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## ► To cite this version:

Sébastien Fourey. Simple points and generic axiomatized digital surface-structures. Proceedings of the 10th international workshop IWCIA 2004, Dec 2004, Auckland, New Zealand. pp.307-317. <hal-00338922>

**HAL Id: hal-00338922**

**<https://hal.science/hal-00338922v1>**

Submitted on 12 Jun 2024

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# Simple Points and Generic Axiomatized Digital Surface-Structures

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**Abstract.** We present a characterization of topology preservation within digital axiomatized digital surface structures (GADS), a generic theoretical framework for digital topology introduced in [2]. This characterization is based on the digital fundamental group that has been classically used for that purpose. More briefly, we define here simple points within GADS and give the meaning of the words: preserving the topology within GADS.

## 1 Introduction

In [2], a generic framework for digital topology has been introduced. This framework is in fact a whole axiomatic theory that allows us to prove results that become valid for any two dimensional digital space that satisfies the axioms of the theory. Some results such as a generic Jordan theorem has already been proved within this framework.

In this paper, we address a classical problem of digital topology: the characterization of topology preservation [15, 12, 18]. The main question being: When can we say that the deletion of one or several pixels (or voxels) from an image preserves the topology? In all cases, the answer to this question comes with the definition of *simple pixels/voxels*.

On the other hand, the digital fundamental group ([11]) has proved to be a convenient tool in order to characterize topology preservation in digital surfaces (see [16, 5]) as well as in the classical three dimensional digital space  $\mathbb{Z}^3$  (see [6]). Here, we state in a very straightforward way a definition of the digital fundamental group of a GADS (Generic Axiomatized Digital Surface-Structure). Then, we present a characterization of topology preservation within a GADS, by removal of a simple point, based on the fundamental group.

## 2 Definition of GADS and pGADS

We recall here the basic notions and definitions from [2]. We should first summarize the motivation for the definition of a GADS. This starts with

an observation: many results in digital topology come with a proof that *depends* on the digital space that is considered. For example, a proof of a Jordan curve theorem exists for the space  $\mathbb{Z}^2$  with the classical (4, 8) or (8, 4) pairs of adjacency relations. A similar result holds for the hexagonal grid, as well as  $\mathbb{Z}^2$  with the Khalimsky adjacency relation. What is unsatisfactory here is that the proof of such a result has to be written for each of the considered spaces. The axiomatic definition of what actually is an admissible digital space is a response to this observation. The purpose of GADS as introduced in [2] is to define a generic framework that allows to state and prove results of digital topology which becomes valid for any admissible digital space. Thus, a single result would no longer need (sometimes similar) multiple proofs.

## 2.1 Basic Concepts and Notations

For any set  $P$  we denote by  $P^{\{2\}}$  the set of all unordered pairs of distinct elements of  $P$  (equivalently, the set of all subsets of  $P$  with exactly two elements). Let  $P$  be any set and let  $\rho \subseteq P^{\{2\}}$ .<sup>1</sup> Two elements  $a$  and  $b$  of  $P$  [respectively, two subsets  $A$  and  $B$  of  $P$ ] are said to be  $\rho$ -adjacent if  $\{a, b\} \in \rho$  [respectively, if there exist  $a \in A$  and  $b \in B$  with  $\{a, b\} \in \rho$ ]. If  $x \in P$  we denote by  $N_\rho(x)$  the set of elements of  $P$  which are  $\rho$ -adjacent to  $x$ ; these elements are also called the  $\rho$ -neighbors of  $x$ . We call  $N_\rho(x)$  the *punctured  $\rho$ -neighborhood* of  $x$ .

A  $\rho$ -path from  $a \in P$  to  $b \in P$  is a finite sequence  $(x_0, \dots, x_l)$  of one or more elements of  $P$  such that  $x_0 = a$ ,  $x_l = b$  and, for all  $i \in \{0, \dots, l-1\}$ ,  $\{x_i, x_{i+1}\} \in \rho$ . The nonnegative integer  $l$  is the *length* of the path. A  $\rho$ -path of length 0 is called a *one-point path*. For all integers  $m, n$ ,  $0 \leq m \leq n \leq l$ , the subsequence  $(x_m, \dots, x_n)$  of  $(x_0, \dots, x_l)$  is called an *interval* or *segment* of the path. For all  $i \in \{1, \dots, l\}$  we say that the elements  $x_{i-1}$  and  $x_i$  are *consecutive* on the path, and also that  $x_{i-1}$  *precedes*  $x_i$  and  $x_i$  *follows*  $x_{i-1}$  on the path. Note that consecutive elements of a  $\rho$ -path can never be equal.

