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► **To cite this version:**

Said Mesloub, Abdelouahab Mansour. A mixed problem for a Boussinesq hyperbolic equation with integral condition. 2005. hal-00338641

**HAL Id: hal-00338641**

**<https://hal.science/hal-00338641>**

Submitted on 14 Nov 2008

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# A MIXED PROBLEM FOR A BOUSSINESQ HYPERBOLIC EQUATION WITH INTEGRAL CONDITION

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ABSTRACT. A hyperbolic problem wich combines a classical(Dirichlet) and a non-local constraint is considered.The existence and uniqueness of strong solutions are proved,we use a fonctionnal analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

## 1.INTRODUCTION

The first study of evolution problems with a nonlocal condition - the so called energy specification - goes back to Cannon[5] ,1963 Using an integral condition ,we proved the existence and uniqueness of the solution of a mixed problem wich combine a classical ( Dirichlet )and an integral condition for the equation . Problems involving local and integral condition for hyperbolic equations are investigated by the energy inequalities method in [1] , [6] , [7] , [8] , [9] , [10] , [11] , [12] .In this paper ,we prove the existence and uniqueness of the solution for the mixed problem (1) – (5) .Our proof is based on a priori estimate and on the fact that the range of the operator generated by the considered problem is dense.

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*Date:* Octobre 30, 2008.

*Key words and phrases.* hyperbolic equation, integral condition, a priori estimate.

## 2. Formulation of the problem

In the region  $Q = (0, l) \times (0, T)$ , with  $l < \infty$  and  $T < \infty$ , we shall consider the problem

$$(1) \quad Lu = u_{tt} - (b(x, t)u_x)_x - \beta \frac{\partial^4 u}{\partial t^2 \partial x^2} = f(x, t), \forall (x, t) \in Q$$

$$(2) \quad l_1 u = u(x, 0) = \varphi_1(x), \quad x \in (0, l)$$

$$(3) \quad l_2 u = u_t(x, 0) = \varphi_2(x), \quad x \in (0, l)$$

$$(4) \quad u(0, t) = 0, \quad t \in (0, T)$$

$$(5) \quad \int_0^l x u(x, t) dx = 0, \quad t \in (0, T)$$

where  $\beta \in IR_+^*$  and  $b(x, t)$  and its derivatives satisfy the conditions:

$C_1 : b_0 \leq b(x, t) \leq b_1$ ,  $b_t(x, t) \leq b_2$ ,  $b_x(x, t) \leq b_3$ ,  
for any  $(x, t) \in \overline{Q}$ ,

$C_2 : b_{tt}(x, t) \leq b_4$ ,  $b_{xt}(x, t) \leq b_5$ , for any  $(x, t) \in \overline{Q}$ .

The functions  $f$ ,  $\varphi_1$  and  $\varphi_2$  are known functions which satisfy the compatibility conditions:

$$\varphi_1(0) = \varphi_2(0) = \int_0^l x \varphi_1(x) dx = \int_0^l x \varphi_2(x) dx = 0.$$

### 3. functional Spaces

The problem (1)-(5) can be put in the following operator form:  $Lu = \mathcal{F}$ ,  $u \in D(L)$ , where:

$$Lu = (\mathcal{L}u, l_1u, l_2u) \text{ and } \mathcal{F} = (f, \varphi_1, \varphi_2).$$

The operator  $L$  is considered from  $B$  to  $H$ , where  $B$  is the Banach space consisting of functions  $u \in L^2(Q)$ , satisfying conditions (4) and (5) with the finite norm:

$$\|u\|_B^2 = \sup_{0 \leq \tau \leq T} \left[ \|u(\cdot, \tau)\|_{L^2(0,l)}^2 + \|u_t(\cdot, \tau)\|_{L^2(0,l)}^2 \right]$$

and  $F$  is the Hilbert space  $L^2(Q) \times L^2(0, l) \times L^2(0, l)$  equipped with the norm:

$$\|\mathcal{F}\|_H^2 = \|f\|_{L^2(Q^\tau)}^2 + \|\varphi_1\|_{L^2(0,l)}^2 + \|\varphi_2\|_{L^2(0,l)}^2.$$

Let  $D(L)$  denote the domain of  $L$ , which is the set of all functions  $u \in L^2(Q)$  for which  $u_t, u_x, u_{tx}, u_{tt}, u_{ttx} \in L^2(Q)$  and satisfying conditions (4) and (5).

