

Clifford groups of quantum gates, BN-pairs and smooth cubic surfaces

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Abstract. The recent proposal (M Planat and M Kibler, Preprint 0807.3650 [quant-ph]) of representing Clifford quantum gates in terms of unitary reflections is revisited. In this essay, the geometry of a Clifford group G is expressed as a BN-pair, i.e. a pair of subgroups B and N that generate G , is such that intersection $H = B \cap N$ is normal in G , the group $W = N/H$ is a Coxeter group and two extra axioms are satisfied by the double cosets acting on B . The BN-pair used in this decomposition relies on the *swap* and *match* gates already introduced for classically simulating quantum circuits (R Jozsa and A Miyake, Preprint 0804.4050 [quant-ph]). The two- and three-qubit steps are related to the configuration with 27 lines on a smooth cubic surface.

Introduction

Euclidean real reflection groups (Coxeter groups) are an important ingredient for representing quantum computations [1]. Coxeter groups are finite set of involutions and specific pairwise relations. As a result, they provide a distinguished class of quantum Boolean functions [2] possessing inherent cristallographic properties. But complex reflections are more appropriate for modeling the Clifford unitaries. For instance, the single qubit Pauli group \mathcal{P}_1 (generated by the ordinary Pauli spin matrices σ_x , σ_y and σ_z) is the imprimitive reflection group $G(4, 2, 2)$. Its normalizer in the unitary group $U(2)$, the so-called Clifford group \mathcal{C}_1 , is isomorphic (but is not the same than) the reflection group number 9 in the Shephard-Todd list [3]. The n -qubit Clifford group \mathcal{C}_n is the normalizer in $U(2^n)$ of the tensor product of n Pauli spin matrices [4]. It originally appeared in the context of doubly-even self-dual classical codes [5], where it was discovered that the space of homogeneous invariants of \mathcal{C}_n is spanned by the complex weight enumerators of the codes. Group \mathcal{C}_2 contains a maximal subgroup (of half its size) which is the Shephard-Todd group number 31, but the connection to unitary reflection groups becomes more tenuous as far as $n \geq 3$.

In this paper, one shows that Clifford groups may be seen as aggregates of Coxeter groups with the structure of BN-pairs, also named Tits systems. It is noticeable that

such a construction also vindicates a connection of Clifford group geometry to smooth cubic surfaces, already pointed out in our earlier work [1].

BN-pairs

Henceforth, G is finite group, B and N two subgroups of G generating G , $H = B \cap N$ is a normal subgroup of G and the quotient group $W = N/H$ is generated by a set $S \subset W$ of order 2 elements (involutions). In the next section, we shall observe that such a pairing easily follows from the structure of the Clifford group $G \equiv \mathcal{C}_n$, when it is divided into its *local* component, the local Clifford group $B \equiv \mathcal{C}_n^L$, and its *topological* component $N \equiv \mathcal{B}_n$.

In 1962, Jacques Tits coined the concept of a BN-*pair* for characterizing groups resembling the general linear group over a field [6, 7, 8]. A group G is said to have a BN-pair iff it is generated as above and two extra relations (i) and (ii) are satisfied by the double cosets ‡

$$(i) \text{ For any } s \in S \text{ and } w \in W, sBw \subseteq (BwB) \cup (BswB),$$

$$(ii) \text{ For any } s \in S, sBs \not\subseteq B.$$

A peculiar example is $G = GL_n(K)$ (the general linear group over a field K). One takes B to be the upper triangular matrices, H to be the diagonal matrices and N to be the matrices with exactly one non-zero element in each row and column. There are $n - 1$ generators s , represented by the matrices obtained by swapping two adjacent rows of a diagonal matrix. More generally, any group of Lie type has the structure of a BN-pair, and BN-pairs can be used to prove that most groups of Lie type are simple.

An important consequence of the axioms (i) and (ii) is that the group G with a BN-pair may be partitioned into the double cosets as $G = BWB$. The application from w to $C(w) = BwB$ is a bijection from W to the set $B \backslash G/B$ of double cosets of G along B [7].

Let us recall that a group W is a *Coxeter group* if it is finitely generated by a subset $S \subset W$ of involutions and pairwise relations

$$W = \langle s \in S | (ss')^{m_{ss'}} = 1 \rangle,$$

where $m_{ss} = 1$ and $m_{ss'} \in \{2, 3, \dots\} \cup \{\infty\}$ if $s \neq s'$. The pair (W, S) is a Coxeter system, of rank $|S|$ equal to the number of generators.

