Moments, cumulants and diagram formulae for non-linear functionals of random measures

Giovanni Peccati, Murad Taqqu

Moments, cumulants and diagram formulae for non-linear functionals of random measures

Giovanni PECCATI* and Murad S. TAQQU†

November 11, 2008

Abstract

This survey provides a unified discussion of multiple integrals, moments, cumulants and diagram formulae associated with functionals of completely random measures. Our approach is combinatorial, as it is based on the algebraic formalism of partition lattices and Möbius functions. Gaussian and Poisson measures are treated in great detail. We also present several combinatorial interpretations of some recent CLTs involving sequences of random variables belonging to a fixed Wiener chaos.

Key Words – Central Limit Theorems; Cumulants; Diagonal measures; Diagram formulae; Gaussian fields; Möbius function; Moments; Multiple integrals; Partitions; Poisson measures; Random Measures.

AMS Subject Classification – 05A17; 05A18; 60C05; 60F05; 60H05.

Contents

*Equipe Modal'X, Université Paris Ouest Nanterre La Défense, 200 Avenue de la République, 92000 Nanterre and LSTA, Université Paris VI, France. E-mail: giovanni.peccati@gmail.com

†Boston University, Departement of Mathematics, 111 Cummington Road, Boston (MA), USA. E-mail: murad@math.bu.edu

1 Introduction

1.1 Overview

The aim of this survey is to provide a unified treatment of moments and cumulants associated with non-linear functionals of completely random measures, such as Gaussian, Poisson or Gamma measures. We will focus on multiple stochastic integrals, and we shall mainly adopt a combinatorial point of view. In particular, our main inspiration is a truly remarkable paper by Rota and Wallstrom [106], building (among many others) on earlier works by Itô [30], Meyer [59, 60] and, most importantly, Engel [18] (see also Bitcheler [8], Kussmaul [42], Linde [47], Masani [56], Neveu [62] and Tsilevich and Vershik [127] for related works). In particular, in [106] the authors point out a crucial connection between the machinery of multiple stochastic integration and the structure of the lattice of partitions of a finite set, with specific emphasis on the role played by the associated Möbius function (see e.g. [2], as well as Section 2 below). As we will see later on, the connection between multiple stochastic integration and partitions is given by the natural isomorphism between the partitions of the set $\{1, ..., n\}$ and the *diagonal sets* associated with the Cartesian product of n measurable spaces (a diagonal set is just a subset of the Cartesian product consisting of points that have two or more coordinates equal).

The best description of the approach to stochastic integration followed in the present survey is still given by the following sentences, taken from [106]:

> The basic difficulty of stochastic integration is the following. We are given a measure φ on a set S, and we wish to extend such a measure to the product set S^n. There is a well-known and established way of carrying out such an extension, namely, taking the product measure. While the product measure is adequate in most instances dealing with a scalar valued measure, it turns out to be woefully inadequate when the measure is vector-valued, or, in the case dealt with presently, random-valued. The product measure of a nonatomic scalar measure will vanish on sets supported by lower-dimensional linear subspaces of S^n. This is not the case, however, for random measures. The problem therefore arises of modifying the definition of product measure of a random measure in such a way that the resulting measure will vanish on lower-dimensional subsets of S^n, or diagonal sets, as we call them.

As pointed out in [106], as well as in Section 5 below, the combinatorics of partition lattices provide the correct framework in order to define a satisfactory stochastic product measure.

As apparent from the title, in the subsequent sections a prominent role will be played by moments and cumulants. In particular, the principal aims of our work are the following:

– **Put diagram formulae in a proper algebraic setting**. Diagram formulae are mnemonic devices, allowing to compute moments and cumulants associated with one or more random variables. These tools have been developed and applied in a variety of frameworks: see e.g. [113, 122] for diagram formulae associated with general random variables; see [9, 11, 24, 51] for non-linear functionals of Gaussian fields; see [121] for non-linear functionals of Poisson measures. They can be quite useful in the obtention of Central Limit Theorem (CLTs) by means of the so-called *method of moments and cumulants* (see e.g. [49]). Inspired by the works by McCullagh [57], Rota and Shen [105] and Speed [117], we shall show that all

diagram formulae quoted above can be put in a unified framework, based on the use of partitions of finite sets. Although somewhat implicit in the previously quoted references, this clear algebraic interpretation of diagrams is new. In particular, in Section 4 we will show that all diagrams encountered in the probabilistic literature (such as Gaussian, non-flat and connected diagrams) admit a neat translation in the combinatorial language of partition lattices.

– **Illustrate the Engel-Rota-Wallstrom theory**. We shall show that the theory developed in [18] and [106] allows to recover several crucial results of stochastic analysis, such as multiplication formulae for multiple Gaussian and Poisson integrals see [38, 75, 121]. This extends the content of [106], which basically dealt with product measures. See also [19] for other results in this direction.

– **Enlight the combinatorial implications of new CLTs**. In a recent series of papers (see [52, 65, 69, 70, 71, 76, 79, 83, 86, 89, 90, 91]), a new set of tools has been developed, allowing to deduce simple CLTs involving random variables having the form of multiple stochastic integrals. All these results can be seen as simplifications of the method of moments and cumulants. In Section 9, we will illustrate these results from a combinatorial standpoint, by providing some neat interpretations in terms of diagrams and graphs. In particular, we will prove that in these limit theorems a fundamental role is played by the so-called *circular diagrams*, that is, connected Gaussian diagrams whose edges only connect subsequent rows.

We will develop the necessary combinatorial tools related to partitions, diagram and graphs from first principles in Section 2 and Section 4. Section 3 provides a self-contained treament of moments and cumulants from a combinatorial point of view. Stochastic integration is introduced in Section 5. Section 6 and Section 7 deal, respectively, with product formulae and diagram formulae. In Section 8 one can find an introduction to the concept of isonormal Gaussian process. Finally, Section 9 deals with CLTs on Wiener chaos.

1.2 Some related topics

In this survey, we choose to follow a very precise path, namely starting with the basic properties of partition lattices and diagrams, and develop from there as many as possible of the formulae associated with products, moments and cumulants in the theory of stochastic integration with respect to completely random measures. In order to keep the length of the present work within bounds, several crucial topics are not included (or are just mentioned) in the discussion to follow. One remarkable omission is of course a complete discussion of the connections between multiple stochastic integrals and orthogonal polynomials. This topic is partially treated in Section 8 below, in the particular case of Gaussian processes. For recent references on more general stochastic processes (such as Lévy processes), see e.g. the monograph by Schoutens [111] and the two papers by Solé and Utzet [115, 116]. Other related (and missing) topics are detailed in the next list, whose entries are followed by a brief discussion.

– *Wick products*. Wick products are intimately related to chaotic expansions. A complete treatment of this topic can be found e.g. in Janson's book [36].

– *Malliavin calculus.* See the two monographs by Nualart [74, 75] for Malliavin calculus in a Gaussian setting. A good introduction to Malliavin calculus for Poisson measures is contained in the classic papers by Nualart and Vives [81], Privault [98] and Privault and Wu [101]. A fundamental connection between Malliavin operators and limit theorems has been first pointed out in [77]. See [65, 67, 69, 86] for further developments.

– *Hu-Meyer formulae.* Hu-Meyer formulae connect Stratonovich multiple integrals and multiple Wiener-Itô integrals. See [75] for a standard discussion of this topic in a Gaussian setting. Hu-Meyer formulae for general Lévy processes can be naturally obtained by means of the theory described in the present survey: see the excellent paper by Farria *et al.* [19] for a complete treatment of this point.

– *Stein's method.* Stein's method for normal and non-normal approximation can be a very powerful tool, in order to obtain central and non-central limit theorems for non-linear functionals of random fields. See [119] for a classic reference on the subject. See [69, 70, 71] for several limit theorems involving functionals of Gaussian fields, obtained by means of Stein's method and Malliavin calculus. See [86] for an application of Stein's method to functionals of Poisson measures.

– *Free probability.* The properties of the lattice of (non-crossing) partitions and the corresponding Möbius function are crucial in free probability. See the monograph by Nica and Speicher [63] for a valuable introduction to the combinatorial aspects of free probability. See Anshelevich [3, 4] for some instances of a "free" theory of multiple stochastic integration.

2 The lattice of partitions of a finite set

In this section we recall some combinatorial results concerning the lattice of partitions of a finite set. These objects play an important role in the obtention of the *diagram formulae* presented in Section 5. The reader is referred to Stanley [118, Chapter 3] and Aigner [2] for a detailed presentation of (finite) partially ordered sets and Möbius inversion formulae.

2.1 Partitions of a positive integer

Given an integer $n \geq 1$, we define the set $\Lambda(n)$ of *partitions* of n as the collection of all vectors of the type $\lambda = (\lambda_1, ..., \lambda_k)$ $(k \geq 1)$, where:

$$\begin{array}{ll} \text{(i)} & \lambda_j \text{ is an integer for every } \ j = 1, ..., k, \\ \text{(ii)} & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1, \\ \text{(iii)} & \lambda_1 + \cdots + \lambda_k = n. \end{array} \tag{2.1}$$

We call k the *length* of λ. It is sometimes convenient to write a partition $\lambda = (\lambda_1, ..., \lambda_k) \in \Lambda(n)$ in the form $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n})$. This representation (which encodes all information about λ) simply indicates that, for every $i = 1, ..., n$, the vector λ contains exactly r_i (≥ 0) components equal to i. Clearly, if $\lambda = (\lambda_1, ..., \lambda_k) = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \in \Lambda(n)$, then

$$1r_1 + \cdots + nr_n = n \tag{2.2}$$

and $r_1 + \cdots + r_n = k$. We will sometimes use the (more conventional) notation

$$\lambda \vdash n \quad \text{instead of} \quad \lambda \in \Lambda(n).$$

Examples. (i) If $n = 5$, one can e.g. have $5 = 4 + 1$ or $5 = 1 + 1 + 1 + 1 + 1$. In the first case the length is $k = 2$, with $\lambda_1 = 4$ and $\lambda_2 = 1$, and the partition is $\lambda = \left(1^1 2^0 3^0 4^1 5^0\right)$. In the second case, the length is $k = 5$ with $\lambda_1 = ... = \lambda_5 = 1$, and the partition is $\lambda = \left(1^5 2^0 3^0 4^0 5^0\right)$.

(ii) One can go easily from one representation to the other. Thus $\lambda = \left(1^2 2^3 3^0 4^2\right)$ corresponds to

$$n = (1 \times 2) + (2 \times 3) + (3 \times 0) + (4 \times 2) = 16,$$

that is, to the decomposition $16 = 4 + 4 + 2 + 2 + 2 + 1 + 1$, and thus to

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (4, 4, 2, 2, 2, 1, 1).$$

2.2 Partitions of a set

Let b denote a finite nonempty set and let

$$\mathcal{P}(b) \text{ be the set of } \textit{partitions} \text{ of } b.$$

By definition, an element π of $\mathcal{P}(b)$ is a collection of nonempty and disjoint subsets of b (called *blocks*), such that their union equals b. The symbol $|\pi|$ indicates the number of blocks (or the *size*) of the partition π.

Remark on notation. For each pair $i, j \in b$ and for each $\pi \in \mathcal{P}(b)$, we write $i \sim_\pi j$ whenever i and j belong to the same block of π.

We now define a partial ordering on $\mathcal{P}(b)$. For every $\sigma, \pi \in \mathcal{P}(b)$, we write $\sigma \leq \pi$ if, and only if, each block of σ is contained in a block of π. Borrowing from the terminology used in topology one also says that π is *coarser* than σ. It is clear that $\leq$ is a *partial ordering relation*, that is, $\leq$ is a *binary*, *reflexive*, *antisymmetric* and *transitive* relation on $\mathcal{P}(b)$ (see e.g. [118, p. 98]). Also, $\leq$ induces on $\mathcal{P}(b)$ a *lattice* structure. Recall that a lattice is a partially ordered set such that each pair of elements has a least upper bound and a greatest lower bound (see [118, p. 102]). In particular, the partition $\sigma \wedge \pi$, *meet* of $\sigma, \pi \in \mathcal{P}(b)$, is the partition of b such that each block of $\sigma \wedge \pi$ is a nonempty intersection between one block of σ and one block of π. On the other hand, the partition $\sigma \vee \pi$, *join* of $\sigma, \pi \in \mathcal{P}(b)$, is the element of $\mathcal{P}(b)$ whose blocks are constructed by taking the non-disjoint unions of the blocks of σ and π, that is, by taking the union of those blocks that have at least one element in common.

Remarks. (a) Whenever $\pi_1 \leq \pi_2$, one has $|\pi_1| \geq |\pi_2|$. In particular, $|\sigma \wedge \pi| \geq |\sigma \vee \pi|$.

(b) The partition $\sigma \wedge \pi$ is the greatest lower bound associated with the pair (σ, π). As such, $\sigma \wedge \pi$ is completely characterized by the property of being the unique element of $\mathcal{P}(b)$ such that: (i) $\sigma \wedge \pi \leq \sigma$, (ii) $\sigma \wedge \pi \leq \pi$, and (iii) $\rho \leq \sigma \wedge \pi$ for every $\rho \in \mathcal{P}(b)$ such that $\rho \leq \sigma, \pi$.

(c) Analogously, the partition $\sigma \vee \pi$ is the least upper bound associated with the pair (σ, π). It follows that $\sigma \vee \pi$ is completely characterized by the property of being the unique element of $\mathcal{P}(b)$ such that: (i) $\sigma \leq \sigma \vee \pi$, (ii) $\pi \leq \sigma \vee \pi$, and (iii) $\sigma \vee \pi \leq \rho$ for every $\rho \in \mathcal{P}(b)$ such that $\sigma, \pi \leq \rho$.

Examples. (i) Take $b = \{1,2,3,4,5\}$. If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$. Then, $\sigma \leq \pi$ (because each block of σ is contained in a block of π) and

$$\sigma \wedge \pi = \sigma \quad \text{and} \quad \sigma \vee \pi = \pi.$$

A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$$\begin{array}{rcl} \pi & = & \boxed{1\ 2\ \ 3}\boxed{4\ 5} \\ \sigma & = & \boxed{1\ 2}\boxed{3}\boxed{4\ 5} \\ \sigma \wedge \pi & = & \boxed{1\ 2}\boxed{3}\boxed{4\ 5} \\ \sigma \vee \pi & = & \boxed{1\ \ 2\ \ 3}\boxed{4\ 5} \end{array}$$

(ii) If $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3,4,5\}\}$, then π and σ are not ordered and

$$\sigma \wedge \pi = \{\{1,2\},\{3\},\{4,5\}\} \quad \text{and} \quad \sigma \vee \pi = \{b\} = \{\{1,2,3,4,5\}\}.$$

A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$$\begin{array}{rcl} \pi & = & \boxed{1\ 2\ \ 3}\boxed{4\ 5} \\ \sigma & = & \boxed{1\ 2}\boxed{3\ \ 4\ 5} \\ \sigma \wedge \pi & = & \boxed{1\ 2}\boxed{3}\boxed{4\ 5} \\ \sigma \vee \pi & = & \boxed{1\ 2\ 3\ 4\ 5} \end{array}$$

(iii) A convenient way to build $\sigma \vee \pi$ is to do it in successive steps. Take the union of two blocks with a common element and let it be a new block of π. See if it shares an element with another block of σ. If yes, repeat. For instance, suppose that $\pi = \{\{1,2\},\{3\},\{4\}\}$ and $\sigma = \{\{1,3\},\{2,4\}\}$. Then, π and σ are not ordered and

$$\sigma \wedge \pi = \{\{1\},\{2\},\{3\},\{4\}\} \quad \text{and} \quad \sigma \vee \pi = \{\{1,2,3,4\}\}.$$

One now obtains $\sigma \vee \pi$ by noting that the element 2 is common to $\{1,2\} \in \pi$ and $\{2,4\} \in \sigma$, and the "merged" block $\{1,2,4\}$ shares the element 1 with the block $\{1,3\} \in \sigma$, thus implying the conclusion. A graphical representation of π, σ, $\sigma \wedge \pi$ and $\sigma \vee \pi$ is:

$$\begin{array}{rcl} \pi & = & \boxed{1\ 2}\boxed{3}\boxed{\ \ 4} \\ \sigma & = & \boxed{1\ 3}\boxed{2\ \ 4} \\ \sigma \wedge \pi & = & \boxed{1}\boxed{2}\boxed{3}\boxed{\ \ 4} \\ \sigma \vee \pi & = & \boxed{1\ 2\ 3\ \ 4} \end{array}$$

Remark on notation. When displaying a partition π of $\{1,...,n\}$ ($n \geq 1$), the blocks $b_1,...,b_k \in \pi$ will always be listed in the following way: b_1 will always contain the element 1, and

$$\min\{i : i \in b_j\} < \min\{i : i \in b_{j+1}\}, \ \ j = 1,...,k-1.$$

Also, the elements within each block will be always listed in increasing order. For instance, if $n = 6$ and the partition π involves the blocks $\{2\},\{4\},\{1,6\}$ and $\{3,5\}$, we will write $\pi = \{\{1,6\},\{2\},\{3,5\},\{4\}\}$.

The *maximal element* of $\mathcal{P}(b)$ is the trivial partition $\hat{1} = \{b\}$. The *minimal element* of $\mathcal{P}(b)$ is the partition $\hat{0}$, such that each block of $\hat{0}$ contains exactly one element of b. Observe that $\left|\hat{1}\right| = 1$ and $\left|\hat{0}\right| = |b|$, and also $\hat{0} \leq \hat{1}$. If $\sigma \leq \pi$, we write $[\sigma, \pi]$ to indicate the *segment* $\{\rho \in \mathcal{P}(b) : \sigma \leq \rho \leq \pi\}$, which is a subset of partitions of b. Plainly, $\mathcal{P}(b) = \left[\hat{0}, \hat{1}\right]$.

2.3 Relations between partitions of a set and partitions of an integer

We now focus on the notion of *class*, which associates with a segment of partitions a partition of an integer. In particular, the *class* of a segment $[\sigma, \pi]$ ($\sigma \leq \pi$), denoted $\lambda(\sigma, \pi)$, is defined as the partition of the integer $|\sigma|$ given by

$$\lambda(\sigma, \pi) = \left(1^{r_1} 2^{r_2} \cdots |\sigma|^{r_{|\sigma|}}\right), \tag{2.3}$$

where r_i indicates the number of blocks of π that contain exactly i blocks of σ. We stress that necessarily $|\sigma| \geq |\pi|$, and also

$$|\sigma| = 1r_1 + 2r_2 + \cdots + |\sigma| r_{|\sigma|} \quad \text{and} \quad |\pi| = r_1 + \cdots + r_{|\sigma|}.$$

For instance, if $\pi = \{\{1,2,3\},\{4,5\}\}$ and $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$, then since $\{1,2\}$ and $\{3\}$ are contained in $\{1,2,3\}$ and $\{4,5\}$ in $\{4,5\}$, we have $r_1 = 1$, $r_2 = 1$, $r_3 = 0$, that is, $\lambda(\sigma, \pi) = \left(1^1 2^1 3^0\right)$, corresponding to the partition of the integer $3 = 2 + 1$. In view of (2.1), one may suppress the terms $r_i = 0$ in (2.3), and write for instance $\lambda(\sigma, \pi) = \left(1^1 2^0 3^2\right) = \left(1^1 3^2\right)$ for the class of the segment $[\sigma, \pi]$, associated with the two partitions $\sigma = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$ and $\pi = \{\{1\},\{2,3,4\},\{5,6,7\}\}$.

Now fix a set b such that $|b| = n \geq 1$. Then, for a fixed $\lambda = \left(1^{r_1} 2^{r_2} \cdots n^{r_n}\right) \vdash n$, the number of partitions $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ is given by

$$\left[\begin{array}{c} n \\ \lambda \end{array}\right] = \left[\begin{array}{c} n \\ r_1, ..., r_n \end{array}\right] = \frac{n!}{(1!)^{r_1} r_1! \, (2!)^{r_2} r_2! \cdots (n!)^{r_n} r_n!} \tag{2.4}$$

(see e.g. [118]). The requirement that $\lambda\left(\hat{0}, \pi\right) = \lambda = \left(1^{r_1} 2^{r_2} \cdots n^{r_n}\right)$ simply means that, for each $i = 1, ..., n$, the partition π must have exactly r_i blocks containing i elements of b. Recall that $r_1, ..., r_n$ must satisfy (2.2).

From now on, we let

$$\boxed{[n] = \{1, ..., n\}, \quad n \geq 1.} \tag{2.5}$$

With this notation, the maximal and minimal element of the set $\mathcal{P}([n])$ are given, respectively, by

$$\boxed{\hat{1} = \{[n]\} = \{\{1, ..., n\}\} \quad \text{and} \quad \hat{0} = \{\{1\}, ..., \{n\}\}.} \tag{2.6}$$

Examples. (i) For any finite set b, one has always that

$$\lambda\left(\hat{0}, \hat{1}\right) = \left(1^0 2^0 \cdots |b|^1\right),$$

because $\hat{1}$ has only one block, namely b, and that block contains $|b|$ blocks of $\hat{0}$.

(ii) Fix $k \geq 1$ and let b be such that $|b| = n \geq k+1$. Consider $\lambda = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \vdash n$ be such that $r_k = r_{n-k} = 1$ and $r_j = 0$ for every $j \neq k, n-k$. For instance, if $n = 5$ and $k = 2$, then $\lambda = \left(1^0 2^1 3^1 4^0 5^0\right)$. Then, each partition $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ has only one block of k elements and one block of $n-k$ elements. To construct such a partition, it is sufficient to specify the block of k elements. This implies that there exists a bijection between the set of partitions $\pi \in \mathcal{P}(b)$ such that $\lambda\left(\hat{0}, \pi\right) = \lambda$ and the collection of the subsets of b having exactly k elements. In particular, (2.4) gives

$$\left[\begin{array}{c} n \\ \lambda \end{array} \right] = \binom{n}{k} = n! / \left(k! \left(n-k\right)!\right).$$

(iii) Let $b = [7] = \{1, ..., 7\}$ and $\lambda = \left(1^1 2^3 3^0 4^0 5^0 6^0 7^0\right)$. Then, (2.4) implies that there are exactly $\frac{7!}{3!(2!)^3} = 105$ partitions $\pi \in \mathcal{P}(b)$, such that $\lambda\left(\hat{0}, \pi\right) = \lambda$. One of these partitions is $\{\{1\}, \{2,3\}, \{4,5\}, \{6,7\}\}$. Another is $\{\{1,7\}, \{2\}, \{3,4\}, \{5,6\}\}$.