### 4. A priori estimate and its consequences

**Theorem 1:** For any function  $u \in D(L)$  satisfies conditions C<sub>1</sub>-C<sub>2</sub> there exists a positive constant  $c$ , such that

$$(7) \quad \|u\|_B \leq c \|Lu\|_H,$$

**Proof :** We consider the scalar product in  $L^2(Q^\tau)$  of the operator  $\mathcal{L}u$  and  $Mu$ , where  $Mu = x\mathfrak{S}_x^*u_t - \mathfrak{S}x^*(\rho u_t)$ , with  $Q^\tau = (0, l) \times (0, \tau)$ ,  $0 \leq \tau \leq T$ , and  $\mathfrak{S}_x^*v = \int_x^l v(\xi, t)d\xi$ , we obtain

$$\begin{aligned}
(8) \quad (\mathcal{L}u, Mu)_{L^2(Q^\tau)} &= (u_{tt}, x\mathfrak{S}_x^*u_t)_{L^2(Q^\tau)} - ((b(x, t)u_x)_x, x\mathfrak{S}_x^*u_t)_{L^2(Q^\tau)} \\
&\quad - \beta(u_{ttxx}, x\mathfrak{S}_x^*u_t)_{L^2(Q^\tau)} - (u_{tt}, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)} \\
&\quad + ((b(x, t)u_x)_x, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)} + \beta(u_{ttxx}, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)}.
\end{aligned}$$

Making use of conditions (2)-(5) and integrating by parts we establish the equalities:

$$\begin{aligned}
(9) \quad (u_{tt}, x\mathfrak{S}_x^*u_t)_{L^2(Q^\tau)} &= \frac{1}{2} \|\mathfrak{S}_x^*u_t(\cdot, \tau)\|_{L^2(0, l)}^2 \\
&\quad - \frac{1}{2} \|\mathfrak{S}_x^*\varphi_2\|_{L^2(0, l)}^2 - (\mathfrak{S}_x^*u_{tt}, u_t)_{L^2_\rho(Q^\tau)},
\end{aligned}$$

$$\begin{aligned}
(10) \quad & - ((b(x, t)u_x)_x, x\mathfrak{S}_x^*(u_t))_{L^2(Q^\tau)} \\
&= \frac{1}{2} \left\| \sqrt{b(\cdot, \tau)}u(\cdot, \tau) \right\|_{L^2(0, l)}^2 - \frac{1}{2} \left\| \sqrt{b(\cdot, 0)}\varphi_1 \right\|_{L^2(0, l)}^2 \\
&\quad - \frac{1}{2} \left\| \sqrt{b_t(\cdot, t)}u \right\|_{L^2(Q^\tau)}^2 - (b_x(x, t)u, \mathfrak{S}_x^*u_t)_{L^2(Q^\tau)} \\
&\quad - (b(x, t)u_x, u_t)_{L^2_\rho(Q^\tau)},
\end{aligned}$$

$$\begin{aligned}
(11) \quad -\beta(u_{ttxx}, x\mathfrak{S}_x^*(u_t))_{L^2(Q^\tau)} &= \frac{\beta}{2} \|u_t(\cdot, \tau)\|_{L^2(0, l)}^2 \\
&\quad - \frac{\beta}{2} \|\varphi_2\|_{L^2(0, l)}^2 - \beta(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}.
\end{aligned}$$

$$(12) \quad - (u_{tt}, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)} = (\mathfrak{S}_x^*u_{tt}, u_t)_{L^2_\rho(Q^\tau)},$$

$$(13) \quad ((b(x, t)u_x)_x, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)} = (b(x, t)u_x, u_t)_{L^2_\rho(Q^\tau)},$$

$$(14) \quad \beta(u_{ttxx}, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q^\tau)} = \beta(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}.$$

