# A continuous semigroup of notions of independence between the classical and the free one

Florent Benaych-Georges, Thierry Lévy

# A CONTINUOUS SEMIGROUP OF NOTIONS OF INDEPENDENCE BETWEEN THE CLASSICAL AND THE FREE ONE

FLORENT BENAYCH-GEORGES AND THIERRY LÉVY

Abstract. In this paper, we investigate a continuous family of notions of independence which interpolates between the classical and free ones for non-commutative random variables. These notions are related to the liberation process introduced by D. Voiculescu. To each notion of independence correspond new convolutions of probability measures, for which we establish formulae and of which we compute simple examples. We prove that there exists no reasonable analogue of classical and free cumulants associated to these notions of independence.

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## Introduction

Free probability is the non-commutative probability theory built in parallel with the classical one, but upon the notion of independence called freeness instead of the classical independence. Concretely, independent random variables are numbers arising randomly with no influence on each other, whereas free random variables are elements of an operator algebra endowed with a state which do not satisfy any algebraic relation, as far as can be observed with the algebra's state. The point of view of random matrices, which is, like most of the bases of this theory, due to Dan Voiculescu, allows to somehow unify both theories: large diagonal random matrices conjugated by independent uniformly distributed random *permutation matrices* will, as far as traces are concerned, behave like *independent* random variables, whereas large diagonal random matrices conjugated by independent uniformly distributed random *unitary matrices* will behave like *free* random variables. Free probability theory, a very active field of mathematics during

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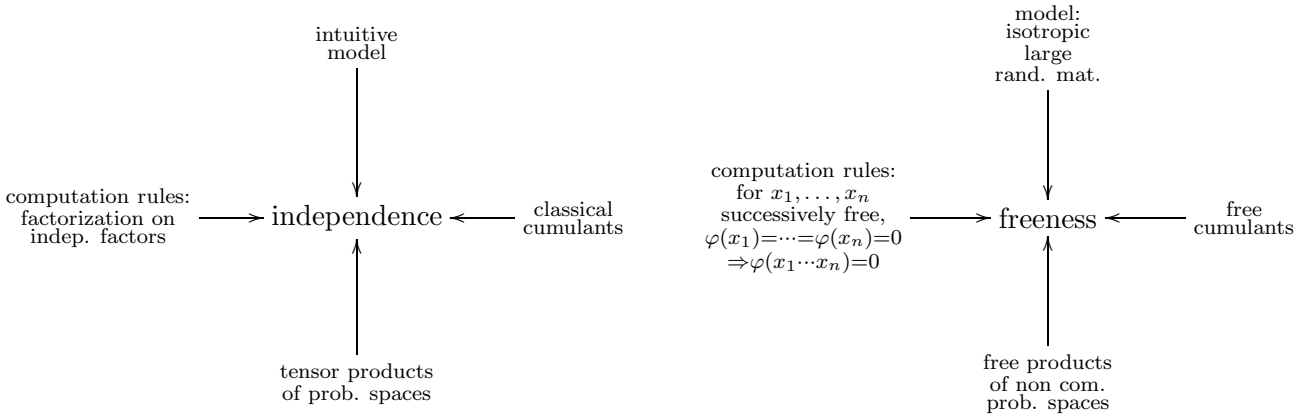
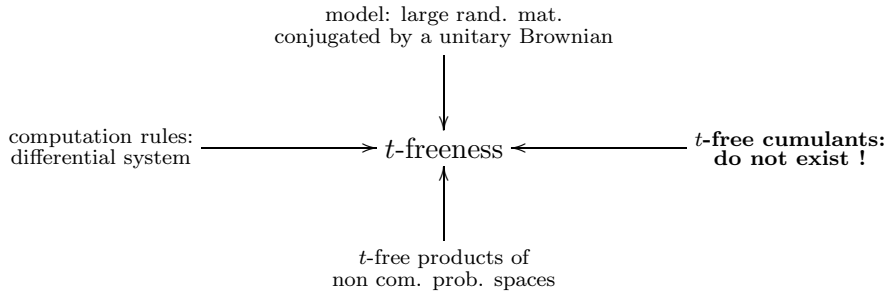
the last two decades, has been constructed in deep analogy with the classical one. It follows that there is a kind of dictionary between objects of both theories: many fundamental notions or results of classical probability theory, like the Law of Large Numbers, the Central Limit Theorem, the Gaussian distribution, the convolution, the cumulants, the infinite divisibility, have a precise analogue in free probability theory. Thus it seems natural to try interpolate between both theories and to find a continuous way to pass from one to the other. But the research of such a continuum has been broken off by a paper of Roland Speicher in 1997 [S97], where it is shown that no other notion of independence than the classical one and the freeness can be the base of a reasonable probability theory. However, even though the axioms in the definition given in this paper for a "reasonable probability theory" seemed very natural, one of them, the *associativity*[1], seems to be too restrictive. Indeed, it appeared to us that replacing associativity by a "semigroup version" of associativity allows one to build a continuous family of "notions of independence" indexed by $[0, +\infty]$ which passes from the classical independence (case $t = 0$) to the free one (case $t = +\infty$). More specifically, we shall define, for all $t \geq 0$, a notion called $t$-freeness such that for all $s, t \geq 0$, if $X, Y, Z$ are three random variables such that $X$ is $t$-free with $Y$ and $Y$ is $s$-free with $Z$, then under certain additional hypotheses, $X$ will be $(s + t)$-free with $Z$. In a non-commutative probability space, two elements will be said to be $t$-free if they can be obtained from two classically independent elements by conjugating one of them by a free unitary Brownian motion taken at time $t$. Equivalently, the general random matrix model for a pair of $t$-free elements, is obtained at the limit when the dimension $n$ of the matrices tends to infinity, from a pair of diagonal random matrices by conjugating one of them by a unitary Brownian motion taken at time $t$, with initial law the uniform law on the group of permutation matrices. This can be considered as a particular case of the so-called *liberation process* introduced by Voiculescu [V99]. The notion of $t$-freeness gives rise to additive and multiplicative convolutions, denoted respectively by $*_t, \odot_t$, and called additive and multiplicative $t$-free convolution, which pass from the classical convolutions to the free ones as $t$ goes from zero to $+\infty$.

There are several ways to characterize and deal with independence and freeness. The first one is to consider what it models: independence concerns numbers chosen randomly independently and freeness concerns matrices chosen randomly, in an independent and isotropic way. The second one is given by the computation rules: the expectation factorizes on independence, whereas the expectation of a product of free elements can be computed using the fact that if $x_1, \ldots, x_n$ are centered and successively free, then their product is centered. The third one is the use of integral transforms (Fourier transform or $R$-transform) i.e. the use of cumulants. The last one, a bit more abstract, is to consider tensor of free product: a family of random variables is independent (resp. free) if it can be realized on a tensor (resp. free) product of probability spaces. This can be summarized in the following diagrams.

[1]Roughly speaking, the associativity property states that if $X, Y, Z$ are three random variables such that $X, Y$ are independent and $Z$ is independent of $(X, Y)$, then $X$ is independent of $(Y, Z)$.




In the present paper, we complete such a diagram for the notion of $t$-freeness. We begin, in Section 2, by giving the definition of a $t$-free product and presenting the random matrix model, mentioned in the previous paragraph. Then, in Section 3, we state the computation rules, which can be understood as a family of differential equations. Finally, in Section 4, we prove that no notion of cumulants can be associated to the notion of $t$-freeness. This can be summarized in the following diagram.




We also study certain examples, like the the $t$-convolutions of symmetric Bernoulli laws.

## 1. Preliminaries

1.1. **General definitions related to non-commutative probability theory.** Non commutative probability is based on the following generalization of the notion of probability space.

**Definition 1.1** (Non commutative probability space)**.** It is a pair $(\mathcal{A}, \varphi)$, where:

- $\mathcal{A}$ is an algebra over $\mathbb{C}$ with a unit element denoted by 1, endowed with an operation of adjunction $x \mapsto x^*$ which is $\mathbb{C}$-antilinear, involutive and satisfies $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{A}$,
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear form on $\mathcal{A}$, satisfying $\varphi(1) = 1$, $\varphi(xy) = \varphi(yx)$, $\varphi(x^*) = \overline{\varphi(x)}$ and $\varphi(xx^*) \geq 0$ for all $x, y \in \mathcal{A}$.

The linear form $\varphi$ is often called the *expectation* of the non-commutative probability space.

Here are two very natural examples of non-commutative probability spaces.

**Example 1.2** (Classical probability space)**.** Let $(\Omega, \Sigma, \mathbb{P})$ be a classical probability space where the expectation is denoted by $\mathbb{E}$. Then the algebra of complex-valued random variables defined on this space having moments to all orders, endowed with the involution $X^* = \overline{X}$ and with the

linear form $\mathbb{E}$ is a non-commutative probability space, of which we will say that it is *inherited* from a classical probability space.

**Example 1.3** (Space of matrices endowed with the normalize trace)**.** Let $n \geq 1$ be an integer. Then the algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ endowed with the linear form $\varphi$ which is the normalized trace is a non-commutative probability space.

The second notion from classical probability which has a natural non-commutative generalization is the one of distribution of a family of random variables.

**Definition 1.4** (Non commutative distribution)**.** The *non-commutative distribution* of a family $(a_1, \ldots, a_n)$ of elements of a non-commutative probability space $(\mathcal{A}, \varphi)$ is the linear map defined on the space of polynomials in the non-commutative variables $X_1, X_1^*, \ldots, X_n, X_n^*$ which maps any such polynomial $P$ to $\varphi(P(a_1, a_1^*, \ldots, a_n, a_n^*))$.

**Remark 1.5.** Consider a self-adjoint element $a$ (i.e. such that $a = a^*$) in a non-commutative probability space $(\mathcal{A}, \varphi)$. Then by the hypothesis $\varphi(xx^*) \geq 0$ for all $x \in \mathcal{A}$, the distribution of $a$ is a linear form on $\mathbb{C}[X]$ which is non negative on the polynomials which are non negative on the real line. Hence it is the integration with respect to a probability measure on the real line. Note that this measure is unique if and only if it is determined by its moments, which is the case when the measure has compact support, i.e. when there is a constant $M$ such that for all $n \geq 0$, one has $\varphi(a^{2n}) \leq M^{2n}$.

**Remark 1.6.** The content of the previous remark can be generalized to any unitary element: its distribution is the integration with respect to a probability measure on the unit circle of $\mathbb{C}$.

**Remark 1.7.** Note, as a generalization of Remark 1.5, that in the case of a non-commutative probability space inherited from a classical one (see Example 1.2), the non-commutative distribution of a family of random variables is the integration of polynomials with respect to the law of the family (defined in the classical way).

## 1.2. **Independence, freeness and random matrices.**

1.2.1. *Definitions and basic properties.* After the notions of probability space and of distribution, one of the most important notion in probability theory is the independence. Here, we shall present two definitions. One of them is the translation of the classical notion of independence in the context of non-commutative probability spaces, which shall coincide with independence in the case of a non-commutative probability space inherited from a classical one (see Example 1.2). The second one is another notion, defined by Voiculescu [VDN91], which is called freeness. In the present paper, we shall define a "continuum of notions" between those two ones.

In this paper, by a subalgebra of the algebra of a non-commutative probability space, we shall always mean a subalgebra which contains 1 and which is stable under the operation $x \mapsto x^*$.

**Definition 1.8** (Independence and freeness)**.** Let $(\mathcal{M}, \varphi)$ be a non-commutative probability space. The kernel of $\varphi$ will be called the set of *centered elements*. Consider a family $(\mathcal{A}_i)_{i \in I}$ of subalgebras of $\mathcal{M}$.

- The family $(\mathcal{A}_i)_{i \in I}$ is said to be *independent* if
  - (i) for all $i \neq j \in I$, $\mathcal{A}_i$ and $\mathcal{A}_j$ commute,
  - (ii) for all $n \geq 1$, $i_1, \ldots, i_n \in I$ pairwise distinct, for all family $(a_1, \ldots, a_n) \in \mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_n}$ of centered elements, the product $a_1 \cdots a_n$ is also centered.
- The family $(\mathcal{A}_i)_{i \in I}$ is said to be *free* if for all $n \geq 1$, $i_1, \ldots, i_n \in I$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$, for all family $(a_1, \ldots, a_n) \in \mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_n}$ of centered elements, the product $a_1 \cdots a_n$ is also centered.

Note that in the case of a non-commutative probability space inherited from a classical one (see Example 1.2), the notion of independence as it is presented here is exactly the classical notion of independence: with the notation of Example 1.2, let us define, for all family $(\Sigma_i)_{i\in I}$ of sub-$\sigma$-fields of $\Sigma$, and for all $i \in I$, the set $\mathcal{A}_i$ of random variables of $\mathcal{A}$ which are $\Sigma_i$-measurable. Then the family $(\Sigma_i)_{i\in I}$ is independent in the classical sense if and only if the family $(\mathcal{A}_i)_{i\in I}$ is independent in the sense of Definition 1.8.

It is well known that a family of random variables is independent if and only if its joint law is the tensor product of the individual ones. In the following definition and proposition, we translate this statement to our vocabulary, and give its analogue for freeness. These definitions shall also be needed to define $t$-freeness in the following.