(iv) Let $b = [5] = \{1, ..., 5\}$, $\sigma = \{\{1\}, \{2\}, \{3\}, \{4,5\}\}$ and $\pi = \{\{1,2,3\}, \{4,5\}\}$. Then, $\sigma \leq \pi$ and the set of partitions defined by the interval $[\sigma, \pi]$ is $\{\sigma, \pi, \rho_1, \rho_2, \rho_3\}$, where

$$\begin{aligned} \rho_1 &= \{\{1,2\}, \{3\}, \{4,5\}\} \\ \rho_2 &= \{\{1,3\}, \{2\}, \{4,5\}\} \\ \rho_3 &= \{\{1\}, \{2,3\}, \{4,5\}\}. \end{aligned}$$

The partitions ρ_1, ρ_2 and ρ_3 are not ordered (i.e., for every $1 \leq i \neq j \leq 3$, one cannot write $\rho_i \leq \rho_j$), and are built by taking unions of blocks of σ in such a way that they are contained in blocks of π. Moreover, $\lambda(\sigma, \pi) = \left(1^1 2^0 3^1 4^0 5^0\right)$, since there is exactly one block of π containing one block of σ, and one block of π containing three blocks of σ.

(v) This example is related to the techniques developed in Section 6.1. Fix $n \geq 2$, as well as a partition $\gamma = (\gamma_1, ..., \gamma_k) \in \Lambda(n)$ such that $\gamma_k \geq 2$. Recall that, by definition, one has that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$ and $\gamma_1 + \cdots + \gamma_k = n$. Now consider the segment $\left[\hat{0}, \pi\right]$, where

$$\begin{aligned} \hat{0} &= \{\{1\}, \{2\}, ..., \{n\}\}, \text{ and} \\ \pi &= \{\{1, ..., \gamma_1\}, \{\gamma_1 + 1, ..., \gamma_1 + \gamma_2\}, ..., \{\gamma_1 + \cdots + \gamma_{k-1} + 1, ..., n\}\}. \end{aligned}$$

Then, the jth block of π contains exactly γ_j blocks of $\hat{0}$, $j = 1, ..., k$, thus giving that the class $\lambda\left(\hat{0}, \pi\right)$ is such that $\lambda\left(\hat{0}, \pi\right) = \left(\gamma_1^1 \gamma_2^1 \cdots \gamma_k^1\right) = \gamma$, after suppressing the indicators of the type r^0.

2.4 Möbius functions and Möbius inversion formulae

For $\sigma, \pi \in \mathcal{P}(b)$, we denote by $\mu(\sigma, \pi)$ the *Möbius function* associated with the lattice $\mathcal{P}(b)$. It is defined as follows. If $\sigma \not\leq \pi$ (that is, if the relation $\sigma \leq \pi$ does not hold), then $\mu(\sigma, \pi) = 0$. If $\sigma \leq \pi$, then the quantity $\mu(\sigma, \pi)$ depends only on the class $\lambda(\sigma, \pi)$ of the segment $[\sigma, \pi]$, and is given by (see [2])

$$\begin{aligned} \mu(\sigma, \pi) &= (-1)^{n-r} (2!)^{r_3} (3!)^{r_4} \cdots ((n-1)!)^{r_n} \quad (2.7) \\ &= (-1)^{n-r} \prod_{j=0}^{n-1} (j!)^{r_{j+1}}, \quad (2.8) \end{aligned}$$

where $n = |\sigma|$, $r = |\pi|$, and $\lambda(\sigma, \pi) = (1^{r_1} 2^{r_2} \cdots n^{r_n})$ (that is, there are exactly r_i blocks of π containing exactly i blocks of σ). Since $0! = 1! = 1$, expressions (2.7) and (2.8) do not depend

on the specific values of r_1 (the number of blocks of π containing exactly 1 block of σ) and r_2 (the number of blocks of π containing exactly two blocks of σ).

Examples. (i) If $|b| = n \geq 1$ and $\sigma \in \mathcal{P}(b)$ is such that $|\sigma| = k$ ($\leq n$), then

$$\mu\left(\sigma, \hat{1}\right) = (-1)^{k-1}(k-1)!. \tag{2.9}$$

Indeed, in (2.7) $r_k = 1$, since $\hat{1}$ has a single block which contains the k blocks of σ. In particular, $\mu\left(\hat{0}, \{b\}\right) = \mu\left(\hat{0}, \hat{1}\right) = (-1)^{n-1}(n-1)!$.

(ii) For every $\pi \in \mathcal{P}(b)$, one has $\mu(\pi, \pi) = 1$. Indeed, since (trivially) each element of π contains exactly one element of π, one has $\lambda(\pi, \pi) = \left(1^{|\pi|}2^0 3^0 \cdots n^0\right)$.

The next result is crucial for the obtention of the combinatorial formulae found in Section 3 and Section 5 below. For every pair of functions G, F, from $\mathcal{P}(b)$ into $\mathbb{C}$ and such that $\forall \sigma \in \mathcal{P}(b)$,

$$G(\sigma) = \sum_{\hat{0} \leq \pi \leq \sigma} F(\pi) \quad \left(\text{resp. } G(\sigma) = \sum_{\sigma \leq \pi \leq \hat{1}} F(\pi)\right) \tag{2.10}$$

one has the following *Möbius inversion formula*: $\forall \pi \in \mathcal{P}(b)$,

$$F(\pi) = \sum_{\hat{0} \leq \sigma \leq \pi} \mu(\sigma, \pi) G(\sigma) \quad \left(\text{resp. } F(\pi) = \sum_{\pi \leq \sigma \leq \hat{1}} \mu(\pi, \sigma) G(\sigma)\right), \tag{2.11}$$

where $\mu(\cdot, \cdot)$ is the Möbius function given in (2.7). For a proof of (2.11), see e.g. [118, Section 3.7] and [2]. To understand (2.11) as inversion formulae, one can interpret the sum $\sum_{\hat{0} \leq \pi \leq \sigma} F(\pi)$ as an integral of the type $\int_{\hat{0}}^{\sigma} F(\pi)\, d\pi$ (and analogously for the other sums appearing in (2.10) and (2.11)).

In general (see [118, Section 3.7]), the Möbius function is defined by recursion on any finite partially ordered set by the following relations:

$$\begin{array}{ll} \mu(x, x) = 1 & \forall x \in P, \\ \mu(x, y) = -\sum_{x \preceq z \prec y} \mu(x, z), & \forall\ x, y \in P : x \prec y, \\ \mu(x, y) = 0 & \forall\ x, y \in P : x \not\preceq y, \end{array} \tag{2.12}$$

where P is a finite partially ordered set, with partial order $\preceq$, and we write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$. For instance, P could be a set of subsets, with $\preceq$ equal to the inclusion relation $\subseteq$. In our context $P = \mathcal{P}(b)$, the set of partitions of b, and $\preceq$ is the partial order $\leq$ considered above, so that (2.12) becomes

$$\begin{array}{ll} \mu(\sigma, \sigma) = 1 & \forall \sigma \in \mathcal{P}(b), \\ \mu(\sigma, \pi) = -\sum_{\sigma \leq \rho < \pi} \mu(\sigma, \rho), & \forall\ \sigma, \pi \in \mathcal{P}(b) : \sigma < \pi, \\ \mu(\sigma, \pi) = 0 & \forall\ \sigma, \pi \in \mathcal{P}(b) : \sigma \not\leq \pi, \end{array} \tag{2.13}$$

where we write $\sigma < \pi$ to indicate that $\sigma \leq \pi$ and $\sigma \neq \pi$ (and similarly for $\rho < \pi$). The recursion formula (2.12) has the following consequence: for each $x \preceq y$,

$$\sum_{x \preceq z \preceq y} \mu(z, y) = \sum_{x \preceq z \preceq y} \mu(x, z) = \begin{cases} 0 & \text{if } x \neq y, \\ \mu(x, x)\ (=1) & \text{if } x = y, \end{cases} \tag{2.14}$$

which will be used in the sequel. The second equality in (2.14) is an immediate consequence of (2.12). To prove the first equality in (2.14), fix x and write $G(z) = \mathbf{1}_{x \preceq z}$. Since, trivially, $G(z) = \sum_{y \preceq z} \mathbf{1}_{y=x}$, one can let $F(y) = \mathbf{1}_{y=x}$ in (2.10) and use the inversion formula (2.11) to deduce that

$$\mathbf{1}_{y=x} = \sum_{z \preceq y} \mu(z,y)\, G(z) = \sum_{x \preceq z \preceq y} \mu(z,y),$$

which is equivalent to (2.14).

Now consider two finite partially ordered sets P, Q, whose order relations are noted, respecively, $\preceq_P$ and $\preceq_Q$. The *lattice product* of P and Q is defined as the cartesian product $P \times Q$, endowed with the following partial order relation: $(x,y) \preceq_{P\times Q} (x',y')$ if, and only if, $x \preceq_P x'$ and $y \preceq_Q y'$. Lattice products of more than two partially ordered sets are defined analogously. We say (see e.g. [118, p. 98]) that P and Q are *isomorphic* if there exists a bijection $\psi : P \to Q$ which is order-preserving and such that the inverse of ψ is also order-preserving; this requirement on the bijection ψ is equivalent to saying that, for every $x, x' \in P$,

$$x \preceq_P x' \text{ if and only if } \psi(x) \preceq_Q \psi\left(x'\right). \tag{2.15}$$

Of course, two isomorphic partially ordered sets have the same cardinality. The following result is quite useful for explicitly computing Möbius functions. It states that the Möbius function is invariant under isomorphisms, and that the Möbius function of a lattice product is the product of the associated Möbius functions. Point 1 is an immediate consequence of (2.12), for a proof of Point 2, see e.g. [118, Section 3.8].

Proposition 2.1 *Let P, Q be two partially ordered sets, and let μ_P and μ_Q denote their Möbius functions. Then,*

1. *If P and Q are isomorphic, then $\mu_P(x,y) = \mu_Q(\psi(x), \psi(y))$ for every $x, y \in P$, where ψ is the bijection appearing in (2.15).*
2. *The Möbius function associated with the partially ordered set $P \times Q$ is given by:*

$$\mu_{P\times Q}\left[(x,y),\left(x',y'\right)\right] = \mu_P\left(x,x'\right) \times \mu_Q\left(y,y'\right).$$

The next result is used in the proof of Theorem 6.1.

Proposition 2.2 *Let b be a finite set, and let $\pi, \sigma \in \mathcal{P}(b)$ be such that: (i) $\sigma \leq \pi$, and (ii) the segment $[\sigma, \pi]$ has class $(\lambda_1, ..., \lambda_k) \vdash |\sigma|$. Then, $[\sigma, \pi]$ is a partially ordered set isomorphic to the lattice product of the k sets $\mathcal{P}([\lambda_i])$, $i = 1, ..., k$.*

Proof. To prove the statement, we shall use the fact that each partition in $[\sigma, \pi]$ is obtained by taking unions of the blocks of σ that are contained in the same block of π. Start by observing that $(\lambda_1, ..., \lambda_k)$ is the class of $[\sigma, \pi]$ if and only if for every $i = 1, ..., k$, there is a block $b_i \in \pi$ such that b_i contains exactly λ_i blocks of σ. In particular, $k = |\pi|$. We now construct a bijection ψ, between $[\sigma, \pi]$ and the lattice products of the $\mathcal{P}([\lambda_i])$'s, as follows.

i) For $i = 1, ..., k$, write $b_{i,j}$, $j = 1, ..., \lambda_i$, to indicate the blocks of σ contained in b_i.

ii) For every partition $\rho \in [\sigma, \pi]$ and every $i = 1, ..., k$, construct a partition $\zeta(i, \rho)$ of $[\lambda_i] = \{1, ..., \lambda_i\}$ by the following rule: for every $j, l \in \{1, ..., \lambda_i\}$, $j \sim_{\zeta(i,\rho)} l$ (that is, j and l belong to the same block of $\zeta(i, \rho)$) if and only if the union $b_{i,j} \cup b_{i,l}$ is contained in a block of ρ.

iii) Define the application $\psi : [\sigma, \pi] \to \mathcal{P}([\lambda_1]) \times \cdots \times \mathcal{P}([\lambda_k])$ as

$$\rho \mapsto \psi(\rho) := (\zeta(1, \rho), ..., \zeta(k, \rho)). \tag{2.16}$$

It is easily seen that the application ψ in (2.16) is indeed an order-preserving bijection, verifying (2.15) for $P = [\sigma, \pi]$ and $Q = \mathcal{P}([\lambda_1]) \times \cdots \times \mathcal{P}([\lambda_k])$. ■

3 Combinatorial expressions of cumulants and moments

We recall here the definition of *cumulant*, and we present several of its properties. A detailed discussion of cumulants is contained in the book by Shiryayev [113]; see also the papers by Rota and Shen [105], Speed [117] and Surgailis [122]. An analysis of cumulants involving different combinatorial structures can be found in [97, pp. 20-23] and the references therein.

3.1 Cumulants

For $n \geq 1$, we consider a vector of real-valued random variables $\mathbf{X}_{[n]} = (X_1, ..., X_n)$ such that $\mathbb{E}|X_j|^n < \infty$, $\forall j = 1, ..., n$. For every subset $b = \{j_1, ..., j_k\} \subseteq [n] = \{1, ..., n\}$, we write

$$\boxed{\mathbf{X}_b = (X_{j_1}, ..., X_{j_k})} \quad \text{and} \quad \boxed{\mathbf{X}^b = X_{j_1} \times \cdots \times X_{j_k},} \tag{3.1}$$

where $\times$ denotes the usual product. For instance, $\forall m \leq n$,

$$\mathbf{X}_{[m]} = (X_1, .., X_m) \quad \text{and} \quad \mathbf{X}^{[m]} = X_1 \times \cdots \times X_m.$$

For every $b = \{j_1, ..., j_k\} \subseteq [n]$ and $(z_1, ..., z_k) \in \mathbb{R}^k$, we let $g_{\mathbf{X}_b}(z_1, .., z_k) = \mathbb{E}\left[\exp\left(i \sum_{\ell=1}^k z_\ell X_{j_\ell}\right)\right]$. The *joint cumulant* of the components of the vector $\mathbf{X}_b$ is defined as

$$\chi(\mathbf{X}_b) = (-i)^k \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log g_{\mathbf{X}_b}(z_1, .., z_k) \mid_{z_1 = ... = z_k = 0}, \tag{3.2}$$

thus

$$\chi(X_1, ..., X_k) = (-i)^k \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log \mathbb{E}\left[\exp\left(i \sum_{\ell=1}^k z_l X_l\right)\right]\Bigg|_{z_1 = ... = z_k = 0}.$$

We recall the following facts.

(i) The application $\mathbf{X}_b \mapsto \chi(\mathbf{X}_b)$ is *homogeneous*, that is, for every $\mathbf{h} = (h_1, ..., h_k) \in \mathbb{R}^k$,

$$\chi(h_1 X_{j_1}, ..., h_k X_{j_k}) = (\Pi_{\ell=1}^k h_\ell) \times \chi(\mathbf{X}_b);$$

(ii) The application $\mathbf{X}_b \mapsto \chi(\mathbf{X}_b)$ is invariant with respect to the permutations of b;

(iii) $\chi(\mathbf{X}_b) = 0$, if the vector $\mathbf{X}_b$ has the form $\mathbf{X}_b = \mathbf{X}_{b'} \cup \mathbf{X}_{b''}$, with $b', b'' \neq \varnothing$, $b' \cap b'' = \varnothing$ and $\mathbf{X}_{b'}$ and $\mathbf{X}_{b''}$ independent;

(iv) if $\mathbf{Y} = \{Y_j : j \in J\}$ is a Gaussian family and if $\mathbf{X}_{[n]}$ is a vector obtained by juxtaposing $n \geq 3$ elements of $\mathbf{Y}$ (with possible repetitions), then $\chi\left(\mathbf{X}_{[n]}\right) = 0$.

Properties (i) and (ii) follow immediately from (3.2). To see how to deduce (iii) from (3.2), just observe that, if $\mathbf{X}_b$ has the structure described in (iii), then

$$\log g_{\mathbf{X}_b}(z_1, .., z_k) = \log g_{\mathbf{X}_{b'}}\left(z_\ell : j_\ell \in b'\right) + \log g_{\mathbf{X}_{b''}}\left(z_\ell : j_\ell \in b''\right)$$

(by independence), so that

$$\begin{aligned} & \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log g_{\mathbf{X}_b}(z_1, .., z_k) \\ = \quad & \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log g_{\mathbf{X}_{b'}}\left(z_\ell : j_\ell \in b'\right) + \frac{\partial^k}{\partial z_1 \cdots \partial z_k} \log g_{\mathbf{X}_{b''}}\left(z_\ell : j_\ell \in b''\right) = 0. \end{aligned}$$

Finally, property (iv) is proved by using the fact that, if $\mathbf{X}_{[n]}$ is obtained by juxtaposing $n \geq 3$ elements of a Gaussian family (even with repetitions), then $\log g_{\mathbf{X}_b}(z_1, .., z_k)$ has necessarily the form $\sum_l a(l) z_l + \sum_{i,j} b(i,j) z_i z_j$, where $a(k)$ and $b(i,j)$ are coefficients not depending on the z_l's. All the derivatives of order higher than 2 are then zero.

When $|b| = n$, one says that the cumulant $\chi(\mathbf{X}_b)$, given by (3.2), *has order* n. When $\mathbf{X}_{[n]} = (X_1, ..., X_n)$ is such that $X_j = X$, $\forall j = 1, ..., n$, where X is a random variable in $L^n(\mathbb{P})$, we write

$$\boxed{\chi\left(\mathbf{X}_{[n]}\right) = \chi_n(X)} \tag{3.3}$$

and we say that $\chi_n(X)$ is the nth *cumulant* (or the *cumulant of order* n) of X. Of course, in this case one has that

$$\chi_n(X) = (-i)^n \frac{\partial^n}{\partial z^n} \log g_X(z)\Big|_{z=0},$$

where $g_X(z) = \mathbb{E}[\exp(izX)]$. Note that, if $X, Y \in L^n(\mathbb{P})$ $(n \geq 1)$ are independent random variables, then (3.2) implies that

$$\chi_n(X + Y) = \chi_n(X) + \chi_n(Y),$$

since $\chi_n(X + Y)$ involve the derivative of $\mathbb{E}\left[\exp\left(i(X + Y)\Sigma_{j=1}^n z_j\right)\right]$ with respect to $z_1, ..., z_n$.

3.2 Relations between moments and cumulants, and between cumulants

We want to relate expectations of products of random variables, such as $\mathbb{E}[X_1 X_2 X_3]$, to cumulants of vectors of random variables, such as $\chi(X_1, X_2, X_3)$. Note the disymmetry: moments involve products, while cumulants involve vectors. We will have, for example, $\chi(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$, and hence $\chi(X_1, X_2) = \mathbf{Cov}(X_1, X_2)$, the covariance of the vector (X_1, X_2). Conversely, we will have $\mathbb{E}[X_1 X_2] = \chi(X_1)\chi(X_2) + \chi(X_1, X_2)$. Thus, using the notation introduced above, we will establish precise relations between objects of the type $\chi(\mathbf{X}_b) = \chi(X_j : j \in b)$ and $\mathbb{E}\left[\mathbf{X}^b\right] = \mathbb{E}[\Pi_{j \in b} X_j]$. We can do this also for random variables that

are products of other random variables: for instance, to obtain $\chi(Y_1Y_2, Y_3)$, we apply the previous formula with $X_1 = Y_1Y_2$ and $X_2 = Y_3$, and get $\chi(Y_1Y_2, Y_3) = \mathbb{E}[Y_1Y_2Y_3] - \mathbb{E}[Y_1Y_2]\,\mathbb{E}[Y_3]$. We shall also state a formula, due to Malyshev, which expresses $\chi(Y_1Y_2, Y_3)$ in terms of other cumulants, namely in this case

$$\chi(Y_1Y_2, Y_3) = \chi(Y_1, Y_3)\,\chi(Y_2) + \chi(Y_1)\,\chi(Y_2, Y_3) + \chi(Y_1, Y_2, Y_3).$$

The next result, first proved in [45] (for Parts 1 and 2) and [50] (for Part 3), contains three crucial relations, linking the cumulants and the moments associated with a random vector $\mathbf{X}_{[n]}$. We use the properties of the lattices of partitions, as introduced in the previous section.

Proposition 3.1 (Leonov and Shiryayev [45] and Malyshev [50]) *For every $b \subseteq [n]$,*

1.

$$\mathbb{E}\left[\mathbf{X}^b\right] = \sum_{\pi=\{b_1,\ldots,b_k\}\in\mathcal{P}(b)} \chi(\mathbf{X}_{b_1}) \cdots \chi(\mathbf{X}_{b_k}); \tag{3.4}$$

2.

$$\chi(\mathbf{X}_b) = \sum_{\sigma=\{a_1,\ldots,a_r\}\in\mathcal{P}(b)} (-1)^{r-1}(r-1)!\,\mathbb{E}(\mathbf{X}^{a_1}) \cdots \mathbb{E}(\mathbf{X}^{a_r}); \tag{3.5}$$

3. $\forall \sigma = \{b_1, ..., b_k\} \in \mathcal{P}(b)$,

$$\chi\left(\mathbf{X}^{b_1}, ..., \mathbf{X}^{b_k}\right) = \sum_{\substack{\tau=\{t_1,\ldots,t_s\}\in\mathcal{P}(b)\\ \tau\vee\sigma=\hat{1}}} \chi(\mathbf{X}_{t_1}) \cdots \chi(\mathbf{X}_{t_s}). \tag{3.6}$$

Remark. To the best of our knowledge, our forthcoming proof of equation (3.6) (which is known as *Malyshev's formula*) is new. As an illustration of (3.6), consider the cumulant $\chi(X_1X_2, X_3)$, in which case one has $\sigma = \{b_1, b_2\}$, with $b_1 = \{1, 2\}$ and $b_2 = \{3\}$. There are three partitions $\tau \in \mathcal{P}([3])$ such that $\tau \vee \sigma = \hat{1} = \{1, 2, 3\}$, namely $\tau_1 = \hat{1}$, $\tau_2 = \{\{1, 3\}, \{2\}\}$ and $\tau_3 = \{\{1\}, \{2, 3\}\}$, from which it follows that $\chi(X_1X_2, X_3) = \chi(X_1, X_2, X_3) + \chi(X_1X_3)\,\chi(X_2) + \chi(X_1)\,\chi(X_2X_3)$.