A  $\rho$ -path  $(x_0, \dots, x_l)$  is said to be *simple* if  $x_i \neq x_j$  for all distinct  $i$  and  $j$  in  $\{0, \dots, l\}$ . It is said to be *closed* if  $x_0 = x_l$ , so that  $x_0$  follows  $x_{l-1}$ . It is called a  $\rho$ -cycle if it is closed and  $x_i \neq x_j$  for all distinct  $i$  and  $j$  in  $\{1, \dots, l\}$ . One-point paths are the simplest  $\rho$ -cycles. Two  $\rho$ -cycles  $c_1 = (x_0, \dots, x_l)$  and  $c_2 = (y_0, \dots, y_l)$  are said to be *equivalent* if there exists an integer  $k$ ,  $0 \leq k \leq l-1$ , such that  $x_i = y_{(i+k) \bmod l}$  for all  $i \in \{0, \dots, l\}$ .

<sup>1</sup>  $\rho$  can be viewed as a binary, symmetric and irreflexive relation on  $P$ , and  $(P, \rho)$  as an undirected simple graph.

If  $S \subseteq P$ , two elements  $a$  and  $b$  of  $S$  are said to be  $\rho$ -connected in  $S$  if there exists a  $\rho$ -path from  $a$  to  $b$  that consists only of points in  $S$ .  $\rho$ -connectedness in  $S$  is an equivalence relation on  $S$ ; its equivalence classes are called the  $\rho$ -components of  $S$ . The set  $S$  is said to be  $\rho$ -connected if there is just one  $\rho$ -component of  $S$ .

Given two sequences  $c_1 = (x_0, \dots, x_m)$  and  $c_2 = (y_0, \dots, y_n)$  such that  $x_m = y_0$ , we denote by  $c_1.c_2$  the sequence  $(x_0, \dots, x_m, y_1, \dots, y_n)$ , which we call the *concatenation of  $c_1$  and  $c_2$* . Whenever we use the notation  $c_1.c_2$ , we are also implicitly saying that the last element of  $c_1$  is the same as the first element of  $c_2$ . It is clear that if  $c_1$  and  $c_2$  are  $\rho$ -paths of lengths  $l_1$  and  $l_2$ , then  $c_1.c_2$  is a  $\rho$ -path of length  $l_1 + l_2$ .

For any sequence  $c = (x_0, \dots, x_m)$ , the *reverse of  $c$* , denoted by  $c^{-1}$ , is the sequence  $(y_0, \dots, y_m)$  such that  $y_k = x_{m-k}$  for all  $k \in \{0, \dots, m\}$ . It is clear that if  $c$  is a  $\rho$ -path of length  $l$  then so is  $c^{-1}$ .

A *simple closed  $\rho$ -curve* is a nonempty finite  $\rho$ -connected set  $C$  such that each element of  $C$  has exactly two  $\rho$ -neighbors in  $C$ . (Note that a simple closed  $\rho$ -curve must have at least three elements.) A  $\rho$ -cycle  $c$  of length  $|C|$  that contains every element of a simple closed  $\rho$ -curve  $C$  is called a  $\rho$ -parameterization of  $C$ . Note that if  $c$  and  $c'$  are  $\rho$ -parameterizations of a simple closed  $\rho$ -curve  $C$ , then  $c'$  is equivalent to  $c$  or to  $c^{-1}$ .

If  $x$  and  $y$  are  $\rho$ -adjacent elements of a simple closed  $\rho$ -curve  $C$ , then we may say that  $x$  and  $y$  are  $\rho$ -consecutive on  $C$ . If  $x$  and  $y$  are distinct elements of a simple closed  $\rho$ -curve  $C$  that are not  $\rho$ -consecutive on  $C$ , then each of the two  $\rho$ -components of  $C \setminus \{x, y\}$  is called a  $\rho$ -cut-interval (of  $C$ ) associated with  $x$  and  $y$ .