**Definition 1.9** (Tensor and free product)**.** Let $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ be two non-commutative probability spaces.

- Their *tensor product*, denoted by $(\mathcal{A}_1, \varphi_1) \otimes (\mathcal{A}_2, \varphi_2)$, is the non-commutative probability space with algebra the tensor product of unital algebras $\mathcal{A}_1 \otimes \mathcal{A}_2$, on which the adjoint operation and the expectation are defined by
$$\forall (x_1, x_2) \in \mathcal{A}_1 \times \mathcal{A}_2, (x_1 \otimes x_2)^* = x_1^* \otimes x_2^*, \quad \varphi(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2).$$
- Their *free product*, denoted by $(\mathcal{A}_1, \varphi_1) * (\mathcal{A}_2, \varphi_2)$, is the non-commutative probability space with algebra the free product of unital algebras $\mathcal{A}_1 * \mathcal{A}_2$, with adjoint operation and expectation defined uniquely by the fact that for all $n \geq 1$, for all $i_1 \neq \cdots \neq i_n \in \{1, 2\}$, for all $(x_1, \ldots, x_n) \in \mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_n}$,
$$(x_1 \cdots x_n)^* = x_n^* \cdots x_1^*$$
and $x_1 \cdots x_n$ is centered whenever all $x_i$'s are.

Note that this definition can easily be extended to tensor and free products of finite families of non-commutative probability spaces. It can also easily be extended to families indexed by any set, but in order to avoid cumbersome notation, we have restricted the definitions to what is needed in this article.

The link between the previous notions and the classical characterization of independence in terms of distributions is made in the following proposition.

**Proposition 1.10** (Characterization of independence and freeness)**.** *Let $(\mathcal{M}, \varphi)$ be a non-commutative probability space. Let $\mathcal{A}_1, \mathcal{A}_2$ be subalgebras of $\mathcal{A}$. Then the family $(\mathcal{A}_1, \mathcal{A}_2)$ is*

- *independent if and only if $\mathcal{A}_1$ commutes with $\mathcal{A}_2$ and the unique algebra morphism defined from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{M}$ which, for all $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, maps $a_1 \otimes 1$ to $a_1$ and $1 \otimes a_2$ to $a_2$, preserves the expectation from $(\mathcal{A}_1, \varphi_{|\mathcal{A}_1}) \otimes (\mathcal{A}_2, \varphi_{|\mathcal{A}_2})$ to $(\mathcal{M}, \varphi)$,*
- *free if and only if the unique algebra morphism defined from the unital algebras tensor product $\mathcal{A}_1 * \mathcal{A}_2$ to $\mathcal{M}$ which, restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$ is the canonical injection, preserves the expectation from $(\mathcal{A}_1, \varphi_{|\mathcal{A}_1}) * (\mathcal{A}_2, \varphi_{|\mathcal{A}_2})$ to $(\mathcal{M}, \varphi)$.*

In the following definition, the free analogue of the classical convolution is defined.

**Definition 1.11** (Additive free convolution)**.** A consequence of the last proposition is that the distribution of the sum of two free self adjoint elements with respective distributions $\mu, \nu$ only depends on $\mu$ and $\nu$ and will be called the *free additive convolution* of $\mu$ and $\nu$, and be denoted by $\mu \boxplus \nu$.

1.2.2. *Asymptotic behavior of random matrices.*

**Definition 1.12** (Convergence in non-commutative distribution)**.** Let $p$ be a positive integer and let, for each $n \geq 1$, $(M(1,n), \ldots, M(p,n))$ be a family of $n$ by $n$ random matrices. This family is said to converge in non-commutative distribution if its non-commutative distribution converges in probability to a non random one, i.e. if the normalized trace of any word in the $M(i,n)$'s and the $M(i,n)^*$'s converges in probability to a constant.

As mentioned above, random variables which are independent in the usual sense are elements of independent algebras in the sense developed in this paper. It is not so easy to construct free random variables without using explicitly the free product of definition 1.9. However, in 91, Voiculescu, in [V91], proved that random matrices conjugated by independent unitary random matrices with Haar distribution are asymptotically free. In the following theorem, we present this result and its analogue for freeness, where unitary matrices are replaced by permutation matrices.

**Theorem 1.13** (Asymptotic independence and asymptotic freeness)**.** *Let us fix $p, q \geq 1$. Let for each $n \geq 1$, $\mathcal{F}_n = (A(1,n), \ldots, A(p,n), B(1,n), \ldots, B(q,n))$ be a family of $n$ by $n$ random matrices which converges in non-commutative distribution. We suppose that for all $r \geq 1$, the entries of these random matrices are uniformly bounded in $L^r$.*

- *Suppose these matrices to be diagonal and consider, for each $n$, the matrix $S_n$ of a uniformly distributed random permutation of $\{1, \ldots, n\}$ independent of the family $\mathcal{F}_n$. Then the family*
$$(A(1,n), \ldots, A(p,n), S_n B(1,n) S_n^{-1}, \ldots, S_n B(q,n) S_n^{-1})$$
*converges in distribution to the distribution of a commutative family $(a_1, \ldots, a_p, b_1, \ldots, b_q)$ of elements of a non-commutative probability space such that the algebras generated by $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_q\}$ are independent.*
- *Consider, for each $n$, the matrix $U_n$ of a uniformly distributed random unitary $n$ by $n$ matrix independent of the family $\mathcal{F}_n$. Then the family*
$$(A(1,n), \ldots, A(p,n), U_n B(1,n) U_n^{-1}, \ldots, U_n B(q,n) U_n^{-1})$$
*converges in distribution to the distribution of a family $(a_1, \ldots, a_p, b_1, \ldots, b_q)$ of elements of a non-commutative probability space such that the algebras generated by $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_q\}$ are free.*

The first part of this theorem is much simpler than the second but seems to be also less well-known. It is in any case harder to locate a proof in the literature, so that we offer one. We shall need the following lemma.

**Lemma 1.14.** *Let, for each $n \geq 1$, $x(n) = (x_{n,1}, \ldots, x_{n,n})$ and $y(n) = (y_{n,1}, \ldots, y_{n,n})$ be two complex random vectors defined on the same probability space such that the random variables*
$$\overline{x(n)} := \frac{x_{n,1} + \cdots + x_{n,n}}{n}, \quad \overline{y(n)} := \frac{y_{n,1} + \cdots + y_{n,n}}{n}$$
*converge in probability to constant limits $x, y$ as $n$ tends to infinity. Suppose moreover that the sequences $\frac{1}{n}||x(n)||_2^2$ and $\frac{1}{n}||y(n)||_2^2$ are bounded in $L^2$. Consider, for all $n$, a uniformly distributed random permutation $\sigma_n$ of $\{1, \ldots, n\}$, independent of $(x(n), y(n))$, and define $y_{\sigma_n}(n) := (y_{n,\sigma_n(1)}, \ldots, y_{n,\sigma_n(n)})$. Then the scalar product*
$$\frac{1}{n}\langle x(n), y_{\sigma_n}(n)\rangle := \frac{1}{n}(x_{n,1} y_{n,\sigma_n(1)} + \cdots + x_{n,n} y_{n,\sigma_n(n)})$$
*converges in probability to $xy$ as $n$ tends to infinity.*

**Proof.** First of all, note that one can suppose that for all $n$, $\overline{x(n)} = \overline{y(n)} = 0$ almost surely. Indeed, if the result is proved under this additional hypothesis, then since for all $n$, one has

$$\frac{1}{n}\langle x(n), y_{\sigma_n}(n)\rangle = \frac{1}{n}\langle x(n) - \overline{x(n)}\cdot 1_n, y_{\sigma_n}(n) - \overline{y(n)}\cdot 1_n\rangle + \overline{x(n)}\cdot\overline{y(n)}, \quad (\text{with } 1_n = (1,\dots,1)),$$

the result holds for general $x(n), y(n)$. So we suppose that for all $n$, $\overline{x(n)} = \overline{y(n)} = 0$. It implies that for all $n$, for all $i, j = 1, \dots, n$,

$$\mathbb{E}[y_{n,\sigma_n(i)} y_{n,\sigma_n(j)} \mid x(n), y(n)] = \begin{cases} \frac{1}{n}||y(n)||_2^2 & \text{if } i = j, \\ -\frac{1}{n(n-1)}||y(n)||_2^2 & \text{if } i \neq j. \end{cases}$$

Thus, using the fact that $\overline{x(n)} = 0$, we have

$$\begin{aligned} \mathbb{E}\left[\frac{1}{n^2}\langle x(n), y_{\sigma_n}(n)\rangle^2\right] &= \mathbb{E}\left[\frac{1}{n^3(n-1)}||x(n)||_2^2||y(n)||_2^2 + \frac{1}{n^3}||x(n)||_2^2||y(n)||_2^2\right] \\ &= O\left(\frac{1}{n}\right), \end{aligned}$$

which completes the proof. □

**Proof of Theorem 1.13.** The second point is a well-known result of Voiculescu (see [VDN91]). To prove the first one, we shall prove that the normalized trace any word in the random matrices $A(1,n), \dots, A(p,n), S_n B(1,n) S_n^{-1}, \dots, S_n B(q,n) S_n^{-1}$ converges to a constant which factorizes in two terms: the limiting normalized trace of the $A(i,n)$'s and the $A(i,n)^*$'s of the product on one hand and the limiting normalized trace of the $B(j,n)$'s and the $B(j,n)^*$'s of the product on the other hand. Since the $A(i,n)$'s, the $A(i,n)^*$'s, the $S_n B(j,n) S_n^{-1}$'s and the $S_n B(j,n)^* S_n^{-1}$'s commute, are uniformly bounded and their non-commutative distribution converges, this amounts to proving that if $M(n), N(n)$ are two diagonal random matrices having entries uniformly bounded in $L^r$ for all $r \geq 1$, whose normalized traces converge in probability to constants $m, n$, then for $S_n$ the matrix of a uniform random permutation of $\{1, \dots, n\}$ independent of $(M(n), N(n))$, the normalized trace of $M(n) S_n N(n) S_n^{-1}$ converges to $mn$. This follows directly from the previous lemma. Thus the proof is complete. □

**Corollary 1.15** (Matricial model for classical and free convolutions)**.** *Let $\mu, \nu$ be two probability measures on the real line. Let, for each $n \geq 1$, $M_n, N_n$ be $n$ by $n$ diagonal random matrices with empirical spectral measures converging weakly in probability to $\mu$ and $\nu$respectively. For each $n \geq 1$, let $S_n$ (resp. $U_n$) be a uniformly distributed $n$ by $n$ permutation (resp. unitary) random matrix independent of $(M_n, N_n)$. Then*

- *the empirical spectral measure of $M_n + S_n N_n S_n^{-1}$ converges weakly in probability to the classical convolution $\mu * \nu$ of $\mu$ and $\nu$,*
- *the empirical spectral measure of $M_n + U_n N_n U_n^{-1}$ converges weakly in probability to the free convolution $\mu \boxplus \nu$ of $\mu$ and $\nu$.*

**Proof.** In the case where $\mu, \nu$ have compact supports and the entries of the diagonal matrices $M_n, N_n$ are uniformly bounded, it is a direct consequence of the previous theorem. The general case can easily be deduced using functional calculus, like in the proof of Theorem 3.13 of [B07]. □

1.3. **Unitary Brownian motion, free unitary Brownian motion.** In this paragraph, we give a brief survey of the definition and the main convergence result for the Brownian motion on the unitary group.

Let $n \geq 1$ be an integer. Let $\mathcal{H}_n$ denote the $n^2$-dimensional real linear subspace of $\mathcal{M}_n(\mathbb{C})$ which consists of anti-Hermitian matrices. On $\mathcal{M}_n(\mathbb{C})$, we denote by Tr the usual trace and by $\mathrm{tr} = \frac{1}{n}\mathrm{Tr}$ the normalized trace. Let us endow $\mathcal{H}_n$ with the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\forall A, B \in \mathcal{H}_n \ , \ \ \langle A, B \rangle = n\,\mathrm{Tr}(A^* B) = -n\,\mathrm{Tr}(AB).$$

There is a linear Brownian motion canonically attached to the Euclidean space $(\mathcal{H}_n, \langle \cdot, \cdot \rangle)$. It is the unique Gaussian process $H$ indexed by $\mathbb{R}_+$ with values in $\mathcal{H}_n$ such that for all $s, t \in \mathbb{R}_+$ and all $A, B \in \mathcal{H}_n$, one has

$$\mathbb{E}[\langle H_s, A \rangle \langle H_t, B \rangle] = \min(s,t)\langle A, B \rangle.$$

Let us consider the following stochastic differential equation:

$$U_0 = I_n \ , \ \ \mathrm{d}U_t = i(\mathrm{d}H_t)U_t - \frac{1}{2}U_t \mathrm{d}t,$$

where $(U_t)_{t \geq 0}$ is a stochastic process with values in $\mathcal{M}_n(\mathbb{C})$. This linear equation admits a strong solution. The process $(U_t^*)_{t \geq 0}$, where $U_t^*$ denotes the adjoint of $U_t$, satisfies the stochastic differential equation

$$U_0^* = I_n \ , \ \ \mathrm{d}U_t^* = -iU_t^* \mathrm{d}H_t - \frac{1}{2}U_t^* \mathrm{d}t.$$

An application of Itô's formula to the process $U_t U_t^*$ shows that, for all $t \geq 0$, $U_t U_t^* = I_n$. This proves that the process $(U_t)_{t \geq 0}$ takes its values in the unitary group $U(n)$.