Proof of Proposition 3.1. The proof of (3.4) is obtained by differentiating the characteristic function and its logarithm, and by identifying corresponding terms (see [113], [105, Section 6] or [117]). We now show how to obtain (3.5) and (3.6) from (3.4). Relation (3.4) implies that, $\forall \sigma = \{a_1, ..., a_r\} \in \mathcal{P}(b)$,

$$\prod_{j=1}^{r} \mathbb{E}[\mathbf{X}^{a_j}] = \sum_{\substack{\pi=\{b_1,\ldots,b_k\}\leq\sigma\\ \pi\in\mathcal{P}(b)}} \chi(\mathbf{X}_{b_1}) \cdots \chi(\mathbf{X}_{b_k}). \tag{3.7}$$

We can therefore set $G(\sigma) = \prod_{j=1}^{r} \mathbb{E}[\mathbf{X}^{a_j}]$ and $F(\pi) = \chi(\mathbf{X}_{b_1}) \cdots \chi(\mathbf{X}_{b_k})$ in (2.10) and (2.11), so as to deduce that, for every $\pi = \{b_1, ..., b_k\} \in \mathcal{P}(b)$,

$$\chi(\mathbf{X}_{b_1}) \cdots \chi(\mathbf{X}_{b_k}) = \sum_{\sigma=\{a_1,\ldots,a_r\}\leq\pi} \mu(\sigma, \pi) \prod_{j=1}^{r} \mathbb{E}[\mathbf{X}^{a_j}]. \tag{3.8}$$

Relation (3.5) is therefore a particular case of (3.8), obtained by setting $\pi = \hat{1}$ and by using the equality $\mu\left(\sigma, \hat{1}\right) = (-1)^{|\sigma|-1}(|\sigma|-1)!$, which is a consequence of (2.7).

To deduce Malyshev's formula (3.6) from (3.5) and (3.7), write $\mathbf{X}^{b_j} = Y_j$, $j = 1, ..., k$ (recall that the $\mathbf{X}^{b_j}$ are random variables defined in (3.1)), and apply (3.5) to the vector $\mathbf{Y} = (Y_1, ..., Y_k)$ to obtain that

$$\begin{aligned} \chi\left(\mathbf{X}^{b_1}, ..., \mathbf{X}^{b_k}\right) &= \chi(Y_1, ..., Y_k) = \chi(\mathbf{Y}) \\ &= \sum_{\beta=\{p_1,...,p_r\}\in\mathcal{P}([k])} (-1)^{r-1}(r-1)!\mathbb{E}(\mathbf{Y}^{p_1})\cdots\mathbb{E}(\mathbf{Y}^{p_r}). \end{aligned} \tag{3.9}$$

Now write $\sigma = \{b_1, ..., b_k\}$, and observe that σ is a partition of the set b, while the partitions β in (3.9) are partitions of the first k integers. Now fix $\beta \in \{p_1, ..., p_r\} \in \mathcal{P}([k])$. For $i = 1, ...r$, take the union of the blocks $b_j \in \sigma$ having $j \in p_i$, and call this union u_i. One obtains therefore a partition $\pi = \{u_1, ..., u_r\} \in \mathcal{P}(b)$ such that $|\pi| = |\beta| = r$. Thanks to (2.7) and (2.9),

$$(-1)^{r-1}(r-1)! = \mu\left(\beta, \hat{1}\right) = \mu\left(\pi, \hat{1}\right) \tag{3.10}$$

(note that the two Möbius functions appearing in (3.10) are associated with different lattices: indeed, $\mu\left(\beta, \hat{1}\right)$ refers to $\mathcal{P}([k])$, whereas $\mu\left(\pi, \hat{1}\right)$ is associated with $\mathcal{P}(b)$). With this notation, one has also that $\mathbb{E}(\mathbf{Y}^{p_1})\cdots\mathbb{E}(\mathbf{Y}^{p_r}) = \mathbb{E}(\mathbf{X}^{u_1})\cdots\mathbb{E}(\mathbf{X}^{u_r})$, so that, by (3.7),

$$\mathbb{E}(\mathbf{Y}^{p_1})\cdots\mathbb{E}(\mathbf{Y}^{p_r}) = \mathbb{E}(\mathbf{X}^{u_1})\cdots\mathbb{E}(\mathbf{X}^{u_r}) = \sum_{\substack{\tau=\{t_1,...,t_s\}\leq\pi \\ \tau\in\mathcal{P}(b)}} \chi(\mathbf{X}_{t_1})\cdots\chi(\mathbf{X}_{t_s}). \tag{3.11}$$

By plugging (3.10) and (3.11) into (3.9) we obtain finally that

$$\begin{aligned} \chi\left(\mathbf{X}^{b_1}, ..., \mathbf{X}^{b_k}\right) &= \sum_{\sigma\leq\pi\leq\hat{1}} \mu\left(\pi, \hat{1}\right) \sum_{\tau=\{t_1,...,t_s\}:\tau\leq\pi} \chi(\mathbf{X}_{t_1})\cdots\chi(\mathbf{X}_{t_s}) \\ &= \sum_{\tau\in\mathcal{P}(b)} \chi(\mathbf{X}_{t_1})\cdots\chi(\mathbf{X}_{t_s}) \sum_{\pi\in\left[\tau\vee\sigma,\hat{1}\right]} \mu\left(\pi, \hat{1}\right) = \sum_{\tau:\tau\vee\sigma=\hat{1}} \chi(\mathbf{X}_{t_1})\cdots\chi(\mathbf{X}_{t_s}), \end{aligned}$$

where the last equality is a consequence of (2.14), since

$$\sum_{\pi\in\left[\tau\vee\sigma,\hat{1}\right]} \mu\left(\pi, \hat{1}\right) = \begin{cases} 1 & \text{if } \tau\vee\sigma = \hat{1} \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

■

For a single random variable X, one has (3.3): hence, Proposition 3.1 implies

Corollary 3.1 *Let X be a random variable such that $\mathbb{E}|X|^n < \infty$. Then,*

$$\mathbb{E}[X^n] = \sum_{\pi=\{b_1,...,b_k\}\in\mathcal{P}([n])} \chi_{|b_1|}(X)\cdots\chi_{|b_k|}(X) \tag{3.13}$$

$$\chi_n(X) = \sum_{\sigma=\{a_1,...,a_r\}\in\mathcal{P}(b)} (-1)^{r-1}(r-1)!\mathbb{E}\left(X^{|a_1|}\right)\cdots\mathbb{E}\left(X^{|a_r|}\right) \tag{3.14}$$

Examples. (i) Formula (3.5), applied respectively to $b = \{1\}$ and to $b = \{1,2\}$, gives immediately the well-known relations

$$\chi(X) = \mathbb{E}(X) \quad \text{and} \quad \chi(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbf{Cov}(X,Y). \tag{3.15}$$

(ii) One has that

$$\begin{aligned}\chi(X_1,X_2,X_3) &= \mathbb{E}(X_1X_2X_3) - \mathbb{E}(X_1X_2)\mathbb{E}(X_3) \\ &\quad -\mathbb{E}(X_1X_3)\mathbb{E}(X_2) - \mathbb{E}(X_2X_3)\mathbb{E}(X_1) \\ &\quad +2\mathbb{E}(X_1)\mathbb{E}(X_2)\mathbb{E}(X_3),\end{aligned}$$

so that, in particular,

$$\chi_3(X) = \mathbb{E}\left(X^3\right) - 3\mathbb{E}\left(X^2\right)\mathbb{E}(X) + 2\mathbb{E}(X)^3.$$

(iii) Let $\mathbf{G}_{[n]} = (G_1,...,G_n)$, $n \geq 3$, be a Gaussian vector such that $\mathbb{E}(G_i) = 0$, $i = 1,...,n$. Then, for every $b \subseteq [n]$ such that $|b| \geq 3$, we know from Section 3.1 that

$$\chi(\mathbf{G}_b) = \chi(G_i : i \in b) = 0.$$

By applying this relation and formulae (3.4) and (3.15) to $\mathbf{G}_{[n]}$, one therefore obtains the well-known relation

$$\begin{aligned}&\mathbb{E}[G_1 \times G_2 \times \cdots \times G_n] \\ &= \begin{cases} \sum_{\pi=\{\{i_1,j_1\},...,\{i_k,j_k\}\} \in \mathcal{P}([n])} \mathbb{E}(G_{i_1}G_{j_1}) \cdots \mathbb{E}(G_{i_k}G_{j_k}), & n \text{ even} \\ 0, & n \text{ odd.}\end{cases}\end{aligned} \tag{3.16}$$

Observe that, on the RHS of (3.16), the sum is taken over all partitions π such that each block of π contains exactly two elements.

4 Diagrams and graphs

In this section, we translate part of the notions presented in Section 2 into the language of diagrams and graphs, which are often used in order to compute cumulants and moments of non-linear functionals of random fields (see e.g. [9, 11, 24, 25, 51, 121, 122]).

4.1 Diagrams

Consider a finite set b. A *diagram* is a graphical representation of a pair of partitions $(\pi,\sigma) \subseteq \mathcal{P}(b)$, such that $\pi = \{b_1,...,b_k\}$ and $\sigma = \{t_1,...,t_l\}$. It is obtained as follows.

1. Order the elements of each block b_i, for $i = 1,...,k$;
2. Associate with each block $b_i \in \pi$ a row of $|b_i|$ *vertices* (represented as dots), in such a way that the jth vertex of the ith row corresponds to the jth element of the the block b_i;
3. For every $a = 1,...,l$, draw a closed curve around the vertices corresponding to the elements of the block $t_a \in \sigma$.

We will denote by $\Gamma(\pi,\sigma)$ the diagram of a pair of partitions (π,σ).

Examples. (i) If $b = [3]$ and $\pi = \sigma = \{\{1,2\},\{3\}\}$, then $\Gamma(\pi,\sigma)$ is represented in Fig. 1.

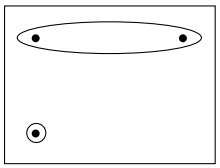

Figure 1: *A simple diagram*

(ii) If $b = [8]$, and $\pi = \{\{1,2,3\},\{4,5\},\{6,7,8\}\}$ and $\sigma = \{\{1,4,6\},\{2,5\},\{3,7,8\}\}$, then $\Gamma(\pi,\sigma)$ is represented in Fig. 2.

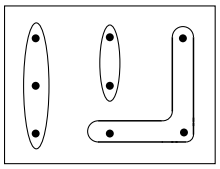

Figure 2: *A diagram built from two three-block partitions*

Hence, the rows in $\Gamma(\pi,\sigma)$ indicate the sets in π and the curves indicate the sets in σ.

Remarks. (a) We use the terms "element" and "vertex" interchangeably.

(b) Note that the diagram generated by the pair (π,σ) is different, in general, from the diagram generated by (σ,π).

(c) Each diagram is a finite *hypergraph*. We recall that a finite hypergraph is an object of the type (V,E), where V is a finite set of vertices, and E is a collection of (not necessarily disjoint) nonempty subsets of V. The elements of E are usually called *edges*. In our setting, these are the blocks of σ.

(d) Note that, once a partition π is specified, the diagram $\Gamma(\pi,\sigma)$ encodes all the information on σ.

Now fix a finite set b. In what follows, we will list and describe several type of diagrams. They can be all characterized in terms of the lattice structure of $\mathcal{P}(b)$, namely the partial ordering $\leq$ and the join and meet operations $\vee$ and $\wedge$, as described in Section 2. Recall that $\hat{1} = \{b\}$, and $\hat{0}$ is the partition whose elements are the singletons of b.

Connected Diagrams. The diagram $\Gamma(\pi,\sigma)$ associated with two partitions (π,σ) is said to be *connected* if $\pi \vee \sigma = \hat{1}$, that is, if the only partition ρ such that $\pi \leq \rho$ and $\sigma \leq \rho$ is the maximal partition $\hat{1}$. The diagram appearing in Fig. 2 is connected, whereas the one in Fig. 1 is not (indeed, in this case $\pi \vee \sigma = \pi \vee \pi = \pi \neq \hat{1}$). Another example of a non-connected diagram (see Fig. 3) is obtained by taking $b = [4]$, $\pi = \{\{1,2\},\{3\},\{4\}\}$ and $\sigma = \{\{1,2\},\{3,4\}\}$, so that $\pi \leq \sigma$ (each block of π is contained in a block of σ) and $\pi \vee \sigma = \sigma \neq \hat{1}$.

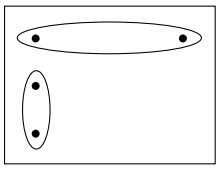

Figure 3: *A non-connected diagram*

In other words, $\Gamma(\pi, \sigma)$ *is connected if and only if the rows of the diagram (the blocks of* π*) cannot be divided into two subsets, each defining a separate diagram.* Fig. 4 shows that the diagram in Fig. 3 can be so divided.

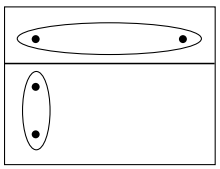

Figure 4: *Dividing a non-connected diagram*

The diagram in Fig. 5, which has the same partition π, but $\sigma = \{\{1,3,4\},\{2\}\}$, is connected.

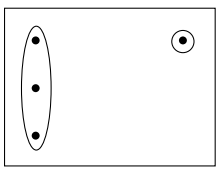

Figure 5: *A connected diagram*

Note that we do not use the term 'connected' as one usually does in graph theory (indeed, the diagrams we consider in this section are always *non-connected hypergraphs*, since their edges are disjoint by construction).

Non-flat Diagrams. The diagram $\Gamma(\pi, \sigma)$ is *non-flat* if

$$\pi \wedge \sigma = \hat{0},$$

that is, if the only partition ρ such that $\rho \leq \pi$ and $\rho \leq \sigma$ is the minimal partition $\hat{0}$. It is easily seen that $\pi \wedge \sigma = \hat{0}$ if and only if for any two blocks $b_j \in \pi$, $t_a \in \sigma$, the intersection $b_j \cap t_a$ either is empty or contains exactly one element. Graphically, a non-flat graph is such that the closed curves defining the blocks of σ cannot join two vertices in the same row (thus having a 'flat' or 'horizontal' portion). The diagrams in Fig. 1-3 are all flat, whereas the diagram in Fig. 5 is non-flat. Another non-flat diagram is given in Fig. 6, and is obtained by taking $b = [7]$, $\pi = \{\{1,2,3\},\{4\},\{5,6,7\}\}$ and $\sigma = \{\{1,4,5\},\{2,7\},\{3,6\}\}$.

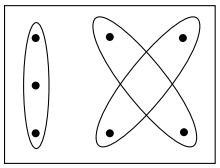

Figure 6: *A non-flat diagram*

Gaussian Diagrams. We say that the diagram $\Gamma(\pi,\sigma)$ is *Gaussian*, whenever every block of σ contains exactly two elements. Plainly, Gaussian diagrams exists only if there is an even number of vertices. When a diagram is Gaussian, one usually represents the blocks of σ not by closed curves, but by segments connecting two vertices (which are viewed as the edges of the resulting graph). For instance, a Gaussian (non-flat and connected) diagram is obtained in Fig. 7, where we have taken $b=[6]$, $\pi=\{\{1,2,3\},\{4\},\{5,6\}\}$ and $\sigma=\{\{1,4\},\{2,5\},\{3,6\}\}$.

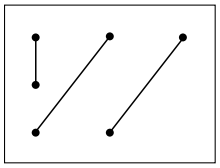

Figure 7: *A Gaussian diagram*

In the terminology of graph theory, a Gaussian diagram is a *non-connected (non-directed) graph*. Since every vertex is connected with exactly another vertex, one usually says that such a graph is a *perfect matching*.

Circular (Gaussian) Diagrams. Consider two partitions $\pi=\{b_1,...,b_k\}$ and $\sigma=\{t_1,...,t_l\}$ such that the blocks of σ have size $|t_a|=2$ for every $a=1,...,l$. Then, the diagram $\Gamma(\pi,\sigma)$ (which is Gaussian) is said to be *circular* if each row of $\Gamma(\pi,\sigma)$ is linked to both the previous and the next row, with no other possible links except for the first and the last row, which should also be linked together. This implies that the diagram is connected. Formally, the diagram $\Gamma(\pi,\sigma)$ is circular (Gaussian) whenever the following properties hold (recall that $i\sim_\sigma j$ means that i and j belong to the same block of σ): (i) for every $p=2,...,k-1$ there exist $j_1\sim_\sigma i_1$ and $j_2\sim_\sigma i_2$ such that $j_1,j_2\in b_p$, $i_1\in b_{p-1}$ and $i_2\in b_{p+1}$, (ii) for every $p=2,...,k-1$ and every $j\in b_p$, $j\sim_\sigma i$ implies that $i\in b_{p-1}\cup b_{p+1}$, (iii) there exist $j_1\sim_\sigma i_1$ and $j_2\sim_\sigma i_2$ such that $j_1,j_2\in b_k$, $i_1\in b_{k-1}$ and $i_2\in b_1$, (iv) for every $j\in b_k$, $j\sim_\sigma i$ implies that $i\in b_1\cup b_{k-1}$ (v) there exist $j_1\sim_\sigma i_1$ and $j_2\sim_\sigma i_2$ such that $j_1,j_2\in b_1$, $i_1\in b_2$ and $i_2\in b_k$, (vi) for every $j\in b_1$, $j\sim_\sigma i$ implies that $i\in b_k\cup b_2$. For instance, a circular diagram is obtained by taking $b=[10]$ and

$$\begin{aligned}\pi &= \{\{1,2\},\{3,4\},\{5,6\}\,\{7,8\},\{9,10\}\}\\ \sigma &= \{\{1,3\},\{2,9\},\{4,6\},\{5,7\},\{8,10\}\},\end{aligned}$$

which implies that $\Gamma(\pi,\sigma)$ is the diagram in Fig. 8.

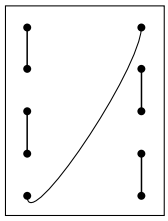

Figure 8: *A circular diagram*

Another example of a circular diagram is given in Fig. 9. It is obtained from $b = [12]$ and

$$\begin{aligned} \pi &= \{\{1,2,3\},\{4,5\},\{6,7\},\{8,9\},\{10,11,12\}\} \\ \sigma &= \{\{1,4\},\{2,11\},\{3,10\},\{5,7\},\{6,8\},\{9,12\}\}. \end{aligned}$$

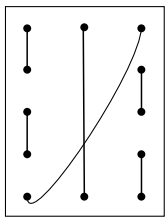

Figure 9: *A circular diagram with rows of different size*

Examples. (i) Thanks to the previous discussion, one can immediately reformulate Malyshev's formula (3.6) as follows. For every finite set b and every $\sigma = \{b_1, ..., b_k\} \in \mathcal{P}(b)$,

$$\chi\left(\mathbf{X}^{b_1}, ..., \mathbf{X}^{b_k}\right) = \sum_{\substack{\tau=\{t_1,...,t_s\}\in\mathcal{P}(b) \\ \Gamma(\sigma,\tau) \text{ is connected}}} \chi(\mathbf{X}_{t_1}) \cdots \chi(\mathbf{X}_{t_s}). \tag{4.17}$$

(ii) Suppose that the random variables X_1, X_2, X_3 are such that $\mathbb{E}|X_i|^3 < \infty$, $i = 1, 2, 3$. We have already applied formula (4.17) in order to compute the cumulant $\chi(X_1X_2, X_3)$. Here, we shall give a graphical demonstration. Recall that, in this case, $b = [3] = \{1, 2, 3\}$, and that the relevant partition is $\sigma = \{\{1, 2\}, \{3\}\}$. There are only three partitions $\tau_1, \tau_2, \tau_3 \in \mathcal{P}([3])$ such that $\Gamma(\sigma, \tau_1)$, $\Gamma(\sigma, \tau_2)$ and $\Gamma(\sigma, \tau_3)$ are connected, namely $\tau_1 = \hat{1}$, $\tau_2 = \{\{1, 3\}, \{2\}\}$ and $\tau_3 = \{\{1\}, \{2, 3\}\}$. The diagrams $\Gamma(\sigma, \tau_1)$, $\Gamma(\sigma, \tau_2)$ and $\Gamma(\sigma, \tau_3)$ are represented in Fig. 10. Relation (4.17) thus implies that

$$\begin{aligned} \chi(X_1X_2, X_3) &= \chi(X_1, X_2, X_3) + \chi(X_1, X_3)\chi(X_2) + \chi(X_1)\chi(X_2, X_3) \\ &= \chi(X_1, X_2, X_3) + \mathbf{Cov}(X_1, X_3)\mathbb{E}(X_3) + \mathbb{E}(X_1)\mathbf{Cov}(X_2, X_3), \end{aligned}$$

where we have used (3.15).

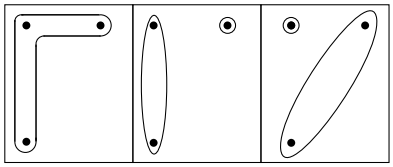

Figure 10: *Computing cumulants by connected diagrams*

4.2 Solving the equation $\sigma \wedge \pi = \hat{0}$

Let π be a partition of $[n] = \{1, ..., n\}$. One is often asked, as will be the case in Section 6.1, to find all partitions $\sigma \in \mathcal{P}([n])$ such that

$$\sigma \wedge \pi = \hat{0}, \tag{4.18}$$

where, as usual, $\hat{0} = \{\{1\}, ..., \{n\}\}$, that is, $\hat{0}$ is the partition made up of singletons. The use of diagrams provides an easy way to solve (4.18), since (4.18) holds if and only if the diagram $\Gamma(\pi, \sigma)$ is non-flat. Hence, proceed as in Section 4.1, by (**1**) ordering the blocks of π, (**2**) associating with each block of π a row of the diagram, the number of points in a row being equal to the number of elements in the block, and (**3**) drawing non-flat closed curves around the points of the diagram.

Examples. (i) Let $n = 2$ and $\pi = \{\{1\}, \{2\}\} = \hat{0}$. Then, $\sigma_1 = \pi = \hat{0}$ and $\sigma_2 = \hat{1}$ (as represented in Fig. 11) solve equation (4.18). Note that $\mathcal{P}([2]) = \{\sigma_1, \sigma_2\}$.

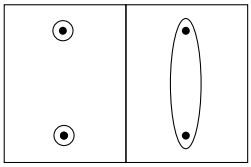

Figure 11: *Solving $\sigma \wedge \pi = \hat{0}$ in the simplest case*

(ii) Let $n = 3$ and $\pi = \{\{1, 2\}, \{3\}\}$. Then, $\sigma_1 = \hat{0}$, $\sigma_2 = \{\{1, 3\}, \{2\}\}$ and $\sigma_3 = \{\{1\}, \{2, 3\}\}$ (see Fig. 12) are the only elements of $\mathcal{P}([3])$ solving (4.18).