## 2.2 Definition of a gads

**Definition 1 (2D digital complex)** A 2D digital complex is an ordered triple  $(V, \pi, \mathcal{L})$ , where

- $V$  is a set whose elements are called vertices or spels,
- $\pi \subseteq V^{\{2\}}$ , and the pairs of vertices in  $\pi$  are called proto-edges,
- $\mathcal{L}$  is a set of simple closed  $\pi$ -curves whose members are called loops,

and the following four conditions hold:

- (i)  $V$  is  $\pi$ -connected and contains more than one vertex.
- (ii) For any two distinct loops  $L_1$  and  $L_2$ ,  $L_1 \cap L_2$  is either empty, or consists of a single vertex, or is a proto-edge.
- (iii) No proto-edge is included in more than two loops.
- (iv) Each vertex belongs to only a finite number of proto-edges.

When specifying a 2D digital complex whose vertex set is the set of points of a grid in  $\mathbb{R}^n$ , a positive integer  $k$  (such as 4, 8 or 6) may be used to denote the set of all unordered pairs of  $k$ -adjacent vertices. We write  $\mathcal{L}_{2 \times 2}$  to denote the set of all unit lattice squares in  $\mathbb{Z}^2$ . The triple  $(\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2})$  is a simple example of a 2D digital complex.

**Definition 2 (GADS)** A generic axiomatized digital surface-structure, or GADS, is a pair  $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$  where  $(V, \pi, \mathcal{L})$  is a 2D digital complex (whose vertices, proto-edges and loops are also referred to as vertices, proto-edges and loops of  $\mathcal{G}$ ) and where  $\kappa$  and  $\lambda$  are subsets of  $V^{\{2\}}$  that satisfy Axioms 1, 2 and 3 below. The pairs of vertices in  $\kappa$  and  $\lambda$  are called  $\kappa$ -edges and  $\lambda$ -edges, respectively.  $(V, \pi, \mathcal{L})$  is called the underlying complex of  $\mathcal{G}$ .

**Axiom 1** Every proto-edge is both a  $\kappa$ -edge and a  $\lambda$ -edge:  $\pi \subseteq \kappa \cap \lambda$ .

**Axiom 2** For all  $e \in (\kappa \cup \lambda) \setminus \pi$ , some loop contains both vertices of  $e$ .

**Axiom 3** If  $x, y \in L \in \mathcal{L}$ , but  $x$  and  $y$  are not  $\pi$ -consecutive on  $L$ , then

- (a)  $\{x, y\}$  is a  $\lambda$ -edge if and only if  $L \setminus \{x, y\}$  is not  $\kappa$ -connected.
- (b)  $\{x, y\}$  is a  $\kappa$ -edge if and only if  $L \setminus \{x, y\}$  is not  $\lambda$ -connected.

Regarding Axiom 2, note that if  $e \in (\kappa \cup \lambda) \setminus \pi$  (i.e.,  $e$  is a  $\kappa$ - or  $\lambda$ -edge that is not a proto-edge) then there can only be one loop that contains both vertices of  $e$ , by condition (ii) in the definition of a 2D digital complex.

As illustrations of Axiom 3, observe that both  $((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (4, 8))$  and  $((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (8, 4))$  satisfy Axiom 3, but  $((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (4, 4))$  violates the “if” parts of the axiom, while  $((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (8, 8))$  violates the “only if” parts of the axiom.

A GADS is said to be *finite* if it has finitely many vertices; otherwise it is said to be *infinite*. The set of all GADS can be ordered as follows:

**Definition 3 ( $\subseteq$  order, subGADS)** Let  $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$  and  $\mathcal{G}' = ((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$  be GADS such that

- $V \subseteq V'$ ,  $\pi \subseteq \pi'$  and  $\mathcal{L} \subseteq \mathcal{L}'$ .
- For all  $L \in \mathcal{L}$ ,  $\kappa \cap L^{\{2\}} = \kappa' \cap L^{\{2\}}$  and  $\lambda \cap L^{\{2\}} = \lambda' \cap L^{\{2\}}$ .

Then we write  $\mathcal{G} \subseteq \mathcal{G}'$  and say that  $\mathcal{G}$  is a subGADS of  $\mathcal{G}'$ . We write  $\mathcal{G} \subsetneq \mathcal{G}'$  to mean  $\mathcal{G} \subseteq \mathcal{G}'$  and  $\mathcal{G} \neq \mathcal{G}'$ . We write  $\mathcal{G} < \mathcal{G}'$  to mean  $\mathcal{G} \subsetneq \mathcal{G}'$  and  $\mathcal{L} \neq \mathcal{L}'$ .

