A digital linking number for discrete curves

Sébastien Fourey, Rémy Malgouyres

A digital linking number for discrete curves

Sébastien Fourey Rémy Malgouyres*

GREYC, ISMRA, 6 bd Maréchal Juin 14000 Caen - France

Fourey@greyc.ismra.fr, Malgouyres@llaic.u-clermont1.fr

Abstract

A topological invariant, analogous to the linking number as defined in knot theory, is defined for pairs of digital closed paths of $\mathbb{Z}^3$. This kind of invariant is very useful for proofs which involve homotopy classes of digital paths. Indeed, it can be used for example in order to state the connection between the tunnels in an object and the ones in its complement. Even if its definition is not as immediate as in the continuous case it has the good property that it is immediately computable from the coordinates of the voxels of the paths with no need of a regular projection. The aim of this paper is to state and prove that the linking number has the same property as its continuous analogue: it is invariant under any homotopic deformation of one of the two paths in the complement of the other.

Keywords: Linking number, link, digital homotopy, fundamental group, binary image, topology preservation.

1 Introduction

The digital fundamental group, as introduced by Kong in [6], involves equivalence classes of paths according to a relation of deformation for digital closed paths. It is an important tool in the field of digital topology and in particular, it is used as a criterion of topology preservation for 3D digital objects (see [7],[2] and [5]). Now, the question remains about the existence of an efficiently computable characterization of the *homotopy* between subsets of $\mathbb{Z}^3$. Here, homotopy links two subsets of $\mathbb{Z}^3$ when one can be ontained from the other by sequential deletion or addition of simple points. Such a difficult question cannot be solved today because of the lack of theoretical tools, notably for studying the topology of three dimensional discrete objects. In particular, we should provide new tools dedicated to the study of homotopy classes of discrete paths.

Several authors have been studying homotopy classes of paths in 2D. Rosenfeld and Nakamura in [10] have, among other things, established the relation between 2D holes and the fact that two curves can or cannot be deformed one into each other. In [8],

*This work has been achieved at the GREYC laboratory, but both authors have now moved. Their actual addresses are : Sébastien Fourey, Center for Computer Sciences and Applied Mathematics, Temple University, 340 Carnell Hall, 1803 N. Broad Street, Philadelphia, PA 19122-6095 - United States ; and, Rémy Malgouyres, LLAIC, IUT Dépt Informatique, B.P. 86, 63172 Aubière - France

Malgouyres gives an algorithm to decide whether two closed paths in 2D are homotopic or not. In [4] and [3] the authors have introduced a new tool which helps in distinguishing homotopy classes of paths drawn on the surface of a 3D object (paths of *surfels*). One purpose of this paper is to provide sufficient conditions under which a discrete closed path in an object $X \subset \mathbb{Z}^3$ cannot be deformed in X into another one.

More precisely, we introduce an analogue to the *linking number* of closed curves defined in classical topology and knot theory (see [9]). Intuitively, the linking number counts the number of times a given closed path is interlaced with another one. The digital linking number has the same properties as its continuous analogue. A very intuitive one is that it is left unchanged when one of the considered paths is *continuously* deformed in the complement of the other. Furthermore, as a step of the proof, we also prove that the linking number well behaves with respect to concatenation of paths, i.e., the linking number between the concatenation of two closed path and a third one is nothing but the sum of their linking numbers with the third one. Because of its invariance property, the linking number can be practically and formally used to distinguish two homotopy classes of paths as soon as one can find a path which does not have the same linking number with two elements, one in each of the considered classes.

Since the digital linking number is expected to be invariant under homotopic deformation of the paths in the complement of each other, it will be defined for paths following the classical duality for adjacencies. Clearly, two closed and linked $26-$paths can be unlinked by an homotopic deformation of one in the complement of the other whereas this cannot occur between a $26-$path and a $6-$path.

Note that in this paper, we chose not to consider the continuous analogues of the dicrete paths in order to prove the main properties of the linking number. On the other hand, the proof given here for the main theorems is self sufficient and only uses the basic notions classically defined in the field of digital topology.

Furthermore, this linking number leads to an intuitive proof of the fact that the number of tunnels in an object $X \subset \mathbb{Z}^3$ is strictly related to the number of tunnels in its complement (this is the subject of [5]). Indeed, the number of tunnels mentionned here must be understood as the number given by the computation of the Euler characteristic. In this case, the equality between the two numbers is immediate. However, the localization of the tunnels is not provided by the Euler characteristic and for this reason, the use of the digital fundamental group is sometimes preferable. But a connection between tunnels of an object and tunnels of its complement is then difficult to obtain and the linking number will help in this case. Indeed, as an example, it has already allowed the authors to state a new characterization of 3D simple points in which no consideration about tunnels in the complement of an object is used. Indeed, the preservation of tunnels in the object, together with connectivity considerations, is then shown to be sufficient [5].

2 Definitions and preliminaries

2.1 Basic notions

In this paper, we consider objects as subsets of the 3 dimensional space $\mathbb{Z}^3$. Elements of $\mathbb{Z}^3$ are called *voxels* (short for "volume elements"). The set of voxels which do not belong to an object $O \subset \mathbb{Z}^3$ constitute the complement of the object and is denoted by $\overline{O}$. Any voxel can be seen as a unit cube centered at a point with integer coordinates $v = (i, j, k) \in \mathbb{Z}^3$. Now, we can define some binary symmetric anti-reflexive relations between voxels. Two voxels are said $6-$*adjacent* if they share a face, $18-$*adjacent* if they share an edge and $26-$*adjacent* if they share a vertex. By transitive closure of these adjacency relations, we can define another one: connectivity between voxels. We first define an *n-path* π with a length l from a voxel a to a voxel b in $O \subset \mathbb{Z}^3$ as a sequence of voxels $(y_i)_{i=0\dots l}$ such that for $0 \le i < l$ the voxel y_i is n-adjacent or equal to y_{i+1}, with $y_0 = a$ and $y_l = b$. The path π is a *closed path* if $y_0 = y_l$ and is called a *simple path* if $y_i \neq y_j$ when $i \neq j$ (except for y_0 and y_l if the path is closed). The voxels y_0 and y_l are called the *extremities* of π even in the case when the path is closed and we denote by π^* the set of voxels of π.

Given a path $\pi = (y_k)_{k=0,\dots,l}$, we denote by π^{-1} the sequence $(y'_k)_{k=0,\dots,l}$ such that $y_k = y'_{l-k}$ for $k \in \{0, \dots, l\}$.

Now we can define connectivity; two voxels a and b are called *n-connected* in an object O if there exists an n-path π from a to b in O. This is an equivalence relation between voxels of O, and the $n-$*connected components* of an object O are equivalence classes of voxels according to this relation. Using this equivalence relation on the complement of an object we can define a *background component* of O as an $\overline{n}-$connected component of $\overline{O}$.

In order to avoid topological paradoxes, we always study the topology of an object using an $n-$adjacency for the object and a complementary adjacency $\overline{n}$ for its complement. We sum up this by the use of a pair $(n, \overline{n}) \in \{(6, 26), (6+, 18), (18, 6+), (26, 6)\}$. Remark that the notation 6+ is used in order to distinguish the $6-$connectivity associated with the $26-$connectivity from the $(6+)-$connectivity associated with the $18-$connectivity.

If $\pi = (y_i)_{i=0,\dots,p}$ and $\pi' = (y'_k)_{k=0,\dots,p'}$ are two $n-$paths such that $y_p = y'_0$ then we denote by $\pi.\pi'$ the path $(y_0, \dots, y_{p-1}, y'_0, \dots, y'_{p'})$ which is the concatenation of the two paths π and π'.

Remark 2.1 *To simplify notations, we will omit to specify that subscripts in a closed $n-$path $c = (x_i)_{i=0,\dots,q}$ must be understood modulo q. For the same reason we will use in this paper the notation of real intervals to denote ranges of integers. Thus, the set of integers greater or equal to n and lower or equal to m will be denoted by $[n, m]$. We also use the notations $]n, m[$, $]n, m]$ and $[n, m[$ to exclude one or both extremities from these intervals.*

2.2 Homotopic deformation of paths

We have mentioned in the introduction the notion of the *continuous deformation* of digital paths. The precise meaning of this notion is given by the two following definitions (see also [6] and [1]).

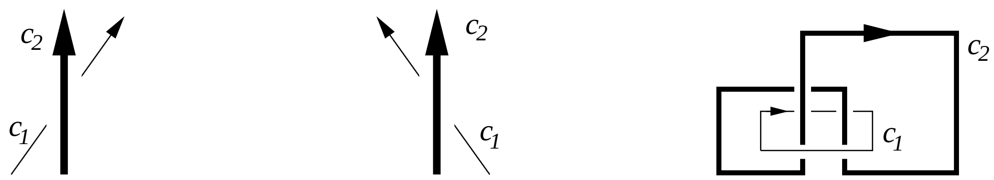
Definition 2.2 (Elementary deformation) *Let c and c' be two $n-$paths in $X \subset \mathbb{Z}^3$. We say that c and c' are the same up to an elementary $n-$deformation in X (denoted by $c \sim_n c'$) if $c = c_1.\gamma.c_2$ and $c' = c_1.\gamma'.c_2$ where γ and γ' have the same extremities and are both included in a 2×2 square for $n = 6$, in a $2 \times 2 \times 2$ cube otherwise.*

In other words, c and c' coincide except inside a 2×2 square if $n = 6$, and a $2 \times 2 \times 2$ cube if $n \in \{6+, 18, 26\}$. Note that the paths γ and γ' in Definition 2.2 are of arbitrary and maybe different lengths.