**Definition 1.16.** The process $(U_t)_{t \geq 0}$ is called the *unitary Brownian motion of dimension n*.

As $n$ tends to infinity, the unitary Brownian motion has a limit in distribution which we now describe. For all $t \geq 0$, the numbers

$$e^{-\frac{kt}{2}} \sum_{j=0}^{k-1} \frac{(-t)^j}{j!} \binom{k}{j+1} k^{j-1} \ , \ \ k \geq 0,$$

are the moments of a unique probability measure on the set $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ invariant by the complex conjugation. We denote this probability measure by $\nu_t$. The following definition was given by P. Biane in [B97a].

**Definition 1.17.** Let $(\mathcal{A}, \tau)$ be a non-commutative probability space. We say that a collection $(u_t)_{t \geq 0}$ of unitary elements of $\mathcal{A}$ is a *free unitary Brownian motion* if the following conditions hold.
1. For all $s, t \geq 0$ such that $s \leq t$, the distribution of $u_t u_s^*$ is the probability measure $\nu_{t-s}$.
2. For all positive integer $m$, for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$, the elements $u_{t_1} u_0^*, u_{t_2} u_{t_1}^*, \ldots, u_{t_m} u_{t_{m-1}}^*$ are free.

In the same paper, P. Biane has proved the following convergence result.

**Theorem 1.18.** *For each $n \geq 1$, let $(U_t^{(n)})_{t \geq 0}$ be a Brownian motion on the unitary group $U(n)$. As $n$ tends to infinity, the collection of random matrices $(U_t^{(n)})_{t \geq 0}$ converges in non-commutative distribution to a free unitary Brownian motion.*

## 2. A continuum of notions of independence

In this section, we shall define a family (indexed by a real number $t \in [0, +\infty]$) of relations between two subalgebras of a non-commutative probability space which passes from the classical independence (case $t = 0$) to freeness (which is the "limit" when $t$ tends to infinity). We have to start with the definition of the $t$-free product of two non-commutative probability spaces. In a few words, it is the space obtained by conjugating one of them, in their tensor product, by a free unitary Brownian motion at time $t$, free with the tensor product.

Fix $t \in [0, +\infty]$ and let $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ be two non-commutative probability spaces. Let $(\mathcal{U}^{(t)}, \varphi_{\mathcal{U}^{(t)}})$ be the non-commutative probability space generated by a single unitary element $u_t$ whose distribution is that of a free unitary Brownian motion at time $t$ (with the convention that a free unitary Brownian motion at time $+\infty$ is a Haar unitary element, i.e. a unitary element whose distribution is the uniform law on the unit circle of $\mathbb{C}$).

**Definition 2.1** ($t$-free product)**.** The $t$-free product of $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$, defined up to an isomorphism of non-commutative probability space, is the non-commutative probability space $(\mathcal{C}, \varphi_{|\mathcal{C}})$, where $\mathcal{C}$ is the subalgebra generated by $\mathcal{A}$ and $u_t \mathcal{B} u_t^*$ in

$$(\mathcal{X}, \varphi) := [(\mathcal{A}, \varphi_{\mathcal{A}}) \otimes (\mathcal{B}, \varphi_{\mathcal{B}})] * (\mathcal{U}^{(t)}, \varphi_{\mathcal{U}^{(t)}}).$$

A few simple observations are in order.

**Remark 2.2.** Both $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ can be identified with subalgebras of the algebra of their $t$-free product (namely with $(\mathcal{A}, \varphi_{|\mathcal{A}})$ and $(u_t \mathcal{B} u_t^*, \varphi_{|u_t \mathcal{B} u_t^*})$). More specifically, if one defines

$$\mathcal{A}_{st} := \{a \in \mathcal{A}\,;\, \varphi_{\mathcal{A}}(a) = 0, \varphi_{\mathcal{A}}(aa^*) = 1\},\ \mathcal{B}_{st} := \{b \in \mathcal{B}\,;\, \varphi_{\mathcal{B}}(b) = 0, \varphi_{\mathcal{B}}(bb^*) = 1\},$$

then any element in the algebra of the $t$-free product $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ can be uniquely written as a constant term plus a linear combination of words in the elements of $\mathcal{A}_{st} \cup u_t \mathcal{B}_{st} u_t^*$ where no two consecutive letters both belong to $\mathcal{A}_{st}$ or to $u_t \mathcal{B}_{st} u_t^*$.

**Remark 2.3.** As a consequence, since $u_t$ is unitary and $(u_t, u_t^*)$ has the same non-commutative distribution as $(u_t^*, u_t)$, the $t$-free product of $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ is clearly isomorphic, as a non-commutative probability space, to the $t$-free product of $(\mathcal{B}, \varphi_{\mathcal{B}})$ and $(\mathcal{A}, \varphi_{\mathcal{A}})$.

**Remark 2.4.** Another consequence of Remark 2.2 is that as a unital algebra, the algebra of the $t$-free product of $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ is isomorphic to the free product of the unital algebras $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$, where $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{B}}$) is the bilateral ideal of the elements $x$ of $\mathcal{A}$ (resp. of $\mathcal{B}$) such that $\varphi_{\mathcal{A}}(xx^*) = 0$ (resp. $\varphi_{\mathcal{B}}(xx^*) = 0$). Thus if $\mathcal{A}$ and $\mathcal{B}$ are subalgebras of the algebra of a non-commutative probability space $(\mathcal{M}, \varphi)$, there is a canonical algebra morphism from the algebra of the $t$-free product of $(\mathcal{A}, \varphi_{|\mathcal{A}})$ and $(\mathcal{B}, \varphi_{|\mathcal{B}})$ to $\mathcal{M}$ whose restriction to $\mathcal{A} \cup \mathcal{B}$ preserves the expectation.

Now, we can give the definition of $t$-freeness. A real $t \in [0, +\infty]$ is fixed.

**Definition 2.5** ($t$-freeness)**.** Let $(\mathcal{M}, \tau)$ be a non-commutative probability space.

- Two subalgebras $\mathcal{A}, \mathcal{B}$ of $\mathcal{M}$ are said to be *$t$-free* if the canonical algebra morphism from the algebra of the $t$-free product of $(\mathcal{A}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \varphi_{\mathcal{B}})$ to $\mathcal{M}$ mentioned in Remark 2.4 preserves the expectation.
- Two subsets $X, Y$ of $\mathcal{M}$ are said to be *$t$-free* if the subalgebras they generate are $t$-free.

**Remark 2.6.** Note that for $t = 0$, $t$-freeness is simply the independence, whereas it follows from [HL00] that in the case where $t = +\infty$, it is the freeness.

The following proposition is obvious from the definition of $t$-freeness.

**Proposition 2.7.** *Let $(\mathcal{M}, \tau)$ be a non-commutative probability space. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_m\}$ be two $t$-free subsets of $\mathcal{M}$. Then the joint non-commutative distribution of the family*

$$(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

*depends only on $t$ and on the distributions of the families $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$.*

**Proposition-Definition 2.8** (Additive and multiplicative $t$-free convolutions)**.** *Let us fix $t \in [0, +\infty)$. Let $\mu, \nu$ be compactly supported probability measures on the real line (resp. on $[0, +\infty)$, on the unit circle). Let $a$, $b$ are $t$-free self-adjoint elements (resp. positive elements, unitary elements) with distributions $\mu$, $\nu$. Then the distribution of $a + b$ (resp. of $\sqrt{b}a\sqrt{b}$, of $ab$) is a compactly supported probability measure on the real line which depends only on $t$, $\mu$ and $\nu$, and which will be denoted by $\mu \boxplus_t \nu$ (resp. $\mu \boxtimes_t \nu$).*

**Proof.** Let us treat the case of the sum of two self-adjoint elements. The other cases can be treated analogously. From Proposition 2.7, it follows that the moments of $a + b$ depend only on $\mu$ and $\nu$. To see that these are the moments of a compactly supported probability measure on the real line, introduce $M > 0$ such that the supports of $\mu$ and $\nu$ are both contained in $[-M, M]$. Then for all $n \geq 1$, by Hölder inequalities in a non-commutative probability space [N74], $\varphi((a + b)^{2n}) \leq 2^{2n} M^{2n}$. By Remark 1.5, the result follows. □

**Proposition 2.9** (Matricial model for the $t$-freeness)**.** *For each $n \geq 1$, let $M_n$ and $N_n$ be diagonal random matrices whose distributions have limits. Let also, for each $n$, $S_n$ be the matrix of a uniform random permutation of $\{1, \ldots, n\}$ and $U_n$ be a random $n$ by $n$ unitary matrix distributed according to the law of a Brownian motion on the unitary group at time $t$. Suppose that for each $n$, the sets of random variables $\{M_n, N_n\}, \{S_n\}, \{U_n\}$ are independent. Then as $n$ tends to infinity, the non-commutative distribution of*

$$(M_n,\ U_n S_n N_n S_n^* U_n^*)$$

*converges in probability to that of a pair $(a, b)$ of self-adjoint elements of a non-commutative probability space which are $t$-free.*

**Proof.** By Theorem 1.13, the non-commutative distribution of $(M_n, S_n N_n S_n^*)$ converges to the one of a pair $(x, y)$ of independent elements. Moreover, since for all $n$, the law of $U_n$ is invariant by conjugation, by Theorems 1.13 and 1.18, the family of sets

$$(\{M_n, S_n N_n S_n^*\}, \{U_n\})$$

is asymptotically free and the limit distribution of $U_n$ is that of a free unitary Brownian motion at time $t$. By definition of $t$-freeness, this concludes the proof. □

**Corollary 2.10.** *With the notation of the previous proposition, in the case where $M_n, N_n$ have real coefficients and where their limit spectral laws, denoted respectively by $\mu, \nu$ are compactly supported, the measure $\mu \boxplus_t \nu$ is the limit spectral law of $M_n + U_n P_n N_n P_n^* U_n^*$.*

## 3. Computation rules for $t$-freeness

### 3.1. **Multivariate free Itô calculus.**

3.1.1. *Technical preliminaries.* In this section, we shall extend some results of [BS98] to the multivariate case. Let us first recall basics of free stochastic calculus. For more involved definitions, the reader should refer to sections 1-2 of [BS98]. Let $(\mathcal{M},\tau)$ be a faithful[2] non-commutative probability space endowed with a filtration $(\mathcal{M}_t)_{t\geq 0}$ and an $(\mathcal{M}_t)_{t\geq 0}$-free additive Brownian motion $(X_t)_{t\geq 0}$. Let $\mathcal{M}^{op}$ be the opposite algebra of $\mathcal{M}$ (it is the same vector space, but it is endowed with the product $a\times_{op} b = ba$). We shall denote by $\sharp$ the left actions of the algebra $\mathcal{M}\otimes\mathcal{M}^{op}$ on $\mathcal{M}$ and $\mathcal{M}\otimes\mathcal{M}$ defined by $(a\otimes b)\sharp u = aub$ and $(a\otimes b)\sharp(u\otimes v) = au\otimes vb$. The algebras $\mathcal{M}$ and $\mathcal{M}\otimes\mathcal{M}^{op}$ are endowed with the inner products defined by $\langle a,b\rangle = \tau(ab^*)$ and $\langle a\otimes b, c\otimes d\rangle = \tau(ac^*)\tau(bd^*)$. The Riemann integral of functions defined on a closed interval with left and right limits at any point with values in the Hilbert space[3] $L^2(\mathcal{M},\tau)$ is a well known notion. Now, we shall recall the definition of the stochastic integral. A simple adapted biprocess is a piecewise constant map $U$ from $[0,+\infty)$ to $\mathcal{M}\otimes\mathcal{M}^{op}$ vanishing for $t$ large enough such that $U_t\in\mathcal{M}_t\otimes\mathcal{M}_t$ for all $t$. The set of simple biprocesses is endowed with the inner product

$$\langle U,V\rangle = \int_0^\infty \langle U_t,V_t\rangle \mathrm{d}t.$$

We shall denote by $\mathcal{B}_2^a$ the closure in $L^2(\mathcal{M},\tau)$ of the set of simple adapted biprocesses. Let $U$ be a simple adapted biprocess. Then there exists times $0=t_0\leq t_1\leq\cdots\leq t_m$ such that $L$ (resp. $U$) is constant on each $[t_i,t_{i+1})$ and vanishes on $[t_m,+\infty)$. Then we define

$$\int_0^\infty U_t\mathrm{d}X_t = \sum_{i=0}^{m-1} U_{t_i}\sharp(X_{t_{i+1}}-X_{t_i}).$$

It can be proved (Corollary 3.1.2 of [BS98]) that the map $U\mapsto\int_0^\infty U_t\mathrm{d}X_t$ can be extended isometrically from $\mathcal{B}_2^a$ to $L^2(\mathcal{M},\tau)$.