The following simple but important property of GADS is an immediate consequence of the symmetry of Axioms 1, 2 and 3 with respect to  $\kappa$  and  $\lambda$ :

**Property 1** *If  $((V, \pi, \mathcal{L}), (\kappa, \lambda))$  is a GADS then  $((V, \pi, \mathcal{L}), (\lambda, \kappa))$  is also a GADS. So any statement which is true of every GADS  $((V, \pi, \mathcal{L}), (\kappa, \lambda))$  remains true when  $\kappa$  is replaced by  $\lambda$  and  $\lambda$  by  $\kappa$ .*

### 2.3 Interior Vertices and pgads

We are particularly interested in those GADS that model a surface without boundary. The next definition gives a name for any such GADS.

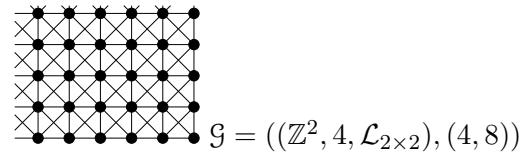
**Definition 4 (pGADS)** *A pGADS is a GADS in which every proto-edge is included in two loops. (The p in pGADS stands for pseudomanifold.)*

A finite pGADS models a closed surface. A pGADS that models the Euclidean plane must be infinite.

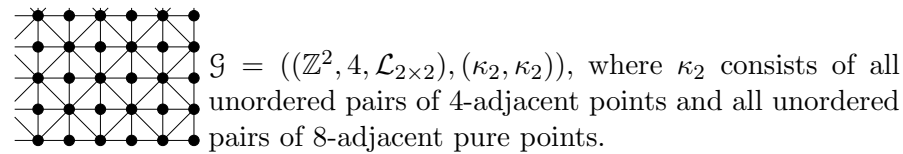
A vertex  $v$  of a GADS  $\mathcal{G}$  is called an *interior vertex* of  $\mathcal{G}$  if every proto-edge of  $\mathcal{G}$  that contains  $v$  is included in two loops of  $\mathcal{G}$ . It follows that a GADS  $\mathcal{G}$  is a pGADS if and only if every vertex of  $\mathcal{G}$  is an interior vertex.

Below are pictures of some pGADS.

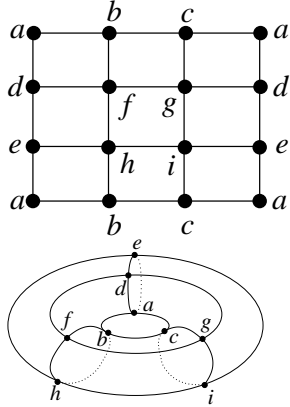
**Example 1**  $\mathbb{Z}^2$  with the 4- and 8-adjacency relations



**Example 2**  $\mathbb{Z}^2$  with Khalimsky's adjacency relation



**Example 3** *A torus-like pGADS*



$$\begin{aligned}
\mathcal{G} &= ((V, \kappa, \mathcal{L}), (\kappa, \lambda)) \\
V &= \{a, b, c, d, e, f, g, h, i\} \\
\kappa &= \{\{a, b\}, \{b, c\}, \{c, a\}, \{d, f\}, \{f, g\}, \{g, d\}, \\
&\quad \{e, h\}, \{h, i\}, \{i, e\}, \{b, f\}, \{c, g\}, \{a, d\}, \\
&\quad \{f, h\}, \{g, i\}, \{d, e\}, \{h, b\}, \{i, c\}, \{e, a\}\} \\
\lambda &= \{\{x, y\} \mid \exists L \in \mathcal{L}, x, y \in L\} \text{ (not shown)} \\
\mathcal{L} &= \{\{a, b, f, d\}, \{d, f, h, e\}, \{e, h, b, a\}, \\
&\quad \{b, c, g, f\}, \{f, g, i, h\}, \{h, i, c, b\}, \\
&\quad \{c, a, d, g\}, \{g, d, e, i\}, \{i, e, a, c\}\}
\end{aligned}$$

In the sequel of this paper,  $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$  is a GADS.

### 3 Simple points in a GADS

In this section, we will define *simple points* in a GADS. In the classical meaning, simple points are points that can be deleted while preserving the topology of an image. By “preserving the topology” we mean preserving connectivity and holes. Proving that our definition is suitable in this sense will be the purpose of Section 5.