Definition 2.3 (Homotopy between paths) *Let c and c' be two $n-$paths in $X \subset \mathbb{Z}^3$. We say that c is $n-$homotopic to c' in X (and we denote by $c \simeq_n c'$) if there exists a sequence $S = (c^0, \ldots, c^l)$ with $c^0 = c$ and $c^l = c'$, such that for $0 \leq i < l$, the $n-$paths c^i and c^{i+1} are the same up to an elementary $n-$deformation in X.*

2.3 The linking number

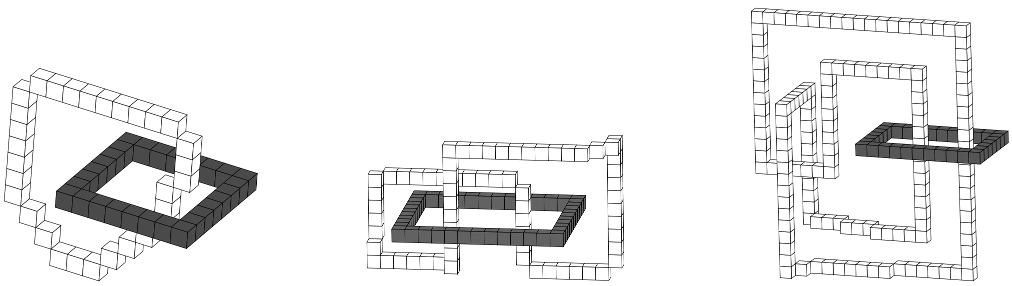
In this section, we define the linking number between two closed paths of voxels which do no intersect one each other. This number is nothing but the linking number of the continuous analogue of the two digital curves as defined in knot theory. This linking number counts the number of times a given closed path is interlaced around another one. Since our further goal is to apply this tool to prove theorems about topology in a digital space, we are interested by the linking number between a closed $n-$path and a closed $\overline{n}-$path where $(n, \overline{n}) \in \{(6+, 18), (6, 26), (18, 6+), (26, 6)\}$ (see the remark in the introduction about two closed $26-$paths). We give three examples of pairs of closed paths and their associated linking numbers in Figure 2. Classically, the linking number is computed by algebraically counting the occurrences of crosses like those depicted in Figure 1 in a 2 dimensional regular projection of the paths (see [9]). In our case, we define the linking number in such a way that it can be immediately obtained by integer only computations using the coordinates of the voxels constituting the paths.



(a) Count -1 for each occurrence of such a cross in the this projection. (b) Count +1 for each cross of this type. (c) The linking number associated with this projection is -2.

Figure 1: The way to compute the classical linking number from a regular projection of two closed curves c_1 and c_2.

For example, the linking number can be computed immediately for paths as depicted by Figure 3. We give the basic idea of the computation in this case. First, we choose to compute the linking number using a "pseudo-2D" projection of the two paths on the plane which contains the first two coordinates axes. Then we observe that the only voxel of the grey path which has a common projection with some white voxels (exactly four ones) is the point x_i. Then, we look for voxels of the white path which have a greater third coordinate than x_i and the same projection as x_i (as the voxels a and b in Figure 3). For each such voxel, a contribution depending on the position of the next and previous voxel of the white path which have a distinct projection from x_i is computed. In this example, the two contributions of the voxels a and b will have opposite signs. The sum of these contributions is the linking number between the two paths, zero in this case.



(a) A closed 18−path and a closed 6−path with a linking number of ±1.

(b) A closed 18−path and a closed 6−path with a linking number of ±2.

(c) The Whitehead's link, the linking number of which is 0.

Figure 2: Three kinds of links.

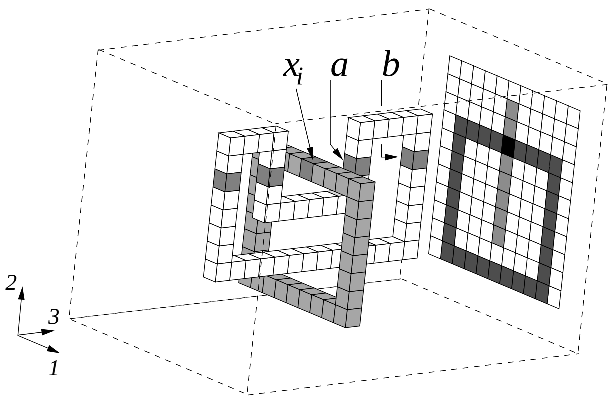


Figure 3: Two 3D curves and their projection, their linking number is 0.

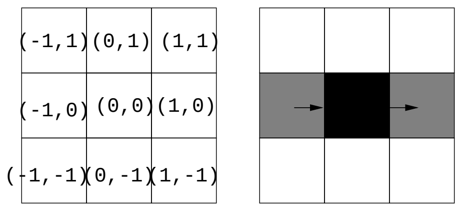


Figure 4: A projective movement.

Notation 2.4 *We will denote by* $\mathcal{P}$ *the following map:*

$$\begin{array}{rccl} \mathcal{P}: & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\ & (x^1,x^2,x^3) & \longmapsto & (x^1,x^2) \end{array}$$

In order to define a single contribution for each sequence of voxels the projection of which is reduced to a point, we first define the predecessor and successor of a voxel in a path.

Definition 2.5 (*Pred* **and** *Succ*) *Let* $c = (x_i)_{i=0,\dots,q}$ *be a closed* $n-$*path and* x_i *be a voxel of* c *for* $i \in [0,q]$. *Then,* $Succ_c(i)$ *is the lowest integer* l *greater than* i *such that* $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$ *; if such an integer* l *does not exist then* $Succ_c(i)$ *is the lowest* $l < i$ *such that* $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$. *If in turn such an* l *does not exist then, clearly* $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ *for all* $l \in [0,q]$ *and we define* $Succ_c(i) = i$.

Similarly, $Pred_c(i)$ *is the preceding subscript* l *of* i *in the cyclic parameterization of* c *such that* $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$, *or* $Pred_c(i) = i$ *if* $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ *for all* $l \in [0,q]$.

Now we can define the *projective movement* associated with a subscript in a path, which depicts the position of the predecessor and the successor of a voxel in the projective plane.

Definition 2.6 (Projective movement) *Let* $c = (x_i)_{i=0,\dots,q}$ *be an* $n-$*path and* $i \in [0,q]$. *Let* V *be the* $8-$*neighborhood of* $(0,0)$ *in the plane:* $(\{-1,0,1\}\times\{-1,0,1\})\backslash\{(0,0)\}$. *We define the* projective movement $P_c(i) \in V \times V$ *associated with the subscript* i *of* c *by:* $P_c(i) = ((x^1_{Pred_\pi(i)} - x^1_i, x^2_{Pred_\pi(i)} - x^2_i), (x^1_{Succ_\pi(i)} - x^1_i, x^2_{Succ_\pi(i)} - x^2_i)) = (P_c(i)^{Pred}, P_c(i)^{Succ})$.

The projective movement represents the position of the previous and the following voxel of x_i *in* c *whose projection does not coincide with the projection of* x_i. *These positions are normalized in a* 3×3 *grid centered at the point* $(0,0)$ *which is associated with the projection of* x_i. *Hence, the projective movement of the voxel* x_i *of Figure 3 is* $((-1,0),(1,0))$ *and can be seen as depicted by Figure 4. Note that this projective movement will be essentially used when* $Pred_c(i) = i-1$.

Then, we must also define the left and right sides in a projective movement in order to define "oriented projective intersections".

Definition 2.7 (Left and right) *Let* $c = (x_i)_{i=0,\dots,q}$ *be an* $n-$*path and* V *be the set introduced in Definition 2.6. One can parameterize the points of* V *using the counter-clockwise order around the point* $(0,0)$. *Then, given a projective movement* $\mathcal{P} = P_c(i)$, *we define the two sets* $Left(\mathcal{P})$ *and* $Right(\mathcal{P})$ *as follows:*

$Right(\mathcal{P})$ *is the set of points met when looking after points of* V *from* $\mathcal{P}^{Pred}$ *to* $\mathcal{P}^{Succ}$ *following the counterclockwise order on* V, *excluding* $\mathcal{P}^{Succ}$ *and* $\mathcal{P}^{Pred}$.

$Left(\mathcal{P})$ *is the set of points met when looking after points of* V *from* $\mathcal{P}^{Succ}$ *to* $\mathcal{P}^{Pred}$ *following the counterclockwise order on* V, *excluding* $\mathcal{P}^{Succ}$ *and* $\mathcal{P}^{Pred}$.

Example: *If* $\mathcal{P} = ((-1,0),(1,-1))$ *then* $Right(\mathcal{P}) = \{(-1,-1),(0,-1)\}$ *and* $Left(\mathcal{P}) = \{(1,0),(1,1),(0,1),(-1,1)\}$.

Notation 2.8 *In the following we say that two paths π and c satisfy the property $\mathcal{H}(\pi, c)$ if π is a closed $n-$path for $n \in \{6, 6+\}$ and c is a closed $\overline{n}-$path such that $c^* \cap \pi^* = \emptyset$.*

In the sequel of the paper we consider $n \in \{6, 6+\}$.