3.1.2. *Free Itô processes.* We shall call a *free Itô process* any process

$$A_t = A_0 + \int_0^t L_s\mathrm{d}s + \int_0^t U_s\mathrm{d}X_s, \tag{1}$$

where $A_0\in\mathcal{M}_0$, $L$ is an adapted process with left and right limit at any point and $U\in\mathcal{B}_2^a$. In this case, we shall denote

$$\mathrm{d}A_t = L_t\mathrm{d}t + U_t\sharp\mathrm{d}X_t. \tag{2}$$

The part $U_t\sharp\mathrm{d}X_t$ of this expression is called the *martingale part* of $A$. Note that the process $A$ is determined by $A_0$ and $\mathrm{d}A_t$.

We shall use the following lemma, which follows from Proposition 2.2.2 of [BS98] and from the linearity of $\tau$.

**Lemma 3.1.** *Let $A_t$ be as in (1). Then $\tau(A_t)=\tau(A_0)+\int_0^t\tau(L_s)\mathrm{d}s$.*

3.1.3. *Multivariate free Itô calculus.* Consider $n$ elements $a_1,\ldots,a_n\in\mathcal{M}$ for some $n\geq 2$. Consider also two elements $u=\sum_k x_k\otimes y_k, v=\sum_l z_l\otimes t_l$ of $\mathcal{M}\otimes\mathcal{M}^{op}$. For all $1\leq i<j\leq n$, we define an element of $\mathcal{M}$ by setting

$$\langle\langle a_1,\ldots,a_{i-1},u,a_{i+1},\ldots,a_{j-1},v,a_{j+1},\ldots,a_n\rangle\rangle_{i,j} = \sum_{k,l} a_1\cdots a_{i-1}x_k\tau(y_ka_{i+1}\cdots a_{j-1}z_l)t_la_{j+1}\cdots a_n.$$

[2]A non-commutative probability space $(\mathcal{M},\tau)$ is said to be *faithful* if for all $x$ in $\mathcal{M}\setminus\{0\}$, $\tau(xx^*)>0$. Any non-commutative probability space can be quotiented by a bilateral ideal into a faithful space.

[3]The Hilbert space $L^2(\mathcal{M},\tau)$ is the completion of $\mathcal{M}$ with respect to the inner product $\langle x,y\rangle=\tau(xy^*)$.

The following theorem follows from Theorem 4.1.12 and the remark following in [BS98].

**Theorem 3.2.** *Let $A_t = A_0 + \int_0^t L_s \mathrm{d}s + \int_0^t U_s \mathrm{d}X_s$ and $B_t = B_0 + \int_0^t K_s \mathrm{d}s + \int_0^t V_s \mathrm{d}X_s$ be two Itô processes with respect to the same free Brownian motion $(X_t)$. Then $AB$ is a free Itô process and with the notations of (2),*

$$\mathrm{d}(AB)_t = A_t \mathrm{d}B_t + (\mathrm{d}A_t)B_t + \langle\langle U_t, V_t\rangle\rangle_{1,2}\mathrm{d}t.$$

In order to prove computation rules for $t$-freeness, we shall need the following theorem.

**Theorem 3.3.** *Let $A_1, \ldots, A_n$ be free Itô processes with respect to the same Brownian motion. For all $k$, denote $A_{k,t} = A_{k,0} + \int_0^t L_{k,s}\mathrm{d}s + \int_0^t U_{k,s}\mathrm{d}X_s$. Then $A_1 \cdots A_n$ is a free Itô process and*

$$\begin{aligned}\mathrm{d}(A_1 \cdots A_n)_t =& \sum_{k=1}^{n} A_{1,t} \cdots A_{k-1,t}(\mathrm{d}A_{k,t})A_{k+1,t} \cdots A_{n,t} \\ &+ \sum_{1 \leq k < l \leq n} \langle\langle A_{1,t}, \ldots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \ldots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \ldots, A_{n,t}\rangle\rangle_{k,l}\mathrm{d}t.\end{aligned}$$

**Proof.** Let us prove this theorem by induction on $n$. For $n = 1$, it is obvious. Let us suppose the result to hold at rank $n$. Then the martingale part of $A_1 \cdots A_n$ is

$$\sum_{k=1}^{n} A_{1,t} \cdots A_{k-1,t}(U_{k,t} \sharp \mathrm{d}X_{k,t})A_{k+1,t} \cdots A_{n,t}.$$

Thus by Theorem 3.2, $A_1 \cdots A_{n+1}$ is a free Itô process and

$$\begin{aligned}\mathrm{d}(A_1 \cdots A_{n+1})_t =& (A_1 \cdots A_n)_t \mathrm{d}A_{n+1,t} + (\mathrm{d}(A_1 \cdots A_n)_t)A_{n+1,t} \\ &+ \sum_{k=1}^{n} \langle\langle A_{1,t}, \ldots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \ldots, A_{n,t}, U_{n+1,t}\rangle\rangle_{k,n+1}\mathrm{d}t \\ =& \sum_{k=1}^{n+1} A_{1,t} \cdots A_{k-1,t}(\mathrm{d}A_{k,t})A_{k+1,t} \cdots A_{n,t} \\ &+ \sum_{1 \leq k < l \leq n} \langle\langle A_{1,t}, \ldots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \ldots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \ldots, A_{n,t}, A_{n+1,t}\rangle\rangle_{k,l}\mathrm{d}t \\ &+ \sum_{k=1}^{n} \langle\langle A_{1,t}, \ldots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \ldots, A_{n,t}, U_{n+1,t}\rangle\rangle_{k,n+1}\mathrm{d}t,\end{aligned}$$

which concludes the proof. □

## 3.2. **Computation rules for $t$-freeness.**

3.2.1. *Main result.* In order to do computations with elements which are $t$-free, we have to find out a formula for the expectation of a product of elements of the type

$$x_1 u_t y_1 u_t^* x_2 u_t y_2 u_t^* \cdots x_n u_r y_n u_t^*, \tag{3}$$

for $\{x_1, \ldots, x_n\}$ independent with $\{y_1, \ldots, y_n\}$ and $\{x_1, y_1, \ldots, x_n, y_n\}$ free with $u_t$, free unitary Brownian motion. Actually, for the result which follows, the independence of the $x_i$'s and the $y_i$'s will not be useful, thus we consider a non-commutative probability space $(\mathcal{M}, \tau)$, an integer $n \geq 1$, $a_1, \ldots, a_{2n} \in \mathcal{M}$ and a free unitary Brownian motion $(u_t)$ which is free with $\{a_1, \ldots, a_{2n}\}$.

In order to have some more concise formulas, it will be useful to multiply the product of (3) by $e^{nt}$. So we define

$$f_{2n}(a_1,\dots,a_{2n},t)=e^{nt}\tau(a_1u_ta_2u_t^*\cdots a_{2n-1}u_ta_{2n}u_t^*).$$

We shall use the convention $f_0(a,t)=\tau(a)$ for all $a\in\mathcal{M}$.

Since $f_{2n}(a_1,\dots,a_{2n},0)=\tau(a_1\cdots a_{2n})$, the following theorem allows us to deduce all functions $f_{2n}(a_1,\dots,a_{2n},t)$ (thus the expectation of any product of the type of (3)) from the joint distribution of the $a_i$'s.

**Theorem 3.4.** *The family of functions $t\mapsto f_{2n}(a_1,\dots,a_{2n},t)$ satisfies the following differential system: for all $n\geq 1$, $a_1,\dots,a_{2n}\in\mathcal{M}$ free with the process $(u_t)$,*

$$\begin{aligned}\frac{\partial}{\partial t}f_{2n}(a_1,\dots,a_{2n},t)=&-\sum_{\substack{1\leq k<l\leq 2n\\ k=l \mod 2}}f_{2n-(l-k)}(a_1,\dots,a_k,a_{l+1},\dots,a_{2n},t)f_{l-k}(a_{k+1},\dots,a_l,t)\\&+e^t\sum_{\substack{1\leq k<l\leq 2n\\ k\neq l \mod 2}}f_{2n-(l-k)-1}(a_1,\dots,a_{k-1},a_ka_{l+1},a_{l+2},\dots,a_{2n},t)f_{l-k-1}(a_la_{k+1},a_{k+2},\dots,a_{l-1},t).\end{aligned}$$

**Proof.** Let us introduce the process $(v_t)$ defined by $v_t=e^{t/2}u_t$ for all $t$. As explained in the beginning of section 2.3 of [B97a], this process can be realized as an Itô process, with the formula

$$v_t=1+i\int_0^t \mathrm{d}X_sv_s.$$

Thus one can realize the family of non-commutative random variables $a_1,\dots,a_{2n},(v_t)_{t\geq 0}$ in a faithful non-commutative probability space $(\mathcal{M},\tau)$ endowed with a filtration $(\mathcal{M}_t)_{t\geq 0}$ and an additive free Brownian motion $(X_t)_{t\geq 0}$ such that $a_1,\dots,a_{2n}\in\mathcal{M}_0$ and for all $t$, $v_t=1+i\int_0^t \mathrm{d}X_sv_s$ and $v_t^*=1-i\int_0^t v_s^*\mathrm{d}X_s$. By definition of $f_{2n}(a_1,\dots,a_{2n},t)$, one has

$$f_{2n}(a_1,\dots,a_{2n},t)=\tau(a_1v_ta_2v_t^*\cdots a_{2n-1}v_ta_{2n}v_t^*).$$

Note that since all $a_i$'s belong to $\mathcal{M}_0$, the processes $A_1:=(a_1v_t),A_2:=(a_2v_t^*),\dots,A_{2n-1}:=(a_{2n-1}v_t),A_{2n}:=(a_{2n}v_t^*)$ are all free Itô processes: if one defines $U_{k,t}=a_k\otimes iv_t$ for $k$ odd and $U_{k,t}=-ia_kv_t^*\otimes 1$ for $k$ even, then for all $k$, $\mathrm{d}A_{k,t}=U_{k,t}\sharp\mathrm{d}X_t$. Thus by theorem 3.3, $A_1\cdots A_{2n}$ is an Itô process such that for all $t$,

$$\begin{aligned}(A_1\cdots A_{2n})_t=&(A_1\cdots A_{2n})_0+\int_0^t\sum_{k=1}^{2n}A_{1,s}\cdots A_{k-1,s}(U_{k,s}\sharp\mathrm{d}X_s)A_{k+1,s}\cdots A_{2n,s}\\&+\int_0^t\sum_{1\leq k<l\leq 2n}\langle\langle A_{1,s},\dots,A_{k-1,s},U_{k,s},A_{k+1,s},\dots,A_{l-1,s},U_{l,s},A_{l+1,s},\dots,A_{2n,s}\rangle\rangle_{k,l}\mathrm{d}s.\end{aligned}$$

Hence by lemma 3.1, for all $t$,

$$\begin{aligned}(4)\qquad\frac{\partial}{\partial t}f_{2n}(a_1,\dots,a_{2n},t)=&\\&\sum_{1\leq k<l\leq 2n}\tau(\langle\langle A_{1,t},\dots,A_{k-1,t},U_{k,t},A_{k+1,t},\dots,A_{l-1,t},U_{l,t},A_{l+1,t},\dots,A_{2n,t}\rangle\rangle_{k,l}).\end{aligned}$$

Now, fix $1\leq k<l\leq 2n$.

• If $k = l \mod 2$. Suppose for example that $k, l$ are both odd (the other case can be treated in the same way). Then $U_{k,t} = a_k \otimes iv_t$ and $U_{l,t} = a_l \otimes iv_t$, which implies that

$$\tau(\langle\langle A_{1,t}, \dots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \dots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \dots, A_{2n,t}\rangle\rangle_{k,l}) = \\ i\tau(a_1 v_t a_2 v_t^* \cdots a_{k-1} v_t^* a_k v_t a_{l+1} v_t \cdots a_{2n} v_t^*) i\tau(v_t a_{k+1} v_t^* \cdots a_{l-1} v_t^* a_l).$$

Note that since $\tau$ is tracial and the joint distribution of $a_1, \dots, a_{2n}, (v_t)_{t\geq 0}$ is the same as the one of $a_1, \dots, a_{2n}, (v_t^*)_{t\geq 0}$, we have $\tau(v_t a_{k+1} v_t^* \cdots a_{l-1} v_t^* a_l) = \tau(a_{k+1} v_t \cdots a_{l-1} v_t a_l v_t^*)$. Hence

$$\tau(\langle\langle A_{1,t}, \dots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \dots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \dots, A_{2n,t}\rangle\rangle_{k,l}) = \\ - f_{2n-(l-k)}(a_1, \dots, a_k, a_{l+1}, \dots, a_{2n}, t) f_{l-k}(a_{k+1}, \dots, a_l, t). \tag{5}$$

• If $k \neq l \mod 2$. Suppose for example $k$ to be odd and $l$ to be even (the other case can be treated in the same way). Then $U_{k,t} = a_k \otimes iv_t$ and $U_{l,t} = -a_l i v_t^* \otimes 1$, which implies that

$$\tau(\langle\langle A_{1,t}, \dots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \dots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \dots, A_{2n,t}\rangle\rangle_{k,l}) = \\ \tau(a_1 v_t a_2 v_t^* \cdots a_{k-1} v_t^* a_k a_{l+1} v_t \cdots a_{2n} v_t^*)(-i^2)\tau(v_t a_{k+1} v_t^* \cdots a_{l-1} v_t a_l v_t^*).$$

Note that since $v_t^* v_t = e^t$, $\tau$ is tracial and the joint distribution of $a_1, \dots, a_{2n}, (v_t)_{t\geq 0}$ is the same as the one of $a_1, \dots, a_{2n}, (v_t^*)_{t\geq 0}$, we have $\tau(v_t a_{k+1} v_t^* \cdots a_{l-1} v_t a_l v_t^*) = e^t \tau(a_l a_{k+1} v_t \cdots a_{l-1} v_t^*)$. Hence

$$\tau(\langle\langle A_{1,t}, \dots, A_{k-1,t}, U_{k,t}, A_{k+1,t}, \dots, A_{l-1,t}, U_{l,t}, A_{l+1,t}, \dots, A_{2n,t}\rangle\rangle_{k,l}) = \\ e^t f_{2n-(l-k)-1}(a_1, \dots, a_{k-1}, a_k a_{l+1}, a_{l+2}, \dots, a_{2n}, t) f_{l-k-1}(a_l a_{k+1}, a_{k+2}, \dots, a_{l-1}, t). \tag{6}$$

Equations (4), (5) and (6) together conclude the proof. □

The following proposition, which we shall use later, is a consequence of the previous theorem.