Several definitions and characterizations have been given for simple points in classical (2D or 3D) digital spaces. See for example [17, 1, 15, 19] for an overview. Our purpose here is to state a definition and a characterization within the generic framework of GADS, thus generalizing this classical notion for any admissible “surface like” digital space.

We can give in intuitive words a first definition of a simple point. Indeed, a point  $x \in X \subset V$  is said to be  $\kappa$ -simple for  $X$  if and only if:

- $X$  and  $X \setminus \{x\}$  have the same number of  $\kappa$ -connected components;
- $\overline{X}$  and  $\overline{X} \cup \{x\}$  have the same number of  $\lambda$ -connected components;
- $X$  and  $X \setminus \{x\}$  have the same holes.

We will now define a few notations that will allow us to state a formal definition of a simple point.

For any vertex  $v$  of  $\mathcal{G}$ , the *punctured loop neighborhood* of  $v$  in  $\mathcal{G}$ , denoted by  $N_{\mathcal{L}}(v)$ , is defined to be the union of all the loops of  $\mathcal{G}$  which contain  $v$ , minus the vertex  $v$  itself.

Let  $x \in V$ . The axioms given in Section 2 somehow guarantee that loops are topological disks. However,  $N_{\mathcal{L}}(x)$  needs not to be a topological disk (see the punctured loop neighborhood of any of the points in Example 3). Thus, we need to define a topology on  $N_{\mathcal{L}}(x)$  under which

it is a topological disk. Let  $y$  and  $y'$  be two points of  $N_{\mathcal{L}}(x) \cup \{x\}$ . We say that  $y$  and  $y'$  are  $\kappa_x$ -adjacent (respectively  $\lambda_x$ -adjacent) if  $\{y, y'\} \in \kappa$  (respectively  $\{y, y'\} \in \lambda$ ) and  $y$  and  $y'$  are both contained in a loop containing  $x$ . If  $X \subset V$ , we denote by  $G_{\kappa}(x, X)$  (resp.  $G_{\lambda}(x, X)$ ) the graph whose vertices are the points of  $N_{\mathcal{L}}(x) \cap X$  and whose edges are pairs of  $\kappa_x$ -adjacent points (resp.  $\lambda_x$ -adjacent points) of  $N_{\mathcal{L}}(x) \cap X$ . Let  $\rho = \kappa$  or  $\rho = \lambda$ . We denote by  $\mathcal{C}_{\rho}^x(G_{\rho}(x, X))$  the set of connected components of  $G_{\rho}(x, X)$  that are  $\rho$ -adjacent to  $x$ . Note that  $\mathcal{C}_{\rho}^x(G_{\rho}(x, X))$  is a set of subsets of points of  $V$  and not a set of points.

**Definition 5** *We call  $x$  a  $\rho$ -isolated point of  $X$  if  $N_{\rho}(x) \cap X = \emptyset$  and a  $\rho$ -interior point if  $N_{\rho}(x) \cap \overline{X} = \emptyset$ .*

We can now state our definition of a simple point, which is also a local characterization.

**Definition 6 (Simple point)** *A point  $x$  is said to be  $\kappa$ -simple in  $X$  if and only if the number  $\text{Card}(\mathcal{C}_{\kappa}^x(G_{\kappa}(x, X)))$  of connected components of  $G_{\kappa}(x, X)$  which are  $\kappa$ -adjacent to  $x$  is equal to 1 and  $x$  is not interior to  $X$ .*

The following Lemma is a first step towards the justification of Definition 6.

**Lemma 1** *Let  $X \subset V$  and  $x \in X$  be a  $\kappa$ -simple point of  $X$ . Then:*

- $X$  and  $X \setminus \{x\}$  have the same number of  $\kappa$ -connected components;
- $\overline{X}$  and  $\overline{X} \cup \{x\}$  have the same number of  $\lambda$ -connected components.

We just give here the main argument for the proof of this Lemma. In order to prove that no  $\kappa$ -connected component is created, just consider two  $\kappa$ -connected points in  $X$  that are no longer connected in  $X \setminus \{x\}$ . If  $c$  is a shortest  $\kappa$ -path in  $X$  between the two points, then the point  $a$  that precedes  $x$  in  $c$  and the point  $b$  that follows  $x$  in  $c$  are both contained in  $C$ , the only connected component of  $G_{\kappa}(x, X)$  which is  $\kappa$ -adjacent to  $x$ . Therefore,  $a$  and  $b$  are  $\kappa$ -connected in  $X \setminus \{x\}$ , like the two initial points.