‡ For G a group, and subgroups A and B of G , each double coset is of form AxB : it is an equivalence class for the equivalence relation defined on G by

$$x \sim y \text{ if there are } a \in A \text{ and } b \in B \text{ with } axb = y.$$

Then G is partitioned into its (A, B) double cosets.

Products of the type sBs in (i) makes sense because W is an equivalence classe modulo H , and as a result is also a subset of G . More generally, for a subset S of W , the product BSB denotes the coset union $\bigcup_{s \in S} BsB$.

The pair (W, S) arising from a BN-pair is a Coxeter system. Denoting $l_s(w)$ for the smallest integer $q \geq 0$ such that w is a product of q elements of S , then (i) may be rewritten as (a) if $l_s(sw) > l_s(w)$ then $C(sw) = C(s).C(w)$, (b) if $l_s(sw) < l_s(w)$ then $C(sw) \cup C(w) = C(s).C(w)$. Such rules are the cell multiplication rules attached to the Bruhat-Tits cells BwB of the Bruhat-Tits decomposition (disjoint union) $G = BWB = \bigcup_{w \in W} BwB$. Axiom (ii) can be rewritten as (c) for any $s \in S$, $C(s).C(s) = B \cup C(s) \neq B$.

Finally let us give the definition of a *split BNpair*. It satisfies the two additional axioms

$$(iii) \quad B = UH,$$

where U is a normal nilpotent subgroup of B such that $U \cap H = 1$, and

$$(iv) \quad H = \bigcap_{n \in \mathbb{N}} nBn^{-1}.$$

BN-pairs from the two-qubit Clifford group

Any action of a Pauli operator $g \in \mathcal{P}_n$ on an n -qubit state $|\psi\rangle$ can be stabilized by a unitary gate U such that $(UgU^\dagger)U|\psi\rangle = U|\psi\rangle$, with the condition $UgU^\dagger \in \mathcal{P}_n$. The n -qubit Clifford group (with matrix multiplication for group law) is defined as the normalizer of \mathcal{P}_n in $U(2^n)$

$$\mathcal{C}_n = \{U \in U(2^n) | U\mathcal{P}_nU^\dagger = \mathcal{P}_n\}.$$

In view of the relation $U^\dagger = U^{-1}$ in the unitary group $U(2^n)$, normal subgroups of Clifford groups are expected to play a leading role in quantum error correction [1, 9]. Let us start with the two-qubit Clifford group \mathcal{C}_2 . The representation

$$\mathcal{C}_2 = \langle \mathcal{C}_1 \otimes \mathcal{C}_1, CZ \rangle$$

[where $CZ = \text{Diag}(1, 1, 1, -1)$ is the "controlled- Z " gate] naturally picks up the local Clifford group

$$\mathcal{C}_2^L = \langle \mathcal{C}_1 \otimes \mathcal{C}_1 \rangle = \langle H \otimes I, I \otimes H, P \otimes I, I \otimes P \rangle,$$

The Hadamard gate $H := 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ occurs in coding theory as the matrix of the MacWilliams transform and the phase gate is $P := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. The weight enumerator of Type II codes is invariant under the group of order 192 generated by P and H , that is \mathcal{C}_1 itself [11]. More generally the weight enumerator of genus n in 2^n variables is invariant under the Clifford group \mathcal{C}_n [5]. The issue of efficient (classical) simulation of quantum circuits [13] as well as the topological approach of quantum

computation [14], suggest another decomposition of \mathcal{C}_2 in terms of the two-qubit gates

$$T := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } R = 1/\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The action of gate T is a *swap* of the two input qubits. It is straightforward to check another representation of the local Clifford group as

$$\mathcal{C}_2^L = \langle H \otimes H, H \otimes P, T \rangle.$$

The action of gate R is a *maximal entanglement* of the two input qubits. Gate R is a *match* gate [13]. It also satisfies the Yang-Baxter equation $(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$ and plays a leading role in the topological approach of quantum computation [14]. It was used in our earlier work to define the *Bell group*

$$\mathcal{B}_2 = \langle H \otimes H, H \otimes P, R \rangle.$$

Both groups \mathcal{C}_2^L and \mathcal{B}_2 are subgroups of order 4608 (with index 20) and 15360 (with index 6) of the Clifford group. The latter may be represented as $\mathcal{C}_2 = \langle H \otimes H, H \otimes P, CZ \rangle$.