Definition 2.9 (Direct Contribution to the linking number) *Let $\pi = (y_k)_{k=0,\dots,p}$ and $c = (x_i)_{i=0,\dots,q}$ be two closed paths such that $\mathcal{H}(\pi, c)$ holds. We define as follows $W_{\pi,c}(k, i)$, the* direct contribution to the linking number *of a couple (k, i), where $0 \leq k \leq p$ and $0 \leq i \leq q$*

- *If $y_k^3 > x_i^3$ or $\mathcal{P}(y_k) \neq \mathcal{P}(x_i)$ or $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ or $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ then $W_{\pi,c}(k, i) = 0$,*
- *otherwise, let $\mathcal{P}_\pi = P_\pi(k)$ and $\mathcal{P}_c = P_c(i)$ be the projective movements associated with the subscripts i and k (note that in this case $Pred_\pi(k) = k - 1$ and $Pred_c(i) = i - 1$):*
  - *If $\mathcal{P}_\pi^{Pred} = \mathcal{P}_\pi^{Succ}$ then $W_{\pi,c}(k, i) = 0$,*
  - *otherwise $W_{\pi,c}(k, i) = W_{\pi,c}^-(k, i) + W_{\pi,c}^+(k, i)$ where*
    $W_{\pi,c}^-(k, i) = -0.5$ if $\mathcal{P}_c^{Pred} \in Left(\mathcal{P}_\pi)$, $W_{\pi,c}^+(k, i) = -0.5$ if $\mathcal{P}_c^{Succ} \in Right(\mathcal{P}_\pi)$,
    $W_{\pi,c}^-(k, i) = 0.5$ if $\mathcal{P}_c^{Pred} \in Right(\mathcal{P}_\pi)$, $W_{\pi,c}^+(k, i) = 0.5$ if $\mathcal{P}_c^{Succ} \in Left(\mathcal{P}_\pi)$,
    $W_{\pi,c}^-(k, i) = 0$ otherwise. $W_{\pi,c}^+(k, i) = 0$ otherwise.

Definition 2.10 (Linking number) *Let $\pi = (y_k)_{k=0,\dots,p}$ and $c = (x_i)_{i=0,\dots,q}$ be two closed paths such that $\mathcal{H}(\pi, c)$ holds. The linking number of π and c, denoted by $L_{\pi,c}$), is defined as follows:*

$$L_{\pi,c} = \sum_{k=0}^{p-1} \sum_{i=0}^{q-1} W_{\pi,c}(k, i) \tag{1}$$

Notation 2.11 *Given two closed paths $\pi = (y_k)_{k=0,\dots,p}$ and $c = (x_i)_{i=0,\dots,q}$, we denote:*

$$\text{For } i \in [0, q],\ L_{\pi,c}^{\pi}(i) = \sum_{k=0}^{p-1} W_{\pi,c}(k, i) \text{ and for } k \in [0, p],\ L_{\pi,c}^{c}(k) = \sum_{i=0}^{q-1} W_{\pi,c}(k, i).$$

2.4 Main properties

Now, we state the two main results which are proved in this paper about the invariance of the linking number up to an homotopic deformation of any the two paths. These very intuitive results, again very close to the similar results of the continuous case are proved in this paper by using technical but very simple considerations about integer coordinates of points.

Theorem 2.12 *Let π and π' be two closed $n-$paths ($n \in \{6, 6+\}$) and c be a closed $\overline{n}-$path of $\mathbb{Z}^3$ such that $\pi^* \cap c^* = \emptyset$ and $\pi'^* \cap c^* = \emptyset$. If π is $n-$homotopic to π' in $\mathbb{Z}^3 \setminus c^*$ then $L_{\pi,c} = L_{\pi',c}$.*

Theorem 2.13 *Let π be a closed $n-$path ($n \in \{6, 6+\}$), let c and c' be two closed $\overline{n}-$paths of $\mathbb{Z}^3$ such that $\pi^* \cap c^* = \emptyset$ and $\pi^* \cap c'^* = \emptyset$. If c is $\overline{n}-$homotopic to c' in $\mathbb{Z}^3 \setminus \pi^*$ then $L_{\pi,c} = L_{\pi,c'}$.*

As an illustration, one can be convinced that any $18-$homotopic deformation of the $18-$closed white path of Figure 2(b) in the complement of the $6-$closed grey path cannot change the linking number associated with the two paths.

3 Useful properties

In this section, we give the definition of the indirect contribution to the linking number which allows to compute the linking number by looking after voxels of the 18 or 26−path and counting the crossings with the projection of the 6−path. This leads to an equivalent definition of the linking number, which allows to prove in a very similar way two propositions about an additive property of the linking number by concatenation of the paths (Proposition 3.5 and Proposition 3.4 of this section). This additive property will be used in the next section to prove the two main theorems of this paper.

3.1 An equivalent definition of the linking number

Definition 3.1 (Indirect contribution to the linking number) *Let* $\pi = (y_k)_{k=0,\dots,p}$ *and* $c = (x_i)_{i=0,\dots,q}$ *be two closed paths such that* $\mathcal{H}(\pi, c)$ *holds. We define as follows* $M_{c,\pi}(i,k)$*, the* indirect contribution to the linking number *of a pair* (i,k) *where* $0 \le i \le q$ *and* $0 \le k \le p$*.*

- *If* $y_k^3 > x_i^3$ *or* $\mathcal{P}(y_k) \neq \mathcal{P}(x_i)$ *or* $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ *or* $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ *then* $M_{c,\pi}(i,k) = 0$*,*
- *otherwise, let* $\mathcal{P}_\pi = P_\pi(k)$ *and* $\mathcal{P}_c = P_c(i)$ *be the projective movements associated with the subscripts* i *and* k*:*
  - *If* $\mathcal{P}_c^{Pred} = \mathcal{P}_c^{Succ}$ *then* $M_{c,\pi}(i,k) = 0$*,*
  - *otherwise,* $M_{c,\pi}(i,k) = M^-_{c,\pi}(i,k) + M^+_{c,\pi}(i,k)$ *where*
    $M^-_{c,\pi}(i,k) = +0.5$ *if* $\mathcal{P}_\pi^{Pred} \in Left(\mathcal{P}_c)$*,* $M^+_{c,\pi}(i,k) = +0.5$ *if* $\mathcal{P}_\pi^{Succ} \in Right(\mathcal{P}_c)$*,*
    $M^-_{c,\pi}(i,k) = -0.5$ *if* $\mathcal{P}_\pi^{Pred} \in Right(\mathcal{P}_c)$*,* $M^+_{c,\pi}(i,k) = -0.5$ *if* $\mathcal{P}_\pi^{Succ} \in Left(\mathcal{P}_c)$*,*
    $M^-_{c,\pi}(i,k) = 0$ *otherwise.* $M^+_{c,\pi}(i,k) = 0$ *otherwise.*

Lemma 3.2 *Let* $\pi = (y_k)_{k=0,\dots,p}$ *and* $c = (x_i)_{i=0,\dots,q}$ *be two closed paths such that* $\mathcal{H}(\pi, c)$ *holds. Then* $W_{\pi,c}(k,i) = M_{c,\pi}(i,k)$ *for all* $k \in \{0,\dots,p\}$ *and all* $i \in \{0,\dots,q\}$*.*

Proof: The first condition of Definition 2.9 and 3.1 are identical. Now, if $\mathcal{P}_\pi^{Pred} = \mathcal{P}_\pi^{Succ}$ then $W_{\pi,c}(k,i) = 0$ but it is then clear that whatever be the configuration $\mathcal{P}_c$, we have $M_{c,\pi}(i,k) = 0$ since $\mathcal{P}_\pi^{Pred}$ and $\mathcal{P}_\pi^{Succ}$ will either both belong to the same side of $\mathcal{P}_c$ or be equal to $\mathcal{P}_c^{Pred}$ or $\mathcal{P}_c^{Succ}$. Similarly, if $\mathcal{P}_c^{Pred} = \mathcal{P}_c^{Succ}$ then $W_{\pi,c}(k,i) = M_{c,\pi}(i,k) = 0$.

If $\mathcal{P}_\pi^{Pred} \neq \mathcal{P}_\pi^{Succ}$ and $\mathcal{P}_c^{Pred} \neq \mathcal{P}_c^{Succ}$, we should evaluate $W_{c,\pi}(i,k)$ depending on the positions of the points $\mathcal{P}_c^{Pred}$ and $\mathcal{P}_c^{Succ}$, which immediately gives the positions of $\mathcal{P}_\pi^{Pred}$ and $\mathcal{P}_\pi^{Succ}$ relative to $\mathcal{P}_c$. In all case we only have to observe that $M_{c,\pi}(i,k) = W_{\pi,c}(k,i)$. Figure 5 gives an overview of the fourteen configurations of projective movements which can occur between an n−path and an $\overline{n}$−path. The reader can check that the direct and indirect contributions of the intersection point are equal. □

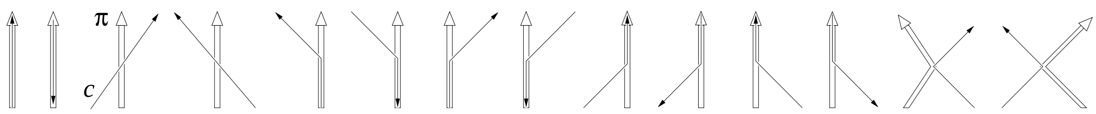


Figure 5: The 14 possible crossing ways in a projective movement.