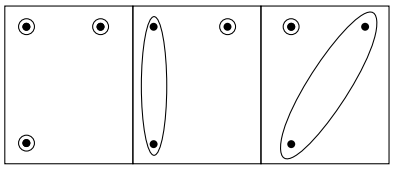

Figure 12: *Solving $\sigma \wedge \pi = \hat{0}$ in a three-vertex diagram*

(iii) Let $n = 4$ and $\pi = \{\{1,2\},\{3,4\}\}$. Then, there are exactly seven $\sigma \in \mathcal{P}([4])$ solving (4.18). They are all represented in Fig. 13.

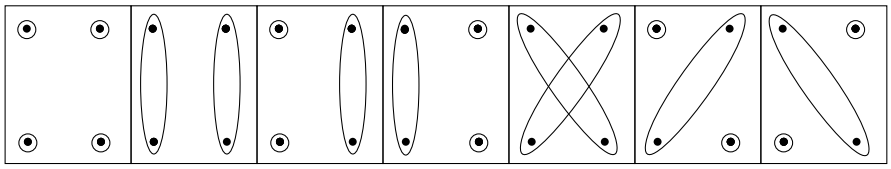

Figure 13: *The seven solutions of* $\sigma \wedge \pi = \hat{0}$ *in a four-vertex diagram*

(iv) Let $n = 4$ and $\pi = \{\{1,2\},\{3\},\{4\}\}$. Then, there are ten $\sigma \in \mathcal{P}([4])$ that are solutions of (4.18). They are all represented in Fig. 14.

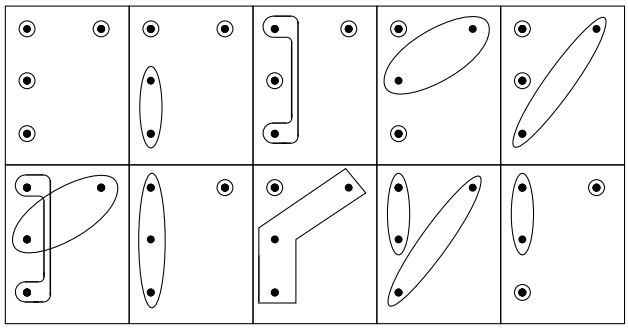

Figure 14: *The ten solutions of* $\sigma \wedge \pi = \hat{0}$ *in a three-row diagram*

In what follows (see e.g. Theorem 7.1 below), we will sometimes be called to solve jointly the equations $\sigma \wedge \pi = \hat{0}$ and $\sigma \vee \pi = \hat{1}$, that is, given π, to find all diagrams $\Gamma(\pi, \sigma)$ that are non-flat ($\sigma \wedge \pi = \hat{0}$) and connected ($\sigma \vee \pi = \hat{1}$). Having found, as before, all those that are non-flat, one just has to choose among them those that are connected, that is, the diagrams whose rows cannot be divided into two subset, each defining a separate diagram. These are: the second diagram in Fig. 11, the last two in Fig. 12, the last six in Fig. 13, the sixth to ninth of Fig. 14. Again as an example, observe that the second diagram in Fig. 14 is not connected: indeed, in this case, $\pi = \{\{1,2\},\{3\},\{4\}\}$, $\sigma = \{\{1\},\{2\},\{3,4\}\}$, and $\pi \vee \sigma = \{\{1,2\},\{3,4\}\} \neq \hat{1}$.

4.3 From Gaussian diagrams to multigraphs

A *multigraph* is a graph in which (a) two vertices can be connected by more than one edge, and (b) loops (that is, edges connecting one vertex to itself) are allowed. Such objects are sometimes called "pseudographs", but we will avoid this terminology. In what follows, we show how a multigraph can be derived from a Gaussian diagram. This representation of Gaussian diagrams can be used in the computation of moments and cumulants (see [25] or [51]).

Fix a set b and consider partitions $\pi, \sigma \in \mathcal{P}(b)$ such that $\Gamma(\pi, \sigma)$ is Gaussian and $\pi = \{b_1, ..., b_k\}$. Then, the multigraph $\hat{\Gamma}(\pi, \sigma)$, with k vertices and $|b|/2$ edges, is obtained from

$\Gamma(\pi,\sigma)$ as follows.

1. Identify each row of $\Gamma(\pi,\sigma)$ with a vertex of $\hat{\Gamma}(\pi,\sigma)$, in such a way that the ith row of $\Gamma(\pi,\sigma)$ corresponds to the ith vertex v_i of $\hat{\Gamma}(\pi,\sigma)$.

2. Draw an edge linking v_i and v_j for every pair (x,y) such that $x \in b_i$, $y \in b_j$ and $x \sim_\sigma y$.

Examples. (i) The multigraph obtained from the diagram in Fig. 7 is given in Fig. 15.

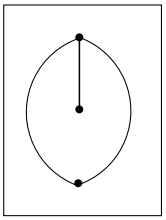

Figure 15: *A multigraph with three vertices*

(ii) The multigraph associated with Fig. 8 is given in Fig. 16 (note that this graph has been obtained from a circular diagram).

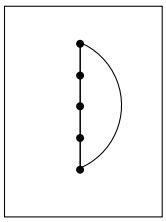

Figure 16: *A multigraph built from a circular diagram*

The following result is easily verified: it shows how the nature of a Gaussian diagram can be deduced from its graph representation.

Proposition 4.1 *Fix a finite set b, as well as a pair of partitions $(\pi,\sigma) \subseteq \mathcal{P}(b)$ such that the diagram $\Gamma(\pi,\sigma)$ is Gaussian and $|\pi| = k$. Then,*

1. *$\Gamma(\pi,\sigma)$ is connected if and only if $\hat{\Gamma}(\pi,\sigma)$ is a connected multigraph.*

2. *$\Gamma(\pi,\sigma)$ is non-flat if and only if $\hat{\Gamma}(\pi,\sigma)$ has no loops.*

3. *$\Gamma(\pi,\sigma)$ is circular if and only if the vertices $v_1,...,v_k$ of $\hat{\Gamma}(\pi,\sigma)$ are such that: (i) there is an edge linking v_i and v_{i+1} for every $i = 1,...,k-1$, and (ii) there is an edge linking v_k and v_1.*

As an illustration, in Fig. 17 we present the picture of a flat and non-connected diagram (on the left), whose graph (on the right) is non-connected and displays three loops.

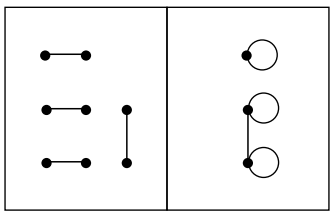

Figure 17: *A non-connected flat diagram and its multigraph*

This situation corresponds to the case $b = [8]$,

$$\begin{aligned} \pi &= \{\{1,2\},\{3,4,5\},\{6,7,8\}\}, \text{ and} \\ \sigma &= \{\{1,2\},\{3,4\},\{5,8\},\{6,7\}\}. \end{aligned}$$

5 Completely random measures and Wiener-Itô stochastic integrals

We will now introduce the notion of a *completely random measure* on a measurable space $(Z, \mathcal{Z})$, as well as those of a *stochastic measure of order* $n \geq 2$, a *diagonal measure* and a *multiple (stochastic) Wiener-Itô integral*. All these concepts can be unified by means of the formalism introduced in Sections 2–4. We stress by now that the domain of the multiple stochastic integrals defined in this section can be extended to more general (and possibly random) classes of integrands. We refer the interested reader to the paper by Kallenberg ans Szulga [39], as well as to the monographs by Kussmaul [42], Kwapień and Woyczyński [44, Ch. 10] and Linde [47], for several results in this direction.

5.1 Completely random measures

Diagonals and subdiagonals play an important role in the context of multiple integrals. The following definitions provide a convenient way to specify them. In what follows, we will denote by $(Z, \mathcal{Z})$ a Polish space, where $\mathcal{Z}$ is the associated Borel σ-field.

Definition 1 *For every $n \geq 1$, we write $(Z^n, \mathcal{Z}^n) = (Z^{\otimes n}, \mathcal{Z}^{\otimes n})$, with $Z^1 = Z$. For every partition $\pi \in \mathcal{P}([n])$ and every $B \in \mathcal{Z}^n$, we set*

$$\boxed{Z^n_\pi \triangleq \{(z_1,...,z_n) \in Z^n : z_i = z_j \quad \textit{if and only if} \quad i \sim_\pi j\}} \quad \textit{and} \quad B_\pi \triangleq B \cap Z^n_\pi. \tag{5.1}$$

Recall that $i \sim_\pi j$ means that the elements i and j belong to the same block of the partition π. Relation (5.1) states that the variables z_i and z_j should be equated if and only if i and j belong to the same block of π.

Examples. (i) Since $\hat{0} = \{\{1\}, ..., \{n\}\}$, no two elements can belong to the same block, and therefore $B_{\hat{0}}$ coincides with the collection of all vectors $(z_1, ..., z_n) \in B$ such that $z_i \neq z_j$, $\forall i \neq j$.

(ii) Since $\hat{1} = \{\{1, ..., n\}\}$, all elements belong to the same block and therefore

$$B_{\hat{1}} = \{(z_1, ..., z_n) \in B : z_1 = z_2 = ... = z_n\}.$$

A set such as $B_{\hat{1}}$ is said to be *purely diagonal*.

(iii) Suppose $n = 3$ and $\pi = \{\{1\}, \{2, 3\}\}$. Then, $B_\pi = \{(z_1, z_2, z_3) \in B : z_2 = z_3,\ z_1 \neq z_2\}$.

The following decomposition lemma (whose proof is immediate and left to the reader) will be used a number of times.

Lemma 5.1 *For every set $B \in \mathcal{Z}^n$,*

$$B = \cup_{\sigma \in \mathcal{P}([n])} B_\sigma = \cup_{\sigma \geq \hat{0}} B_\sigma.$$

Moreover $B_\pi \cap B_\sigma = \varnothing$ if $\pi \neq \sigma$.

One has also that

$$(A_1 \times \cdots \times A_n)_{\hat{1}} = (\underbrace{(\cap_{i=1}^n A_i) \times \cdots \times (\cap_{i=1}^n A_i)}_{n \text{ times}})_{\hat{1}}; \tag{5.2}$$

indeed, since all coordinates are equal in the LHS of (5.2), their common value must be contained in the intersection of the sets.

Example. As an illustration of (5.2), let $A_1 = [0, 1]$ and $A_2 = [0, 2]$ be intervals in $\mathbb{R}^1$, and draw the rectangle $A_1 \times A_2 \in \mathbb{R}^2$. The set $(A_1 \times A_2)_{\hat{1}}$ (that is, the subset of $A_1 \times A_2$ composed of vectors whose coordinates are equal) is therefore identical to the diagonal of the square $(A_1 \cap A_2) \times (A_1 \cap A_2) = [0, 1]^2$. The set $(A_1 \times A_2)_{\hat{1}}$ can be visualized as the thick diagonal segment in Fig. 18.

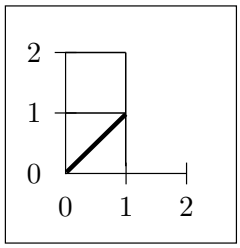

Figure 18: *A diagonal set*

We shall now define a "completely random measure" φ, often called an "independently scattered random measure". It has two characteristics: it is a measure and it takes values in a space of random variables. It will be denoted with its arguments as $\varphi(B, \omega)$, where B is a Borel set and ω is a point in the underlying sample space Ω. The "size" of φ will be controlled

by a non-random, σ-finite and non-atomic measure ν, where $\nu(B) = \mathbb{E}\varphi(B)^2$. The fact that ν is non-atomic means that $\nu(\{z\}) = 0$ for every $z \in Z$. The measure φ will be used to define multiple integrals, where one integrates either over a whole subset of Z^p, $p \geq 1$, or over a subset "without diagonals". In the first case we will need to suppose $\mathbb{E}|\varphi(B)|^p < \infty$; in the second case, one may suppose as little as $\mathbb{E}\varphi(B)^2 < \infty$. Since $p \geq 1$ will be arbitrary, in order to deal with the first case we shall suppose that $\mathbb{E}|\varphi(B)|^p < \infty$, $\forall p \geq 1$, that is, $\varphi \in \cap_{p\geq 1} L^p(\mathbb{P})$. We now present the formal definition of φ.

Definition 2 *(1) Let ν be a positive, σ-finite and non-atomic measure on $(Z, \mathcal{Z})$, and let*

$$\mathcal{Z}_\nu = \{B \in \mathcal{Z} : \nu(B) < \infty\}. \tag{5.3}$$

*A centered **completely random measure** (in $\cap_{p\geq 1} L^p(\mathbb{P})$) on $(Z, \mathcal{Z})$ with **control measure** ν, is a function $\varphi(\cdot,\cdot)$, from $\mathcal{Z}_\nu \times \Omega$ to $\mathbb{R}$, such that*

(i) *For every fixed $B \in \mathcal{Z}_\nu$, the application $\omega \mapsto \varphi(B,\omega)$ is a random variable;*

(ii) *For every fixed $B \in \mathcal{Z}_\nu$, $\varphi(B) \in \cap_{p\geq 1} L^p(\mathbb{P})$;*

(iii) *For every fixed $B \in \mathcal{Z}_\nu$, $\mathbb{E}[\varphi(B)] = 0$;*

(iv) *$\varphi(\varnothing) = 0$;*

(v) *For every collection of disjoint elements of $\mathcal{Z}_\nu$, $B_1, ..., B_n$, the variables $\varphi(B_1)$,..., $\varphi(B_n)$ are independent;*

(vi) *For every $B, C \in \mathcal{Z}_\nu$, $\mathbb{E}[\varphi(B)\varphi(C)] = \nu(B \cap C)$.*

*(2) When $\varphi(\cdot)$ verifies the properties (i) and (iii)–(vi) above and $\varphi(B) \in L^2(\mathbb{P})$, $\forall B \in \mathcal{Z}_\nu$ (so that it is not necessarily true that $\varphi(B) \in L^p(\mathbb{P})$, $p \geq 3$), we say that φ is a **completely random measure in** $L^2(\mathbb{P})$.*

Two crucial remarks. (*On additivity*) (a) Let $B_1, ..., B_n, ...$ be a sequence of *disjoint* elements of $\mathcal{Z}_\nu$, and let φ be a completely random measure on $(Z, \mathcal{Z})$ with control ν. Then, for every finite $N \geq 2$, one has that $\cup_{n=1}^N B_n \in \mathcal{Z}_\nu$, and, by using Properties (iii), (v) and (vi) in Definition 2, one has that

$$\mathbb{E}\left[\left(\varphi\left(\cup_{n=1}^N B_i\right) - \sum_{n=1}^N \varphi(B_n)\right)^2\right] = \nu\left(\cup_{n=1}^N B_n\right) - \sum_{n=1}^N \nu(B_n) = 0, \tag{5.4}$$

because ν is a measure, and therefore it is finitely additive. Relation (5.4) implies in particular that

$$\varphi\left(\cup_{n=1}^N B_n\right) = \sum_{n=1}^N \varphi(B_n), \quad \text{a.s.-}\mathbb{P}. \tag{5.5}$$

Now suppose that $\cup_{n=1}^{\infty} B_n \in \mathcal{Z}_\nu$. Then, by (5.5) and again by virtue of Properties (iii), (v) and (vi) in Definition 2,

$$\begin{aligned}\mathbb{E}\left[\left(\varphi\left(\cup_{n=1}^{\infty} B_n\right) - \sum_{n=1}^{N} \varphi\left(B_n\right)\right)^2\right] &= \mathbb{E}\left[\left(\varphi\left(\cup_{n=1}^{\infty} B_n\right) - \varphi\left(\cup_{n=1}^{N} B_i\right)\right)^2\right] \\ &= \nu\left(\cup_{n=N+1}^{\infty} B_n\right) \underset{N\to\infty}{\to} 0,\end{aligned}$$

because ν is σ-additive. This entails in turn that

$$\boxed{\varphi\left(\cup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \varphi\left(B_n\right), \quad \text{a.s.-}\mathbb{P},} \tag{5.6}$$

where the series on the RHS converges in $L^2(\mathbb{P})$. Relation (5.6) simply means that the application

$$\mathcal{Z}_\nu \to L^2(\mathbb{P}) : B \mapsto \varphi(B),$$

is σ-additive, and therefore that *every completely random measure is a σ-additive measure with values in the Hilbert space $L^2(\mathbb{P})$*. See e.g. Engel [18], Kussmaul [42] or Linde [47] for further discussions on vector-valued measures.

(b) In general, it is *not* true that, for a completely random measure φ and for a fixed $\omega \in \Omega$, the application

$$\mathcal{Z}_\nu \to \mathbb{R} : B \mapsto \varphi(B, \omega)$$

is a σ-additive real-valued (signed) measure. The most remarkable example of this phenomenon is given by Gaussian completely random measures. See the discussion below for more details on this point.

Remark on notation. We consider the spaces $(Z, \mathcal{Z})$ and $(Z^n, \mathcal{Z}^n) = (Z^{\otimes n}, \mathcal{Z}^{\otimes n})$. Do not confuse the subset Z_π^n in (5.1), where π denotes a partition, with the σ-field $\mathcal{Z}_\nu^n$ in (5.3), where ν denotes a control measure.

Now fix a completely random measure φ. For every $n \geq 2$ and every rectangle $C = C_1 \times \cdots \times C_n$, $C_j \in \mathcal{Z}_\nu$, we define $\varphi^{[n]}(C) \triangleq \varphi(C_1) \times \cdots \times \varphi(C_n)$, so that the application $C \mapsto \varphi^{[n]}(C)$ defines a finitely additive application on the ring of rectangular sets contained in Z^n, with values in the set of $\sigma(\varphi)$-measurable random variables. In the next definition we focus on those completely random measures such that the application $\varphi^{[n]}$ admits a unique infinitely additive (and square-integrable) extension on $\mathcal{Z}^n$. Here, the infinite additivity is in the sense of the $L^1(\mathbb{P})$ convergence. Note that we write $\varphi^{[n]}$ to emphasize the dependence of $\varphi^{[n]}$ not only on n, but also on the set $[n] = \{1, ..., n\}$, whose lattice of partitions will be considered later.

Definition 3 *For $n \geq 2$, we write $\mathcal{Z}_\nu^n = \{C \in \mathcal{Z}^n : \nu^n(C) < \infty\}$. A completely random measure φ, verifying points (i)–(vi) of Definition 2, is said to be **good** if, for every fixed $n \geq 2$, there exists a (unique) collection of random variables $\varphi^{[n]} = \left\{\varphi^{[n]}(C) : C \in \mathcal{Z}^n\right\}$ such that*

(i) $\left\{\varphi^{[n]}(C) : C \in \mathcal{Z}_\nu^n\right\} \subseteq L^2(\mathbb{P})$ *;*

(ii) *For every rectangle $C = C_1 \times \cdots \times C_n$, $C_j \in \mathcal{Z}_\nu$,*

$$\boxed{\varphi^{[n]}(C) = \varphi(C_1) \cdots \varphi(C_n);} \tag{5.7}$$

(iii) *$\varphi^{[n]}$ is a σ-additive random measure in the following sense: if $C \in \mathcal{Z}_\nu^n$ is such that $C = \cup_{j=1}^{\infty} C_j$, with the $\{C_j\}$ disjoints, then*

$$\varphi^{[n]}(C) = \sum_{j=1}^{\infty} \varphi^{[n]}(C_j), \quad \textit{with convergence (at least) in } L^1(\mathbb{P}). \tag{5.8}$$

Note that, in the case $n = 1$, the σ-additivity of φ follows from point (vi) of Definition 2 and the σ-additivity of ν (through L^2 convergence). In the case $n \geq 2$, the assumption that the measure φ is good implies σ-additivity in the sense of (5.8).

Remark. The notion of a "completely random measure" can be traced back to Kingman's seminal paper [40]. For further references on completely random measures, see also the two surveys by Surgailis [122] and [123] (note that, in such references, completely random measures are called "independently scattered measures"). The use of the term "good", to indicate completely random measures satisfying the requirements of Definition 3, is taken from Rota and Wallstrom [106]. Existence of good measures is discussed in Engel [18] and Kwapień and Woyczyński [44, Ch. 10]. For further generalizations of Engel's results the reader is referred e.g. to [39] and [107].

Examples. The following two examples of good completely random measures will play a crucial role in the subsequent sections.

(i) A *centered Gaussian random measure* with control ν is a collection $G = \{G(B) : B \in \mathcal{Z}_\nu\}$ of jointly Gaussian random variables, centered and such that, for every $B, C \in \mathcal{Z}_\nu$, $\mathbb{E}[G(C) G(B)] = \nu(C \cap B)$. The family G is clearly a completely random measure. The fact that G is also good is classic, and can be seen as a special case of the main results in [18].

(ii) A *compensated Poisson measure with control* ν is a completely random measure $\hat{N} = \{\hat{N}(B) : B \in \mathcal{Z}_\nu\}$, as in Definition 2, such that, $\forall B \in \mathcal{Z}_\nu$, $\hat{N}(B) \stackrel{law}{=} N(B) - \nu(B)$, where $N(B)$ is a Poisson random variable with parameter $\nu(B) = \mathbb{E}N(B) = \mathbb{E}N(B)^2$. The fact that $\hat{N}$ is also good derives once again from the main findings of [18]. A more direct proof of this last fact can be obtained by observing that, for almost every ω, $\hat{N}^{[n]}(\cdot, \omega)$ must necessarily coincide with the canonical product (signed) measure (on $(Z^n, \mathcal{Z}^n)$) associated with the signed measure on $(Z, \mathcal{Z})$ given by $\hat{N}(\cdot, \omega) = N(\cdot, \omega) - \nu(\cdot)$ (indeed, such a canonical product measure satisfies necessarily (5.7)). Note that a direct proof of this type cannot be obtained in the Gaussian case. Indeed, if G is a Gaussian measure as in Point (i), one has that, for almost every ω, the mapping $B \mapsto G(B, \omega)$ *does not define* a signed measure (see e.g. [36, Ch. 1]).

5.2 Single integrals and infinite divisibility

Let φ be a completely random measure in the sense of Definition 2, with control measure ν. Our aim in this section is twofolds: (i) we shall define (single) Wiener-Itô integrals with respect to φ, and (ii) we shall give a characterization of these integrals as infinitely divisible random variables.