The search of the BN-pairs

Clearly, the Clifford group is generated by the local Clifford group \mathcal{C}_2^L and Bell group \mathcal{B}_2 . Their intersection is the Pauli group \mathcal{P}_2 , of order 64, that is isomorphic to the central product $E_{32}^+ * \mathbb{Z}_4$ (where E_{32}^+ is the extraspecial 2–group of order 32 and type +). The Pauli group \mathcal{P}_2 is normal in the Clifford and Bell groups but neither of the quotient groups $\mathcal{C}_2^L/\mathcal{P}_2$ and $\mathcal{B}_2/\mathcal{P}_2 \cong \mathbb{Z}_2 \times S_5$ is a Coxeter group, so that the pair $(\mathcal{C}_2^L, \mathcal{B}_2)$ cannot be of the BN-type.

Let us search a BN-pair candidate by selecting the subgroup $N \equiv \mathcal{B}_2$ and reducing the size of \mathcal{C}_2^L to a subgroup B so that the intersection group $H = N \cap B$ is a subgroup of B and N/H is a Coxeter group. One gets

$$B \cong W(F_4), \quad N \equiv \mathcal{B}_2, \quad H \equiv Z(\mathcal{B}_2) \cong \mathbb{Z}_8 \text{ and } W \cong W(D_5),$$

in which B is the unique subgroup of \mathcal{C}_2^L which is both of order 1152 and isomorphic to the Coxeter group of type F_4 (the symmetry group of the 24-cell), N is \mathcal{B}_2 , H is the center $Z(\mathcal{B}_2)$ and W , of order 1920, is isomorphic to the Coxeter group of type D_5 .

The above pair of groups is of the BN type seeing that conditions (i) and (ii) are satisfied. Axiom (i) directly follows from the Coxeter group structure of W . For (ii), which is equivalent to (c), it is enough to discover an element in the double coset $C(s)$ which does not lie in group B . Elements of the coset $C(s) = BsB$ arise from elements of the coset $\mathcal{C}_2^L g \mathcal{C}_2^L$, $g \in \mathcal{B}_2$. The latter coset contains the entangling match gate $R' = TRT$, which lies in \mathcal{B}_2 but not in \mathcal{C}_2^L . Thus (c) cannot be satisfied. The BN pair does not split because there is none normal subgroup of order $|B|/|H| = 144$ within the group B .

A split BN-pair

A further structure may be displayed in the two-qubit Clifford group. Let us denote \hat{G} the central quotient of the derived subgroup of G . One immediately checks that $\hat{\mathcal{C}}_2 = \langle \hat{\mathcal{C}}_2^{(L)}, \hat{\mathcal{B}}_2 \rangle \cong U_6$, $\hat{\mathcal{B}}_2 \cong M_{20}$ and $\hat{\mathcal{C}}_2^{(L)} \cong \hat{W}(F_4)$. Group $U_6 = \mathbb{Z}_2^4 \rtimes A_6$, of order 5760, appears in several disguises. The full automorphism group of the Pauli group \mathcal{P}_2 possesses a derived subgroup isomorphic to U_6 (see relation (7) in [1]). Geometrically, it corresponds to the stabilizer of an hexad in the Mathieu group M_{22} (see Sec 4.2 in [1]). Group $M_{20} = \mathbb{Z}_2^4 \rtimes A_5$, of order 960, is isomorphic to the derived subgroup of the imprimitive reflection group $G(2, 2, 5)$ (see Sec 3.5 in [1]). Incidentally, M_{20} is the smallest perfect group for which the set of commutators departs from the commutator subgroup [9]. Remarkably, the group $\hat{\mathcal{C}}_2$ forms the split BN-pair

$$B \equiv \hat{\mathcal{C}}_2^{(L)}, \quad N \equiv \hat{\mathcal{B}}_2, \quad H \equiv \tilde{\mathcal{P}}_2 \cong \mathbb{Z}_2^4, \quad W \cong A_5, \quad \text{and} \quad U \cong \mathbb{Z}_3^2.$$