Remark 3.3 *From Lemma 3.2 we have:* $L_{\pi,c} = \sum_{i=0}^{q-1} \sum_{k=0}^{p-1} M_{c,\pi}(i,k)$*. Furthermore, it is readily seen that the linking number is not dependent to the choice of a parameterization for any of the two paths as soon as its orientation is preserved.*

3.2 The concatenation property

Proposition 3.4 *Let* π_1*,* π_2 *be two closed* $n-$*paths with the same extremity and* c *be a closed* $\overline{n}-$*path such that* $\mathcal{H}(c, \pi_1)$ *and* $\mathcal{H}(c, \pi_2)$ *hold. Then,* $L_{\pi_1.\pi_2,c} = L_{\pi_1,c} + L_{\pi_2,c}$.

The proof of Proposition 3.4 is based on an identification between terms in the expressions of the three linking numbers $L_{\pi_1.\pi_2,c}$, $L_{\pi_1,c}$ and $L_{\pi_2,c}$ as double sums of indirect contributions (see Remark 3.3). In order to increase the readability of this paper, this proof is given in the appendix.

Proposition 3.5 *Let* c_1 *and* c_2 *be two closed paths with the same extremities and* π *be a closed path such that* $\mathcal{H}(c_1, \pi)$ *and* $\mathcal{H}(c_1, \pi)$ *hold. Then,* $L_{\pi,c_1.c_2} = L_{\pi,c_1} + L_{\pi,c_2}$.

Proof: The proof is similar to the proof of Proposition 3.4 (see Appendix) but uses the definition of the linking number as the sum of the direct contributions $W_{\pi,c}(k, i)$ instead of the indirect contributions $M_{c,\pi}(i, k)$. □

4 Proof of the main theorems

4.1 Independence up to a deformation of the $6/(6+)-$path

In this section we will prove Theorem 2.12 in the case when $(n, \overline{n}) \in \{(6+, 18), (6, 26)\}$. The main idea of the proof is that a homotopic deformation of $6-$paths or $(6+)-$paths can be achieved by insertion/deletion of simple closed loops included in a cube or a square (like depicted in Figure 6). Then, proving that such small $n-$paths have a linking number of 0 with any other $\overline{n}-$path will be sufficient to prove the main theorem by using Proposition 3.4.

Definition 4.1 *Let* π *and* π' *be two closed* $n-$*paths (*$n \in \{6, 6+, 18, 26\}$*) in* $X \subset \mathbb{Z}^3$*. We say that* π *and* π' *are the* same up to a simple $n-$loop insertion/deletion *if* $\pi = \pi_1.(p).\pi_2$ *where* p *is a voxel, and* $\pi' = \pi_1.\gamma.\pi_2$ *where* γ *is a simple closed* $n-$*path from* p *to* p *included in a* $2\times2\times2$ *cube (a* 2×2 *square if* $(n, \overline{n}) = (6, 26)$*); or if* $\pi = \pi_1.\gamma.\pi_2$ *and* $\pi' = \pi_1.(p).\pi_2$.

Proposition 4.2 *Let* π *and* π' *be two* $n-$*paths (*$n \in \{6, 6+, 18, 26\}$*) of* $X \subset \mathbb{Z}^3$*. Then the two following properties are equivalent:*

i) π *is* $n-$*homotopic to* π'*.*

ii) There exists a sequence $S = (\pi^0, \ldots, \pi^l)$ *such that* $\pi^0 = \pi$*,* $\pi^l = \pi'$ *and for* $h = 1 \ldots l$*, the two paths* π^{h-1} *and* π^h *are the same up to a simple* $n-$*loop insertion/deletion*

In case ii) is satisfied, we denote $\pi \equiv_{SL} \pi'$.

Proof: $ii) \Rightarrow i)$ is obvious from the definitions since a simple $n-$loop insertion/deletion is a kind of elementary $n-$deformation. Conversely, it is sufficient to prove $ii)$ assuming that π and π' are the same up to an elementary $n-$deformation. So, suppose that $\pi = \pi_1.\gamma.\pi_2$ and $\pi' = \pi_1.\gamma'.\pi_2$. Where γ and γ' have the same extremities (say p and q) and are included in a $2 \times 2 \times 2$ cube $\mathcal{C}$ (a 2×2 square if $(n, \overline{n}) = (6, 26)$).

We give the sequence S: First, by inserting or deleting simple loops of the form (x, y, x) in π we get: $\pi = \pi_1.\gamma.\pi_2 \equiv_{SL} \pi_1.\gamma.\gamma'^{-1}.\gamma'.\pi_2$. Now, consider the closed path $\gamma.\gamma'^{-1}$ from p to p. One can sequentially remove minimal sub-paths of the loop $\gamma.\gamma'^{-1}$ which are simple loops until the resulting path is itself a simple loop and can be fully removed. Finally, $\pi \equiv_{SL} \pi_1.\gamma.\gamma'^{-1}.\gamma'.\pi_2 \equiv_{SL} \pi_1.\gamma'.\pi_2 = \pi'$. □

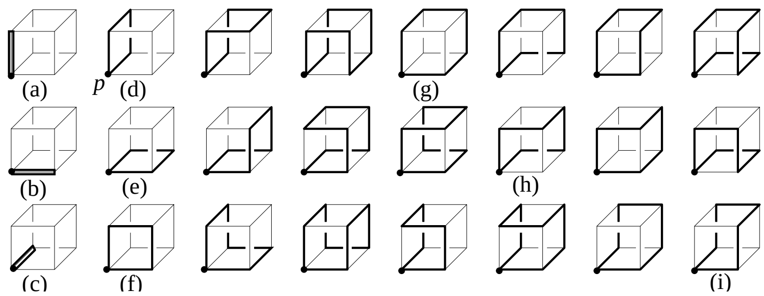


Figure 6: The 24 closed 6−loops from p in a $2 \times 2 \times 2$ cube.

Lemma 4.3 *Let π be a simple $6-$loop included in a 2×2 square and c be a $26-$path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

The proof of this intuitive lemma is given in the appendix. It consists in the computation of the linking number between two paths π and c following Definition 2.10, when π is one of the possible simple $6-$loops from a given point p to p and included in a 2×2 square which are depicted in Figures 6(a) to 6(f).

Lemma 4.4 *If π is a simple $(6+)-$loop included in a $2 \times 2 \times 2$ cube and c is a closed $18-$path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

Again, all the simple $(6+)-$loops in a $2 \times 2 \times 2$ cube from a point p are depicted in Figure 6 up to a choice of p and a choice of a parameterization. The proof, also given in the appendix, is then similar to the one of Lemma 4.3.

Proof of Theorem 2.12: From Proposition 4.2 it is sufficient to prove Theorem 2.12 in the case when π and π' are the same up to a simple $n-$loop insertion/deletion ($n \in \{6, 6+\}$). In this case, let us suppose that $\pi = \pi_1.(x).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a simple loop from x to x included in a $2 \times 2 \times 2$ cube $\mathcal{C}$ (in a 2×2 square if $(n, \overline{n}) = (6, 26)$). Since the linking number is invariant under any order preserving change of parameterization we have $L_{\pi',c} = L_{\pi_1.\gamma.\pi_2,c} = L_{\gamma.\pi_2.\pi_1,c}$. From Proposition 3.4, $L_{\gamma.\pi_2.\pi_1,c} = L_{\gamma,c} + L_{\pi_2.\pi_1,c}$ ($\pi_2.\pi_1$ is a closed $n-$path from x to x as well as γ). Now, since γ is a simple loop in $\mathcal{C}$ and from Lemma 4.3 or Lemma 4.4 we have $L_{\gamma,c} = 0$. Finally, $L_{\gamma.\pi_2.\pi_1,c} = L_{\pi_2.\pi_1,c} = L_{\pi_1.\pi_2,c} = L_{\pi,c}$. □

4.2 Independence up to a deformation of the $26-$path

Definition 4.5 *Let c and c' be two $26-$paths in $X \subset \mathbb{Z}^3$. We say that c and c' are* the same up to a triangle or a back and forth insertion *if:*
– Either $c = c_1.(x).c_2$ and $c' = c_1.(x, y, z, x).c_2$ where the voxels x, y and z are included in a $2 \times 2 \times 2$ cube $\mathcal{C}$,
– or $c = c_1.(x).c_2$ and $c' = c_1.(x, y, x).c_2$.

We say that c and c' are the same up to a triangle or a back and forth insertion/deletion if either c and c' or c' and c are the same up to a triangle or a back and forth insertion.

Proposition 4.6 *Let c and c' be two $26-$paths in $X \subset \mathbb{Z}^3$. Then the two following properties are equivalent:*

i) c is $26-$homotopic to c' in X.

ii) There exists a sequence $S = (c^0 = c, \ldots, c^k = c')$ of paths in X such that for all $i \in [1, k[$ the paths c^{i-1} and c^i are the same up to a triangle or back and forth insertion/deletion.

If ii) is satisfied, we denote $c \equiv_{TBF} c'$.