Combining equalities (9)-(14) and(8)we obtain :

$$\begin{aligned}
(15) \quad & \frac{1}{2} \|\mathfrak{S}_x^* u_t(\cdot, \tau)\|_{L^2(0,l)}^2 + \frac{1}{2} \left\| \sqrt{b(\cdot, t)} u(\cdot, \tau) \right\|_{L^2(0,l)}^2 \\
& + \frac{\beta}{2} \|u_t(\cdot, \tau)\|_{L^2(0,l)}^2 \\
& = \frac{1}{2} \|\mathfrak{S}_x^* \varphi_2\|_{L^2(0,l)}^2 + \frac{1}{2} \left\| \sqrt{b(\cdot, t)} \varphi_1 \right\|_{L^2(0,l)}^2 \\
& + \frac{\beta}{2} \|\varphi_2\|_{L^2(0,l)}^2 + \frac{1}{2} \left\| \sqrt{b_t} u \right\|_{L^2(Q_\tau)}^2 + (b_x(x, t) u, \mathfrak{S}_x^* u_t)_{L^2(Q_\tau)} \\
& + (\mathcal{L}u, x \mathfrak{S}_x^* u_t)_{L^2(Q_\tau)} - (\mathcal{L}u, \mathfrak{S}_x^*(\rho u_t))_{L^2(Q_\tau)}.
\end{aligned}$$

By applying the Cauchy inequality to the last three terms on the right-hand side of the inequality (15) and making use conditions C<sub>1</sub>, combining with (15), we obtain

$$\begin{aligned}
(16) \quad & \|u(\cdot, \tau)\|_{L^2(0,l)}^2 + \|u_t(\cdot, \tau)\|_{L^2(0,l)}^2 \\
& + \|\mathfrak{S}_x^* u_t(\cdot, \tau)\|_{L^2(0,l)}^2 \\
& \leq k \left[ \|f\|_{L^2(Q_\tau)}^2 + \|\varphi_1\|_{L^2(0,l)}^2 + \|\varphi_2\|_{L^2(0,l)}^2 \right. \\
& \left. \|u\|_{L^2(Q_\tau)}^2 + \|u_t\|_{L^2(Q_\tau)}^2 + \|\mathfrak{S}_x^* u_t\|_{L^2(Q_\tau)}^2 \right].
\end{aligned}$$

where  $k = \frac{\max(2, b_1, \beta + l^2, b_3^2 + b_2, l^4)}{\min(1, b_0, \beta)}$ .

Applying the Gronwall lemma to(16), and eliminating the term  $\|\mathfrak{S}_x^* u_t(\cdot, \tau)\|_{L^2(0,l)}^2$  of the left-hand side of the inequality we obtain

$$\begin{aligned}
(17) \quad & \|u(\cdot, \tau)\|_{L^2(0,l)}^2 + \|u_t(\cdot, \tau)\|_{L^2(0,l)}^2 \\
& \leq k \exp(kT) \left( \|f\|_{L^2(Q_\tau)}^2 + \|\varphi_1\|_{L^2(0,l)}^2 + \|\varphi_2\|_{L^2(0,l)}^2 \right).
\end{aligned}$$

Since the left-hand side of (17) does not depend on  $\tau$ , we take the supremum with  $\tau$  from 0 to  $T$ , then the estimate (7) follows with  $c = \sqrt{k} \exp(k\frac{T}{2})$ .

## 5. Solvability of the problem

**Proposition 1.** The operator  $L$  acting from  $B$  to  $H$  have a closure.

**Proof.** ( see [3] )

Let be  $\bar{L}$  the closure of  $L$ ,  $D(\bar{L})$  its domain .

**Definition .**The solution of  $\bar{L}u = F$  for any  $u \in D(\bar{L})$  is strong solution of problem(1)-(5). we take the limit in the inequality (7) ,we obtain  $\|u\|_B \leq c \|\bar{L}u\|_H, \forall u \in D(\bar{L})$ .

From the inequality we have:

**Corollary 1:** The strong solution of problem (1)-(5) when it exists, it's unique, and depends continuly of data  $f, \varphi_1, \varphi_2$ .

**Corollary 2 :**The set of values  $R(\bar{L})$  of the operator  $\bar{L}$  is equal to the closure  $\overline{R(L)}$  of  $R(L)$ .

