d – to a game B, we define $\sigma^{\dagger(n)}$ is defined by the following commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{d_n} & A^{\otimes n} \\ {\scriptstyle \sigma^{\dagger(n)}}\downarrow & & \downarrow{\scriptstyle \sigma^{\otimes n}} \\ !_e B & \xleftarrow{i_n} & B^{\otimes n} \end{array}$$

where $d_0 = e$, d_1 is the identity on A and d_n denotes the $n-1$ applications of the comultiplication d. The following diagram shows that $\sigma^{\dagger(n)} \subset \sigma^{\dagger(n+1)}$



In the diagram above, all the faces commute except the bottom one which is just an inclusion and the extremal clockwise path is equal to $\sigma^{\dagger(n)}$. Thus, we can define the comonoidal lifting strategy $\sigma^\dagger$ by

$$\sigma^\dagger \quad \stackrel{\text{def}}{=} \quad \bigcup_n \sigma^{\dagger(n)}$$

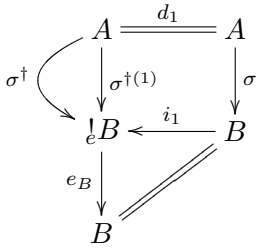
We will now show that $\sigma^\dagger$ is comonoidal. We have the inclusion chain

$$\delta_B \circ \sigma^{\dagger(n)} \;\subseteq\; (\sigma^{\dagger(n)} \otimes \sigma^{\dagger(n)}) \circ d \;\subseteq\; \delta_B \circ \sigma^{\dagger(2n)}.$$

Taking the limit of this inclusion chain gives us:

$$\delta_B \circ \sigma^\dagger \;\subseteq\; (\sigma^\dagger \otimes \sigma^\dagger) \circ d \;\subseteq\; \delta_B \circ \sigma^\dagger,$$

which is the required equality. Remark that $!_e B \xrightarrow{e_B^{\otimes n} \circ d_B^n} B^{\otimes n}$ coequalizes $\sigma^\dagger$ and $\sigma^{\dagger(n)}$, where – as for d_n – the notation d_B^0 stands for e_B, d_B^1 is the identity on $!_e B$ and d_B^n denotes the $n-1$ applications of the comultiplication d_B . Using this remark for $n = 1$ gives the commutative diagrams

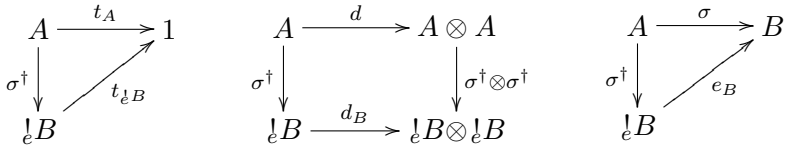


which indicates that

$$e_B \circ \sigma^\dagger = \sigma.$$

It remains to show the uniqueness of this lifting.

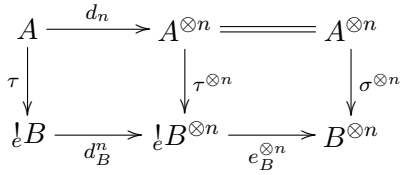
Proposition 10 *Given a strategy $\sigma : A \to B$ from a commutative comonoid A to a payoff game B, $\sigma^\dagger : A \to !_e B$ is the unique strategy that satisfies :*



Proof: We have already shown that $\sigma^\dagger$ was a good candidate. It remains to show that it is the only one. Take another candidate τ. We define $\tau^{(n)}$ as the sub-strategy of τ that plays only on the first n copies of $!_e B$, that is

$$\tau^{(n)} = i_n \circ (e_B^{\otimes n} \circ d_B^n) \circ \tau \qquad \text{and} \qquad \tau \quad = \quad \bigcup_n \tau^{(n)}.$$

Consider now the commutative diagrams



Postcomposing this diagrams with i_n directly gives the equality

$$\tau^{(n)} \quad = \quad \sigma^{\dagger(n)}$$

which means that $\tau = \sigma^\dagger$. $\square$

The only things that are missing to have a model of tensorial logic are additives. Unfortunately, the category $\mathcal{P}$ does not have coproducts. We will use the *family construction*, described by Abramsky et McCusker in [4], to add them freely to our category.