Proof: $ii) \Rightarrow i')$ is obvious since a triangle insertion is a particular case of elementary $26-$deformation.

$i') \Rightarrow ii)$ Conversely, from Definition 2.3, it is sufficient to prove $ii)$ if c and c' are the same up to an elementary deformation. We suppose that $c = c_1.\gamma.c_2$ and $c' = c_1.\gamma'.c_2$ where γ and γ' have the same extremities and are included in a $2 \times 2 \times 2$ cube $\mathcal{C}$. By an induction on the length of γ we show that there exists a sequence of triangle or back and forth insertions/deletions which leads from γ to the path reduced to its extremities.

Suppose that $\gamma = \gamma^0$ has a length $l \geq 2$. Then we have $\gamma^0 = \gamma_1^0.(x, y, z).\gamma_2^0$. Now, by a back and forth insertion we can obtain the path $\gamma_1^0.(x, y, z).(z, x, z).\gamma_2^0 = \gamma_1^0.(x, y, z, x, z).\gamma_2^0$ and then by a triangle deletion we obtain the path $\gamma_1^0.(x, z).\gamma_2^0 = \gamma^1$ which has a length of $l - 1 < l$. By induction, we can obtain a path $\gamma^k = (p, q)$ with a length of 1 where p and q are the common extremities of γ and γ'.

By the same way we can obtain the path γ' from the path (p, q) by a sequence of triangle insertions and back and forth deletions. Finally, any elementary $26-$deformation can be done by a sequence of triangle or back and forth insertions/deletions. □

Lemma 4.7 *If c is a $26-$triangle, then for any $6-$path π such that $\mathcal{H}(\pi, c)$ holds we have $L_{\pi,c} = 0$.*

The proof of this technical but again intuitive lemma is given in the appendix.

Proof of Theorem 2.13 in the case $(6, 26)$: The proof is similar to the proof of Theorem 2.12 using Proposition 4.6 instead of Proposition 4.2 and Proposition 3.5 instead of Proposition 3.4. Lemma 4.7 shows that $L_{\pi,\gamma} = 0$ when γ is a $26-$triangle. The case when γ is a back and forth is obvious and we also have $L_{\pi,\gamma} = 0$. □

4.3 Independence up to a deformation of the $18-$path

Definition 4.8 *Let c and c' be two closed $18-$paths in $X \subset \mathbb{Z}^3$. We say that c and c' are* the same up to a triangle, back and forth or square insertion *respectively if:*
– $c = c_1.(x).c_2$ and $c' = c_1.(x,y,z,x).c_2$ where the voxels x, y and z are included in a $2 \times 2 \times 2$ cube $\mathcal{C}$,
– $c = c_1.(x).c_2$ and $c' = c_1.(x,y,x).c_2$,
– $c = c_1.(x).c_2$ and $c' = c_1.\gamma.c_2$ where γ is one of the closed paths depicted in Figure 8 (up to a parameterization),

We say that c and c' are the same up to a triangle, back and forth or square insertion/deletion if either c and c' or c' and c are the same up to a triangle, back and forth or square insertion.

Proposition 4.9 *Let c and c' be two closed $18-$paths in $X \subset \mathbb{Z}^3$. Then the two following properties are equivalent:*

i) c is $18-$homotopic to c' in X.

ii) There exists a sequence $S = (c^0, \ldots, c^k)$ of paths in X with $c^0 = c$ and $c^k = c'$, such that for all $i \in [1,k]$, the paths c^{i-1} and c^i are the same up to a triangle, back and forth or square insertion/deletion.

In case ii) is satisfied we denote $c \equiv_{TBS} c'$.

Proof: $i) \Leftarrow ii)$ is obvious since the insertion/deletion of a back and forth, a triangle or a square is a particular case of elementary $18-$deformation.

$i) \Rightarrow ii)$ Conversely, from Proposition 4.2, if c' and c are $18-$homotopic, then there exists a sequence of simple $18-$loop insertions/deletions which leads from c to c'. We prove that each step of simple $18-$loop insertion/deletion can be achieved by a sequence of back and forth, triangle or square insertions/deletions.

Let γ be a simple $18-$loop in a $2 \times 2 \times 2$ cube. By an induction on the length of γ, we show that γ can be reduced to a single voxel by a sequence of back and forth, triangle or square insertions/deletions. This will indeed prove that any simple $18-$loop can be obtained by a sequence of insertions/deletions of back and forths, squares or triangles in a path reduced to a single voxel. Note that each step of this sequence only involves voxels of the loop γ so that the intermediate paths do belong to X.

Let $\gamma = \gamma^0$ be any simple closed $18-$loop, then given γ^k we must distinguish several cases:

• γ^k is reduced to a single voxel, there is nothing to prove in this case.

• If γ^k has a length 2, say $\gamma^k = (x,y,x)$, then $\gamma^{k+1} = (x)$ can be obtained by a back and forth deletion.

• If γ^k has a length 3, say $\gamma^k = (x,y,z,x)$, then $\gamma^{k+1} = (x)$ can be obtained by a triangle deletion.

• If γ^k has a length $l > 3$, then we distinguish two cases:

– If there exists x, y and z in γ^k such that $\gamma^k = \gamma_1^k.(x,y,z).\gamma_2^k$ where x is $18-$adjacent to z. In this case, the path $\gamma_1^k.(x,y,z).(z,x,z).\gamma_2^k = \gamma_1^k.(x,y,z,x,z).\gamma_2^k$ can be obtained by a back and forth insertion and then the path $\gamma^{k+1} = \gamma_1^k.(x,z).\gamma_2^k$ is obtained by a triangle deletion. The path γ^{k+1} has a length equal to $l-1$.

– If there exists no subsequence (x, y, z) in γ^k such that x is 18−adjacent to z. Then, in this case we prove that γ is a 18−square (i.e., one of the loops of Figure 8). Indeed, we have depicted in Figure 7 a $2 \times 2 \times 2$ cube. Suppose that γ has a length $l > 3$ and no triangle. Let us consider any two consecutive voxels of γ; up to a rotation these two voxels may have the configuration of the couple (a, b) or (a, h) of Figure 7. First, suppose that the two consecutive voxels have the same configuration as a and b in Figure 7 and try to extend this part of a simple 18−loop taking care not to add a voxel which would be 18−adjacent to the predecessor of its predecessor. Then, the only kind of loop you can obtain is the loop depicted in Figure 8(c). By the same way, trying to extend the sequence (a, h) into a simple 18−loop will also lead to the path depicted in Figure 8(c). Finally, γ^k is a square which obviously can be removed by a square deletion into a path γ^{k+1} reduced to a single voxel.

In all cases, we can obtain a path γ^{k+1} with a length either lower than l or equal to 1 by insertions/deletions of back and forths, triangles or squares. By induction, it is clear that there must exist an integer h such that γ^h is reduced to a single voxel. Then, any simple 18−loop can be inserted of removed in a closed 18−path by a sequence of triangle, back and forth or square insertions/deletions. This finishes proving that $i) \Rightarrow ii)$.

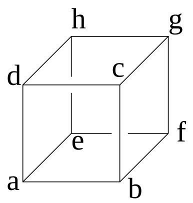


Figure 7: A $2 \times 2 \times 2$ cube.

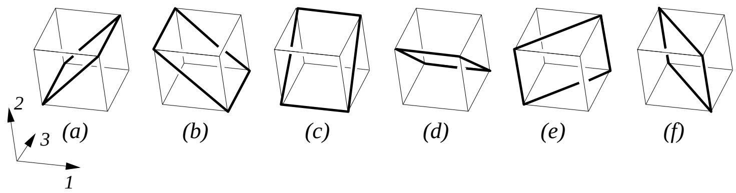


Figure 8: Possible simple 18−loops with no triangle in a $2 \times 2 \times 2$ cube.

□

Lemma 4.10 *If c is a 18−square and π is a closed (6+)−path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

Proof: Let $c = (x_0, x_1, x_2, x_3, x_0)$.

– Case of a square of the kind depicted in Figure 8(a) and Figure 8(b).

In this case, we have $L_{\pi,c} = L^{\pi}_{\pi,c}(0) + L^{\pi}_{\pi,c}(1) + L^{\pi}_{\pi,c}(2) + L^{\pi}_{\pi,c}(3)$ but for $i = 0, \ldots, 3$ $L^{\pi}_{\pi,c}(i) = 0$ either since $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ or since $P_c(i)^{Pred} = P_c(i)^{Succ}$.

– Case of a square of the kind depicted in Figures 8(c), 8(d), 8(e) and 8(f).

The proof of the Lemma in these cases is similar to the case of Figure 6(f) in the proof of Lemma 4.3 but a little less tricky since the case when two consecutive voxels of π have 8−adjacent projections which are not 4−adjacent cannot occur since c is a 6−path or a (6+)−path. Note that, with respect to Lemma 4.3, we must use $M_{c,\pi}$ here, instead of $W_{\pi,c}$ as in Lemma 4.3. □

Proof of Theorem 2.13 in the case $(6, 18)$**:** Again, the proof is similar to the proof of Theorem 2.12 using Proposition 4.9 instead of Proposition 4.2 and Proposition 3.5 instead of Proposition 3.4. Lemma 4.7 shows that $L_{\pi,\gamma} = 0$ when γ is a 18−triangle (which is also a 26−triangle). The case when γ is a back and forth is obvious and we also have $L_{\pi,\gamma} = 0$. Finally, Lemma 4.10 is used to prove that $L_{\pi,\gamma} = 0$ when γ is a square. □

5 Conclusion

A new tool for studying topological properties of objects in $\mathbb{Z}^3$ has been introduced. This tool, the linking number, has the same properties as its continuous analogue. A proof of its most important properties is given with no need of the use of notions of the continuous case. Indeed, the proof given here needs no more tools than those exposed in the first section of the paper. The very few notions of digital topology which are used here show that some strong properties can be proved with the only use of the digital theoretical framework. Furthermore, an application of the linking number to prove a new – and more concise – characterization of 3D simple points can be found in [5] and gives an effective example of the use of this new tool in order to prove results in the field of digital topology.