BN-pairs from the three-qubit Clifford group

The local Clifford group

$$\mathcal{C}_3^{(L)} = \{\mathcal{C}_1 \otimes \mathcal{C}_1 \otimes \mathcal{C}_1\},$$

and the three-qubit Bell group

$$\mathcal{B}_3 = \langle H \otimes H \otimes P, H \otimes R, R \otimes H \rangle,$$

are subgroups of index 6720 and 56, respectively, of the three-qubit Clifford group (of order 743 178 240). It may be generated as

$$\mathcal{C}_3 = \langle H \otimes H \otimes P, H \otimes CZ, CZ \otimes H \rangle.$$

The central quotients $\tilde{\mathcal{C}}_3$ and $\tilde{\mathcal{B}}_3$ may be expressed as semi-direct products

$$\tilde{\mathcal{C}}_3 = \mathbb{Z}_2^6 \rtimes W'(E_7) \quad \text{and} \quad \tilde{\mathcal{B}}_3 = \mathbb{Z}_2^6 \rtimes W'(E_6),$$

in which $W'(E_7) \equiv \text{Sp}(6, 2)$ and $W'(E_6)$ are the reflection groups of type E_7 and E_6 , respectively [1]. Following the intuition gained at the previous section, one immediately gets the *non-split* § BN-pair

$$B \equiv \tilde{\mathcal{C}}_3^{(L)}, \quad N \equiv \tilde{\mathcal{B}}_3, \quad H \equiv \tilde{\mathcal{P}}_3 \cong \mathbb{Z}_2^6, \quad W \cong W'(E_6), \quad \text{and} \quad \tilde{\mathcal{C}}_3^{(L)}/H \equiv V,$$

in which $V \cong S_3^3$ (S_3 is the symmetric group on three letters).

§ The pair is not split since the quotient group V is not normal in $\tilde{\mathcal{C}}_3^{(L)}$, not nilpotent and $V \cap H \neq 1$.

BN-pairs and a smooth cubic surface

The occurrence of reflection groups $W(F_4)$ and $W(D_5)$ in the decomposition of the two-qubit Clifford group, and of $W(E_6)$ in the decomposition of the three-qubit Clifford group, can be grasped in a different perspective from the structure of a *smooth cubic surface* \mathcal{S} embedded into the three-dimensional complex projective space $\mathbb{P}^3(\mathbb{C})$ [15]. The surface contains a maximum of 27 lines in general position and 45 sets of tritangent planes. The group of permutations of the 27 lines is $W(E_6)$, the stabilizer of a line is $W(D_5)$ (observe that $|W(E_6)|/|W(D_5)| = 27$) and the stabilizer of a tritangent plane is $W(F_4)$. Thus the BN-pairs happens to be reflected into the geometry of such a cubic surface.

Other “coincidences ” occur as follows. The number 216 of pairs of skew lines of \mathcal{S} equals the cardinality of the quotient group V entering in the decomposition of $\tilde{\mathcal{C}}_3$. There are 36 double sixes, each one stabilized by the group $g_6 := A_6.\mathbb{Z}_2^2$ of order 1440 (the symbol $.$ means that the group extension does not split). The latter group can be displayed in the context of the two-qubit Clifford group. Let us observe that the quotients of \mathcal{C}_2 and \mathcal{B}_2 by the Pauli group \mathcal{P}_2 are isomorphic to g_6 and $g_5 := A_5.\mathbb{Z}_2^2$, respectively. For three-qubits, one checks that the quotients of \mathcal{C}_3 and \mathcal{B}_3 by the Pauli group \mathcal{P}_3 are isomorphic to $W(E_7)$ and $W(E_6)$. Groups $W'(E_6)$, $W(D_5)$, $W(F_4)$ and g_6 , which correspond to the permutations of the 27 lines, the stabilizer of a line, a tritangent plane and a double six, respectively, are among the six maximal subgroups of $W(E_6)$. The remaining two are of order 1296 and index 40, corresponding to the size of double cosets BwB , $B \cong W(F_4)$ and $w \in W \cong W(D_5)$, in the BN-pair decomposition of the two-qubit Clifford group.

To conclude, a smooth cubic surface is a particular instance of a K_3 surface, a concept playing a founding role in string theory. Further work is necessary to explore the interface between quantum computing, graded rings and K_3 surfaces [16].

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