It remains to be proved that the removal of a simple point preserves holes. This will be the purpose of Section 5.

## 4 Homotopic Paths and the Digital Fundamental Group of a GADS

In this section,  $\rho$  is a subset of  $V^{\{2\}}$  such that  $\rho \in \{\kappa, \lambda, \pi\}$ , and  $X$  is a  $\rho$ -connected subset of  $V$ .



Loosely speaking, two  $\rho$ -paths in  $X$  with the same initial and the same final vertices are said to be  $\rho$ -homotopic within  $X$  in  $\mathcal{G}$  if one of the paths can be transformed into the other by a sequence of small local deformations within  $X$ . The initial and final vertices of the path must remain fixed throughout the deformation process. The next two definitions make this notion precise.

**Definition 7 (elementary  $\mathcal{G}$ -deformation)** *Two finite vertex sequences  $c$  and  $c'$  of  $\mathcal{G}$  with the same initial and the same final vertices are said to be the same up to an elementary  $\mathcal{G}$ -deformation if there exist vertex sequences  $c_1, c_2, \gamma$  and  $\gamma'$  such that  $c = c_1.\gamma.c_2$ ,  $c' = c_1.\gamma'.c_2$ , and either there is a proto-edge  $\{x, y\}$  for which one of  $\gamma$  and  $\gamma'$  is  $(x)$  and the other is  $(x, y, x)$ , or there is a loop of  $\mathcal{G}$  that contains all of the vertices in  $\gamma$  and  $\gamma'$ .*

**Definition 8 (homotopic  $\rho$ -paths)** *Two  $\rho$ -paths  $c$  and  $c'$  in  $X$  with the same initial and the same final vertices are  $\rho$ -homotopic within  $X$  in  $\mathcal{G}$  if there exists a sequence of  $\rho$ -paths  $c_0, \dots, c_n$  in  $X$  such that  $c_0 = c$ ,  $c_n = c'$  and, for  $0 \leq i \leq n-1$ ,  $c_i$  and  $c_{i+1}$  are the same up to an elementary  $\mathcal{G}$ -deformation. Two  $\rho$ -paths with the same initial and the same final vertices are said to be  $\rho$ -homotopic in  $\mathcal{G}$  if they are  $\rho$ -homotopic within  $V$  in  $\mathcal{G}$ .*

The next proposition states a useful characterization of  $\rho$ -homotopy that is based on a more restrictive kind of local deformation than was considered above, which allows only the insertion or removal of either a “ $\rho$ -back-and-forth” or a cycle that parameterizes a simple closed  $\rho$ -curve in a loop of  $\mathcal{G}$ .

**Definition 9 (minimal  $\rho$ -deformation)** *Two  $\rho$ -paths  $c$  and  $c'$  with the same initial and the same final vertices are said to be the same up to a minimal  $\rho$ -deformation in  $\mathcal{G}$  if there exist  $\rho$ -paths  $c_1, c_2$  and  $\gamma$  such that one of  $c$  and  $c'$  is  $c_1.\gamma.c_2$ , the other of  $c$  and  $c'$  is  $c_1.c_2$ , and either  $\gamma = (x, y, x)$  for some  $\rho$ -edge  $\{x, y\}$  or  $\gamma$  is a  $\rho$ -parameterization of a simple closed  $\rho$ -curve whose vertices are contained in a single loop of  $\mathcal{G}$ .*

This concept of deformation is particularly simple when  $\rho = \pi$ , because a simple closed  $\pi$ -curve whose vertices are contained in a single loop of  $\mathcal{G}$  must in fact be a loop of  $\mathcal{G}$ , since a loop of  $\mathcal{G}$  is a simple closed  $\pi$ -curve.

**Proposition 1** *Two  $\rho$ -paths  $c$  and  $c'$  in  $X$  with the same initial and the same final vertices are  $\rho$ -homotopic within  $X$  in  $\mathcal{G}$  if and only if there is*

a sequence of  $\rho$ -paths  $c_0, \dots, c_n$  in  $X$  such that  $c_0 = c$ ,  $c_n = c'$  and, for  $0 \leq i \leq n-1$ ,  $c_i$  and  $c_{i+1}$  are the same up to a minimal  $\rho$ -deformation in  $\mathcal{G}$ .

The proof of this proposition is not particularly difficult, and we leave it to the interested reader.