Appendix

Proof of Proposition 3.4: Let $c = (x_0, \ldots, x_q)$, $\pi_1 = (z_0, \ldots, z_{p_1})$, $\pi_2 = (t_0, \ldots, t_{p_2})$ and let $\pi_1.\pi_2 = (y_0, \ldots, y_{p_1+p_2})$. From Definition 2.10, we have to prove that

$$\sum_{i=0}^{q-1} L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) = \sum_{i=0}^{q-1} L^{\pi_1}_{\pi_1,c}(i) + \sum_{i=0}^{q-1} L^{\pi_2}_{\pi_2,c}(i).$$

More precisely, it is sufficient to prove that $L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) = L^{\pi_1}_{\pi_1,c}(i) + L^{\pi_2}_{\pi_2,c}(i)$ for any $i \in [0, q-1]$. From Definition 2.9, both terms of the previous equality are equal to zero if $\mathcal{P}(x_{i-1}) = \mathcal{P}(x_i)$ or if $\mathcal{P}(x_{i-1}) = \mathcal{P}(x_{Succ_c(i)})$. Therefore, we have to investigate the case when the projective movement $\mathcal{P} = P_c(i)$ (see Definition 2.9) is not trivial in the sense that $\mathcal{P}^{Pred} \neq \mathcal{P}^{Succ}$.

In this case, we prove that: $\displaystyle\sum_{k=0}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k) = \sum_{k=0}^{p_1-1} M_{c,\pi_1}(i,k) + \sum_{k=0}^{p_2-1} M_{c,\pi_2}(i,k)$.

• If both π_1 and π_2 are closed paths the projection of which is reduced to a single point, i.e., $\mathcal{P}(z_0) = \mathcal{P}(z_k) = \mathcal{P}(t_0)$ for any $k \in [0, p_1]$ and $\mathcal{P}(t_0) = \mathcal{P}(t_k)$ for any $k \in [0, p_2]$ then it is immediate that $L_{\pi_1,c} = L_{\pi_2,c} = L_{\pi_1.\pi_2,c} = 0$.

• If the path π_2 has a projection reduced to a single point (i.e., $Succ_{\pi_2}(0) = 0$) and π_1 has a projection which is not reduced to a single point (i.e., $Succ_{\pi_1}(0) \neq 0$) then $L_{\pi_2,c} = 0$ and we prove that $L_{\pi_1,c} = L_{\pi_1.\pi_2,c}$. Indeed, in this case and for any $i \in [0, q]$:

$$L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) = \sum_{k=0}^{Succ_{\pi_1.\pi_2}(0)-1} M_{c,\pi_1.\pi_2}(i,k) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(0)} M_{c,\pi_1.\pi_2}(i,k) + \sum_{k=Pred_{\pi_1.\pi_2}(0)}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k)$$

But from the definition of $Succ_{\pi_1.\pi_2}(0)$ and $Pred_{\pi_1.\pi_2}(0)$ and due to the fact that the indirect contribution of a couple (k, i) is equal to 0 in the case when $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ we obtain that:

$$\sum_{k=Pred_{\pi_1.\pi_2}(0)}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k) + \sum_{k=0}^{Succ_{\pi_1.\pi_2}(0)-1} M_{c,\pi_1.\pi_2}(i,k) = M_{c,\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1,k)$$

We also observe that: $Pred_{\pi_1.\pi_2}(0) = Pred_{\pi_1}(0) \in]0,p_1[$ since $\mathcal{P}(y_k) = \mathcal{P}(y_{p_1})$ for $k \in [p_1, p_1+p_2]$. But $y_j = z_j$ for all $j \in [0,\dots,p_1]$ so that $y_{Pred_{\pi_1.\pi_2}(0)} = z_{Pred_{\pi_1}(0)}$ and $y_{Pred_{\pi_1.\pi_2}(0)+1} = z_{Pred_{\pi_1}(0)+1}$. On the other hand, $Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1) = Succ_{\pi_1.\pi_2}(0)$ from the definition of *Succ* and *Pred*. But $Succ_{\pi_1.\pi_2}(0) \in]0,p_1[$, so that $Succ_{\pi_1.\pi_2}(0) = Succ_{\pi_1}(0) = Succ_{\pi_1}(Pred_{\pi_1}(0)+1)$. Finally, $y_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1)} = z_{Succ_{\pi_1}(Pred_{\pi_1}(0)+1)}$. From the definition of the contribution to the linking number we obtain:
$M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) = M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1)$.

From the definition of $Succ_{\pi_1}(0)$ and $Pred_{\pi_1}(0)$ and the fact that the contribution of a couple (k,i) is equal to 0 in the case when $\mathcal{P}(z_k) = \mathcal{P}(z_{k-1})$ we have:

$$M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) = \sum_{k=Pred_{\pi_1}(0)+1}^{p_1-1} M_{c,\pi_1}(i,k) + \sum_{k=0}^{Succ_{\pi_1}(0)-1} M_{c,\pi_1}(i,k).$$

Due to the expression of $L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i)$ set above, and due to the fact that the sequence of voxels of π_1 appears in $\pi_1.\pi_2$ between $Succ_{\pi_1.\pi_2}(0)$ and $Pred_{\pi_1.\pi_2}(0)$:

$$\begin{aligned}
L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) &= M_{c,\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1,k) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(0)} M_{c,\pi_1.\pi_2}(i,k) \\
&= M_{c,\pi_1}(Pred_{\pi_1}(0)+1,k) + \sum_{k=Succ_{\pi_1}(0)}^{Pred_{\pi_1}(0)} M_{c,\pi_1}(i,k) \\
&= L^{\pi_1}_{\pi_1,c}(i)
\end{aligned}$$

• The case when $Succ_{\pi_1}(0) = 0$ and $Succ_{\pi_2}(0) \neq 0$ is similar.

• In the case when none of the paths π_1 and π_2 has a projection reduced to a single point (i.e., $Succ_{\pi_1}(0) \neq 0$ and $Succ_{\pi_2}(0) \neq 0$).

Then, following the same considerations as in the previous case we show that:

$$\begin{aligned}
L^{\pi_1}_{\pi_1,c}(i) &= M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + \sum_{k=Succ_{\pi_1}(0)}^{Pred_{\pi_1}(0)} M_{c,\pi_1}(i,k) && (2.1) \\
L^{\pi_2}_{\pi_2,c}(i) &= M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) + \sum_{k=Succ_{\pi_2}(0)}^{Pred_{\pi_2}(0)} M_{c,\pi_2}(i,k) && (2.2) \\
L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) &= M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(p_1)} M_{c,\pi_1.\pi_2}(i,k) && (2.3) \\
&+ M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1) + \sum_{k=Succ_{\pi_1.\pi_2}(p_1)}^{Pred_{\pi_1.\pi_2}(0)} M_{c,\pi_1.\pi_2}(i,k)
\end{aligned}$$

For $k \in [Succ_{\pi_1}(0), Pred_{\pi_1}(0)] = [Succ_{\pi_1.\pi_2}(0), Pred_{\pi_1.\pi_2}(p_1)] \subset]0,p_1[$ we have $z_k = y_k$, $z_{k-1} = y_{k-1}$ and $z_{Succ_{\pi_1}(k)} = y_{Succ_{\pi_1.\pi_2}(k)}$, so $M_{c,\pi_1}(i,k) = M_{c,\pi_1.\pi_2}(i,k)$. So the sum in equation (2.1) is equal to the first sum in (2.3).

For $k \in [Succ_{\pi_2}(0), Pred_{\pi_2}(0)] = [Succ_{\pi_1.\pi_2}(p_1) - p_1, Pred_{\pi_1.\pi_2}(0) - p_1] \subset]0, p_2[$ we have $t_k = y_{k+p_1}$, $t_{k-1} = y_{k+p_1-1}$ and $t_{Succ_{\pi_1}(k)} = y_{Succ_{\pi_1.\pi_2}(k)+p_1}$, so $M_{c,\pi_2}(i,k) = W_{c,\pi_1.\pi_2}(i, k{+}p_1)$. Then, the sum in equation (2.2) is equal to the second sum in (2.3).

There remains to prove that $M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) + M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$.