**Proposition 3.5.** *In a non-commutative probability space $(\mathcal{M}, \tau)$, consider two independent normal elements $a, b$ with symmetric compactly supported laws. Let $(u_t)$ be a free unitary Brownian motion which is free with $\{a, b\}$. Then the function*

$$G(t, z) = \sum_{n\geq 1} \tau((au_t b u_t^*)^{2n}) e^{2nt} z^n$$

*is the only solution, in a neighborhood of $(0, 0)$ in $[0, +\infty) \times \mathbb{C}$, to the nonlinear, first order partial differential equation*

$$\frac{\partial G}{\partial t} + 4zG\frac{\partial G}{\partial z} = 0 \tag{7}$$

$$G(0, z) = \sum_{n\geq 1} \tau(a^{2n})\tau(b^{2n}) z^n. \tag{8}$$

**Proof.** Let us define, for all $n \geq 1$, $g_n(t) = \tau((au_t b u_t^*)^n) e^{nt}$. For $n = 0$, we set $g_0(t) = 0$. Let us fix $n \geq 1$. In order to apply the previous theorem, let us define, for $i = 1, \dots, 2n$, $a_i = a$ if $i$ is odd and $a_i = b$ if $i$ is even. By the previous theorem, for all $n \geq 1$, we have

$$\frac{\partial}{\partial t} g_n(t) = -\sum_{\substack{1\leq k<l\leq 2n \\ k=l \mod 2}} g_{n-(l-k)/2}(t) g_{(l-k)/2}(t) \\ + e^t \sum_{\substack{1\leq k<l\leq 2n \\ k\neq l \mod 2}} f_{2n-(l-k)-1}(a_1, \dots, a_{k-1}, a_k a_{l+1}, a_{l+2}, \dots, a_{2n}, t) f_{l-k-1}(a_l a_{k+1}, a_{k+2}, \dots, a_{l-1}, t). \tag{9}$$

Now, note that since for any $\varepsilon, \varepsilon' = \pm 1$, the joint distribution of $(a, b, u_t)$ is the same as the one of $(\varepsilon a, \varepsilon' b, u_t)$, $g_p(t) = 0$ when $p$ is odd. Thus in the first sum of (9) only pairs $(k, l)$ such that $k = l \mod 4$ have a non null contribution. For the same reason, all terms in the second sum are null. Indeed, for any $1 \leq k < l \leq 2n$ such that $k \neq l \mod 2$, the set $\{k+1, k+2, \ldots, l\}$, whose cardinality is odd, has either an odd number of odd elements or an odd number of even elements. To sum up, for all $n \geq 1$, we have

$$\begin{aligned}\frac{\partial}{\partial t} g_{2n}(t) &= -\sum_{\substack{1 \leq k < l \leq 4n \\ k = l \mod 4}} g_{2n-(l-k)/2}(t) g_{(l-k)/2}(t) = -4 \sum_{1 \leq i < j \leq n} g_{2(n-(j-i))}(t) g_{2(j-i)}(t) \\ &= -4 \sum_{m=1}^{n-1} m g_{2(n-m)}(t) g_{2m}(t) = -2 \sum_{m=1}^{n-1} m g_{2(n-m)}(t) g_{2m}(t) - 2 \sum_{m=1}^{n-1} (n-m) g_{2(n-m)}(t) g_{2m}(t) \\ &= -2n \sum_{m=1}^{n-1} g_{2(n-m)}(t) g_{2m}(t) = -2n \sum_{m=0}^{n} g_{2(n-m)}(t) g_{2m}(t).\end{aligned}$$

Thus since $G(t, z) = \sum_{n \geq 1} g_{2n}(t) z^n = \sum_{n \geq 0} g_{2n}(t) z^n$, we have

$$\frac{\partial G}{\partial t} = -2z \frac{\partial G^2}{\partial z},$$

which proves (7). The formula (8) is obvious.

To prove the uniqueness, let $H(t, z) = \sum_{n \geq 0} h_n(t) z^n$ be another solution of (7) and (8). By (8), for all $n \geq 0$, we have $h_n(0) = g_{2n}(0)$ and by (8), for all $n \geq 0$, we have $\frac{\partial}{\partial t} h_n(t) = -2n \sum_{m=0}^{n} h_{n-m}(t) h_m(t)$, which implies that $h_0 = 0$ and that by induction on $n$, $h_n = g_{2n}$. □

3.2.2. *Examples.* Let $\mathcal{A}, \mathcal{B}$ be two independent subalgebras of a non-commutative probability space $(\mathcal{M}, \tau)$ and $(u_t)$ be a free unitary Brownian motion free from $\mathcal{A} \cup \mathcal{B}$.

1) For $a \in \mathcal{A}, b \in \mathcal{B}$, for all $t \geq 0$, we have

$$\tau(a u_t b u_t^*) = \tau(a)\tau(b). \tag{10}$$

(In fact, it even follows from theorem 3.4 that without the assumption that $a$ and $b$ are independent, for all $t$, we have $\tau(a u_t b u_t^*) = e^{-t}\tau(ab) + (1 - e^{-t})\tau(a)\tau(b)$).

2) For $a, a' \in \mathcal{A}, b, b' \in \mathcal{B}$, for all $t \geq 0$, we have

$$\tau(a u_t b u_t^* a' u_t b' u_t^*) = \tag{11}$$

$$(\tau(a)\tau(a')\tau(bb') + \tau(aa')\tau(b)\tau(b') - \tau(a)\tau(a')\tau(b)\tau(b'))(1 - e^{-2t}) + \tau(aa')\tau(bb')e^{-2t}.$$

3) For $a, a', a'' \in \mathcal{A}$ and $b, b', b'' \in \mathcal{B}$, we have

$$\begin{aligned}
&\tau(au_tbu_t^*a'u_tb'u_t^*a''u_tb''u_t^*) \qquad (12)\\
=\ &\tau(a)\tau(a')\tau(a'')\tau(b)\tau(b')\tau(b'')(2-6e^{-2t}+4e^{-3t})\\
&-(1-3e^{-2t}+2e^{-3t})\tau(a)\tau(a')\tau(a'')\tau(b)\tau(b'b'')\\
&-(1-3e^{-2t}+2e^{-3t})\tau(a)\tau(a')\tau(a'')\tau(bb')\tau(b'')\\
&-(1-3e^{-2t}+2e^{-3t})\tau(a)\tau(a')\tau(a'')\tau(bb'')\tau(b')\\
&-(1-3e^{-2t}+2e^{-3t})\tau(aa')\tau(a'')\tau(b)\tau(b')\tau(b'')\\
&-(1-3e^{-2t}+2e^{-3t})\tau(aa'')\tau(a')\tau(b)\tau(b')\tau(b'')\\
&-(1-3e^{-2t}+2e^{-3t})\tau(a)\tau(a'a'')\tau(b)\tau(b')\tau(b'')\\
&+(1-3e^{-2t}+2e^{-3t})[\tau(a)\tau(a')\tau(a'')\tau(bb'b'')+\tau(aa'a'')\tau(b)\tau(b')\tau(b'')]\\
&-(e^{-2t}-e^{-3t})[\tau(aa')\tau(a'')\tau(bb')\tau(b'')+\tau(aa')\tau(a'')\tau(bb'')\tau(b')+\tau(aa'')\tau(a')\tau(bb'')\tau(b')\\
&+\tau(aa'')\tau(a')\tau(b)\tau(b'b'')+\tau(a)\tau(a'a'')\tau(b'')\tau(bb')+\tau(a)\tau(a'a'')\tau(b)\tau(b'b'')]\\
&+(1-2e^{-2t}+e^{-3t})[\tau(aa')\tau(a'')\tau(b'b'')\tau(b)+\tau(aa'')\tau(a')\tau(bb')\tau(b'')+\tau(a)\tau(a'a'')\tau(bb'')\tau(b')]\\
&+(e^{-2t}-e^{-3t})\tau(bb'b'')[\tau(a)\tau(a'a'')+\tau(aa')\tau(a'')+\tau(aa'')\tau(a')]\\
&+(e^{-2t}-e^{-3t})\tau(aa'a'')[\tau(b)\tau(b'b'')+\tau(bb')\tau(b'')+\tau(bb'')\tau(b')]\\
&+e^{-3t}\tau(aa'a'')\tau(bb'b'')
\end{aligned}$$

It can be verified that the last formula actually corresponds to the formula of $\mathbb{E}(aba'b'a''b'')$ with $\{a, a', a''\}$ and $\{b, b', b''\}$ independent when $t = 0$, and to the formula of $\tau(aba'b'a''b'')$ with $\{a, a', a''\}$ and $\{b, b', b''\}$ free when $t$ tends to infinity.

### 3.3. **Multiplicative and additive $t$-free convolutions of two symmetric Bernoulli laws.**

3.3.1. *Multiplicative case.* Here, we shall compute the multiplicative $t$-free convolution of $\frac{\delta_{-1}+\delta_1}{2}$ (considered as a law on the unit circle) with itself.

**Theorem 3.6.** *For all $t \geq 0$, $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ is the only law on the unit circle which is invariant under the symmetries with respect to the real and imaginary axes and whose push-forward by the map $z \mapsto z^2$ has the law of $u_{4t}$, a free unitary Brownian motion taken at time $4t$.*

**Remark 3.7.** The moments of $u_{4t}$ have been computed by P. Biane at Lemma 1 of [B97a]: for all $n \geq 1$,

$$\tau(u_{4t}^n) = \frac{e^{-2nt}}{n} \sum_{k=0}^{n-1} \frac{(-4nt)^k}{k!} \binom{n}{k+1}.$$

**Proof.** In a non-commutative probability space $(\mathcal{M}, \tau)$, consider two independent normal elements $a, b$ with law $\frac{\delta_{-1}+\delta_1}{2}$. Let $(u_t)$ be a free unitary Brownian motion which is free with $\{a, b\}$. Then $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ is the distribution of the unitary element $au_tbu_t^*$. Since the joint distribution of $(a, b, u_t)$ is the same as the one of $(-a, b, u_t)$, $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ is invariant under the transformation $z \mapsto -z$. Moreover, $(au_tbu_t^*)^* = u_tbu_t^*a$ has the same distribution as $au_tbu_t^*$ (because $\tau$ is tracial and $u_t$ has the same law as $u_t^*$), hence $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ is invariant under the transformation $z \mapsto \bar{z}$. This proves that $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ is invariant under the symmetries with respect to the real and imaginary axes.

Since any distribution on the unit circle is determined by its moments, to prove that the push-forward of $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ by the map $z \mapsto z^2$ is the law of $u_{4t}$, it suffices to prove that

for all $n \geq 1$,

$$\tau((au_t bu_t^*)^{2n}) = \tau(u_{4t}^n),$$

i.e. to prove that the functions

$$F_1(t,z) = \sum_{n\geq 1} \tau((au_t bu_t^*)^{2n}) e^{2nt} z^n \quad \text{and} \quad F_2(t,z) = \sum_{n\geq 1} \tau(u_{4t}^n) e^{2nt} z^n$$

are equal. It follows from Proposition 3.5 that $F_1$ is the only solution, in a neighborhood of $(0,0)$ in $[0,+\infty)\times\mathbb{C}$, to equation (7) satisfying $F_1(0,z) = \frac{z}{1-z}$. But it follows from Lemma 1 of [B97a] that $F_2$ is also a solution of (7) with the same initial conditions. By uniqueness, it closes the proof. □

For all $t \in [0,1]$, let us define $\beta(t) = 2\sqrt{t(1-t)} + \arccos(1-2t)$. Then $\beta(t)$ is an increasing function of $t$ which goes from 0 to $\pi$ when $t$ goes from 0 to 1. P. Biane has proved in [B97b, Prop. 10] that the distribution of $u_{4t}$ is absolutely continuous with respect to the Lebesgue measure on the unit circle, that its support is the full unit circle for $t \geq 1$, and the set $\{e^{i\theta} : |\theta| \leq \beta(t)\}$ for all $t \in [0,1]$. Moreover, the density of this distribution with respect to the uniform probability measure on the unit circle, which we denote by $\rho_{4t}$, is positive and analytic on the interior of its support for all $t \geq 0$, except at $-1$ for $t = 1$. From these facts, one deduces easily the next result.