**Theorem 2:** If the conditions  $C_1$ - $C_2$  are satisfying ,then for any  $\mathcal{F} = (f, \varphi_1, \varphi_2) \in H$ , there exists a strong unique solution  $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$  of the probleme (1)-(5) where the estimate  $\|u\|_B \leq c \|\mathcal{F}\|_H$  is satisfying ,where  $c$  is a positive constant does not depends of  $u$ .

**Proof:** From(22) we conclude that the operator  $\bar{L}$  acting from  $D(\bar{L})$  in  $R(\bar{L})$  have an inverse  $\bar{L}^{-1}$ , and from corollary 2, we conclude that the range  $R(\bar{L})$  of the operator  $\bar{L}$  is closed. Then we will be prove the density of the set  $R(L)$  in the space  $H$  (i.e)  $\overline{R(L)} = H$ .

For this we need the following proposition :

**Proposition 2:** If, for all functions  $u \in D_0(L)$ , where

$$D_0(L) = \{u/u \in D(L) : l_1 u = l_2 u = 0\}$$

and for some function  $\omega \in L^2(Q)$ , we have

$$(18) \quad (\mathcal{L}u, \omega)_{L^2(Q)} = 0,$$

then  $\omega$  vanishes almost everywhere in  $Q$ .

**Proof of the proposition 2 :** The relation (18) is given for all  $u \in D_0(L)$ , we can express it in a particular form .Let  $u_{tt}$  be a solution of :

$$(19) \quad b(\sigma, t) [x \mathfrak{S}_x^* u_{tt} - \mathfrak{S}_x^*(\rho u_{tt})] = h(x, t),$$

where  $\sigma$  is a constant in  $(0, l)$  and  $h(x, t) = \int_t^T \omega(x, \tau) d\tau$ .

And let  $u$  be the fonction defined by:

$$(20) \quad u = \begin{cases} 0, & \text{si } 0 \leq t \leq s, \\ \int_s^t (t - \tau) u_{\tau\tau} d\tau, & \text{si } s \leq t \leq T. \end{cases}$$

(19) and (20) follows  $u$  is in  $D_0(L)$  and:

$$(21) \quad \begin{aligned} \omega(x, t) &= \mathfrak{S}_x^{*-1} h \\ &= - [b(\sigma, t) (x \mathfrak{S}_x^* u_{tt} - \mathfrak{S}_x^*(\rho u_{tt}))]_t \\ &= [b(\sigma, t) \mathfrak{S}_x^*(\rho - x) u_{tt}]_t. \end{aligned}$$

To continue the proof we need the following lemma :

**Lemma 2.** The function  $\omega$  defined by (21), belongs to the space  $L^2(Q)$ .

**Proof of lemma 2 :** We start with the proof of this inequality  $\|\mathfrak{S}_x^*(\rho - x) u_{tt}\|_{L^2(0, l)}^2 \leq \frac{l^4}{12} \|u_{tt}\|_{L^2(0, 1)}^2$ .

From this inequality and since the conditions  $C_1$  are satisfied we conclude that  $b_t(\sigma, t) \mathfrak{S}_x^*(\rho - x) u_{tt}$  belongs to  $L^2(Q)$ .

Because  $\omega(x, t) = [b(\sigma, t) \mathfrak{S}_x^*(\rho - x) u_{tt}]_t = b_t(\sigma, t) \mathfrak{S}_x^*(\rho - x) u_{tt} + b(\sigma, t) \mathfrak{S}_x^*(\rho - x) u_{ttt}$ , then we will



be proved that:  $b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{ttt} \in L^2(Q)$ .