Now, $M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M^-_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M^+_{c,\pi_1}(i, Pred_{c_1}(0)+1) + M^-_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) + M^+_{c,\pi_2}(i, Pred_{\pi_2}(0)+1)$.But,
$M^-_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) = M^-_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$ since $z_{Pred_{\pi_1}(0)} = y_{Pred_{\pi_1.\pi_2}(p_1)}$.
$M^+_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) = M^+_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1)$ since $z_{Succ_{\pi_1}(Pred_{\pi_1}(0)+1)} = z_{Succ_{\pi_1}(0)} = y_{Succ_{\pi_1.\pi_2}(0)} = y_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1)}$.
$M^-_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M^-_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1)$ since $t_{Pred_{\pi_2}(0)} = y_{Pred_{\pi_1.\pi_2}(0)}$.
$M^+_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M^+_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$ since $t_{Succ_{\pi_2}(Pred_{\pi_2}(0)+1)} = t_{Succ_{\pi_2}(0)} = t_{Succ_{\pi_1.\pi_2}(p_1)} = t_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(p_1)+1)}$.

Finally, $M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) + M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$. $\square$

Proof of Lemma 4.3: From Figure 6(a) to Figure 6(f), all the possible simple $6-$loops included in a 2×2 square from a given point p to p are depicted up to a choice of an orientation. In this proof, we will only investigate the cases of simple loops from a fixed given point p. By changing the parametrization of the loop, it is clear that the proof is similar for the 3 other positions of the voxel p in a 2×2 square.

– Cases of the figures 6(a) and 6(b): In this case, $\pi = (x, y, x)$ and $L_{\pi,c} = L^c_{\pi,c}(0) + L^c_{\pi,c}(1) = 0$ from the very definition of the contribution to the linking number.

– Case of Figure 6(c): In this case, $\pi = (x, y, x)$ and $\mathcal{P}(x) = \mathcal{P}(y)$ so it is clear that $L_{\pi,c} = 0$.

– Cases of the figures 6(d) and 6(e)

Let $\pi = (y_0, y_1, y_2, y_3, y_4 = y_0)$. In both cases and for any choice of a parameterization, one can easily check that $L^c_{\pi,c}(k) = 0$ for $k = 0, \dots, 3$ either because $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ or because $P_\pi(k)^{Pred} = P_\pi(k)^{Succ}$.

– Case of Figure 6(f)

We set $c = (x_i)_{i=0,\dots,q}$, $\pi = (y_0, y_1, y_2, y_3, y_4 = y_0)$, $P = \{\mathcal{P}(y_0), \mathcal{P}(y_1), \mathcal{P}(y_2), \mathcal{P}(y_3)\}$ and let $I = \{[i_1, i_2] |\ \mathcal{P}(x_{i_1-1}) \notin P,\ \mathcal{P}(x_{i_2+1}) \notin P$ and $\mathcal{P}(x_i) \in P$ for all $i \in [i_1, i_2]\}$. In the case when $\mathcal{P}(x_i) \in P$ for all $i \in [0, q]$ then $I = \{[0, q-1]\}$.

Then, it is clear that $L_{\pi,c} = \displaystyle\sum_{[i_1,i_2] \in I} \sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i)$.

It is sufficient to prove prove that for any $[i_1, i_2] \in I$ the sum $\sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i)$ is equal to 0. One can choose an orientation for the path π but in all cases we observe that for all $i \in [i_1, i_2]$ there exists a single subscript $k(i) \in \{0, 1, 2, 3\}$ such that $L^\pi_{\pi,c}(i) = W_{\pi,c}(k(i), i)$. First, for a given interval $[i_1, i_2]$, either $x^3_i < y^3_{k(i)}$ for all $i \in [i_1, i_2]$ or $x^3_i > y^3_{k(i)}$ for all $i \in [i_1, i_2]$. In the first case, $L^\pi_{\pi,c}(i) = 0$ for all $i \in [i_1, i_2]$ and there is nothing else to prove.

In the second case one can consider some $a_0 < \dots < a_{l+1}$ such that: $[i_1, i_2] = [a_0, a_1[\cup [a_1, a_2[\cup \dots \cup [a_l, a_{l+1}]$, with $\mathcal{P}(x_{a_i}) \neq \mathcal{P}(x_{a_{i-1}})$ for any $i = 1, \dots, l$ and $\forall i = 0, \dots, l$, $\forall j \in [a_i, a_{i+1}[$ we have $\mathcal{P}(x_j) = \mathcal{P}(x_{a_i})$. Then, for any $i = 0, \dots, l-1$ we have:

$$\sum_{j=a_i}^{a_{i+1}-1} W_{\pi,c}(k(j),j) = W_{\pi,c}(k(a_i),a_i) \text{ and } \sum_{j=a_l}^{a_{l+1}} W_{\pi,c}(k(j),j) = W_{\pi,c}(k(a_l),a_l).$$

By construction of the intervals $[a_i, a_{i+1}[$, we have:

$$\sum_{i\in[i_1,i_2]} W_{\pi,c}(k(i),i) = \sum_{i=1,\dots,l} W_{\pi,c}(k(a_i),a_i).$$

Now, we prove that for $i = 1,\dots,l$, $W^+_{\pi,c}(k(a_i),a_i) + W^-_{\pi,c}(k(a_{i+1}),a_{i+1}) = 0$. Indeed, let $\beta_i = P_c(a_i)$ and $\alpha_i = P_\pi(k(a_i))$. Then $W^+_{\pi,c}(k(a_i),a_i)$ depends on the position of the point β_i^{Succ} with respect to the projective movement α_i. If $\beta_i^{Succ} \in \{\alpha^{Pred}, \alpha^{Succ}\}$ then we have $W^+_{\pi,c}(k(a_i),a_i) = 0$ and $W^-_{\pi,c}(k(a_{i+1}),a_{i+1}) = 0$ since $Pred_c(a_{i+1}) = a_i$. If $\beta_i^{Succ} \notin \{\alpha^{Pred}, \alpha^{Succ}\}$ then β^{Succ} is a point which is $8-$adjacent but not $4-$adjacent to $(0,0)$. But in this case, and depending on the orientation of the loop π, if $\beta_i^{Succ} \in Right(\alpha_i)$ then $\beta_{i+1}^{Pred} \in Right(\alpha_{i+1})$ and if $\beta_i^{Succ} \in Left(\alpha_i)$ then $\beta_{i+1}^{Pred} \in Left(\alpha_{i+1})$.

We also see that $W^-_{\pi,c}(k(a_0),a_0) + W^+_{\pi,c}(k(a_l),a_l) = 0$. Indeed, up to a choice of an orientation for the loop reduced to a 2×2 square, it is clear that the projection of the voxels $P_c(a_0)^{Pred}$ and $P_c(a_l)^{Succ}$ will both belong either respectively to $Right(P_\pi(k(a_0)))$ and $Right(P_\pi(k(a_l)))$ or to $Left(P_\pi(k(a_0)))$ and $Left(P_\pi(k(a_l)))$. In the case when $I = \{[0, q-1]\}$ then $W^-_{\pi,c}(k(a_0),a_0) + W^+_{\pi,c}(k(a_l),a_l) = 0$ either because both terms are equal to 0 or for the same reason as explained above for the intervals $[a_i, a_{i+1}[$ (contributions with opposite signs).

Finally, for any $[i_1, i_2] \in I$,

$$\begin{aligned}
\sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i) &= \sum_{i=1}^{l} \left(W^-_{\pi,c}(k(a_i),a_i) + W^+_{\pi,c}(k(a_i),a_i)\right) \\
&= W^-_{\pi,c}(k(a_1),a_1) + \sum_{i=1}^{l} \left(W^+_{\pi,c}(k(a_i),a_i) + W^-_{\pi,c}(k(a_{i+1}),a_{i+1})\right) \\
&\quad + W^+_{\pi,c}(k(a_l),a_l) \\
&= 0
\end{aligned}$$

□

Proof of Lemma 4.4: In Figure 6 are depicted up to a choice of a point p and a choice of a parameterization all the simple loops in a $2\times 2\times 2$ cube from p to p (the proof when p is any one of the 7 other voxels in the cube is similar).

In this proof we only have to show that $L_{\pi,c} = 0$ when π is one of the loops $(a)\dots(i)$. Then, using Proposition 3.4 and the fact that any other loop of Figure 6 can be obtained by insertion/deletion of loops $(a)\dots(i)$ we will achieve to prove that $L_{\pi,c} = 0$ when π is any of the loops of Figure 6.

In the proof of Lemma 4.3, we have already proved that $L_{\pi,c} = 0$ when π is one of the loops of figures (a), (b), (c), (d), (e) and (f) (indeed, c, as an $18-$path, is also a $26-$path).

– Cases of the figures 6(g), (h) and (i):

The proof in these cases is similar to the case of Figure 6(f) (see Proof of Proposition 4.3). Thus, we still observe the existence of an integer $k(i)$ such that $L^\pi_{\pi,c}(i) = W_{\pi,c}(k(i),i)$. Indeed, only one of any two voxels of π which have the same projection may have a non zero contribution. We also still use the fact that for a given interval $[i_1, i_2] \in I$ (as defined in the previous case), either $x_i^3 < y_{k(i)}^3$ for all $i \in [i_1, i_2]$ or $x_i^3 > y_{k(i)}^3$ for all

$i \in [i_1, i_2]$. Indeed, this comes from the fact that the two points of the cube which are not points of π are not 18−adjacent, so that the path c cannot have intersection intervals with voxels in the two sides of π according to their third coordinates. The end of the proof is similar.