Now, let  $b \in X$  be a point called the *base point*. We denote by  $A_b^\rho(X)$  the set of all closed  $\rho$ -paths  $c = (x_0, \dots, x_l)$  which are contained in  $X$  such that  $x_0 = x_l = b$ . The  $\rho$ -homotopy relation is an equivalence relation on  $A_b^\rho(X)$ , and we denote by  $\Pi_1^\rho(X, b)$  the set of equivalence classes of this equivalence relation. The concatenation of paths is compatible with the  $\rho$ -homotopy relation, hence it defines an operation on  $\Pi_1^\rho(X, b)$  which to the class of  $c_1$  and  $c_2$  associates the class of  $c_1.c_2$ . This operation provides  $\Pi_1^\rho(X, b)$  with a group structure. We call this group the *digital  $\rho$ -fundamental group of  $X$* .

Now, we consider  $Y \subset X \subset V$  and  $b \in X$  a base point. Any closed  $\rho$ -path in  $Y$  is a particular case of a closed  $\rho$ -path in  $X$ . Furthermore, if two closed  $\rho$ -paths in  $Y$  are  $\rho$ -homotopic in  $Y$ , then they are also  $\rho$ -homotopic in  $X$ . These two properties enable us to define a canonical morphism  $i_* : \Pi_1^\rho(Y, b) \longrightarrow \Pi_1^\rho(X, b)$  induced by the inclusion map  $i : Y \longrightarrow X$ . To the class of a closed  $\rho$ -path  $c \in A_b^\rho(Y)$  in  $\Pi_1^\rho(Y, b)$  the morphism  $i_*$  associates the class of the same  $\rho$ -path in  $\Pi_1^\rho(X, b)$ .

## 5 Simple points and the digital fundamental group

Here, we show that simple points have been properly defined. For this purpose, we use the formalism of the digital fundamental group. Indeed, it allows us to prove that “holes are preserved” when one removes a simple point from  $X$ , a subset of the set of points of a GADS.

In this section,  $\rho$  is either equal to  $\kappa$  or equal to  $\lambda$ , and  $X$  is a  $\rho$ -connected subset of  $V$ .

**Lemma 2** *Let  $b \in X$  and let  $x \in X$  be a  $\rho$ -simple point distinct from  $b$ . Then any  $\rho$ -path of  $A_b^\rho(X)$  is  $\rho$ -homotopic to a  $\rho$ -path contained in  $X \setminus \{x\}$ .*

**Proof:** Let  $c = (x_0, \dots, x_p)$  be a  $\rho$ -path in  $X$  such that  $x_0 \neq x$  and  $x_p \neq x$ . We define a  $\rho$ -path  $P(c)$  as follows: For any maximal sequence  $\gamma = (x_k, \dots, x_l)$  with  $0 \leq k \leq l \leq p$  of points of  $c$  such that for all  $i = k, \dots, l$  we have  $x_i \neq x$ , we define  $s(\gamma) = \gamma$ . For any maximal sequence  $\gamma = (x_k, \dots, x_l)$  with  $0 \leq k \leq l \leq p$  of points of  $c$  such that for  $i = k, \dots, l$

we have  $x_i = x$ , we define  $s(\gamma)$  as equal to the shortest  $\rho$ -path from  $x_{l-1}$  to  $x_{k+1}$  in the single connected component of  $G_\rho(x, X)$ . Now  $P(c)$  is the concatenation of all  $s(\gamma)$  for all maximal sequences  $\gamma = (x_k, \dots, x_l)$  of points of  $c$  such that either for  $i = k, \dots, l$  we have  $x_i \neq x$  or for  $i = k, \dots, l$  we have  $x_i = x$ . Now, it is readily seen that  $c$  is  $\rho$ -homotopic to  $P(c)$  in  $X$ .  $\square$

**Remark 1** *If  $x \in X$  is a  $\rho$ -simple point in  $X$  and  $C$  is the single connected component of  $G_\rho(x, X)$ , then any two  $\rho$ -paths in  $C$  with the same extremities are  $\rho$ -homotopic in  $X \setminus \{x\}$ .*

**Lemma 3** *Let  $b \in X$  and let  $x \in X$  be a  $\rho$ -simple point distinct from  $b$ . If two closed  $\rho$ -paths  $c_1$  and  $c_2$  of  $A_b^\rho(X \setminus \{x\})$  are  $\rho$ -homotopic in  $X$ , then they are  $\rho$ -homotopic in  $X \setminus \{x\}$ .*