**Corollary 3.8.** *For all $t > 0$, the measure $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$ has a density with respect to the uniform probability measure on the unit circle, which we shall denote by $\sigma_t$ and which is given by the formula $\sigma_t(z) = \rho_{4t}(z^2)$ for all $z$ in the unit circle. In particular, the support of this measure is the full unit circle for $t \geq 1$ and the set $\{e^{i\theta} : |\theta| \leq \frac{1}{2}\beta(t)$ or $|\pi - \theta| \leq \frac{1}{2}\beta(t)\}$ for $t \in [0,1]$. The density $\sigma_t$ is positive and analytic on the interior of its support for all $t \geq 0$, except at $\pm i$ for $t = 1$.*

3.3.2. *Additive case.* Here, we shall compute the additive $t$-free convolution of $\frac{\delta_{-1}+\delta_1}{2}$ (considered as a law on the real line) with itself.

**Theorem 3.9.** *For all $t \geq 0$, $\frac{\delta_{-1}+\delta_1}{2} *_t \frac{\delta_{-1}+\delta_1}{2}$ is the only symmetric law on the real line whose push-forward by the map $x \mapsto x^2$ has the law of $2 + v + v^*$, with $v$ unitary element distributed according to $\frac{\delta_{-1}+\delta_1}{2} \odot_t \frac{\delta_{-1}+\delta_1}{2}$.*

**Remark 3.10.** One can recover the fact that as $t$ goes from 0 to $+\infty$, $\frac{\delta_{-1}+\delta_1}{2} *_t \frac{\delta_{-1}+\delta_1}{2}$ passes from $\frac{\delta_{-1}+\delta_1}{2} * \frac{\delta_{-1}+\delta_1}{2} = \frac{1}{4}(\delta_{-2} + 2\delta_0 + \delta_2)$ to $\frac{\delta_{-1}+\delta_1}{2} \boxplus \frac{\delta_{-1}+\delta_1}{2} = 1_{[-2,2]}(x)\frac{\mathrm{d}x}{\pi\sqrt{2-x^2}}$.

**Proof.** In a non-commutative probability space $(\mathcal{M},\tau)$, consider two independents normal elements $a, b$ with law $\frac{\delta_{-1}+\delta_1}{2}$. Let $(u_t)$ be a free unitary Brownian motion which is free with $\{a,b\}$. Then $\frac{\delta_{-1}+\delta_1}{2} *_t \frac{\delta_{-1}+\delta_1}{2}$ is the distribution of $a + u_t bu_t^*$. Since the joint distribution of $(a,b,u_t)$ is the same as the one of $(-a,-b,u_t)$, $\frac{\delta_{-1}+\delta_1}{2} *_t \frac{\delta_{-1}+\delta_1}{2}$ is symmetric. Note that since $a^2$ and $b^2$ have $\delta_1$ for distribution, one can suppose that $a^2 = b^2 = 1$. In this case,

$$(a + u_t bu_t^*)^2 = 2 + au_t bu_t^* + u_t bu_t^* a = 2 + au_t bu_t^* + (au_t bu_t^*)^*,$$

and the result is obvious by definition of $\odot_t$. □

From the last result and Proposition 3.8, we deduce the following.

**Corollary 3.11.** *For all $t > 0$, the measure $\frac{\delta_{-1}+\delta_1}{2} *_t \frac{\delta_{-1}+\delta_1}{2}$ has a density with respect to the Lebesgue measure on $[-2,2]$, which we shall denote by $\eta_t$ and which is given by the formula*

$$\forall x \in [-2,2]\ ,\ \ \eta_t(x) = \rho_{4t}(e^{4i \arccos \frac{x}{2}}) \frac{1}{\pi\sqrt{4-x^2}}.$$

*The support of this measure is the interval $[-2,2]$ for $t \geq 1$, and the set*

$$\left[-2, -2\cos\frac{\beta(t)}{4}\right] \cup \left[-2\sin\frac{\beta(t)}{4}, 2\sin\frac{\beta(t)}{4}\right] \cup \left[2\cos\frac{\beta(t)}{4}, 2\right]$$

*for $t \in [0,1]$. The density $\eta_t$ is positive and analytic on the interior of its support for all $t \geq 0$, except at $\pm\sqrt{2}$ for $t = 1$.*

## 4. The lack of cumulants

In this section, we investigate the existence of an analogue of classical and free cumulants in the context of $t$-freeness. Informally, the problem is to find multilinear forms defined on any non-commutative probability space which vanish when evaluated on a family of elements which can be split into two non-empty subfamilies which are $t$-free.

More precisely, given a non-commutative probability space $(\mathcal{M}, \varphi)$, we would like to know if there exists a family $(k_n)_{n\geq 2}$ of multilinear forms on $\mathcal{M}$, with $k_n$ an $n$-linear form for all $n \geq 2$, such that, for all $n \geq 2$, all $n_1, n_2 > 0$ such that $n_1 + n_2 = n$, all $m_1, \ldots, m_n$ in $\mathcal{M}$ such that $\{m_1, \ldots, m_{n_1}\}$ and $\{m_{n_1+1}, \ldots, m_{n_1+n_2}\}$ are $t$-free, and finally for all $\sigma \in \mathfrak{S}_n$, one has $k_n(m_{\sigma(1)}, \ldots, m_{\sigma(n)}) = 0$.

Our main result is negative: there does not exist in general such a family of multilinear forms, at least in a large class which we describe now.

**Definition 4.1.** *Let $(\mathcal{M}, \varphi)$ be a non-commutative probability space. Let $n \geq 1$ be an integer. Let $\sigma$ be an element of $\mathfrak{S}_n$. We define the $n$-linear form $\varphi_\sigma$ on $\mathcal{M}$ as follows:*

$$\forall m_1, \ldots, m_n \in \mathcal{M}\ ,\ \ \varphi_\sigma(m_1, \ldots, m_n) = \prod_{\substack{c \text{ cycle of } \sigma \\ c=(i_1 \ldots i_r)}} \varphi(m_{i_1} \ldots m_{i_r}).$$

Using only the algebra structure of $\mathcal{M}$ and the linear form $\varphi$, a linear combination of the forms $\{\varphi_\sigma : \sigma \in \mathfrak{S}_n\}$ seems to be the most general $n$-linear form that one can construct on $\mathcal{M}$. We seek cumulants within this wide class of $n$-linear forms. Our definition does not require that the vanishing of cumulants characterize $t$-freeness. We only insist that mixed cumulants of $t$-free variable vanish.

**Definition 4.2.** *Let $n \geq 2$ be an integer. Let $t \geq 0$ be a real number. A* $t$-free cumulant of order *$n$ is a collection $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ of complex numbers such that $\sum_{\sigma\ n-\text{cycle}} c(\sigma) \neq 0$ and the following properties hold for every non-commutative probability space $(\mathcal{M}, \varphi)$.*
*Let $\mathcal{A}$ and $\mathcal{B}$ be two sub-algebras of $\mathcal{M}$ which are $t$-free with respect to $\varphi$. Let $m_1, \ldots, m_n$ be $n$ elements of $\mathcal{A} \cup \mathcal{B}$, which do not all belong to $\mathcal{A}$, and not all to $\mathcal{B}$. Then*

$$\sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\varphi_\sigma(m_1, \ldots, m_n) = 0. \tag{13}$$

Let us emphasize that what we call *cumulant* is not a specific multilinear form, but rather a collection of coefficients which allows one to define a multilinear form on any non-commutative probability space.

If $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ is a $t$-free cumulant of order $n$ and $m_1,\dots,m_n$ are elements of a non-commutative probability space $(\mathcal{M},\varphi)$, at least one of which is equal to 1, then

$$\sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\varphi_\sigma(m_1,\dots,m_n)=0. \tag{14}$$

Indeed, the subalgebra $\mathbb{C}.1$ of $\mathcal{M}$ is $t$-free with any subalgebra of $\mathcal{M}$.

We extend the previous definition by including the free case $t=+\infty$. We will mainly consider collections $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ with the property that $c(\rho\sigma\rho^{-1})=c(\sigma)$ for all $\sigma,\rho\in\mathfrak{S}_n$. We call such collections *conjugation-invariant*. They are in fact indexed by conjugacy classes of $\mathfrak{S}_n$, that is, integer partitions of $n$. Thus, we write use as well the notation $(c_\lambda)_{\lambda\vdash n}$ for a conjugation-invariant collection.

Our main results are the following.

**Theorem 4.3.** *For all $t\in[0,+\infty]$ and all $n\in\{2,3,4,5,6\}$, there exists, up to scaling, a unique conjugation-invariant $t$-free cumulant of order $n$.*

**Theorem 4.4.** *There exists a $t$-free cumulant of order 7 if and only if $t=0$ or $t=+\infty$.*

Let us start by proving that we lose nothing by focusing on conjugation-invariant $t$-free cumulants.

**Lemma 4.5.** *If for some $t$ and some $n$ there exists a $t$-free cumulant of order $n$, then there exists such a cumulant $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ such that moreover $c(\sigma)=c(\rho\sigma\rho^{-1})$ for all $\sigma,\rho\in\mathfrak{S}_n$.*

*Proof.* The point is that the order of the arguments is arbitrary in (13). Hence, if (13) is satisfied, then for all $\rho\in\mathfrak{S}_n$,

$$\begin{aligned}0=\sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\varphi_\sigma(m_{\rho^{-1}(1)},\dots,m_{\rho^{-1}(n)}) &= \sum_{\sigma\in\mathfrak{S}_n} c(\sigma)\varphi_{\rho^{-1}\sigma\rho}(m_1,\dots,m_n)\\ &= \sum_{\sigma\in\mathfrak{S}_n} c(\rho\sigma\rho^{-1})\varphi_\sigma(m_1,\dots,m_n).\end{aligned}$$

Hence, if $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ is a $t$-free cumulant, then so is $(c(\rho\sigma\rho^{-1}))_{\sigma\in\mathfrak{S}_n}$. By averaging over $\rho$, we get a conjugation-invariant cumulant. □

Observe that the assumption made in the definition of a cumulant that the sum of $c(\sigma)$ when $\sigma$ spans the $n$-cycles is nonzero implies that $c_n\neq 0$ for any conjugation-invariant cumulant.

Let us introduce some notation. Given a permutation $\sigma$ of $\{1,\dots,n\}$, we denote by $\{\{\sigma\}\}$ the partition of $\{1,\dots,n\}$ whose blocks are the sets underlying the cycles of $\sigma$. Let $\mathcal{P}(n)$ denote the set of partitions of the set $\{1,\dots,n\}$. Let $(\mathcal{A},\varphi)$ be a commutative non-commutative probability space. For each partition $\pi\in\mathcal{P}(n)$, we define an $n$-linear form $\varphi_\pi$ on $\mathcal{A}$ by setting $\varphi_\pi=\varphi_\sigma$, where $\sigma$ is any permutation of $\{1,\dots,n\}$ such that $\{\{\sigma\}\}=\pi$. Since $\mathcal{A}$ is commutative, this definition does not depend on the choice of $\sigma$. Finally, when $\sigma$ is a permutation of $\{1,\dots,n\}$, we say that $i,j\in\{1,\dots,n\}$ are *consecutive* in a cycle of $\sigma$ if $\sigma(i)=j$ or $\sigma(j)=i$. We will use repeatedly the following fact, which is a consequence of Proposition 2.7 and Proposition 1 of [B98].

**Lemma 4.6.** *Choose two integers $k,l>0$ and set $n=k+l$.*
*1. There exists universal coefficients $(C(\sigma,\pi,\pi'))_{\sigma\in\mathfrak{S}_n,\pi\in\mathcal{P}(k),\pi'\in\mathcal{P}(l)}$ such that the following property holds:*

*Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative sub-algebras of some non-commutative probability space $(\mathcal{M},\varphi)$ which are $t$-free with respect to $\varphi$. Consider $\sigma\in\mathfrak{S}_n$. For all $a_1,\dots,a_k\in\mathcal{A}$ and all*

$b_1, \ldots, b_l \in \mathcal{B}$,

$$\varphi_\sigma(a_1, \ldots, a_k, b_1, \ldots, b_l) = \sum_{\pi \in \mathcal{P}(k), \pi' \in \mathcal{P}(l)} C(\sigma, \pi, \pi') \varphi_\pi(a_1, \ldots, a_k) \varphi_{\pi'}(b_1, \ldots, b_l). \tag{15}$$

*2. The coefficient $C(\sigma, \pi, \pi')$ can be non-zero only if every block of $\pi$ is contained in a block of $\{\{\sigma\}\}$.*
*3. If two elements $i$ and $j$ of $\{1, \ldots, k\}$ are consecutive in a cycle of $\sigma$, then $C(\sigma, \pi, \pi')$ can be non-zero only if $i$ and $j$ are in the same block of $\pi$.*

With the notation of the lemma above, we associate to every collection $(c(\sigma))_{\sigma \in \mathfrak{S}_n}$ the following family of coefficients:

$$\forall \pi \in \mathcal{P}(k), \pi' \in \mathcal{P}(l), \quad D_c(\pi, \pi') = \sum_{\sigma \in \mathfrak{S}_n} c(\sigma) C(\sigma, \pi, \pi'), \tag{16}$$

which will play an important role in the proofs of Theorems 4.4 and 4.3.