– Other cases: Now, if π is a simple closed loop in $\mathcal{C}$ such that $L_{\pi,c} = 0$ and π' is a simple loop in $\mathcal{C}$ obtained by the insertion of any of the loops $(a), \dots, (i)$ in π, then $L_{\pi,c} = L_{\pi',c} = 0$. Indeed, if $\pi = \pi_1.(x).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a loop from x to x of some form in $(a), \dots, (i)$, then $L_{\pi_1.\gamma.\pi_2,c} = L_{\pi_2.\pi_1.\gamma,c}$ because the linking number is invariant under any orientation preserving change of parameterization. Furthermore, from Proposition 3.4 we have $L_{\pi_2.\pi_1.\gamma,c} = L_{\pi_2.\pi_1,c} + L_{\gamma,c}$. Since we have proved that $L_{\gamma,c} = 0$ when γ is of type $(a), \dots, (i)$ then $L_{\pi_2.\pi_1.\gamma,c} = L_{\pi_2.\pi_1,c}$ and again we have $L_{\pi_2.\pi_1,c} = L_{\pi_1.\pi_2,c}$. Finally, $L_{\pi',c} = L_{\pi,c}$.

Now, it is left to the reader to check that any of the 15 other simple loops can be obtained by a sequence of insertions or deletions of loops like $(a), \dots, (i)$. Then, we obtain that $L_{\pi,c} = 0$ when π is any kind of loop depicted in Figure 6.

As an example, we only give here the sequence of simple loops insertion/deletion of the kind $(a) \dots (i)$ which leads from the path reduced to the voxel a to the path depicted in Figure 9.

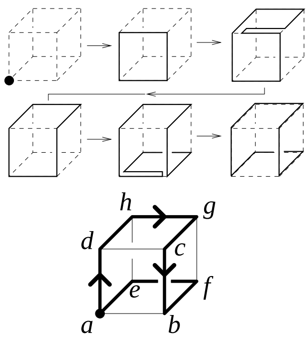


Figure 9: A sequence of simple loop insertions/deletions.

$(\underline{a, d, c, b, a})$ can be obtained from $(\underline{a})$ by insertion of a loop like Figure 6(f).
$(a, \underline{d, h, g, c, d}, c, b, a)$ can be obtained from $(a, \underline{d}, c, b, a)$ by insertion of a loop like Figure 6(e).
$(a, d, h, g, \underline{c}, b, a)$ can be obtained from $(a, d, h, g, \underline{c, d, c}, b, a)$ by deletion of a loop like Figure 6(b).
$(a, d, h, g, c, \underline{b, f, e, a, b}, a)$ can be obtained from $(a, d, h, g, c, \underline{b}, a)$ by insertion of a loop like Figure 6(e).
$(a, d, h, g, c, b, f, e, \underline{a})$ can be obtained from $(a, d, h, g, c, b, f, e, \underline{a, b, a})$ by deletion of a loop like Figure 6(b). □

Proof of Lemma 4.7: Let c be a 26−triangle. We first consider the case when exactly two voxels of c have the same projection (the case when all the voxels have the same projection immediately implies that $L_{\pi,c} = 0$).

We suppose that two voxels of $c = (x_0, x_1, x_2, x_0)$ have the same projection. Without loss of generality, we suppose that $\mathcal{P}(x_1) = \mathcal{P}(x_2)$. Now, for any 6−path π we have $L_{\pi,c} = L^{\pi}_{\pi,c}(0) + L^{\pi}_{\pi,c}(1) + L^{\pi}_{\pi,c}(2)$. But from Definition 2.9 we have $L^{\pi}_{\pi,c}(0) = 0$ since $Succ_c(0) = 1$ and $Pred_c(0) = 2$ and $\mathcal{P}(x_1) = \mathcal{P}(x_2)$. We also have $L^{\pi}_{\pi,c}(1) = 0$ since

$Succ_c(1) = Pred_c(1) = 0$. Finally, $L^{\pi}_{\pi,c}(2) = 0$ since $\mathcal{P}(x_1) = \mathcal{P}(x_2)$.

Now, we assume that the three voxels of c have pairwise distinct projections.

Let $c = (x_0, x_1, x_2, x_3 = x_0)$ and $\pi = (y_k)_{k=0,\ldots,p}$. Let $P = \{\mathcal{P}(x_0), \mathcal{P}(x_1), \mathcal{P}(x_2)\}$ and $K = \{[k_1, k_2] |\ \mathcal{P}(y_{k_1-1}) \notin P,\ \mathcal{P}(y_{k_2+1}) \notin P$ and $\mathcal{P}(y_i) \in P$ for all $i \in [k_1, k_2]\}$. If $\mathcal{P}(y_k) \in P$ for all $k \in [0, p]$ then $K = \{[0, p-1]\}$. It is clear that $L_{\pi,c} = \sum_{[k_1,k_2]\in K} \sum_{k=k_1}^{k_2} L^c_{\pi,c}(k)$.

For any $[k_1, k_2] \in K$ and any $k \in [k_1, k_2]$ we denote by $i(k)$ the only subscript of voxels of c such that $\mathcal{P}(x_{i(k)}) = \mathcal{P}(y_k)$. Then, for any such k, we have $L^c_{\pi,c}(k) = W_{\pi,c}(k, i(k))$. So, $L_{\pi,c} = \sum_{[k_1,k_2]\in K} \sum_{k=k_1}^{k_2} W_{\pi,c}(k, i(k))$.

Now, from the definition of the contribution to the linking number, it is clear that for any $[k_1, k_2] \in K$: $\sum_{k=k_1}^{k_2} W_{\pi,c}(k, i(k)) = \sum_{k=k_1}^{Pred_\pi(k_2)+1} W_{\pi,c}(k, i(k))$.

But for $k \in [k_1 + 1, Pred_\pi(k_2)]$, $W_{\pi,c}(k, i(k)) = 0$ either because $\{P_\pi(k)^{Pred}, P_\pi(k)^{Succ}\} \subset \{P_c(i(k))^{Pred}, P_c(i(k))^{Succ}\}$ or because $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$.

Similarly, we observe that $W^+_{\pi,c}(k_1, i(k_1)) = W^-_{\pi,c}(Pred_\pi(k_2) + 1, i(Pred_\pi(k_2) + 1)) = 0$. On the other hand, $W^-_{\pi,c}(k_1, i(k_1)) + W^+_{\pi,c}(Pred_\pi(k_2) + 1, i(Pred_\pi(k_2) + 1)) = 0$. Indeed, depending on a choice of an orientation for the triangle c, it is clear that the projections of the voxels $P_\pi(a_1)^{Pred}$ and $P_\pi(Pred_\pi(k_2) + 1)^{Succ}$ will either belong repsectively to $Right(P_c(i(a_1)))$ and $Right(P_c(i(Pred_\pi(k_2) + 1)))$ or belong respectively to $Left(P_c(i(a_1)))$ and $Left(P_c(i(Pred_\pi(k_2) + 1)))$. If $K = \{[0, p-1]\}$ then $W^-_{\pi,c}(0, i(0)) = W^+_{\pi,c}(Pred_\pi(p) + 1, i(Pred_\pi(p) + 1)) = 0$ from the definition of $W_{\pi,c}(k, i)$.

Thus, $\sum_{k=k_1}^{k_2} W_{\pi,c}(k, i(k)) = W_{\pi,c}(k_1, i(k_1)) + \sum_{k=k_1+1}^{Pred_\pi(k_2)} W_{\pi,c}(k, ik)) + W_{\pi,c}(Pred_\pi(k_2) + 1, i(Pred_\pi(k_2) + 1)) = 0$ and finally $L_{\pi,c} = 0$. □

References

[1] G. Bertrand. A new topological classification of points in 3d images. *2cnd European Conference Computer Vision*, pages 710–714, 1992.

[2] G. Bertrand. Simple points, topological numbers and geodesic neighborhoods in cubics grids. *Patterns Recognition Letters*, 15:1003–1011, 1994.

[3] S. Fourey and R. Malgouyres. Intersection number and topology preservation within digital surfaces. In *Proceedings of the Sixth International Workshop on Parallel Image Processing and Analysis (IWPIPA'99)*, pages 138–158, January 1999. Madras, India.

[4] S. Fourey and R. Malgouyres. Intersection number of paths lying on a digital surface and a new jordan theorem. In *Proceedings of the 8th International Conference Discrete Geometry for Computer Imagery (DGCI'99)*, volume 1568 of *Lecture Notes in Computer Science*, pages 104–117. Springer, March 1999. Marne la Valle, France.

[5] S. Fourey and R. Malgouyres. A concise characterization of 3d simple points. In *Proceedings of the 9th International Conference Discrete Geometry for Computer Imagery (DGCI'00)*, volume 1953 of *Lecture Notes in Computer Science*, pages 27–36. Springer, December 2000. Uppsala, Sweden.

[6] T. Yung Kong. A digital fundamental group. *Computer Graphics*, 13:159–166, 1989.

[7] T. Yung Kong and Azriel Rosenfeld. Digital topology : introduction and survey. *Computer Vision, Graphics and Image Processing*, 48:357–393, 1989.

[8] R. Malgouyres. Homotopy in 2-dimensional digital images. *Theoretical Computer Science*, 230:221–233, 2000.

[9] Dale Rolfsen. *Knots and Links.* Mathematics Lecture Series. University of British Columbia.

[10] Azriel Rosenfeld and A. Nakamura. Local deformation of digital curves. *Pattern Recognition Letters*, 18:613–620, 1997.