The fact that single Wiener-Itô integrals are infinitely divisible should not come as a surprise. Indeed, observe that, since $(Z, \mathcal{Z})$ is a Polish space and ν is non-atomic, the law of any random variable of the type $\varphi(B)$, $B \in \mathcal{Z}_\nu$, is infinitely divisible. Infinitely divisible laws are introduced

in many textbooks, see e.g. Billingsley [7]. In particular, for every $B \in \mathcal{Z}_\nu$ there exists a unique pair $\left(c^2(B), \alpha_B\right)$ such that $c^2(B) \in [0, \infty)$ and α_B is a measure on $\mathbb{R}$ satisfying

$$\alpha_B(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} u^2 \alpha_B(du) < \infty, \tag{5.9}$$

and, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(i\lambda\varphi(B)\right)\right] = \exp\left[-\frac{c^2(B)\lambda^2}{2} + \int_{\mathbb{R}} \left(\exp(i\lambda u) - 1 - i\lambda u\right) \alpha_B(du)\right]. \tag{5.10}$$

The measure α_B is called a *Lévy measure*, and the components of the pair $(c^2(B), \alpha_B)$ are called the *Lévy-Khintchine exponent characteristics* associated with $\varphi(B)$. Also, the exponent

$$-\frac{c^2(B)\lambda^2}{2} + \int_{\mathbb{R}} \left(\exp(i\lambda u) - 1 - i\lambda u\right) \alpha_B(du)$$

is known as the *Lévy-Khintchine exponent* associated with $\varphi(B)$. Plainly, if φ is Gaussian, then $\alpha_B = 0$ for every $B \in \mathcal{Z}_\nu$ (the reader is referred e.g. to [110] for an exhaustive discussion of infinitely divisible laws).

We now establish the existence of single Wiener-Itô integrals with respect to a completely random measure φ.

Proposition 5.1 *Let φ be a completely random measure in $L^2(\mathbb{P})$, with σ-finite control measure ν. Then, there exists a unique continuous linear operator $h \mapsto \varphi(h)$, from $L^2(\nu)$ into $L^2(\mathbb{P})$, such that*

$$\varphi(h) = \sum_{j=1}^{m} c_j \varphi(B_j) \tag{5.11}$$

for every elementary function of the type

$$h(z) = \sum_{j=1}^{m} c_j \mathbf{1}_{B_j}(z), \tag{5.12}$$

where $c_j \in \mathbb{R}$ and the sets B_j are in $\mathcal{Z}_\nu$ and disjoint.

Proof. In what follows, we call *simple kernel* a kernel h as in (5.12). For every simple kernel h, set $\varphi(h)$ to be equal to (5.11). Then, by using Properties (iii), (v) and (vi) in Definition 2, one has that, for every pair of simple kernels h, h',

$$\mathbb{E}\left[\varphi(h)\,\varphi\left(h'\right)\right] = \int_Z h(z)\, h'(z)\, \nu(dz). \tag{5.13}$$

Since simple kernels are dense in $L^2(\nu)$, the proof is completed by the following (standard) approximation argument. If $h \in L^2(\nu)$ and $\{h_n\}$ is a sequence of simple kernels converging to h, then (5.13) implies that $\{\varphi(h_n)\}$ is a Cauchy sequence in $L^2(\mathbb{P})$, and one defines $\varphi(h)$ to be the $L^2(\mathbb{P})$ limit of $\varphi(h_n)$. One easily verifies that the definition of $\varphi(h)$ does not depend on the chosen approximating sequence $\{h_n\}$. The application $h \mapsto \varphi(h)$ is therefore well-defined, and (by virtue of (5.13)) it is an isomorphism from $L^2(\nu)$ into $L^2(\mathbb{P})$. ■

The random variable $\varphi(h)$ is usually written as

$$\int_Z h(z)\,\varphi(dz), \quad \int_Z h d\varphi \quad \text{or} \quad I_1^{\varphi}(h), \tag{5.14}$$

and it is called the *Wiener-Itô stochastic integral* of h with respect to φ. By inspection of the previous proof, one sees that Wiener-Itô integrals verify the isometric relation

$$\boxed{\mathbb{E}[\varphi(h)\,\varphi(g)] = \int_Z h(z)\,g(z)\,\nu(dz) = (g,h)_{L^2(\nu)}, \quad \forall g,h \in L^2(\nu).} \tag{5.15}$$

Observe also that $\boxed{\mathbb{E}\varphi(h) = 0.}$ If $B \in \mathcal{Z}_\nu$, we write interchangeably $\varphi(B)$ or $\varphi(\mathbf{1}_B)$ (the two objects coincide, thanks to (5.11)). For every $h \in L^2(\nu)$, the law of the random variable $\varphi(h)$ is also infinitely divisible. The following result provides a description of the Lévy-Khintchine exponent of $\varphi(h)$. The proof is taken from [88] and uses arguments and techniques developed in [103] (see also [43, Section 5]). Following the proof, we present an interpretation of the result.

Proposition 5.2 *For every $B \in \mathcal{Z}_\nu$, let $\left(c^2(B), \alpha_B\right)$ denote the pair such that $c^2(B) \in [0,\infty)$, α_B verifies (5.9) and*

$$\mathbb{E}[\exp(i\lambda\varphi(B))] = \exp\left[-\frac{c^2(B)\,\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x)\,\alpha_B(dx)\right]. \tag{5.16}$$

Then, the following holds

1. *The application $B \mapsto c^2(B)$, from $\mathcal{Z}_\nu$ to $[0,\infty)$, extends to a unique σ-finite measure $c^2(dz)$ on $(Z,\mathcal{Z})$, such that $c^2(dz) \ll \nu(dz)$.*

2. *There exists a unique measure α on $(Z\times\mathbb{R}, \mathcal{Z}\times\mathcal{B}(\mathbb{R}))$ such that $\alpha(B\times C) = \alpha_B(C)$, for every $B \in \mathcal{Z}_\nu$ and $C \in \mathcal{B}(\mathbb{R})$.*

3. *There exists a function $\rho_\nu : Z\times\mathcal{B}(\mathbb{R}) \mapsto [0,\infty]$ such that (i) for every $z \in Z$, $\rho_\nu(z,\cdot)$ is a Lévy measure[1] on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ satisfying $\int_Z x^2\rho_\nu(z,dx) < \infty$, (ii) for every $C \in \mathcal{B}(\mathbb{R})$, $\rho_\nu(\cdot,C)$ is a Borel measurable function, (iii) for every positive function $g(z,x) \in \mathcal{Z}\otimes\mathcal{B}(\mathbb{R})$,*

$$\int_Z\int_{\mathbb{R}} g(z,x)\,\rho_\nu(z,dx)\,\nu(dz) = \int_Z\int_{\mathbb{R}} g(z,x)\,\alpha(dz,dx). \tag{5.17}$$

4. *For every $(\lambda,z) \in \mathbb{R}\times Z$, define*

$$K_\nu(\lambda,z) = -\frac{\lambda^2}{2}\sigma_\nu^2(z) + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right)\rho_\nu(z,dx), \tag{5.18}$$

where $\sigma_\nu^2(z) = \frac{dc^2}{d\nu}(z)$; then, for every $h \in L^2(\nu)$, $\int_Z |K_\nu(\lambda h(z),z)|\,\nu(dz) < \infty$ and

$$\begin{aligned} & \mathbb{E}[\exp(i\lambda\varphi(h))] \qquad (5.19)\\ = \;& \exp\left[\int_Z K_\nu(\lambda h(z),z)\,\nu(dz)\right] \\ = \;& \exp\left[-\frac{\lambda^2}{2}\int_Z h^2(z)\,\sigma_\nu^2(z)\,\nu(dz) + \int_Z\int_{\mathbb{R}} \left(e^{i\lambda h(z)x} - 1 - i\lambda h(z)\,x\right)\rho_\nu(z,dx)\,\nu(dz)\right]. \end{aligned}$$

[1] That is, $\rho_\nu(z,\{0\}) = 0$ and $\int_{\mathbb{R}} \min\left(1,x^2\right)\rho_\nu(z,dx) < \infty$.

Proof. The proof follows from results contained in [103, Section II]. Point 1 is indeed a direct consequence of [103, Proposition 2.1 (a)]. In particular, whenever $B \in \mathcal{Z}$ is such that $\nu(B) = 0$, then $\mathbb{E}[\varphi(B)^2] = 0$ (due to Point (vi) of Definition 2) and therefore $c^2(B) = 0$, thus implying $c^2 \ll \nu$. Point 2 follows from the first part of the statement of [103, Lemma 2.3]. To establish Point 3 define, as in [103, p. 456],

$$\gamma(B) = c^2(B) + \int_{\mathbb{R}} \min\left(1, x^2\right) \alpha_B(dx) = c^2(B) + \int_{\mathbb{R}} \min\left(1, x^2\right) \alpha(B, dx),$$

whenever $B \in \mathcal{Z}_\nu$, and observe (see [103, Definition 2.2]) that $\gamma(\cdot)$ can be canonically extended to a σ-finite and positive measure on $(Z, \mathcal{Z})$. Moreover, since $\nu(B) = 0$ implies $\varphi(B) = 0$ a.s.-$\mathbb{P}$, the uniqueness of the Lévy-Khinchine characteristics implies as before $\gamma(B) = 0$, and therefore $\gamma(dz) \ll \nu(dz)$. Observe also that, by standard arguments, one can select a version of the density $(d\gamma/d\nu)(z)$ such that $(d\gamma/d\nu)(z) < \infty$ for every $z \in Z$. According to [103, Lemma 2.3], there exists a function $\rho : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, \infty]$, such that: (a) $\rho(z, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ for every $z \in Z$, (b) $\rho(\cdot, C)$ is a Borel measurable function for every $C \in \mathcal{B}(\mathbb{R})$, (c) for every positive function $g(z, x) \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$,

$$\int_Z \int_{\mathbb{R}} g(z, x) \rho(z, dx) \gamma(dz) = \int_Z \int_{\mathbb{R}} g(z, x) \alpha(dz, dx). \tag{5.20}$$

In particular, by using (5.20) in the case $g(z, x) = \mathbf{1}_A(z) x^2$ for $A \in \mathcal{Z}_\mu$,

$$\int_A \int_{\mathbb{R}} x^2 \rho(z, dx) \gamma(dz) = \int_{\mathbb{R}} x^2 \alpha_A(dx) < \infty,$$

since $\varphi(A) \in L^2(\mathbb{P})$, and we deduce that ρ can be chosen in such a way that, for every $z \in Z$, $\int_{\mathbb{R}} x^2 \rho(z, dx) < \infty$. Now define, for every $z \in Z$ and $C \in \mathcal{B}(\mathbb{R})$,

$$\rho_\nu(z, C) = \frac{d\gamma}{d\nu}(z) \rho(z, C),$$

and observe that, due to the previous discussion, the application $\rho_\nu : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, \infty]$ trivially satisfies properties (i)-(iii) in the statement of Point 3, which is therefore proved. To prove Point 4, first define (as before) a function $h \in L^2(\nu)$ to be *simple* if $h(z) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(z)$, where $a_i \in \mathbb{R}$, and $(A_1, ..., A_n)$ is a finite collection of disjoint elements of $\mathcal{Z}_\nu$. Of course, the class of simple functions (which is a linear space) is dense in $L^2(\nu)$, and therefore for every $L^2(\nu)$ there exists a sequence h_n, $n \geq 1$, of simple functions such that $\int_Z (h_n(z) - h(z))^2 \nu(dz) \to 0$. As a consequence, since ν is σ-finite there exists a subsequence n_k such that $h_{n_k}(z) \to h(z)$ for ν-a.e. $z \in Z$ (and therefore for γ-a.e. $z \in Z$) and moreover, for every $A \in \mathcal{Z}$, the random sequence $\varphi(\mathbf{1}_A h_n)$ is a Cauchy sequence in $L^2(\mathbb{P})$, and hence it converges in probability. In the terminology of [103, p. 460], this implies that every $h \in L^2(\nu)$ is φ-integrable, and that, for every $A \in \mathcal{Z}$, the random variable $\varphi(h\mathbf{1}_A)$, defined according to Proposition 5.1, coincides with $\int_A h(z) \varphi(dz)$, i.e. the integral of h with respect to the restriction of $\varphi(\cdot)$ to A, as defined in [103, p. 460]. As a consequence, by using a slight modification of [103, Proposition 2.6][2], the function K_0 on $\mathbb{R} \times Z$ given by

$$K_0(\lambda, z) = -\frac{\lambda^2}{2}\sigma_0^2(z) + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \rho(z, dx),$$

[2]The difference lies in the choice of the truncation.

where $\sigma_0^2(z) = \left(dc^2/d\gamma\right)(z)$, is such that $\int_Z |K_0(\lambda h(z), z)|\,\gamma(dz) < \infty$ for every $h \in L^2(\nu)$, and also

$$\mathbb{E}\left[\exp\left(i\lambda\varphi(h)\right)\right] = \int_Z K_0(\lambda h(z), z)\,\gamma(dz).$$

The fact that, by definition, K_ν in (5.18) verifies

$$K_\nu(\lambda h(z), z) = K_0(\lambda h(z), z)\frac{d\gamma}{d\nu}(z), \quad \forall z \in Z, \forall h \in L^2(\nu), \forall \lambda \in \mathbb{R},$$

yields (5.19). ■

Interpretation of Proposition 5.2. Let B be a given set in $\mathcal{Z}_\nu$. The characteristic function of the random variable $\varphi(B)$ or $\varphi(\mathbf{1}_B)$ involves the Lévy characteristic $(c^2(B), \alpha_B)$, where $c^2(B)$ is a non-negative constant and $\alpha_B(dx)$ is a Lévy measure on $\mathbb{R}$. We want now to view $B \in \mathcal{Z}_\nu$ as a "variable" and thus to extend $c^2(B)$ to a measure $c^2(dz)$ on $(Z, \mathcal{Z})$, and $\alpha_B(dx)$ to a measure $\alpha(dz, dx)$ on $\mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$. Consider first α_B. According to Proposition 5.2, it is possible to extend it to a measure $\alpha(dz, dx)$ on $\mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$, which can be expressed as

$$\alpha(dz, dx) = \rho_\nu(z, dx)\nu(dz), \tag{5.21}$$

where ρ_ν is a function on $Z \times \mathbb{R}$, with the property that $\rho_\nu(z, \cdot)$ is a Lévy measure for every $z \in Z$. In view of (5.21), the measure $\alpha(dz, dx)$ is thus obtained as a "mixture" of the Lévy measures $\rho_\nu(z, \cdot)$ over the variable z, using the control measure ν as a mixing measure. A similar approach is applied to the Gaussian part of the exponent in (5.16), involving $c^2(B)$. The coefficient $c^2(B)$ can be extended to a measure $c^2(dz)$, and this measure can be moreover expressed as

$$c^2(dz) = \sigma_\nu^2(z)\nu(dz), \tag{5.22}$$

where σ_ν^2 is the density of c^2 with respect to ν. This allows the to represent the characteristic function of the Wiener-Itô integral $\varphi(h)$ as in (5.19). In that expression, the function $h(z)$ appears explicitly in the Lévy-Khinchine exponent as a factor to the argument λ of the characteristic function.

Examples. (i) If $\varphi = G$ is a centered Gaussian measure with control measure ν, then $\alpha = 0$ and $c^2 = \nu$ (therefore $\sigma_\nu^2 = 1$) and, for $h \in L^2(\nu)$,

$$\mathbb{E}\left[\exp\left(i\lambda G(h)\right)\right] = \exp\left[-\frac{\lambda^2}{2}\int_Z h^2(z)\,\nu(dz)\right].$$

(ii) If $\varphi = \hat{N}$ is a compensated Poisson measure with control measure ν, then $c^2(\cdot) = 0$ and $\rho_\nu(z, dx) = \delta_1(dx)$ for all $z \in Z$, where δ_1 is the Dirac mass at $x = 1$. It follows that, for $h \in L^2(\nu)$,

$$\mathbb{E}\left[\exp\left(i\lambda\hat{N}(h)\right)\right] = \int_Z \left(e^{i\lambda h(z)} - 1 - i\lambda h(z)\right)\nu(dz).$$

(iii) Let $(Z, \mathcal{Z})$ be a measurable space, and let $\hat{N}$ be a centered Poisson random measure on $\mathbb{R}\times Z$ (endowed with the usual product σ-field) with σ-finite control measure $\nu(du, dz)$. Define the measure μ on $(Z, \mathcal{Z})$ by

$$\mu(B) = \int_{\mathbb{R}}\int_Z u^2\mathbf{1}_B(z)\,\nu(du, dz).$$

Then, by setting $k_B(u,z) = u\mathbf{1}_B(z)$, the mapping

$$B \mapsto \varphi(B) = \int_{\mathbb{R}}\int_Z k_B(u,z)\,\hat{N}(du,dz) = \int_{\mathbb{R}}\int_Z u\mathbf{1}_B(z)\,\hat{N}(du,dz), \tag{5.23}$$

where $B \in \mathcal{Z}_\mu = \{B \in \mathcal{Z} : \mu(B) < \infty\}$, is a completely random measure on $(Z,\mathcal{Z})$, with control measure μ. In particular, by setting $k_B(u,z) = u\mathbf{1}_B(z)$, one has that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\lambda\varphi(B)\right)\right] &= \mathbb{E}\left[\exp\left(i\lambda\hat{N}(k_B)\right)\right] \\
&= \exp\left[\int_{\mathbb{R}}\int_Z \left(e^{i\lambda k_B(u,z)} - 1 - i\lambda k_B(u,z)\right)\nu(du,dz)\right] \\
&= \exp\left[\int_{\mathbb{R}}\int_Z \left(e^{i\lambda u\mathbf{1}_B(z)} - 1 - i\lambda u\mathbf{1}_B(z)\right)\nu(du,dz)\right] \\
&= \exp\left[\int_{\mathbb{R}}\int_Z \left(e^{i\lambda u} - 1 - i\lambda u\right)\mathbf{1}_B(z)\,\nu(du,dz)\right] \\
&= \exp\left[\int_{\mathbb{R}} \left(e^{i\lambda u} - 1 - i\lambda u\right)\alpha_B(du)\right],
\end{aligned} \tag{5.24}$$

where $\alpha_B(du) = \int_Z \mathbf{1}_B(z)\,\nu(du,dz)$ (compare with (5.10)).

(iv) Keep the framework of Point (iii). When the measure ν is a product measure of the type $\nu(du,dx) = \rho(du)\,\beta(dx)$, where β is σ-finite and $\rho(du)$ verifies $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} u^2\rho(du) < \infty$ (and therefore $\alpha_B(du) = \beta(B)\,\rho(du)$), one says that the completely random measure φ in (5.23) is *homogeneous* (see e.g. [85]). In particular, for a homogeneous measure φ, relation (5.24) gives

$$\mathbb{E}\left[\exp\left(i\lambda\varphi(B)\right)\right] = \exp\left[\beta(B)\int_{\mathbb{R}}\left(e^{i\lambda u} - 1 - i\lambda u\right)\rho(du)\right]. \tag{5.25}$$

(v) From (5.25) and the classic results on infinitely divisible random variables (see e.g. [110]), one deduces that a centered and square-integrable random variable Y is infinitely divisible if and only if the following holds: there exists a homogeneous completely random measure φ on some space $(Z,\mathcal{Z})$, as well as an independent centered standard Gaussian random variable G, such that

$$Y \overset{law}{=} aG + \varphi(B), \text{ for some } a \geq 0 \text{ and } B \in \mathcal{Z}.$$

(vi) Let the framework and notation of the previous Point (iii) prevail, and assume moreover that: (1) $(Z,\mathcal{Z}) = ([0,\infty),\mathcal{B}([0,\infty)))$, and (2) $\nu(du,dx) = \rho(du)\,dx$, where dx stands for the restriction of the Lebesgue measure on $[0,\infty)$, and ρ verifies $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} u^2\rho(du) < \infty$. Then, the process

$$t \mapsto \varphi([0,t]) = \int_{\mathbb{R}}\int_{[0,t]} u\hat{N}(du,dz), \quad t \geq 0, \tag{5.26}$$

is a *centered and square-integrable Lévy process* (with no Gaussian component) started from zero: in particular, the stochastic process $t \mapsto \varphi([0,t])$ has independent and stationary increments.

(vii) Conversely, every centered and square-integrable Lévy process Z_t with no Gaussian component is such that $Z_t \overset{law}{=} \varphi([0,t])$ (in the sense of stochastic processes) for some $\varphi([0,t])$ defined as in (5.26). To see this, just use the fact that, for every t,

$$\mathbb{E}\left[\exp\left(i\lambda Z_t\right)\right] = \exp\left[t\int_{\mathbb{R}}\left(e^{i\lambda u} - 1 - i\lambda u\right)\rho(du)\right],$$

where the Lévy measure verifies $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} u^2 \rho(du) < \infty$, and observe that this last relation implies that $\varphi([0,t])$ and Z_t have the same finite-dimensional distributions. This fact is the starting point of the paper by Farras *et al.* [19], concerning Hu-Meyer formulae for Lévy processes.

Remark. Let $(Z, \mathcal{Z})$ be a measurable space. Point 4 in Proposition 5.2 implies that every centered completely random measure φ on $(Z, \mathcal{Z})$ has the same law as a random mapping of the type

$$B \mapsto G(B) + \int_{\mathbb{R}} \int_{Z} u \mathbf{1}_B(z) \hat{N}(du, dz),$$

where G and $\hat{N}$ are, respectively, a Gaussian measure on Z and an independent compensated Poisson measure on $\mathbb{R} \times Z$.

Examples. (i) (*Gamma random measures*) Let $(Z, \mathcal{Z})$ be a measurable space, and let $\hat{N}$ be a centered Poisson random measure on $\mathbb{R} \times Z$ with σ-finite control measure

$$\nu(du, dz) = \frac{\exp(-u)}{u} \mathbf{1}_{u>0} du \beta(dz),$$

where $\beta(dz)$ is a σ-finite measure on $(Z, \mathcal{Z})$. Now define the completely random measure φ according to (5.23). By using (5.24) and the fact that

$$\alpha_B(du) = \beta(B) \frac{\exp(-u)}{u} \mathbf{1}_{u>0} du,$$

one infers that, for every $B \in \mathcal{Z}$ such that $\beta(B) < \infty$ and every real λ,

$$\begin{aligned} \mathbb{E}[\exp(i\lambda\varphi(B))] &= \exp\left[\beta(B) \int_0^\infty \left(e^{i\lambda u} - 1 - i\lambda u\right) \frac{\exp(-u)}{u} du\right] \\ &= \exp\left[\beta(B) \int_0^\infty \left(e^{i\lambda u} - 1\right) \frac{\exp(-u)}{u} du\right] \exp(-i\lambda\beta(B)) \\ &= \frac{1}{(1-i\lambda)^{\beta(B)}} \exp(-i\lambda\beta(B)), \end{aligned}$$

thus yielding that $\varphi(B)$ is a centered Gamma random variable, with unitary scale parameter and shape parameter $\beta(B)$. The completely random measure $\varphi(B)$ has control measure β, and it is called a (centered) *Gamma random measure*. Note that $\varphi(B) + \beta(B) > 0$, a.s.-$\mathbb{P}$, whenever $0 < \beta(B) < \infty$. See e.g. [27, 28, 35, 125], and the references therein, for recent results on (non-centered) Gamma random measures.