*Proof.* (Theorem 4.4) Let us choose $t > 0$ a positive real. We prove by contradiction that there exists no $t$-free cumulant of order 7. So, let us assume that there exists one and let $(c(\sigma))_{\sigma \in \mathfrak{S}_7}$ be one of them, which we choose to be conjugation-invariant thanks to Lemma 4.5. Thus, we denote it also by $(c_\lambda)_{\lambda \vdash 7}$. Since $c_7 \neq 0$, we may and will assume that $c_7 = 1$. Then, we proceed as follows.

Let us consider a non-commutative probability space $(\mathcal{M}, \varphi)$ and two commutative subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{M}$ which are $t$-free with respect to $\varphi$. Let us choose $a_1, a_2, a_3 \in \mathcal{A}$ and $b_1, b_2, b_3, b_4, b_5 \in \mathcal{B}$, which we assume to be all centered. Set $k_7 = \sum_{\sigma \in \mathfrak{S}_7} c(\sigma) \varphi_\sigma$. By using the $t$-freeness of $\mathcal{A}$ and $\mathcal{B}$, we will express $k_7(a_1, a_2, b_1, b_2, b_3, b_4, b_5)$ and $k_7(a_1, a_2, a_3, b_1, b_2, b_3, b_4)$ in terms of the coefficients $(c_\lambda)_{\lambda \vdash 7}$, the joint moments of $a_1, a_2, a_3$, and the joint moments of $b_1, b_2, b_3, b_4, b_5$. By the assumption that $k_7$ is a $t$-free cumulant, the two expressions that we get must vanish. Since the distribution of the $a$'s and $b$'s is arbitrary, every coefficient of a given product of moments of the $a$'s and $b$'s must vanish. This gives us linear relations on the coefficients $(c_\lambda)_{\lambda \vdash 7}$, which will turn out to be incompatible.

Let us start with $k_7(a_1, a_2, b_1, b_2, b_3, b_4, b_5)$. By Lemma 4.6, this quantity can be written as

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_7, \pi \in \mathcal{P}(2), \pi' \in \mathcal{P}(5)} c(\sigma) C(\sigma, \pi, \pi') \varphi_\pi(a_1, a_2) \varphi_{\pi'}(b_1, b_2, b_3, b_4, b_5) \\ &\qquad = \sum_{\pi \in \mathcal{P}(2), \pi' \in \mathcal{P}(5)} D_c(\pi, \pi') \varphi_\pi(a_1, a_2) \varphi_{\pi'}(b_1, b_2, b_3, b_4, b_5). \end{aligned} \tag{17}$$

We are thus interested in computing, for each pair $(\pi, \pi')$, the coefficient $D_c(\pi, \pi')$. It turns out to be convenient to think of $b_1, \ldots, b_5$ as occupying the slots 3 to 7 rather than 1 to 5 and to see $\pi'$ as a partition of the set $\{3, \ldots, 7\}$ accordingly. We hope that no ambiguity will result from this change in our conventions.

Since we have chosen to consider elements which are centered, the sum (17) can be restricted to pairs of partitions without singletons. This leaves us with the following pairs $(\pi, \pi')$: $(\{\{1, 2\}\}, \{\{3, 4, 5, 6, 7\}\})$, $(\{\{1, 2\}\}, \{\{3, 4\}, \{5, 6, 7\}\})$ and those which are deduced from the latter by permuting $3, 4, 5, 6, 7$.

Let us compute $D_c(\{\{1, 2\}\}, \{\{3, 4, 5, 6, 7\}\})$. By the second assertion of Lemma 4.6, the permutations $\sigma$ which contribute to this term must have $1, 2$ on one hand, and $3, 4, 5, 6, 7$ on the

other hand, in the same cycle. This can occur if $\sigma$ is either a 7-cycle or the product of a 2-cycle and a 5-cycle.

Let us first compute the contribution of 7-cycles. The coefficient $C(\sigma, \{1,2\}, \{3,4,5,6,7\})$ is not the same for all 7-cycles $\sigma$. We must distinguish between those in which 1 and 2 are consecutive and those in which they are not. There are 2!5! 7-cycles in which 1 and 2 are consecutive. For each such cycle $\sigma$, $C(\sigma, \{1,2\}, \{3,4,5,6,7\}) = 1$, thanks to (10). In a cycle where 1 and 2 are not consecutive, there may be one, two, three or four elements of $\{3,4,5,6,7\}$ between 1 and 2. In each case, there are 5! cycles, each contributing a factor $e^{-2t}$, thanks to (11).

Let us now compute the contribution of products of a transposition and a 5-cycle. There are 1!4! permutations with two cycles, one which contains $1,2$ and the other $3,4,5,6,7$. Each such permutation contributes a factor $c_{52}$.

Altogether, we have found that

$$D_c(\{\{1,2\}\}, \{\{3,4,5,6,7\}\}) = 24(c_{52} + 10(1 + 2e^{-2t})). \tag{18}$$

Let us now compute $D_c(\{\{1,2\}\}, \{\{3,4\}, \{5,6,7\}\})$. By the second assertion of Lemma 4.6, there are five possibilities for the partition $\{\{\sigma\}\}$ underlying a permutation $\sigma$ which contributes to this coefficient. We study them one after the other.

•$\{\{\sigma\}\} = \{\{1,2,3,4,5,6,7\}\}$. Since, by the third assertion of Lemma 4.6, any two elements of $\{3,4,5,6,7\}$ which are consecutive in $\sigma$ must be in the same block of $\pi' = \{\{3,4\}, \{5,6,7\}\}$, no element of $\{3,4\}$ can be consecutive to an element of $\{5,6,7\}$ in $\sigma$. Since there are only two $a$'s, the only possibility is that 3 and 4 on one hand, and 5, 6, and 7 on the other hand, are consecutive in $\sigma$ and separated by 1 and 2. There are 2!2!3! 7-cycles with this property. Each of them contributes to the sum with a factor $1 - e^{-2t}$, according to (11).

•$\{\{\sigma\}\} = \{\{1,2\}, \{3,4,5,6,7\}\}$. By the third assertion of Lemma 4.6, these permutations do not contribute.

•$\{\{\sigma\}\} = \{\{1,2,3,4\}, \{5,6,7\}\}$. There are two possible structures for the 4-cycle of $\sigma$ in this case. Either the $a$'s and the $b$'s are consecutive, or they are intertwined. In the first situation, there are 2!2!2! permutations, each of which contributes $c_{43}$, thanks to (10). In the second situation, there are 2!2! permutations, because of a higher symmetry, each of which contributes $e^{-2t}c_{43}$, thanks to (11).

•$\{\{\sigma\}\} = \{\{1,2,5,6,7\}, \{3,4\}\}$. Again, there are two possible structures for the 5-cycle of $\sigma$, depending on whether the $a$'s are consecutive or not. There are 2!3! permutations where they are, and each contributes $c_{52}$. There are also 2!3! permutations where they are not, each contributing $e^{-2t}c_{52}$.

•$\{\{\sigma\}\} = \{\{1,2\}, \{3,4\}, \{5,6,7\}\}$. This is the simplest situation. There are 2 permutations with this cycle structure and each contributes $c_{322}$.

Finally,

$$D_c(\{\{1,2\}\}, \{\{3,4\}, \{5,6,7\}\}) = 2\left(c_{322} + 2(2 + e^{-2t})c_{43} + 6(1 + e^{-2t})c_{52} + 12(1 - e^{-2t})\right). \tag{19}$$

Let us perform the same kind of computations on

$$k_7(a_1, a_2, a_3, b_1, b_2, b_3, b_4) = \sum_{\pi \in \mathcal{P}(\{1,2,3\}), \pi' \in \mathcal{P}(\{4,5,6,7\})} D_c(\pi, \pi')\varphi_\pi(a_1, a_2, a_3)\varphi_{\pi'}(b_1, b_2, b_3, b_4).$$

Since our variables are centered, the only pairs of partitions which occur in the sum are $(\{\{1,2,3\}\}, \{\{4,5,6,7\}\})$, $(\{\{1,2,3\}\}, \{\{4,5\}, \{6,7\}\})$ and those which are deduced from the latter by permuting $4,5,6,7$.

Let us compute $D_c(\{\{1,2,3\}\},\{\{4,5,6,7\}\})$. The permutations which contribute to this coefficient are 7-cycles and products of a 3-cycle and a 4-cycle. As before, all 7-cycles do not contribute in the same way. If the $a$'s are consecutive, which is the case for 3!4! 7-cycles, the contribution is simply 1. If two $a$'s are consecutive and the third is on its own, the 7-cycle contributes $e^{-2t}$. In this case, there can be one, two or three $b$'s between the isolated $a$ and the pair of consecutive $a$'s, in the cyclic order. In each case, there are 3!4! possible 7-cycles. Finally, the three $a$'s can be isolated. This happens in 3!4! 7-cycles, and each contributes $e^{-3t}$, thanks to (12). So far, we have a contribution of $144(1+3e^{-2t}+e^{-3t})$. The contribution of products of a 3-cycle and a 4-cycle is much simpler to compute: it is $2!3!c_{43}$. We find

$$D_c(\{\{1,2,3\}\},\{\{4,5,6,7\}\}) = 12(c_{43}+12(1+3e^{-2t}+e^{-3t})). \tag{20}$$

Let us finally compute $D_c(\{\{1,2,3\}\},\{\{4,5\},\{6,7\}\})$. Again, by Lemma 4.6, there are five possibilities for the partition $\{\{\sigma\}\}$, which we examine one after the other.

- $\{\{\sigma\}\} = \{\{1,2,3,4,5,6,7\}\}$. No element of $\{4,5\}$ can be consecutive with an element of $\{6,7\}$ in $\sigma$. Still, there are several possibilities. Let us first consider the 7-cycles where $4,5$ on one hand and $6,7$ on the other hand are consecutive. These two groups must be separated by $a$'s. There are 2!2!2!3! such 7-cycles, and each contributes for $1-e^{-2t}$, according to (11). Since there are only three $a$'s, one at least of the two pairs $\{4,5\}$ and $\{6,7\}$ must be consecutive. However, it can happen that one is not. This happens in 2!2!2!3! 7-cycles, and according to (12), each contributes for $e^{-2t}-e^{-3t}$.
- $\{\{\sigma\}\} = \{\{1,2,3\},\{4,5,6,7\}\}$. These permutations do not contribute.
- $\{\{\sigma\}\} = \{\{1,2,3,4,5\},\{6,7\}\}$. As usual by now, there are two possibilities in the 5-cycle of $\sigma$. Either the two $b$'s are consecutive, which happens in 2!3! cases with the contribution $c_{52}$, or they are not. This happens in 2!3! cases, and each case contributes for $e^{-2t}c_{52}$.
- $\{\{\sigma\}\} = \{\{1,2,3,6,7\},\{4,5\}\}$. By symmetry, this contribution is equal to the one above.
- $\{\{\sigma\}\} = \{\{1,2,3\},\{4,5\},\{6,7\}\}$. There are 2 permutations, each contributing for $c_{322}$.

Finally,

$$D_c(\{\{1,2,3\}\},\{\{4,5\},\{6,7\}\}) = 2(c_{322}+12(1+e^{-2t})c_{52}+24(1-e^{-3t})). \tag{21}$$

Let us summarize our results. We have proved that, if there exists a $t$-free cumulant of order 7, denoted by $k_7$, then for all centered $a_1,a_2,a_3 \in \mathcal{A}$ and $b_1,\dots,b_5 \in \mathcal{B}$, the following equalities hold:

$$\begin{aligned}
k_7(a_1,a_2,b_1,b_2,b_3,b_4,b_5) &= 24(c_{52}+10(1+2e^{-2t}))\varphi(a_1a_2)\varphi(b_1b_2b_3b_4b_5)\\
&\quad + 2\left(c_{322}+2(2+e^{-2t})c_{43}+6(1+e^{-2t})c_{52}+12(1-e^{-2t})\right)\varphi(a_1a_2)\varphi(b_1b_2)\varphi(b_3b_4b_5)\\
&\quad + 2\left(c_{322}+2(2+e^{-2t})c_{43}+6(1+e^{-2t})c_{52}+12(1-e^{-2t})\right)\varphi(a_1a_2)\varphi(b_1b_3)\varphi(b_2b_4b_5)\\
&\quad + \dots,
\end{aligned}$$

where all partitions of $\{b_1,b_2,b_3,b_4,b_5\}$ into a pair and a triple appear, and

$$\begin{aligned}
k_7(a_1,a_2,a_3,b_1,b_2,b_3,b_4) &= 12(c_{43}+12(1+3e^{-2t}+e^{-3t}))\varphi(a_1a_2a_3)\varphi(b_1b_2b_3b_4)\\
&\quad + 2(c_{322}+12(1+e^{-2t})c_{52}+24(1-e^{-3t}))\varphi(a_1a_2a_3)\varphi(b_1b_2)\varphi(b_3b_4)\\
&\quad + 2(c_{322}+12(1+e^{-2t})c_{52}+24(1-e^{-3t}))\varphi(a_1a_2a_3)\varphi(b_1b_3)\varphi(b_2b_4)\\
&\quad + 2(c_{322}+12(1+e^{-2t})c_{52}+24(1-e^{-3t}))\varphi(a_1a_2a_3)\varphi(b_1b_4)\varphi(b_2b_3).
\end{aligned}$$

Since $k_7$ is a $t$-free cumulant, these two expressions must be zero for all choices of $a$'s and $b$'s. Since the distribution of the $a$'s and $b$'s is unspecified, this implies that the coefficients which

appear in these equalities in front of the various products of moments of $a$'s and $b$'s must vanish. This implies the following relations:

$$\begin{aligned} c_{52} &= -10(1+2e^{-2t}),\\ c_{43} &= -12(1+3e^{-2t}+e^{-3t}),\\ c_{322} &= -2(2+e^{-2t})c_{43} - 6(1+e^{-2t})c_{52} - 12(1-e^{-2t}),\\ c_{322} &= -12(1+e^{-2t})c_{52} - 24(1-e^{-3t}). \end{aligned}$$

It does not take a long computation to see that the two expressions of $c_{322}$ are different, since the first involves $e^{-5t}$, whereas the second does not. We leave it to the reader to check that the difference between the two values of $c_{322}$ that we have obtained is equal to $24e^{-3t}(1-e^{-t})^2$. This quantity vanishes only for $t=0$ or $t=+\infty$. □

In order to prove that $t$-free cumulants of order at most 6 exist, we are going to construct them. We prove first a lemma which settles the problem of the coefficients $c_\lambda$ for the partitions $\lambda$ whose smallest part is 1.