**Proof:** Let  $P(c_1)$  and  $P(c_2)$  be the two paths as defined in the proof of Lemma 2. Following Proposition 1, it is sufficient to prove that if  $c_1$  and  $c_2$  are the same up to a minimal  $\rho$ -deformation in  $X$ , then the two  $\rho$ -paths  $P(c_1)$  and  $P(c_2)$  are  $\rho$ -homotopic in  $X \setminus \{x\}$ . Thus, we suppose that  $c_1 = c.\gamma.c'$  and  $c_2 = c.c'$  where  $\gamma$  is either equal to  $(a, b, a)$  (with  $\{a, b\}$  being a  $\rho$ -edge) or is a simple closed  $\rho$ -curve included in a loop  $L$  of  $\mathcal{L}$ . First, if we suppose that  $x$  does not belong to  $\gamma$ , then it is immediate that  $P(c_1)$  and  $P(c_2)$  are  $\rho$ -homotopic. Therefore, we may suppose without loss of generality that  $x$  belongs to  $\gamma$ . Furthermore, in order to clarify the proof, we suppose in the sequel that all paths are such that any two consecutive points are distincts. Let  $C$  be the only connected component of  $G_\rho(x, X)$ .

In a first case, we suppose that  $\gamma = (a, b, a)$  with  $x = a$ . We write  $c_1 = \mu.(y, x, z).\mu'$  with  $c = \mu.(y, x)$  and  $c' = (x, z).\mu'$ . Thus, we have  $c_2 = \mu.(y, x, b, x, z).\mu'$ . Now,  $y, z$  and  $b$  all belong to  $C$ . Let  $\alpha$  be the shortest  $\rho$ -path from  $y$  to  $b$  in  $C$  and let  $\beta$  be the shortest  $\rho$ -path from  $b$  to  $z$  in  $C$ . Finally, let  $\gamma'$  be the shortest  $\rho$ -path from  $y$  to  $z$  in  $C$ . We have  $P(c_1) = P(\mu).\gamma'.P(\mu')$  and  $P(c_2) = P(\mu).\alpha.\beta.P(\mu')$ . From the previous remark and since  $\gamma'$  and  $\alpha.\beta$  are  $\rho$ -paths in  $C$  with the same extremities, we obtain that  $P(c_1)$  and  $P(c_2)$  are  $\rho$ -homotopic in  $X \setminus \{x\}$ .

The proof is similar in the case when  $\gamma$  is a closed  $\rho$ -path from the vertex  $x$  to  $x$ .

In the case when  $\gamma = (a, b, a)$  with  $x = b$  we obtain that  $P(c_1) = P(c_2)$ .

Remains the case when  $\gamma$  is a simple closed  $\rho$ -curve included in a loop and containing  $x$  (not being an extremity). But in this case,  $P(\gamma)$  is included in  $C$  and therefore  $\rho$ -homotopic to the path reduced to its

extremities. Since in this case  $P(c_1) = P(c).P(\gamma).P(c')$  and  $P(c_2) = P(c).P(c')$ , we obtain that  $P(c_1)$  and  $P(c_2)$  are  $\rho$ -homotopic.  $\square$

**Theorem 1** *Let  $b \in X$  and let  $x \in X$  be a  $\rho$ -simple point of  $X$  distinct from  $b$ . The morphism  $i_* : \Pi_1^\rho(X \setminus \{x\}, b) \longrightarrow \Pi_1^\rho(X, b)$  induced by the inclusion map of  $X \setminus \{x\}$  in  $X$  is a group isomorphism.*

**Proof:** Lemma 2 implies that  $i_*$  is onto and Lemma 3 implies that  $i_*$  is one to one.  $\square$

The latter lemma is the main result of this section. Indeed, it states that when one removes a simple point (following Definition 6) from a connected set  $X$ , then no hole is created nor removed. This, added to the fact that the removal of a simple point cannot create some new connected components nor remove any one (see Lemma 1), leads to the justification of the following affirmation:

“Removing a simple point preserves the topology.”

This achieves the justification of the local characterization of simple points in a GADS given by Definition 6.

## 6 Concluding Remarks

We have introduced two new notions in the context of GADS: the digital fundamental group and the notion of simple point. In so doing, we illustrate the power of this axiomatic theory that allows us to prove very general results of digital topology. Indeed, the two previously mentioned notions are now valid for any two dimensional digital space that one can reasonably consider.

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