(ii) (*Dirichlet processes*) Let the notation and assumptions of the previous example prevail, and assume that $0 < \beta(Z) < \infty$ (that is, β is non-zero and finite). Then, $\varphi(Z) + \beta(Z) > 0$ and the mapping

$$B \longmapsto \frac{\varphi(B) + \beta(B)}{\varphi(Z) + \beta(Z)} \tag{5.27}$$

defines a random probability measure on $(Z, \mathcal{Z})$, known as *Dirichlet process with parameter* β. Since the groundbreaking paper by Ferguson [21], Dirichlet processes play a fundamental role in Bayesian non-parametric statistics: see e.g. [34, 58] and the references therein. Note that (5.27) *does not* define a completely random measure (the independence over disjoint sets fails):

however, as shown in [84], one can develop a theory of (multiple) stochastic integration with respect to general Dirichlet processes, by using some appropriate approximations in terms of orthogonal U-statistics. See [97] for a state of the art overview of Dirichlet processes in modern probability.

5.3 Multiple stochastic integrals of elementary functions

We now fix a good completely random measure φ, in the sense of Definition 3 of Section 5.1, and consider what happens when $\varphi^{[n]}$ is applied not to $C \in \mathcal{Z}_\nu^n$ but to its restriction C_π, where π is a partition of $[n] = \{1, ..., n\}$. The set C_π is defined according to (5.1). We shall also apply $\varphi^{[n]}$ to the union $\cup_{\sigma \geq \pi} C_\sigma$.[3] It will be convenient to express the result in terms of C, and thus to view $\varphi^{[n]}(C_\pi)$, for example, not as the map $\varphi^{[n]}$ applied to C_π, but as a suitably restricted map applied to C. This restricted map will be denoted $\mathrm{St}_\pi^{\varphi,[n]}$, where "St" stands for "Stochastic". In this way, the restriction is embodied in the map, that is, the measure, rather than in the set.

Thus, fix a good completely random measure φ, as well as an integer $n \geq 2$.

Definition 4 *For every $\pi \in \mathcal{P}([n])$, we define the two random measures:*[4]

$$\boxed{\mathrm{St}_\pi^{\varphi,[n]}(C) \triangleq \varphi^{[n]}(C_\pi),} \quad C \in \mathcal{Z}_\nu^n, \tag{5.28}$$

and

$$\boxed{\mathrm{St}_{\geq\pi}^{\varphi,[n]}(C) \triangleq \varphi^{[n]}(\cup_{\sigma\geq\pi} C_\sigma) = \textstyle\sum_{\sigma\geq\pi} \mathrm{St}_\sigma^{\varphi,[n]}(C),} \quad C \in \mathcal{Z}_\nu^n, \tag{5.29}$$

that are the restrictions of $\varphi^{[n]}$, respectively to the sets Z_π^n and $\cup_{\sigma\geq\pi} Z_\sigma^n$.

In particular, one has the following relations:

- $\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]} = \varphi^{[n]}$, because the subscript " $\geq \hat{0}$ " involves no restriction. Hence, $\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}$ charges the whole space, and therefore coincides with $\varphi^{[n]}$ (see also Lemma 5.1);
- $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$ does not charge diagonals;
- $\mathrm{St}_{\hat{1}}^{\varphi,[n]}$ charges only the full diagonal set $Z_{\hat{1}}^n$;
- for every $\sigma \in \mathcal{P}([n])$ and every $C \in \mathcal{Z}_\nu^n$, $\mathrm{St}_{\geq\sigma}^{\varphi,[n]}(C) = \mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}(C \cap Z_\sigma^n)$.

We also set

$$\mathrm{St}_{\hat{1}}^{\varphi,[1]}(C) = \mathrm{St}_{\hat{0}}^{\varphi,[1]}(C) = \varphi(C), \quad C \in \mathcal{Z}_\nu. \tag{5.30}$$

Observe that (5.30) is consistent with the trivial fact that the class $\mathcal{P}([1])$ contains uniquely the trivial partition $\{\{1\}\}$, so that, in this case, $\hat{1} = \hat{0} = \{\{1\}\}$.

[3] From here, and for the rest of the paper (for instance, in formula (5.29)), the expressions "$\sigma \geq \pi$" and "$\pi \leq \sigma$" are used interchangeably.

[4] Here, we use a slight variation of the notation introduced by Rota and Wallstrom in [106]. In particular, Rota and Wallstrom write $\mathrm{St}_\pi^{[n]}$ and $\varphi_\pi^{[n]}$, respectively, instead of $\mathrm{St}_\pi^{\varphi,[n]}$ and $\mathrm{St}_{\geq\pi}^{\varphi,[n]}$.

We now define the class $\mathcal{E}(\nu^n)$ of *elementary functions* on Z^n. This is the collection of all functions of the type

$$f(\mathbf{z}_n) = \sum_{j=1}^{m} k_j \mathbf{1}_{C_j}(\mathbf{z}_n), \tag{5.31}$$

where $k_j \in \mathbb{R}$ and every $C_j \in \mathcal{Z}_\nu^n$ has the form $C_j = C_j^1 \times \cdots \times C_j^n$, $C_j^\ell \in \mathcal{Z}_\nu$ $(\ell = 1, ..., n)$. For every $f \in \mathcal{E}(\nu^n)$ as above, we set

$$\mathrm{St}_\pi^{\varphi,[n]}(f) = \int_{Z^n} f d\mathrm{St}_\pi^{\varphi,[n]} = \sum_{j=1}^{m} k_j \mathrm{St}_\pi^{\varphi,[n]}(C_j) \tag{5.32}$$

$$\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f) = \int_{Z^n} f d\mathrm{St}_{\geq\pi}^{\varphi,[n]} = \sum_{j=1}^{m} k_j \mathrm{St}_{\geq\pi}^{\varphi,[n]}(C_j), \tag{5.33}$$

and we say that $\mathrm{St}_\pi^{\varphi,[n]}(f)$ (resp. $\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f)$) is the *stochastic integral* of f with respect to $\mathrm{St}_\pi^{\varphi,[n]}$ (resp. $\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f)$). For $C \in \mathcal{Z}_\nu^n$, we write interchangeably $\mathrm{St}_\pi^{\varphi,[n]}(C)$ and $\mathrm{St}_\pi^{\varphi,[n]}(\mathbf{1}_C)$ (resp. $\mathrm{St}_{\geq\pi}^{\varphi,[n]}(C)$ and $\mathrm{St}_{\geq\pi}^{\varphi,[n]}(\mathbf{1}_C)$). Note that (5.29) yields the relation

$$\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f) = \sum_{\sigma\geq\pi} \mathrm{St}_\sigma^{\varphi,[n]}(f).$$

We can therefore apply the Möbius formula (2.11) in order to deduce the inverse relation

$$\mathrm{St}_\pi^{\varphi,[n]}(f) = \sum_{\sigma\geq\pi} \mu(\pi,\sigma)\, \mathrm{St}_{\geq\sigma}^{\varphi,[n]}(f), \tag{5.34}$$

(see also [106, Proposition 1]).

Remarks. (i) The random variables $\mathrm{St}_\pi^{\varphi,[n]}(f)$ and $\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f)$ are elements of $L^2(\mathbb{P})$ for every $f \in \mathcal{E}(\nu^n)$. While here f is an elementary function, it is neither supposed that f is symmetric nor that it vanishes on the diagonals.

(ii) Because f is elementary, the moments and cumulants of the integrals (5.32) and (5.33) are always defined. They will be computed later via diagram formulae.

5.4 Wiener-Itô stochastic integrals

We consider the extension of the integrals $\mathrm{St}_\pi^{\varphi,[n]}(f)$ to non-elementary functions f in the case $\pi = \hat{0} = \{\{1\}, ..., \{n\}\}$. In view of (5.1) and (5.28), the random measure $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$ does not charge diagonals (see Definition 1, as well as the subsequent examples).

We start with a heuristic presentation. While relation (5.14) involves a simple integral over Z, our goal here is to define integrals over Z^n with respect to $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$, that is, multiple integrals of functions $f : Z^n \mapsto \mathbb{R}$, of the form

$$I_n^\varphi(f) = \int_{Z_{\hat{0}}^n} f(z_1, ..., z_n)\, \varphi(dz_1) \cdots \varphi(dz_n).$$

Since the integration is over $Z_{\hat{0}}^n$ we are excluding diagonals, that is, we are asking that the support of the integrator is restricted to the set of those $(z_1, ..., z_n)$ such that $z_i \neq z_j$ for every

$i \neq j$, $1 \leq i, j \leq n$. To define the multiple integral, we approximate the restriction of f to $Z_{\hat{0}}$ by special elementary functions, namely by finite linear combinations of indicator functions $\mathbf{1}_{C_1 \times \cdots \times C_n}$, where the C_j's are disjoint sets in $\mathcal{Z}_\nu$. This will allow us to define the extension by using isometry, that is, relations of the type

$$\begin{aligned} \mathbb{E}\left[I_n^\varphi(f)^2\right] &= n! \int_{Z^n} f(z_1, ..., z_n)^2 \, \nu(dx_1) \cdots \nu(dx_n) \\ &= n! \int_{Z_{\hat{0}}^n} f(z_1, ..., z_n)^2 \, \nu(dx_1) \cdots \nu(dx_n). \end{aligned} \tag{5.35}$$

Note that the equality (5.35) is due to the fact that the control measure ν is non-atomic, and therefore the associated product measure never charges diagonals. It is enough, moreover, to suppose that f is symmetric, because if

$$\widetilde{f}(z_1, ..., z_n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f\left(z_{w(1)}, ..., z_{w(n)}\right) \tag{5.36}$$

is the canonical symmetrization of f ($\mathfrak{S}_n$ is the group of permutations of $[n]$), then

$$\boxed{I_n^\varphi(f) = I_n^\varphi(\widetilde{f}).} \tag{5.37}$$

This last equality is just a "stochastic equivalent" of the well known fact that integrals with respect to deterministic symmetric measures are invariant with respect to symmetrizations of the integrands. Indeed, an intuitive explanation of (5.37) can be obtained by writing

$$I_n^\varphi(f) = \int_{Z^n} f\left[\mathbf{1}_{Z_{\hat{0}}} d\varphi^{[n]}\right]$$

and by observing that the set $Z_{\hat{0}}$ is symmetric[5], so that $I_n^\varphi(f)$ appears as an integral with respect to the symmetric stochastic measure $\mathbf{1}_{Z_{\hat{0}}} d\varphi^{[n]}$.

From now on, we will denote by $\mathcal{Z}_{s,\nu}^n = \mathcal{Z}_s^n$ (the dependence on ν is dropped, whenever there is no risk of confusion) the *symmetric σ-field* generated by the elements of $\mathcal{Z}_\nu^n$ of the type

$$\widetilde{C} = \bigcup_{w \in \mathfrak{S}_n} C_{w(1)} \times C_{w(2)} \times \cdots \times C_{w(n)}, \tag{5.38}$$

where $\{C_j : j = 1, ..., n\} \subset \mathcal{Z}_\nu$ are *pairwise* disjoint and $\mathfrak{S}_n$ is the group of the permutations of $[n]$.

Remark. One can easily show that $\mathcal{Z}_s^n$ is the σ-field generated by the symmetric functions on Z^n that are square-integrable with respect to ν^n, vanishing on every set Z_π^n such that $\pi \neq \hat{0}$, that is, on all diagonals of Z^n of the type $z_{i_1} = \cdots = z_{i_j}$, $1 \leq i_1 \leq \cdots \leq i_j \leq n$.

By specializing (5.28)-(5.33) to the case $\pi = \hat{0}$, we obtain an intrinsic characterization of *Wiener-Itô multiple stochastic integrals*, as well as of the concept of *stochastic measure of order* $n \geq 2$. The key is the following result, proved in [106, p. 1268].

[5] That is: $(z_1, ..., z_n) \in Z_{\hat{0}}^n$ implies that $(z_{w(1)}, ..., z_{w(n)}) \in Z_{\hat{0}}^n$ for every $w \in \mathfrak{S}_n$

Proposition 5.3 *Let φ be a good completely random measure.*

(A) For every $f \in \mathcal{E}(\nu^n)$,

$$\mathrm{St}_{\hat{0}}^{\varphi,[n]}(f) = \mathrm{St}_{\hat{0}}^{\varphi,[n]}\left(\widetilde{f}\right),$$

where $\widetilde{f}$ is given in (5.36). In other words, the measure $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$ is symmetric.

(B) The collection $\left\{\mathrm{St}_{\hat{0}}^{\varphi,[n]}(C) : C \in \mathcal{Z}_\nu^n\right\}$ is the unique symmetric random measure on $\mathcal{Z}_\nu^n$ verifying the two properties: (i) $\mathrm{St}_{\hat{0}}^{\varphi,[n]}(C) = 0$ for every $C \in \mathcal{Z}_\nu^n$ such that $C \subset Z_\pi^n$ for some $\pi \neq \hat{0}$, and (ii)

$$\mathrm{St}_{\hat{0}}^{\varphi,[n]}\left(\widetilde{C}\right) = \mathrm{St}_{\hat{0}}^{\varphi,[n]}\left(\mathbf{1}_{\widetilde{C}}\right) = n!\varphi(C_1) \times \varphi(C_2) \times \cdots \times \varphi(C_n), \tag{5.39}$$

for every set $\widetilde{C}$ as in (5.38).

Remark. Note that $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$ is defined on the σ-field $\mathcal{Z}_\nu^n$, which also contains non-symmetric sets. The measure $\mathrm{St}_{\hat{0}}^{\varphi,[n]}$ is "symmetric" in the sense that, for every set $C \in \mathcal{Z}_\nu^n$, the following equality holds: $\mathrm{St}_{\hat{0}}^{\varphi,[n]}(C) = \mathrm{St}_{\hat{0}}^{\varphi,[n]}(C_w)$, a.s.-$\mathbb{P}$, where w is a permutation of the set $[n]$ and

$$C_w = \left\{(z_1, ..., z_n) \in Z^n : \left(z_{w(1)}, ..., z_{w(n)}\right) \in C\right\}.$$

We denote by $L_s^2(\nu^n)$ the Hilbert space of symmetric and square integrable functions on Z^n (with respect to ν^n). We also write $\mathcal{E}_{s,0}(\nu^n)$ to indicate the subset of $L_s^2(\nu^n)$ composed of *elementary functions vanishing on diagonals*, that is, the functions of the type $f = \sum_{j=1}^m k_j \mathbf{1}_{\widetilde{C}_j}$, where $k_j \in \mathbb{R}$ and every $\widetilde{C}_j \subset Z_{\hat{0}}^n$ has the form (5.38). The index 0 in $\mathcal{E}_{s,0}(\nu^n)$ refers to the fact that it is a set of functions which equals 0 on the diagonals. Since ν is non-atomic, and ν^n does not charge diagonals, one easily deduces that $\mathcal{E}_{s,0}(\nu^n)$ is dense in $L_s^2(\nu^n)$. Moreover, the relation (5.39) implies that, $\forall n, m \geq 2$,

$$\mathbb{E}\left[\mathrm{St}_{\hat{0}}^{\varphi,[m]}(f)\,\mathrm{St}_{\hat{0}}^{\varphi,[n]}(g)\right] = \delta_{n,m} \times n! \int_{Z^n} f(\mathbf{z}_n)\, g(\mathbf{z}_n)\, \nu^n(d\mathbf{z}_n), \tag{5.40}$$

$\forall f \in \mathcal{E}_{s,0}(\nu^m)$ and $\forall g \in \mathcal{E}_{s,0}(\nu^n)$, where $\delta_{n,m} = 1$ if $n = m$, and $= 0$ otherwise. This immediately yields that, for every $n \geq 2$, the linear operator $f \mapsto \mathrm{St}_{\hat{0}}^{\varphi,[n]}(f)$, from $\mathcal{E}_{s,0}(\nu^n)$ into $L^2(\mathbb{P})$, can be uniquely extendend to a continuous operator from $L_s^2(\nu^n)$ into $L^2(\mathbb{P})$. It is clear that these extensions also enjoy the orthogonality and isometry properties given by (5.40).

Definition 5 *For every $f \in L_s^2(\nu^n)$, the random variable $\mathrm{St}_{\hat{0}}^{\varphi,[n]}(f)$ is the* ***multiple stochastic Wiener-Itô integral*** *(of order n) of f with respect to φ. We also use the classic notation*

$$\boxed{\mathrm{St}_{\hat{0}}^{\varphi,[n]}(f) = I_n^\varphi(f), \quad f \in L_s^2(\nu^n).} \tag{5.41}$$

Note that

$$\mathbb{E}\left[I_m^\varphi(f)\, I_n^\varphi(g)\right] = \delta_{n,m} \times n! \int_{Z^n} f(\mathbf{z}_n)\, g(\mathbf{z}_n)\, \nu^n(d\mathbf{z}_n), \tag{5.42}$$

$\forall f \in L_s^2(\nu^m)$ *and* $\forall g \in L_s^2(\nu^n)$. *For* $n \geq 2$, *the random measure* $\left\{ \mathrm{St}_{\hat{0}}^{\varphi,[n]}(C) : C \in \mathcal{Z}_\nu^n \right\}$ *is called the* ***stochastic measure of order*** n *associated with* φ. *When* $f \in L^2(\nu^n)$ *(not necessarily symmetric), we set*

$$I_n^\varphi(f) = I_n^\varphi(\widetilde{f}), \tag{5.43}$$

where $\widetilde{f}$ *is the symmetrization of* f *given by (5.36).*

We have supposed so far that $\varphi(C) \in L^p(\mathbb{P})$, $p \geq 3$, for every $C \in \mathcal{Z}_\nu$ (see Definition 2). We shall now suppose that $\varphi(C) \in L^2(\mathbb{P})$, $C \in \mathcal{Z}_\nu$. In this case, the notion of "good measure" introduced in Definition 3 and Proposition 5.3 do not apply since, in this case, $\varphi^{[n]}$ may not exist. Indeed, consider (5.7) for example with $C_1 = ... = C_n \in \mathcal{Z}_\nu$. Then, the quantity

$$\mathbb{E}\left|\varphi^{[n]}(C)\right|^2 = \mathbb{E}\left|\varphi(C_1)\right|^{2n}$$

may be infinite (see also [18]). It follows that, for $n \geq 2$, the multiple Wiener-Itô integral cannot be defined as a multiple integral with respect to the restriction to $Z_{\hat{0}}^n$ of the "full stochastic measure" $\varphi^{[n]}$. Nontheless, one can always do as follows.

Definition 6 *Let* φ *be a completely random measure in* $L^2(\mathbb{P})$ *(and not necessarily in* $L^p(\mathbb{P})$, $p \geq 3$*), with non-atomic control measure* ν. *For* $n \geq 2$, *let*

$$I_n^\varphi(f) = n! \sum_{k=1}^m \gamma_k \times \left\{ \varphi\left(C_1^{(k)}\right) \varphi\left(C_2^{(k)}\right) \cdots \varphi\left(C_n^{(k)}\right) \right\}, \tag{5.44}$$

for every simple function $f \in \sum_{k=1}^p \gamma_k \mathbf{1}_{\widetilde{C}^{(k)}} \in \mathcal{E}_{s,0}(\nu^n)$, *where every* $\widetilde{C}^{(k)}$ *is as in (5.38). It is easily seen that the integrals* $I_n^\varphi(f)$ *defined in (5.44) still verify the* $L^2(\mathbb{P})$ *isometry property (5.42). Since the sets of the type* $\widetilde{C}$ *generate* $\mathcal{Z}_s^n$, *and* ν *is non-atomic, the operator* $I_n^\varphi(\cdot)$ *can be extended to a continuous linear operator from* $L_s^2(\nu^n)$ *into* $L^2(\mathbb{P})$, *such that (5.42) is verified. When* $f \in L^2(\nu^n)$ *(not necessarily symmetric), we set*

$$I_n^\varphi(f) = I_n^\varphi(\widetilde{f}),$$

where $\widetilde{f}$ *is given by (5.36).*

Remark. Of course, if $\varphi \in \cap_{p \geq 1} L^p(\mathbb{P})$ (for example, when φ is a Gaussian measure or a compensated Poisson measure), then the definition of I_n^φ obtained from (5.44) coincides with the one given in (5.41).