For this we introduce the  $t$ -averaging operators  $\rho_\varepsilon$  of the form

$$(\rho_\varepsilon f)(x, t) = \frac{1}{\varepsilon} \int_0^T \omega\left(\frac{t-s}{\varepsilon}\right) f(x, s) ds,$$

where  $\omega \in C_0^\infty(0, T)$ ,  $\omega \geq 0$ ,

$$\int_{-\infty}^{+\infty} \omega(s) ds = 1, \omega \equiv 0 \text{ for } t \leq 0 \text{ and } t \geq T,$$

applying the operators  $\rho_\varepsilon$  and  $\frac{\partial}{\partial t}$  to the equation

$$-b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt} = h(x, t),$$

we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} (-b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}) = \\ & \frac{\partial}{\partial t} [-b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt} + \rho_\varepsilon (b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt})] - \frac{\partial}{\partial t} \rho_\varepsilon h. \end{aligned}$$

Then

$$\begin{aligned} & \|b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}\|_{L^2(Q)}^2 \\ & \leq 2 \left\| \frac{\partial}{\partial t} [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt} - \rho_\varepsilon (b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt})] \right\|_{L^2(Q)}^2 \\ & \quad + 2 \left\| \frac{\partial}{\partial t} \rho_\varepsilon h \right\|_{L^2(Q)}^2. \end{aligned}$$

Since  $\rho_\varepsilon f \xrightarrow{\varepsilon \rightarrow 0} f$ , and  $\frac{\partial}{\partial t} (b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt})$  is bounded in  $L^2(Q)$ , then  $\omega \in L^2(Q)$ .

Now we return to the 2<sup>nd</sup> proposition, we replace  $\omega$  in

(18) by its representation given by (21) we have:

$$\begin{aligned} (22) \quad & (u_{tt}, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)} \\ & = ((b(x, t) u_x)_x, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)} + \\ & \quad + \beta (u_{ttxx}, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)}. \end{aligned}$$

Making use conditions(3)-(5),and from the particular forme of  $u$  given by (19) and (20), the equality (22) can be simplified.For this integrating by parts each term of the equality on the sup-domain  $Q_s = (0, l) \times (s, T)$  where  $0 \leq s \leq T$

$$(23) \quad (u_{tt}, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)} \\ = \frac{1}{2} \left\| \sqrt{b(\sigma, s)} \mathfrak{S}_x^* u_{tt}(\cdot, s) \right\|_{L^2(0, l)}^2 - \frac{1}{2} \left\| \sqrt{b_t(\sigma, \cdot)} \mathfrak{S}_x^* u_{tt} \right\|_{L^2(Q_s)}^2,$$

$$(24) \quad ((b(x, t) u_x)_x, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)} \\ = -\frac{1}{2} \left\| \sqrt{b(\cdot, T) b(\sigma, T)} u_t(\cdot, T) \right\|_{L^2(0, l)}^2 \\ + \frac{1}{2} \int_{Q_s} [3b_t(x, t) b(\sigma, t) + b(x, t) b_t(\sigma, t)] (u_t)^2 dxdt \\ - \int_0^l b_t(x, T) b(\sigma, T) u(x, T) u_t(x, T) dx \\ + \int_{Q_s} [b_{tt}(x, t) b(\sigma, t) + b_t(x, t) b_t(\sigma, t)] uu_t dxdt \\ + \int_{Q_s} [b_x(x, t) u_t + b_{xt}(x, t) u] b(\sigma, t) \mathfrak{S}_x^* u_{tt} dxdt,$$

$$(25) \quad \beta (u_{ttxx}, [b(\sigma, t) \mathfrak{S}_x^*(\rho - x)u_{tt}]_t)_{L^2(Q)} \\ = \frac{\beta}{2} \left\| \sqrt{b_t(\sigma, \cdot)} u_{tt} \right\|_{L^2(Q_s)}^2 - \frac{\beta}{2} \left\| \sqrt{b(\sigma, s)} u_{tt}(\cdot, s) \right\|_{L^2(0, l)}^2.$$