Finite products. For our family construction to fit well with the tensorial negation, we need finite products. Given a family $(A_i)_{i\in I}$ of payoff games indexed on a finite set I, the product $\&_{i\in I} A_i$ is defined as

- its underlying graph is obtained by taking the disjoint union of the graphs underlying each A_i, and by merging their root,
- the polarity of moves is directly inherited from the polarity of the moves in A_i;
- the payoff function is inherited from the payoff function of each A_i, except for the root which has payoff $+1$ if all the roots of the A_i are of payoff $+1$, and has payoff 0 else. In particular, the game $\&_{i\in I} A_i$ is affine as soon as one of the game is affine.

The i^{th} projection is given by the obvious copycat strategy on the game A_i.

Free finite coproducts. Let us now construct the free completion under finite coproducts – noted $Fam(\mathcal{P})$ – of the category $\mathcal{P}$.

Given a category $\mathcal{C}$, its free completion under finite coproducts $Fam(\mathcal{C})$ is defined as follows:

- objects: families $\{A_i | i \in I\}$ of objects of $\mathcal{C}$, where I is a finite set,
- morphisms: a morphism from $\{A_i \mid i \in I\}$ to $\{B_j \mid j \in J\}$ consist in a reindexing function $f : I \to J$ together with a family of morphisms $\{f_i : A_i \to B_{f(i)} \mid i \in I\}$ of $\mathcal{C}$.

Finite coproducts in $Fam(\mathcal{C})$ are simply given by disjoint unions of families. This construction extends to a 2-monad. Hyland and Power give this 2-monad as an example of a symmetric pseudocommutative 2-monad in [20]. From this, they deduce that it distributes with the 2-monad for symmetric monoidal categories. Consequently, (a) the category $Fam(\mathcal{C})$ inherits the symmetric monoidal structure of $\mathcal{C}$, (b) finite coproducts in $Fam(\mathcal{C})$ distributes with the tensor product, and (c) Fam preserves monoidal adjunction.

Abramsky and McCusker [4] have shown that this family construction preserves also finite products and the existence of a terminal objects. Thus, we deduce that this construction preserves affine, relevant and exponential modalities; and that the finite products of $\mathcal{C}$ lifts to $Fam(\mathcal{C})$. More concretely, the tensor product and product of $A = \{A_i | i \in I\}$ and $B = \{B_j | j \in J\}$ are defined as

$$\begin{array}{lcl} A \otimes B & \stackrel{\text{def}}{=} & \{A_i \otimes B_j \mid (i,j) \in I \times J\} \\ A \,\&\, B & \stackrel{\text{def}}{=} & \{A_i \,\&\, B_j \mid (i,j) \in I \times J\} \end{array}$$

When C has a tensorial negation, it can also be exported to $Fam(C)$ by associated to any $A = \{A_i | i \in I\}$ the singleton family

$$\neg A = \{\&_i(\neg A_i)\}.$$

Proof: A morphism $A \otimes B \to \neg C$ in $Fam(C)$ is given by morphisms $f_{i,j} : A_i \otimes B_j \to \&_k(\neg A_k)$ (the reindexing function is trivial here). Those morphisms can be projected by $\pi_k \circ f_{i,j} : A_i \otimes B_j \to \neg C_k$. We can now use the tensorial negation of $\mathcal{C}$ to get morphisms $g_{i,j,k} : A_i \to \neg(B_j \otimes C_k)$ on which we can form the product

$$g_i \stackrel{\text{def}}{=} \&_{j,k}\, g_{i,j,k} \quad : \quad A_i \to \&_{j,k}\neg(B_j \otimes C_k).$$

We obtain in this way a morphism $g : A \to \neg(B \otimes C)$ on the category $Fam(\mathcal{C})$. It is easy to check that the transformation defines above induces a one-to-one correspondence between $Fam(C)(A \otimes B, \neg C)$ and $Fam(C)(A, \neg(B \otimes C))$. □

Gathering all those remarks, we obtain that:

Proposition 11 *$Fam(\mathcal{P})$ is a model of tensorial logic.*

6 Conclusion

In this paper, we integrate resource modalities in game semantics, a task which has always appeared extremely difficult to accomplish in the past. The task requires indeed to put many ideas together, and to reunderstand the topography of the field. In particular, linear logic is refined along the way into tensorial logic, where the involutive negation of linear logic is replaced by a tensorial negation. It is then possible to keep the best of linear logic: resource modalities, etc. but to work on games and continuations instead. And linear logic coincides then with tensorial logic with the additional axiom that the continuation monad is commutative. In that sense, tensorial logic is more primitive than linear logic, in the same way that groups are more primitive than abelian groups.

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