5.5 Integral notation

The following "integral notation" is somewhat cumbersome but quite suggestive: for every $n \geq 2$, every $\sigma \in \mathcal{P}([n])$ and every elementary function $f \in \mathcal{E}(\nu^n)$,

$$\boxed{\mathrm{St}_\sigma^{\varphi,[n]}(f) = \int_{Z_\sigma^n} f(z_1, ..., z_n)\, \varphi(dz_1) \cdots \varphi(dz_n),}$$

and

$$\boxed{\mathrm{St}_{\geq\pi}^{\varphi,[n]}(f)=\int_{\cup_{\sigma\geq\pi}Z_{\sigma}^{n}}f(z_{1},...,z_{n})\,\varphi(dz_{1})\cdots\varphi(dz_{n}).}$$

For instance:

- if $n=2$, $f(z_1,z_2)=f_1(z_1)\,f_2(z_2)$ and $\sigma=\hat{0}=\{\{1\},\{2\}\}$, then

$$I_2(f)=\mathrm{St}_{\sigma}^{\varphi,[2]}(f)=\int_{z_1\neq z_2}f_1(z_1)\,f_2(z_2)\,\varphi(dz_1)\,\varphi(dz_2);$$

- if $n=2$, $f(z_1,z_2)=f_1(z_1)\,f_2(z_2)$ and $\sigma=\hat{1}=\{1,2\}$, then

$$\mathrm{St}_{\hat{1}}^{\varphi,[2]}(f)=\int_{z_1=z_2}f_1(z_1)\,f_2(z_2)\,\varphi(dz_1)\,\varphi(dz_2);$$

- if $n=2$,

$$\begin{aligned}\mathrm{St}_{\geq\hat{0}}^{\varphi,[2]}(f) &= \mathrm{St}_{\{\{1\},\{2\}\}}^{\varphi,[2]}(f)+\mathrm{St}_{\{\{1,2\}\}}^{\varphi,[2]}(f)\\ &= \int_{z_1\neq z_2}f(z_1,z_2)\,\varphi(dz_1)\,\varphi(dz_2)+\int_{z_1=z_2}f(z_1,z_2)\,\varphi(dz_1)\,\varphi(dz_2)\\ &= \int_{Z^2}f(z_1,z_2)\,\varphi(dz_1)\,\varphi(dz_2).\end{aligned}$$

- if $n=3$, $f(z_1,z_2,z_3)=f_1(z_1,z_2)\,f_2(z_3)$ and $\sigma=\{\{1,2\},\{3\}\}$, then

$$\mathrm{St}_{\sigma}^{\varphi,[3]}(f)=\int_{\substack{z_3\neq z_1\\ z_1=z_2}}f_1(z_1,z_2)\,f_2(z_3)\,\varphi(dz_1)\,\varphi(dz_2)\,\varphi(dz_3);$$

- if $n=3$ and $f(z_1,z_2,z_3)=f_1(z_1,z_2)\,f_2(z_3)$ and $\sigma=\hat{1}=\{\{1,2,3\}\}$, then

$$\int_{z_1=z_2=z_3}f_1(z_1,z_2)\,f_2(z_3)\,\varphi(dz_1)\,\varphi(dz_2)\,\varphi(dz_3).$$

5.6 Chaotic representation

When φ is a Gaussian measure or a compensated Poisson measure, multiple stochastic Wiener-Itô integrals play a crucial role, due to the *chaotic representation property* enjoyed by φ. Indeed, when φ is Gaussian or compensated Poisson, one can show that every functional $F(\varphi)\in L^2(\mathbb{P})$ of φ, admits a unique *chaotic (Wiener-Itô) decomposition*

$$F(\varphi)=\mathbb{E}[F(\varphi)]+\sum_{n\geq1}I_n^{\varphi}(f_n),\qquad f_n\in L_s^2(\nu^n),\tag{5.45}$$

where the series converges in $L^2(\mathbb{P})$ (see for instance [16], [36] or [49]), and the kernels $\{f_n\}$ are uniquely determined. Formula (5.45) implies that random variables of the type $I_n^{\varphi}(f_n)$ are the basic "building blocks" of the space of square-integrable functionals of φ. In general, for a completely random measure φ, the Hilbert space $C_n^{\varphi}=\left\{I_n^{\varphi}(f):f\in L_s^2(\nu^n)\right\}$, $n\geq1$, is called the nth *Wiener chaos* associated with φ. We set by definition $C_0^{\varphi}=\mathbb{R}$ (that is, C_0^{φ} is the

collection of all non-random constants) so that, in (5.45), $\mathbb{E}[F] \in C_0^{\varphi}$. Observe that relation (5.45) can be reformulated in terms of Hilbert spaces as follows:

$$L^2(\mathbb{P}, \sigma(\varphi)) = \bigoplus_{n=0}^{\infty} C_n^{\varphi},$$

where $\oplus$ indicates an orthogonal sum in the Hilbert space $L^2(\mathbb{P}, \sigma(\varphi))$.

Remarks. (i) If $\varphi = G$ is a Gaussian measure with non-atomic control ν, for every $p > 2$ and every $n \geq 2$, there exists a universal constant $c_{p,n} > 0$, such that

$$\mathbb{E}\left[\left|I_n^G(f)\right|^p\right]^{1/p} \leq c_{n,p}\mathbb{E}\left[I_n^G(f)^2\right]^{1/2}, \tag{5.46}$$

$\forall\, f \in L_s^2(\nu^n)$ (see [36, Ch. V]). Moreover, on every finite sum of Wiener chaoses $\oplus_{j=0}^m C_j^G$ and for every $p \geq 1$, the topology induced by $L^p(\mathbb{P})$ convergence is equivalent to the L^0-topology induced by convergence in probability, that is, convergence in probability is equivalent to convergence in L^p, for every $p \geq 1$ (see e.g. [112]). We refer the reader to [16], [36] or [75, Ch. 1] for an exhaustive analysis of the properties of multiple stochastic Wiener-Itô integrals with respect to a Gaussian measure G.

(ii) The "chaotic representation property" is enjoyed by other processes. One of the most well-known examples is given by the class of *normal martingales*, that is, real-valued martingales on $\mathbb{R}_+$ having a predictable quadratic variation equal to t. See [16] and [48] for a complete discussion of this point.

When φ is Gaussian or compensated Poisson, one can characterize the measures $\mathrm{St}_{\pi}^{\varphi,[n]}$, when $\pi \neq \hat{0}$, that is, the effect of these measures on diagonals. The key fact is the following elementary identity (corresponding to Proposition 2 in [106]).

Proposition 5.4 *Let φ be a good completely random measure. Then, for every $n \geq 2$, every $C_1, ..., C_n \subset Z_\nu$, and every partition $\pi \in \mathcal{P}([n])$,*

$$\begin{aligned}
&\mathrm{St}_{\geq\pi}^{\varphi,[n]}(C_1 \times \cdots \times C_n) \\
&= \prod_{b=\{i_1,\ldots,i_{|b|}\}\in\pi} \mathrm{St}_{\hat{1}}^{\varphi,[|b|]}\left(C_{i_1} \times \cdots \times C_{i_{|b|}}\right) \qquad (5.47)\\
&= \prod_{b=\{i_1,\ldots,i_{|b|}\}\in\pi} \mathrm{St}_{\hat{1}}^{\varphi,[|b|]}(\underbrace{(\cap_{k=1}^{|b|} C_{i_k}) \times \cdots \times (\cap_{k=1}^{|b|} C_{i_k})}_{|b| \ \ times}). \qquad (5.48)
\end{aligned}$$

Proof. Recall that $\mathrm{St}_{\geq\pi}^{\varphi,[n]} = \sum_{\sigma\geq\pi} \mathrm{St}_{\sigma}^{\varphi,[n]}$, by (5.29). To prove the first equality, just observe that both random measures on the LHS and the RHS of (5.47) are the restriction of the product measure $\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}$ to the union of the sets Z_σ^n such that $\sigma \geq \pi$. Equality (5.48) is an application of (5.2). ■

5.7 Computational rules

We are now going to apply our setup to the Gaussian case ($\varphi = G$) and to the Poisson case ($\varphi = \hat{N} = N - \nu$). We always suppose that the control measure ν of either G or $\hat{N}$ is non-atomic. Many of the subsequent formulae can be understood intuitively, by applying the following computational rules:

Gaussian case:

$$G(dx)^2 = \nu(dx) \text{ and } G(dx)^n = 0, \text{ for every } n \geq 3. \tag{5.49}$$

Poisson case:

$$(\hat{N}(dx))^n = (N(dx))^n = N(dx), \text{ for every } n \geq 2. \tag{5.50}$$

5.8 Multiple Gaussian stochastic integrals of elementary functions

Suppose φ is Gaussian. The next result (whose proof is sketched below) can be deduced from [106, Example G, p. 1272, and Proposition 2, 6 and 12].

Theorem 5.1 *Let $\varphi = G$ be a centered Gaussian completely random measure with non-atomic control measure ν. For every $n \geq 2$ and every $A \in \mathcal{Z}_\nu$*

$$\mathrm{St}_{\hat{1}}^{G,[n]}(\underbrace{A \times \cdots \times A}_{n \text{ times}}) \triangleq \Delta_n^G(A) = \begin{cases} 0 & n \geq 3 \\ \nu(A) & n = 2, \end{cases} \tag{5.51}$$

*(the measure $\Delta_n^G(\cdot)$ is called the **diagonal measure** of order n associated with G). More generally, for every $n \geq 2$, $\sigma \in \mathcal{P}([n])$ and $A_1, ..., A_n \in \mathcal{Z}_\nu$,*

$$\begin{aligned} &\mathrm{St}_{\geq\sigma}^{G,[n]}(A_1 \times \cdots \times A_n) \\ &= \begin{cases} 0, & \text{if } \exists b \in \sigma : |b| \geq 3 \\ \prod_{b=\{i,j\}\in\sigma} \nu(A_i \cap A_j) \prod_{\ell=1}^k G(A_{j_\ell}), & \text{otherwise,} \end{cases} \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} &\mathrm{St}_{\sigma}^{G,[n]}(A_1 \times \cdots \times A_n) \\ &= \begin{cases} 0, & \text{if } \exists b \in \sigma : |b| \geq 3 \\ \prod_{b=\{i,j\}\in\sigma} \nu(A_i \cap A_j)\, \mathrm{St}_{\hat{0}}^{G,[k]}(A_{j_1} \times \cdots \times A_{j_k}), & \text{otherwise,} \end{cases} \end{aligned} \tag{5.53}$$

where $j_1, ..., j_k$ are the singletons contained in $\sigma \setminus \{b \in \sigma : |b| = 2\}$.

Proof. Relation (5.51) is classic (for a proof, see e.g. [106, Proposition 6]). Formula (5.52) is obtained by combining (5.51) and (5.47). To prove (5.53), suppose first that $\exists b \in \sigma$ such that $|b| \geq 3$. Then, by using Möbius inversion (5.34),

$$\mathrm{St}_{\sigma}^{G,[n]}(A_1 \times \cdots \times A_n) = \sum_{\sigma \leq \rho} \mu(\sigma, \rho)\, \mathrm{St}_{\geq\rho}^{G,[n]}(A_1 \times \cdots \times A_n) = 0,$$

where the last equality is due to (5.52) and to the fact that, if $\rho \geq \sigma$ and σ contains a block with more than two elements, then ρ must also contain a block with more than two elements. This

proves the first line of (5.53). Now suppose that all the blocks of σ have at most two elements, and observe that, by Definition 4 and (5.7),

$$\prod_{\ell=1}^{k} G\left(A_{j_\ell}\right) = \mathrm{St}_{\geq \hat{0}}^{G,[k]}(A_{j_1} \times \cdots \times A_{j_k}).$$

The proof is concluded by using the following relations:

$$\begin{aligned}
\mathrm{St}_{\sigma}^{G,[n]}\left(A_1 \times \cdots \times A_n\right) &= \sum_{\sigma \leq \rho} \mu(\sigma,\rho) \mathrm{St}_{\geq \rho}^{G,[n]}\left(A_1 \times \cdots \times A_n\right) \\
&= \prod_{b=\{i,j\}\in\sigma} \nu(A_i \cap A_j) \sum_{\sigma \leq \rho} \mu(\sigma,\rho) \prod_{b=\{r,l\}\in\rho\backslash\sigma} \nu(A_r \cap A_l) \times \\
&\qquad \times \mathrm{St}_{\geq \hat{0}}^{G,[m]}(A_{i_1} \times \cdots \times A_{i_m}) \mathbf{1}_{\{\{i_1\},\ldots,\{i_m\} \text{ are the singletons of } \rho\}},
\end{aligned}$$

where we write $b = \{r,l\} \in \rho\backslash\sigma$ to indicate that the block b is in ρ and not in σ (equivalently, b is obtained by merging two singletons of σ). Indeed, by the previous discussion, one has that the partitions ρ involved in the previous sums have uniquely blocks of size one or two, and moreover, by Möbius inversion,

$$\begin{aligned}
&\sum_{\sigma \leq \rho} \mu(\sigma,\rho) \prod_{b=\{r,l\}\in\rho\backslash\sigma} \nu(A_r \cap A_l) \times \mathrm{St}_{\geq \hat{0}}^{G,[m]}(A_{i_1} \times \cdots \times A_{i_m}) \mathbf{1}_{\{\{i_1\},\ldots,\{i_m\} \text{ are the singletons of } \rho\}} \\
&= \sum_{\sigma^* \leq \rho^*} \mu(\sigma^*,\rho^*) \prod_{b=\{r,l\}\in\rho^*} \nu(A_r \cap A_l) \times \mathrm{St}_{\geq \hat{0}}^{G,[m]}(A_{i_1} \times \cdots \times A_{i_m}) \mathbf{1}_{\{\{i_1\},\ldots,\{i_m\} \text{ are the singletons of } \rho^*\}} \\
&= \sum_{\hat{0} \leq \rho^*} \mu(\hat{0},\rho^*) \mathrm{St}_{\geq \rho^*}^{G,[k]}(A_{j_1} \times \cdots \times A_{j_k}) \\
&= \mathrm{St}_{\hat{0}}^{G,[k]}(A_{j_1} \times \cdots \times A_{j_k}),
\end{aligned}$$

where σ^* and ρ^* indicate, respectively, the restriction of σ and ρ to $\{j_1,...,j_k\}$, where $\{j_1\},...,\{j_k\}$ are the singletons of σ (in particular, $\sigma^* = \hat{0}$). Note that the fact that $\mu(\sigma,\rho) = \mu(\sigma^*,\rho^*) = \mu(\hat{0},\rho^*)$ is a consequence of (2.7) and of the fact that ρ has uniquely blocks of size one or two.[6]
■

Examples. (i) One has $\mathrm{St}_{\geq \hat{0}}^{G,[n]}\left(A_1 \times \cdots \times A_n\right) = G\left(A_1\right) \cdots G\left(A_n\right)$, which follows from (5.52), since $\hat{0} = \{\{1\},...,\{n\}\}$, but also directly since the symbol " $\geq \hat{0}$ " entails no restriction on the partition. In integral notation (f is always supposed to be elementary)

$$\mathrm{St}_{\geq \hat{0}}^{G,[n]}\left(f\right) = \int_{Z^n} f\left(z_1,...,z_n\right) G\left(dz_1\right) \cdots G\left(dz_n\right).$$

On the other hand, there is no way to "simplify" an object such as $\mathrm{St}_{\hat{0}}^{G,[n]}\left(A_1 \times \cdots \times A_n\right)$, wich is expressed, in integral notation, as

$$\mathrm{St}_{\hat{0}}^{G,[n]}\left(f\right) = I_n^G\left(f\right) = \int_{z_1 \neq \cdots \neq z_n} f\left(z_1,...,z_n\right) G\left(dz_1\right) \cdots G\left(dz_n\right).$$

[6] Thanks to F. Benaych-Georges for pointing out this argument.

(ii) For $n \geq 3$, one has

$$\mathrm{St}_{\geq \hat{1}}^{G,[n]}(A_1 \times \cdots \times A_n) = \mathrm{St}_{\hat{1}}^{G,[n]}(A_1 \times \cdots \times A_n) = 0,$$

since the partition $\hat{1}$ contains a single block of size ≥ 2. In integral notation,

$$\mathrm{St}_{\hat{1}}^{G,[n]}(f) = \int_{z_1=\cdots=z_n} f(z_1, ..., z_n)\, G(dz_1) \cdots G(dz_n) = 0.$$

When $n = 1$, however, one has that

$$\mathrm{St}_{\hat{1}}^{G,[1]}(f) = \int_Z f(z)\, G(dz) \sim \mathcal{N}\left(0, \int_Z f^2 d\nu\right).$$

When $n = 2$,

$$\mathrm{St}_{\hat{1}}^{G,[2]}(f) = \int_Z f(z,z)\, \nu(dz).$$

(iii) Let $n = 3$ and $\sigma = \{\{1\}, \{2,3\}\}$, then $\mathrm{St}_{\hat{1}}^{G,[3]}(A_1 \times A_2 \times A_n) = 0$, and therefore

$$\begin{aligned} & \mathrm{St}_{\geq \sigma}^{G,[3]}(A_1 \times A_2 \times A_3) \\ = \; & \mathrm{St}_{\sigma}^{G,[3]}(A_1 \times A_2 \times A_3) + \mathrm{St}_{\hat{1}}^{G,[n]}(A_1 \times A_2 \times A_3) \\ = \; & \mathrm{St}_{\sigma}^{G,[3]}(A_1 \times A_2 \times A_3) = G(A_1)\, \nu(A_2 \times A_3). \end{aligned}$$

In integral notation:

$$\mathrm{St}_{\geq \sigma}^{G,[3]}(f) = \mathrm{St}_{\sigma}^{G,[3]}(f) = \int_Z \int_Z f(z,z,x)\, \nu(dz)\, G(dx).$$

(iv) Let $n = 6$, and $\sigma = \{\{1,2\}, \{3\}, \{4\}, \{5,6\}\}$. Then,

$$\mathrm{St}_{\geq \sigma}^{G,[6]}(A_1 \times ... \times A_6) = \nu(A_1 \cap A_2)\, \nu(A_5 \cap A_6)\, G(A_3)\, G(A_4),$$

whereas

$$\mathrm{St}_{\sigma}^{G,[6]}(A_1 \times ... \times A_6) = \nu(A_1 \cap A_2)\, \nu(A_5 \cap A_6)\, \mathrm{St}_{\hat{0}}^{G,[2]}(A_3 \times A_4).$$

These relations can be reformulated in integral notation as

$$\begin{aligned} & \mathrm{St}_{\geq \sigma}^{G,[6]}(A_1 \times ... \times A_6) \\ = \; & \int_Z \int_Z \left\{ \int_Z \int_Z f(x,x,y,y,w,z)\, \nu(dx)\, \nu(dy) \right\} G(dw)\, G(dz) \\ & \mathrm{St}_{\sigma}^{G,[6]}(A_1 \times ... \times A_6) \\ = \; & \int_{w \neq z} \left\{ \int_Z \int_Z f(x,x,y,y,w,z)\, \nu(dx)\, \nu(dy) \right\} G(dw)\, G(dz). \end{aligned}$$

(v) Let $n = 6$, and $\sigma = \{\{1,2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$. Then,

$$\mathrm{St}_{\geq \sigma}^{G,[6]}(A_1 \times ... \times A_6) = \nu(A_1 \cap A_2)\, G(A_3)\, G(A_4)\, G(A_5)\, G(A_6)$$

and also

$$\mathrm{St}_{\sigma}^{G,[6]}(A_1 \times ... \times A_6) = \nu(A_1 \cap A_2)\, \mathrm{St}_{\hat{0}}^{G,[4]}(A_3 \times A_4 \times A_5 \times A_6).$$

5.9 Multiple stochastic Poisson integrals of elementary functions.

A result analogous to Theorem 5.1 holds in the Poisson case. To state this result in a proper way, we shall introduce the following notation. Given $n \geq 2$ and $\sigma \in \mathcal{P}[n]$, we write

$$\mathbf{B}_1(\sigma) = \{b \in \sigma : |b| = 1\},$$

to denote the collection of singleton blocks of σ, and

$$\mathbf{B}_2(\sigma) = \{b = \{i_1, ..., i_\ell\} \in \sigma : \ell = |b| \geq 2\} \tag{5.54}$$

to denote the collection of the blocks of σ containing *two or more* elements. We shall also denote by $\mathbf{PB}_2(\sigma)$ the set of all 2-partitions of $\mathbf{B}_2(\sigma)$, that is, $\mathbf{PB}_2(\sigma)$ is the collection of all ordered pairs $(R_1; R_2)$ of *non-empty* subsets of $\mathbf{B}_2(\sigma)$ such that $R_1, R_2 \subset \mathbf{B}_2(\sigma)$, $R_1 \cap R_2 = \varnothing$, and $R_1 \cup R_2 = \mathbf{B}_2(\sigma)$; whenever $\mathbf{B}_2(\sigma) = \varnothing$, one sets $\mathbf{PB}_2(\sigma) = \varnothing$. We stress that $\mathbf{PB}_2(\sigma)$ is a partition of $\mathbf{B}_2(\sigma)$; the fact that $\mathbf{B}_2(\sigma)$ is also a subset of the partition σ should not create confusion.

Examples. (i) Let $n = 7$, and $\sigma = \{\{1,2\},\{3,4\},\{5,6\},\{7\}\}$. Then,

$$\mathbf{B}_2(\sigma) = \{\{1,2\},\{3,4\},\{5,6\}\}$$

and $\mathbf{PB}_2(\sigma)$ contains the six ordered pairs:

$$\begin{array}{lcl} (\{\{1,2\},\{3,4\}\};\{\{5,6\}\}) & ; & (\{\{5,6\}\};\{\{1,2\},\{3,4\}\}) \\ (\{\{1,2\},\{5,6\}\};\{\{3,4\}\}) & ; & (\{\{3,4\}\};\{\{1,2\},\{5,6\}\}) \\ (\{\{3,4\},\{5,6\}\};\{\{1,2\}\}) & ; & (\{\{1,2\}\};\{\{3,4\},\{5,6\}\}). \end{array}$$

For instance, the first ordered pair is made up of $R_1 = \{\{1,2\},\{3,4\}\}$ and $R_2 = \{\{5,6\}\}$, whose union is $\mathbf{B}_2(\sigma)$.

(ii) If $n = 5$ and $\sigma = \{\{1,2,3\},\{4\},\{5\}\}$, then $\mathbf{B}_1(\sigma) = \{\{4\},\{5\}\}$, $\mathbf{B}_2(\sigma) = \{\{1,2,3\}\}$ and $\mathbf{PB}_2(\sigma) = \varnothing$.

(iii) If $n = 7$ and $\sigma = \{\{1,2,3\},\{4,5\},\{6\},\{7\}\}$, then $\mathbf{B}_1(\sigma) = \{\{6\},\{7\}\}$ and $\mathbf{B}_2(\sigma) = \{\{1,2,3\},\{4,5\}\}$. Also, the set $\mathbf{PB}_2(\sigma)$ contains the two ordered pairs

$$(\{1,2,3\};\{4,5\}) \quad \text{and} \quad (\{4,5\},\{1,2,3\}).$$

We shall now suppose that φ is a compensated Poisson measure.