Let us introduce some notation. Let $\mu = (\mu_1 \geq \ldots \geq \mu_r)$ be a partition of some integer $n$. We denote by $\ell(\mu)$ the number of non-zero parts of $\mu$ and we write $\mu \Vdash n$ if $\mu_{\ell(\mu)} \geq 2$, that is, if $\mu$ has no part equal to 1. Let $i \geq 1$ an integer. We denote by $\mu + \delta_i$ the partition of $n+1$ whose parts are $\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_r$, rearranged in non-increasing order. If $i > \ell(\mu)$, then $\mu + \delta_i$ is simply the partition $\mu$ to which a part equal to 1 has been appended.

**Proposition 4.7.** *Let $n \geq 2$ be an integer. Choose $t \in [0, +\infty]$. A collection $(c_\lambda)_{\lambda \vdash n}$ is a $t$-free cumulant if and only if the following two conditions hold:*
*1. The relation (13) is satisfied for all $m_1, \ldots, m_n$ which are centered.*
*2. For all $\mu \vdash n-1$,*

$$c_{\mu + \delta_{\ell(\mu)+1}} = -\sum_{i=1}^{\ell(\mu)} \mu_i c_{\mu+\delta_i}. \tag{22}$$

*Moreover, a collection of complex numbers $(c_\lambda)_{\lambda \Vdash n}$ which satisfies (13) for all $m_1, \ldots, m_n$ which are centered can be extended in a unique way into a $t$-free cumulant of order $n$.*

When $\sigma$ is a permutation of $\{1, \ldots, n\}$, let us denote by $[\sigma]$ the partition of the integer $n$ given by the lengths of the cycles of $\sigma$.

*Proof.* Let $c$ be a $t$-free cumulant of order $n$. In order to check that (22) is satisfied, let us choose $m_1, \ldots, m_{n-1}$ in some probability space $(\mathcal{M}, \varphi)$ and write the fact that $k_n(m_1, \ldots, m_{n-1}, 1) = 0$. We find

$$\sum_{\lambda \vdash n} c_\lambda \sum_{\substack{\sigma \in \mathfrak{S}_n \\ [\sigma] = \lambda}} \varphi_\sigma(m_1, \ldots, m_{n-1}, 1) = 0. \tag{23}$$

Let $r_n : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$ denote the following function: for all $\sigma \in \mathfrak{S}_n$, $r_n(\sigma)$ is the permutation of $\{1, \ldots, n-1\}$ obtained by removing $n$ from the cycle of $\sigma$ which contains it. For each $\sigma$, we have the equality $\varphi_\sigma(m_1, \ldots, m_{n-1}, 1) = \varphi_{r_n(\sigma)}(m_1, \ldots, m_{n-1})$. Now a permutation $\tau \in \mathfrak{S}_{n-1}$ has exactly $n$ preimages by $r_n$. Moreover, if $[\tau] = \mu = (\mu_1 \geq \ldots \geq \mu_{\ell(\mu)} > 0) \vdash n-1$, then all preimages of $\tau$ belong to one of the conjugacy classes $\mu + \delta_i$ for $i = 1, \ldots, \ell(\mu)+1$. Finally, $r_n^{-1}(\tau)$ contains exactly one element of $\mu + \delta_{\ell(\mu)+1}$ and $\mu_i$ elements of $\mu + \delta_i$ for $i = 1, \ldots, \ell(\mu)$.

We can thus rewrite (23) as follows:

$$\sum_{\mu\vdash n-1}\left(\sum_{i=1}^{\ell(\mu)}\mu_i c_{\mu+\delta_i}+c_{\mu+\delta_{\ell(\mu)+1}}\right)\sum_{\substack{\tau\in\mathfrak{S}_{n-1}\\ [\tau]=\mu}}\varphi_\tau(m_1,\dots,m_{n-1})=0. \tag{24}$$

Since the distribution of $m_1,\dots,m_n$ is arbitrary, all the coefficients between the brackets must vanish. It follows that (22) is satisfied.

Conversely, let $(c(\sigma))_{\sigma\in\mathfrak{S}_n}$ be a collection which satisfies (13) for centered elements and (22). Then, by the computation that we have just done, this collection satisfies (14) and hence, by multilinearity, (13) for arbitrary elements.

Let us prove the last assertion. For any $\lambda\vdash n$ with at least one part equal to 1, the relation (22) expresses the value of $c_\lambda$ in terms of $c_{\lambda'}$ for partitions $\lambda'$ of $n$ which have strictly less parts equal to 1 than $\lambda$. The collection $(c_\lambda)_{\lambda\vdash n}$ is thus completely and uniquely determined by $(c_\lambda)_{\lambda\nvdash n}$. The fact that the resulting collection is a $t$-free cumulant is granted by the first part of the proposition. □

The last result simplifies greatly the search for $t$-free cumulants, since it allows one to restrict to centered elements and partitions in parts at least equal to 2. We apply it to find cumulants of order less than 6.

*Proof.* (Theorem 4.3) Let us prove that there exist $t$-free cumulants up to order 6. We proceed by first establishing enough conditions that their coefficients must satisfy, in order to determine these coefficients. Then, we check that we actually have a $t$-free cumulant.

We will always normalize our cumulants by the condition $c_n=1$.

• $n=2$. By Proposition 4.7, the condition $c_2=1$ suffices to determine the whole cumulant, and $c_{11}=-1$. The relation (10) implies that we have indeed got a $t$-free cumulant.

• $n=3$. Again, the condition $c_3=1$ determines completely the cumulant. Using (22), we find $c_{21}=-2$ and $c_{111}=4$. The relation (10) implies again that the collection thus obtained is a $t$-free cumulant. Indeed, the product of any three centered elements, one being $t$-free with the two others, is centered. Hence, our collection satisfies (13) on centered elements.

• $n=4$. This is the first case where the relation $c_4=1$ does not suffice determine the cumulant. Indeed, we must compute $c_{22}$. For this, let us choose in some probability space elements $a_1,a_2,\dots$ and $b_1,b_2,\dots$, such that $\{a_1,a_2,\dots\}$ and $\{b_1,b_2,\dots\}$ are $t$-free. We will use this notation again in this proof without redefining it. Let us assume that a $t$-free cumulant $c$ of order 4 is given and let us compute $D_c(\{\{1,2\}\},\{\{3,4\}\})$ (see (16)). There are 4-cycles which contribute to this coefficient. In 2!2! of them, 1 and 2 are consecutive and they contribute for 1 each. In 2! others, 1 and 2 are not consecutive and each such cycle contributes for $e^{-2t}$. There is also one product of two 2-cycles, which contributes for $c_{22}$. Finally, $D_c(\{\{1,2\}\},\{\{3,4\}\})=c_{22}+2(2+e^{-2t})$. The nullity of this coefficient implies $c_{22}=-2(2+e^{-2t})$. Using (22), we determine the remaining coefficients, and find

$$c_4=1,\ c_{31}=-3,\ c_{22}=-2(2+e^{-2t}),\ c_{211}=2(5+e^{-2t}), c_{1111}=-6(5+e^{-2t}).$$

Now let us check that the collection thus defined satisfies (13) for centered elements. Set $k_4=\sum_\sigma c(\sigma)\varphi_\sigma$. If we expand $k_4(a_1,b_1,b_2,b_3)$ according to (15), then all terms involve $\varphi(a_1)$ and vanish. Now $k_4(a_1,a_2,b_1,b_2)$ also vanishes, because this is how we have chosen the value of $c_{22}$. Finally, we do have a $t$-free cumulant of order 4.

• $n=5$. Let $c$ be a $t$-free cumulant of order 5. Let us compute $c_{32}$ by writing the nullity of $D_c(\{\{1,2,3\}\},\{\{4,5\}\})$. There are 3!2! 5-cycles in which 4 and 5 are consecutive, and they contribute for 1 each. There are also 3!2! 5-cycles in which they are not consecutive, and each

contributes for $e^{-2t}$. There are finally 2! products of a 3-cycle and a 2-cycle, which contribute for $c_{32}$ each. Hence, we must have $c_{32} = -6(1 + e^{-2t})$. Using as usual (22), we find that the other values of $c$ must be

$$c_5 = 1,\ c_{41} = -4,\ c_{32} = -6(1 + e^{-2t}),\ c_{311} = 6(3 + e^{-2t}), c_{221} = 12(1 + e^{-2t}),$$
$$c_{2111} = -12(5 + 2e^{-2t}),\ c_{11111} = 48(5 + 2e^{-2t}).$$

The fact that $k_5 \sum_\sigma c(\sigma)\varphi_\sigma$ is a cumulant is checked just as in the case $n = 4$: the identity $k_5(a_1, b_1, b_2, b_3, b_4) = 0$ is granted by (15) and $k_5(a_1, a_2, b_1, b_2, b_3) = 0$ by the choice of $c_{32}$.

• $n = 6$. Let $c$ be a $t$-free cumulant of order 5. The value of $c_{42}$, deduced as usual from the nullity of $D_c(\{\{1, 2, 3, 4\}\}, \{\{5, 6\}\})$, is $c_{42} = -4(2 + 3e^{-2t})$. Similarly, $D_c(\{\{1, 2, 3\}\}, \{\{4, 5, 6\}\}) = 0$ gives us $c_{33} = -3(3 + 6e^{-2t} + e^{-3t})$. Finally, $D_c(\{\{1, 2\}\}, \{\{3, 4\}, \{5, 6\}\}) = 0$ implies $c_{222} = 8(7 + 17e^{-2t} + 6e^{-4t})$. The other coefficients follow as usual from (22) and we find

$$c_6 = 1,\ c_{51} = -5,\ c_{42} = -4(2 + 3e^{-2t}),\ c_{411} = 4(7 + 3e^{-2t}),\ c_{33} = -3(3 + 6e^{-2t} + e^{-3t})$$
$$c_{321} = 6(7 + e^{-3t} + 12e^{-2t}),\ c_{3111} = -12(14 + 15e^{-2t} + e^{-3t}),\ c_{222} = 8(7 + 17e^{-2t} + 6e^{-4t})$$
$$c_{2211} = -8(28 + 53e^{-2t} + 3e^{-3t} + 6e^{-4t}),\ c_{21111} = 48(21 + 34e^{-2t} + 2e^{-3t} + 3e^{-4t})$$
$$c_{111111} = -240(21 + 34e^{-2t} + 2e^{-3t} + 3e^{-4t}).$$

Let us set $k_6 = \sum_\sigma c(\sigma)\varphi_\sigma$. The nullity of $k_6(a_1, b_1, \ldots, b_5)$ follows as usual from (15). That of $k_6(a_1, a_2, b_1, b_2, b_3, b_4)$ results from the choices of $c_{42}$ and $c_{222}$. Finally, $k_6(a_1, a_2, a_3, b_1, b_2, b_3) = 0$ is granted by the choice of $c_{33}$.

Nowhere there has been any freedom in the definition of the cumulants. This shows the uniqueness of conjugation-invariant $t$-free cumulants of order at most than 6. □

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Florent Benaych-Georges, LPMA, UPMC Univ Paris 6, Case courier 188, 4, Place Jussieu, 75252 Paris Cedex 05, France, and CMAP, Ecole Polytechnique, route de Saclay, 91128 Palaiseau Cedex, France

*E-mail address*: florent.benaych@gmail.com

Thierry Lévy, CNRS and École Normale Supérieure, DMA, 45, rue d'Ulm, F-75005 Paris
*E-mail address*: levy@dma.ens.fr