Substitution of(23)-(25)into (22) gives

$$\begin{aligned}
(26) \quad & \frac{1}{2} \left\| \sqrt{b(\sigma, s)} \mathfrak{S}_x^* u_{tt}(\cdot, s) \right\|_{L^2(0, l)}^2 \\
& + \frac{1}{2} \left\| \sqrt{b(\cdot, T) b(\sigma, T)} u_t(\cdot, T) \right\|_{L^2(0, l)}^2 \\
& + \frac{\beta}{2} \left\| \sqrt{b(\sigma, s)} u_{tt}(\cdot, s) \right\|_{L^2(0, l)}^2 \\
= & \frac{1}{2} \left\| \sqrt{b_t(\sigma, s)} \mathfrak{S}_x^* u_{tt} \right\|_{L^2(Q_s)}^2 + \frac{\beta}{2} \left\| \sqrt{b_t(\sigma, \cdot)} u_{tt} \right\|_{L^2(Q_s)}^2 \\
& + \frac{1}{2} \int_{Q_s} [3b_t(x, t) b(\sigma, t) + b(x, t) b_t(\sigma, t)] (u_t)^2 dx dt \\
& - \int_0^l b_t(x, T) b(\sigma, T) u(x, T) u_t(x, T) dx \\
& + \int_{Q_s} [b_{tt}(x, t) b(\sigma, t) + b_t(x, t) b_t(\sigma, t)] uu_t dx dt \\
& + \int_{Q_s} [b_x(x, t) u_t + b_{xt}(x, t) u] b(\sigma, t) \mathfrak{S}_x^* u_{tt} dx dt .
\end{aligned}$$

By applying the Cauchy inequality and Cauchy inequality with  $\varepsilon$  to estimate the last three terms on the right-hand side of the inequality (26) and making use conditions  $C_1 - C_2$ , combining the estimates and(26)taking into account

that  $\varepsilon = \frac{b_0^2}{2b_1^2}$  we obtain

$$\begin{aligned}
(27) \quad & \frac{b_0}{2} \left[ \|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \frac{b_0}{2} \|u_t(\cdot, T)\|_{L^2(0,l)}^2 + \right. \\
& \quad \left. + \beta \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 \right] \\
& \leq \left( b_1^2 + \frac{b_2}{2} \right) \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \frac{\beta b_2}{2} \|u_{tt}\|_{L^2(Q_s)}^2 + \\
& \quad + \frac{b_2^2 + b_1^2 + 4b_1 b_2 + b_3^2}{2} \|u_t\|_{L^2(Q_s)}^2 + \\
& \quad + \frac{b_2^2 + b_4^2 + b_5^2}{2} \|u\|_{L^2(Q_s)}^2 + \frac{b_1^2 b_2^2}{b_0^2} \|u(\cdot, T)\|_{L^2(0,l)}^2
\end{aligned}$$

By virtue of the elementary inequality

$$(28) \quad \frac{b_1^2 b_2^2}{b_0^2} \|u(\cdot, T)\|_{L^2(0,l)}^2 \leq \frac{b_1^2 b_2^2}{b_0^2} \|u\|_{L^2(Q_s)}^2 + \frac{b_1^2 b_2^2}{b_0^2} \|u_t\|_{L^2(Q_s)}^2,$$

we estimate the last term of the right-hand side of the inequality(27),we obtain

$$\begin{aligned}
(29) \quad & \|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \\
& \quad + \frac{b_0}{2} \|u_t(\cdot, T)\|_{L^2(0,l)}^2 + \beta \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 \\
& \leq \left( \frac{2b_1^2 + b_2}{b_0} \right) \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \frac{\beta b_2}{b_0} \|u_{tt}\|_{L^2(Q_s)}^2 \\
& \quad + \frac{\frac{2b_1^2 b_2^2}{b_0^2} + 2b_1 b_2 + (b_2 + b_1)^2 + b_3^2}{b_0} \|u_t\|_{L^2(Q_s)}^2 \\
& \quad + \frac{b_0^2 (b_2^2 + b_4^2 + b_5^2) + 2b_1^2 b_2^2}{b_0^3} \|u\|_{L^2(Q_s)}^2
\end{aligned}$$

For estimate the last term of the right-hand side of the inequality(29),we will be prove the inequality  $\|u\|_{L^2(Q_s)}^2 \leq 24T^2 \|u_t\|_{L^2(Q_s)}^2$ , combining the last inequality and (29) we get

$$(30) \quad \begin{aligned} & \|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \\ & \quad + \|u_t(\cdot, T)\|_{L^2(0,l)}^2 \\ & \leq k \left[ \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \|u_{tt}\|_{L^2(Q_s)}^2 + \|u_t\|_{L^2(Q_s)}^2 \right], \end{aligned}$$

$$\text{where } k = \frac{\max(\beta b_2, (2b_1^2 + b_2), b_0 k(b_i, T))}{b_0 \min(1, \beta, \frac{b_0}{2})}.$$

$$k(b_i, T) = \frac{2b_1^2 b_2^2 + b_0^2 [2b_1 b_2 + (b_2 + b_1)^2 + b_3^2] + 24T^2 [b_0^2 (b_2^2 + b_4^2 + b_5^2) + 2b_1^2 b_2^2]}{b_0^3}.$$

To continue, we introduce the new function  $v(x, t) = \int_t^T u_{\tau\tau} d\tau$ , then  $u_t(x, t) = v(x, s) - v(x, t)$ , and  $u_t(x, T) = v(x, s)$ .

The inequality (30) it be

$$(31) \quad \begin{aligned} & \|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \\ & \quad + (1 - 2k(T - s)) \|v(\cdot, s)\|_{L^2(0,l)}^2 \\ & \leq 2k \left( \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \|u_{tt}\|_{L^2(Q_s)}^2 + \|v\|_{L^2(Q_s)}^2 \right). \end{aligned}$$

If  $s_0 > 0$  satisfies  $(1 - 2k(T - s_0)) = \frac{1}{2}$ , then the inequality (31) implies

$$(32) \quad \begin{aligned} & \|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 + \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 \\ & \quad + \|v(\cdot, s)\|_{L^2(0,l)}^2 \\ & \leq 4k \left( \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \|u_{tt}\|_{L^2(Q_s)}^2 + \|v\|_{L^2(Q_s)}^2 \right), \end{aligned}$$

for all  $s \in [T - s_0, T]$ . We denote

$$Y(s) = \|\mathfrak{S}_x^* u_{tt}\|_{L^2(Q_s)}^2 + \|u_{tt}\|_{L^2(Q_s)}^2 + \|v\|_{L^2(Q_s)}^2.$$

We get:

$$Y'(s) = -\|\mathfrak{S}_x^* u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 - \|u_{tt}(\cdot, s)\|_{L^2(0,l)}^2 - \|v(\cdot, s)\|_{L^2(0,l)}^2.$$

Then and from (32) we obtain  $-Y'(s) \leq 4kY(s)$ .  
Then  $-\frac{\partial}{\partial s}(Y(s) \exp(4ks)) \leq 0$ .

Integrating this inequality on  $(s, T)$  and taking into account that  $Y(T) = 0$ , we obtain  $Y(s) \exp(4ks) \leq 0$ .  
Then  $Y(s) = 0$  for all  $s \in [T - s_0, T]$ . Then  $\omega = 0$  almost everywhere in  $Q_{T-s_0}$ , proceeding in this way step by step, we prove that  $\omega = 0$  almost everywhere in  $Q$ .

This achieves the proof of proposition. Now we return to prove the théorème. We will be prove that  $\overline{R(L)} = H$ .  
Since  $H$  is a Hilbert space, the equality  $\overline{R(L)} = H$  is true, if from

$$(33) \quad (Lu, W)_H = (\mathcal{L}u, \omega)_{L^2(Q)} + (l_1u, \omega_1)_{L^2(0,l)} + (l_2u, \omega_2)_{L^2(0,l)} = 0,$$

where  $W = (\omega, \omega_1, \omega_2) \in R(L)^\perp$ , we get  $\omega \equiv 0, \omega_1 \equiv 0$  and  $\omega_2 \equiv 0$  in  $Q$ , for any element of  $D_0(L)$ .

From (33) we obtain  $\forall u \in D_0(L), (\mathcal{L}u, \omega)_{L^2(Q)} = 0$ . Then by virtue of the 2<sup>nd</sup> proposition, we conclude that  $\omega \equiv 0$ .

Then for (33), we obtain  $(l_1u, \omega_1)_{L^2(0,l)} + (l_2u, \omega_2)_{L^2(0,l)} = 0$ .

Since the quantities  $l_1u$  and  $l_2u$  can vanish independently and the ranges of the trace operators  $l_1$  and  $l_2$  are dense in the Hilbert space  $L^2(0, l)$ , then  $\omega_1 = \omega_2 = 0$ . Thus to conclude that  $W = 0$ .

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