Theorem 5.2 *Let $\varphi = \hat{N}$ be a compensated Poisson measure with non-atomic control measure ν, and let $N(\cdot) \triangleq \hat{N}(\cdot) + \nu(\cdot)$. For every $n \geq 2$ and every $A \in \mathcal{Z}_\nu$,*

$$\mathrm{St}_{\hat{1}}^{\hat{N},[n]}(\underbrace{A \times \cdots \times A}_{n \text{ times}}) \triangleq \Delta_n^{\hat{N}}(A) = N(A) \tag{5.55}$$

*($\Delta_n^{\hat{N}}(\cdot)$ is called the **diagonal measure** of order n associated with $\hat{N}$). Moreover, for every $A_1, ..., A_n \in \mathcal{Z}_\nu$,*

$$\mathrm{St}_{\hat{1}}^{\hat{N},[n]}(A_1 \times \cdots \times A_n) = N(A_1 \cap \cdots \cap A_n). \tag{5.56}$$

More generally, for every $n \geq 2$, $\sigma \in \mathcal{P}([n])$ and $A_1, ..., A_n \in \mathcal{Z}_\nu$,

$$\begin{aligned} & \mathrm{St}_{\geq\sigma}^{\hat{N},[n]}(A_1 \times \cdots \times A_n) \qquad (5.57) \\ = & \prod_{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)} N(A_{i_1} \cap \cdots \cap A_{i_\ell}) \prod_{a=1}^{k} \hat{N}(A_{j_a}), \end{aligned}$$

where $\{\{j_1\}, ..., \{j_k\}\} = \mathbf{B}_1(\sigma)$, and also

$$\begin{aligned} & \mathrm{St}_{\sigma}^{\hat{N},[n]}(A_1 \times \cdots \times A_n) \qquad (5.58) \\ = & \sum_{(R_1;R_2)\in\mathbf{PB}_2(\sigma)} \prod_{b=\{i_1,...,i_\ell\}\in R_1} \nu(A_{i_1} \cap \cdots \cap A_{i_\ell}) \times \qquad (5.59) \\ & \times \mathrm{St}_{\hat{0}}^{\hat{N},[|R_2|+k]} \left(\underset{b=\{e_1,...,e_m\}\in R_2}{\mathrm{X}} (A_{e_1} \cap \cdots \cap A_{e_m}) \times A_{j_1} \times \cdots \times A_{j_k} \right) \qquad (5.60) \\ & + \mathrm{St}_{\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]} \left(\underset{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}} (A_{i_1} \cap \cdots \cap A_{i_\ell}) \times A_{j_1} \times \cdots \times A_{j_k} \right) \qquad (5.61) \\ & + \prod_{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)} \nu(A_{i_1} \cap \cdots \cap A_{i_\ell}) \, \mathrm{St}_{\hat{0}}^{\hat{N},[k]}(A_{j_1} \times \cdots \times A_{j_k}), \qquad (5.62) \end{aligned}$$

where $\{\{j_1\}, ..., \{j_k\}\} = \mathbf{B}_1(\sigma)$ and where (by convention) $\Sigma_\varnothing \equiv 0$, $\Pi_\varnothing \equiv 0$ and $\mathrm{St}_{\hat{0}}^{\hat{N},[0]} \equiv 1$. Also, $|R_2|$ and $|\mathbf{B}_2(\sigma)|$ stand, respectively, for the cardinality of R_2 and $\mathbf{B}_2(\sigma)$, and in formula (5.60) we used the notation

$$\underset{b=\{e_1,...,e_m\}\in R_2}{\mathrm{X}} (A_{e_1} \cap \cdots \cap A_{e_m}) = \underset{b\in R_2}{\mathrm{X}} (\cap_{e\in b} A_e) = (\cap_{e\in b_1} A_e) \times \cdots \times (\cap_{e\in b_{|R_2|}} A_e),$$

where $b_1, ..., b_{|R_2|}$ is some enumeration of R_2 (note that, due to the symmetry of $\mathrm{St}_{\hat{0}}^{\hat{N},[|R_2|+k]}$, the choice of the enumeration is immaterial). The summand appearing in formula (5.61) is defined via the same conventions.

Remarks. (a) When writing formula (5.61), we implicitly use the following convention: if $\mathbf{B}_2(\sigma) = \varnothing$, then the symbol $\underset{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}} (A_{i_1} \cap \cdots \cap A_{i_\ell})$ is immaterial, and one should read

$$\underset{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}} (A_{i_1} \cap \cdots \cap A_{i_\ell}) \times A_{j_1} \times \cdots \times A_{j_k} = A_{j_1} \times \cdots \times A_{j_k} = A_1 \times \cdots \times A_n, \quad (5.63)$$

where the last equality follows from the fact that, in this case, $k = n$ and $\{j_1, ..., j_k\} = \{1, ..., n\} = [n]$. To see how the convention (5.63) works, suppose that $\mathbf{B}_2(\sigma) = \varnothing$. Then, $\mathbf{PB}_2(\sigma) = \varnothing$ and consequently, according to the conventions stated in Theorem 5.2, the lines (5.59)–(5.60) and (5.62) are equal to zero (they correspond, respectively, to a sum and a product over the empty set). The equality (5.58) reads therefore

$$\mathrm{St}_{\sigma}^{\hat{N},[n]}(A_1 \times \cdots \times A_n) = \mathrm{St}_{\hat{0}}^{\hat{N},[n]} \left(\underset{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}} (A_{i_1} \cap \cdots \cap A_{i_\ell}) \times A_{j_1} \times \cdots \times A_{j_k} \right). \quad (5.64)$$

By using (5.63) one obtains

$$\begin{aligned} & \mathrm{St}_{\hat{0}}^{\hat{N},[n]}\left(\bigtimes_{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}(A_{i_1}\cap\cdots\cap A_{i_\ell})\times A_{j_1}\times\cdots\times A_{j_k}\right) \\ = \;& \mathrm{St}_{\hat{0}}^{\hat{N},[n]}(A_{j_1}\times\cdots\times A_{j_k}) = \mathrm{St}_{\sigma}^{\hat{N},[n]}(A_1\times\cdots\times A_n), \end{aligned}$$

entailing that, in this case, relation (5.58) is equivalent to the identity $\mathrm{St}_{\hat{0}}^{\hat{N},[n]} = \mathrm{St}_{\hat{0}}^{\hat{N},[n]}$.

(b) If $k=0$ (that is, if $\mathbf{B}_1(\sigma)$ equals the empty set), then, according to the conventions stated in Theorem 5.2, one has $\mathrm{St}_{\hat{0}}^{\hat{N},[k]} = \mathrm{St}_{\hat{0}}^{\hat{N},[0]} = 1$. This yields that, in this case, one should read line (5.62) as follows:

$$\begin{aligned} & \prod_{b=\{i_1,\ldots,i_\ell\}\ \in\mathbf{B}_2(\sigma)}\nu(A_{i_1}\cap\cdots\cap A_{i_\ell})\,\mathrm{St}_{\hat{0}}^{\hat{N},[k]}(A_{j_1}\times\cdots\times A_{j_k}) \\ = \;& \prod_{b=\{i_1,\ldots,i_\ell\}\ \in\mathbf{B}_2(\sigma)}\nu(A_{i_1}\cap\cdots\cap A_{i_\ell}). \end{aligned}$$

(c) If $k=0$, one should also read line (5.60) as

$$\begin{aligned} & \mathrm{St}_{\hat{0}}^{\hat{N},[|R_2|+k]}\left(\bigtimes_{b=\{e_1,\ldots,e_m\}\in R_2}(A_{e_1}\cap\cdots\cap A_{e_m})\times A_{j_1}\times\cdots\times A_{j_k}\right) \\ = \;& \mathrm{St}_{\hat{0}}^{\hat{N},[|R_2|]}\left(\bigtimes_{b=\{e_1,\ldots,e_m\}\in R_2}(A_{e_1}\cap\cdots\cap A_{e_m})\right), \end{aligned}$$

and line (5.61) as

$$\begin{aligned} & \mathrm{St}_{\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]}\left(\bigtimes_{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}(A_{i_1}\cap\cdots\cap A_{i_\ell})\times A_{j_1}\times\cdots\times A_{j_k}\right) \\ = \;& \mathrm{St}_{\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|]}\left(\bigtimes_{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}(A_{i_1}\cap\cdots\cap A_{i_\ell})\right). \end{aligned}$$

Proof of Theorem 5.2. To see that (5.56) must necessarily hold, use the fact that ν is non-atomic by assumption. Therefore,

$$\begin{aligned} & \mathrm{St}_{\hat{1}}^{\hat{N},[n]}(A_1\times\cdots\times A_n) = \mathrm{St}_{\hat{0}}^{\hat{N},[1]}(A_1\cap\cdots\cap A_n) + \nu(A_1\cap\cdots\cap A_n) \\ & = \hat{N}(A_1\cap\cdots\cap A_n) + \nu(A_1\cap\cdots\cap A_n) = N(A_1\cap\cdots\cap A_n). \end{aligned}$$

Observe also that (5.56) implies (5.55). Equation (5.57) is an immediate consequence of (5.47), (5.55) and (5.56). To prove (5.58), use (5.57) and the relation $N=\hat{N}+\nu$, to write

$$\begin{aligned} & \mathrm{St}_{\geq\sigma}^{\hat{N},[n]}(A_1\times\cdots\times A_n) \\ = \;& \prod_{\ell=2}^{n}\prod_{b=\{i_1,\ldots,i_\ell\}\in\sigma}\left[\hat{N}(A_{i_1}\cap\cdots\cap A_{i_\ell}) + \nu(A_{i_1}\cap\cdots\cap A_{i_\ell})\right]\prod_{a=1}^{k}\hat{N}(A_{j_a}), \end{aligned}$$

and the last expression equals

$$\begin{aligned}
&\sum_{(R_1;R_2)\in\mathbf{PB}_2(\sigma)}\prod_{b=\{i_1,\ldots,i_\ell\}\ \in R_1}\nu\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times\\
&\times\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|R_2|+k]}\left(\underset{b=\{e_1,\ldots,e_m\}\in R_2}{\mathrm{X}}\left(A_{e_1}\cap\cdots\cap A_{e_m}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right)\\
&+\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]}\left(\underset{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}}\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right)\\
&+\prod_{b=\{i_1,\ldots,i_\ell\}\ \in\mathbf{B}_2(\sigma)}\nu\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\mathrm{St}_{\geq\hat{0}}^{\hat{N},[k]}\left(A_{j_1}\times\cdots\times A_{j_k}\right),
\end{aligned}$$

since $\mathrm{St}_{\geq\hat{0}}^{\hat{N},[n]}\left(A_1\times\cdots\times A_n\right)=\hat{N}\left(A_1\right)\cdots\hat{N}\left(A_n\right)$. The term before last in the displayed equation corresponds to $R_1=\varnothing$, $R_2=\mathbf{B}_2\left(\sigma\right)$, and the last term to $R_1=\mathbf{B}_2\left(\sigma\right)$ and $R_2=\varnothing$. By definition, these two cases are not involved in $\mathbf{PB}_2\left(\sigma\right)$. The last displayed equation yields

$$\begin{aligned}
&\mathrm{St}_{\sigma}^{\hat{N},[n]}\left(A_1\times\cdots\times A_n\right)=\mathrm{St}_{\geq\sigma}^{\hat{N},[n]}\left(\left(A_1\times\cdots\times A_n\right)\mathbf{1}_{Z_\sigma^n}\right)\\
=&\sum_{(R_1;R_2)\in\mathbf{PB}_2(\sigma)}\prod_{b=\{i_1,\ldots,i_\ell\}\ \in R_1}\nu\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times\\
&\times\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|R_2|+k]}\left(\left[\underset{b=\{e_1,\ldots,e_m\}\in R_2}{\mathrm{X}}\left(A_{e_1}\cap\cdots\cap A_{e_m}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^{|R_2|+k}}\right)\\
&+\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]}\left(\left[\underset{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}}\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^{|\mathbf{B}_2(\sigma)|+k}}\right)\\
&+\prod_{b=\{i_1,\ldots,i_\ell\}\ \in\mathbf{B}_2(\sigma)}\nu\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\mathrm{St}_{\geq\hat{0}}^{\hat{N},[k]}\left(\left[A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^k}\right).
\end{aligned}$$

Since, by definition,

$$\begin{aligned}
&\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|R_2|+k]}\left(\left[\underset{b=\{e_1,\ldots,e_m\}\in R_2}{\mathrm{X}}\left(A_{e_1}\cap\cdots\cap A_{e_m}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^{|R_2|+k}}\right)\\
&=\mathrm{St}_{\hat{0}}^{\hat{N},[|R_2|+k]}\left(\underset{b=\{e_1,\ldots,e_m\}\in R_2}{\mathrm{X}}\left(A_{e_1}\cap\cdots\cap A_{e_m}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{St}_{\geq\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]}\left(\left[\underset{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}}\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^{|\mathbf{B}_2(\sigma)|+k}}\right)\\
&=\mathrm{St}_{\hat{0}}^{\hat{N},[|\mathbf{B}_2(\sigma)|+k]}\left(\underset{b=\{i_1,\ldots,i_\ell\}\in\mathbf{B}_2(\sigma)}{\mathrm{X}}\left(A_{i_1}\cap\cdots\cap A_{i_\ell}\right)\times A_{j_1}\times\cdots\times A_{j_k}\right),
\end{aligned}$$

and

$$\mathrm{St}_{\geq\hat{0}}^{\hat{N},[k]}\left(\left[A_{j_1}\times\cdots\times A_{j_k}\right]\mathbf{1}_{Z_{\hat{0}}^k}\right)=\mathrm{St}_{\hat{0}}^{\hat{N},[k]}\left(A_{j_1}\times\cdots\times A_{j_k}\right),$$

one obtains immediately the desired conclusion. ■

Remark on integral notation. It is instructive to express the results of Theorem 5.2 in integral notation. With $f(z_1,...,z_n) = g(z_1)\cdots g(z_n)$, formula (5.57) becomes

$$\begin{aligned}&\int_{\cup_{\pi\geq\sigma} Z_\pi^n} g(z_1)\cdots g(z_n)\,\hat{N}(dz_1)\cdots\hat{N}(dz_n)\\ =\ &\left(\prod_{b\in\sigma,|b|\geq 2}\int_Z g(z)^{|b|}\,N(dz)\right)\times\left(\int_Z g(z)\,\hat{N}(dz)\right)^k,\end{aligned}$$

where $k = |\mathbf{B}_1(\sigma)|$. Again with $f(z_1,..,z_n) = g(z_1)\cdots g(z_n)$, (5.58)–(5.62) become

$$\begin{aligned}&\int_{Z_\sigma^n} g(z_1)\cdots g(z_n)\,\hat{N}(dz_1)\cdots\hat{N}(dz_n)\\ =\ &\sum_{(R_1;R_2)\in\mathbf{PB}_2(\sigma)}\prod_{b\in R_1}\int_Z g(z)^{|b|}\,\nu(dz)\times\mathbf{1}_{\left\{R_2=\left\{b_1,...,b_{|R_2|}\right\}\right\}}\times\\ &\quad\times\int_{z_1\neq\cdots\neq z_{|R_2|+k}} g(z_1)^{|b_1|}\cdots g\left(z_{|R_2|}\right)^{\left|b_{|R_2|}\right|} g\left(z_{|R_2|+1}\right)\cdots g\left(z_{|R_2|+k}\right)\\ &\qquad\qquad\hat{N}(dz_1)\cdots\hat{N}\left(dz_{|R_2|+k}\right)\\ &+\mathbf{1}_{\left\{\mathbf{B}_2(\sigma)=\left\{b_1,...,b_{|\mathbf{B}_2(\sigma)|}\right\}\right\}}\times\\ &\quad\times\int_{z_1\neq\cdots\neq z_{|\mathbf{B}_2(\sigma)|+k}} g(z_1)^{|b_1|}\cdots g\left(z_{|\mathbf{B}_2(\sigma)|}\right)^{\left|b_{|\mathbf{B}_2(\sigma)|}\right|} g\left(z_{|\mathbf{B}_2(\sigma)|+1}\right)\cdots g\left(z_{|\mathbf{B}_2(\sigma)|+k}\right)\\ &\qquad\qquad\hat{N}(dz_1)\cdots\hat{N}\left(dz_{|R_2|+k}\right)\\ &\quad+\prod_{b\in\mathbf{B}_2(\sigma)}\int_Z g(z)^{|b|}\,\nu(dz)\times\int_{z_1\neq\cdots\neq z_k} g(z_1)\cdots g(z_k)\,\hat{N}(dz_1)\cdots\hat{N}(dz_k),\end{aligned}$$

where $k = |\mathbf{B}_1(\sigma)|$.

Examples. The examples below apply to a compensated Poisson measure $\hat{N}$, and should be compared with those discussed after Theorem 5.1. We suppose throughout that $n \geq 2$.

(i) When $\sigma = \hat{0} = \{\{1\},...,\{n\}\}$ one has, as in the Gaussian case,

$$\mathrm{St}_{\geq\hat{0}}^{\hat{N},[n]}(A_1\times\cdots\times A_n) = \hat{N}(A_1)\cdots\hat{N}(A_n)$$

because the symbol " $\geq \hat{0}$ " entails no restriction on the considered partitions. In integral notation, this becomes

$$\mathrm{St}_{\geq\hat{0}}^{\hat{N},[n]}(f) = \int_{Z^n} f(z_1,...,z_n)\,\hat{N}(dz_1)\cdots\hat{N}(dz_n).$$

The case of equality (5.58) has already been discussed: indeed, since $\sigma = \hat{0}$, and according to the conventions stated therein, one has that $\mathbf{B}_2(\sigma) = \mathbf{PB}_2(\sigma) = \varnothing$ and therefore (5.58) becomes an identity given by (5.62), namely $\mathrm{St}_{\hat{0}}^{\hat{N},[n]}(\cdot) = \mathrm{St}_{\hat{0}}^{\hat{N},[n]}(\cdot)$. Observe that, in integral notation, $\mathrm{St}_{\hat{0}}^{\hat{N},[n]}(\cdot)$ is expressed as

$$\mathrm{St}_{\hat{0}}^{\hat{N},[n]}(f) = \int_{z_1\neq\cdots\neq z_n} f(z_1,...,z_n)\,\hat{N}(dz_1)\cdots\hat{N}(dz_n).$$

(ii) Suppose now $\sigma = \hat{1} = \{\{1, ..., n\}\}$. Then, (5.58) reduces to (5.56). To see this, note that $k = 0$ and that $\mathbf{B}_2\left(\hat{1}\right)$ contains only the block $\{1, ..., n\}$, so that $\mathbf{PB}_2(\sigma) = \varnothing$. Hence the sum appearing in (5.59) vanishes and one has

$$\begin{aligned} & \mathrm{St}_{\hat{1}}^{\hat{N},[n]}(A_1 \times \cdots \times A_n) = \mathrm{St}_{\hat{0}}^{\hat{N},[1]}(A_1 \cap \cdots \cap A_n) + \nu(A_1 \cap \cdots \cap A_n) \\ = \; & \hat{N}(A_1 \cap \cdots \cap A_n) + \nu(A_1 \cap \cdots \cap A_n) = N(A_1 \cap \cdots \cap A_n). \end{aligned}$$

In integral notation,

$$\begin{aligned} \mathrm{St}_{\hat{1}}^{\hat{N},[n]}(f) &= \int_{z_1=\cdots=z_n} f(z_1, ..., z_n)\, \hat{N}(dz_1) \cdots \hat{N}(dz_n) \\ &= \int_Z f(z, ..., z)\, N(dz). \end{aligned}$$

This last relation makes sense heuristically, in view of the computational rule

$$\left(\hat{N}(dx)\right)^2 = (N(dx))^2 - 2N(dx)\,\nu(dx) + (\nu(dx))^2 = N(dx),$$

since $(N(dx))^2 = N(dx)$ and ν is non-atomic.

(iii) Let $n = 3$ and $\sigma = \{\{1\}, \{2, 3\}\}$, so that $\mathbf{B}_2(\sigma) = \{\{2, 3\}\}$ and $\mathbf{PB}_2(\sigma) = \varnothing$. According to (5.57),

$$\mathrm{St}_{\geq\sigma}^{\hat{N},[3]}(A_1 \times A_2 \times A_3) = \hat{N}(A_1)\, N(A_2 \cap A_3). \tag{5.65}$$

On the other hand, (5.58) yields

$$\mathrm{St}_{\sigma}^{\hat{N},[3]}(A_1 \times A_2 \times A_3) = \mathrm{St}_{\hat{0}}^{\hat{N},[2]}(A_1 \times (A_2 \cap A_3)) + \hat{N}(A_1)\,\nu(A_2 \cap A_3). \tag{5.66}$$

In integral form, relation (5.65) becomes

$$\mathrm{St}_{\geq\sigma}^{\hat{N},[3]}(f) = \int_Z \int_Z f(z_1, z_2, z_2)\, \hat{N}(dz_1)\, N(dz_2),$$

and (5.66) becomes

$$\begin{aligned} \mathrm{St}_{\sigma}^{\hat{N},[3]}(f) &= \int_{z_1 \neq z_2,\; z_2 = z_3} f(z_1, z_2, z_3)\, \hat{N}(dz_1)\, \hat{N}(dz_2)\, \hat{N}(dz_3) \\ &= \int_{z_1 \neq z_2} f(z_1, z_2, z_2)\, \hat{N}(dz_1)\, \hat{N}(dz_2) \\ &\quad + \int_{Z^2} f(z_1, z_2, z_2)\, \hat{N}(dz_1)\, \nu(dz_2). \end{aligned}$$

This last relation makes sense heuristically, by noting that

$$\begin{aligned} \hat{N}(dz_1)\, \hat{N}(dz_2)\, \hat{N}(dz_2) &= \hat{N}(dz_1)\, N(dz_2) \\ &= \hat{N}(dz_1)\, \hat{N}(dz_2) + \hat{N}(dz_1)\, \nu(dz_2). \end{aligned}$$

We also stress that $\mathrm{St}_{\sigma}^{\hat{N},[3]}(f)$ can be also be expressed as

$$\mathrm{St}_{\sigma}^{\hat{N},[3]}(f) = I_2^{\hat{N}}(g_1) + I_1^{\hat{N}}(g_2), \tag{5.67}$$

where $g_1(x,y) = f(x,y,y)$ and $g_2(x) = \int f(x,y,y)\,\nu(dy)$. The form (5.67) will be needed later. Since, by (5.56), $\mathrm{St}_{\hat{1}}^{\hat{N},[3]}(A_1 \times A_2 \times A_3) = N(A_1 \cap A_2 \cap A_3)$ and since, for our σ, one has $\mathrm{St}_{\geq\sigma}^{\hat{N},[3]} = \mathrm{St}_{\sigma}^{\hat{N},[3]} + \mathrm{St}_{\hat{1}}^{\hat{N},[3]}$, one also deduces the relation

$$\hat{N}(A_1)\,N(A_2 \cap A_3) = \mathrm{St}_{\hat{0}}^{\hat{N},[2]}(A_1 \times (A_2 \cap A_3)) + N(A_1 \cap A_2 \cap A_3) + \hat{N}(A_1)\,\nu(A_2 \cap A_3),$$

or, equivalently, since $\hat{N} = N + \nu$,

$$\hat{N}(A_1)\,\hat{N}(A_2 \cap A_3) = \mathrm{St}_{\hat{0}}^{\hat{N},[2]}(A_1 \times (A_2 \cap A_3)) + \nu(A_1 \cap A_2 \cap A_3) + \hat{N}(A_1 \cap A_2 \cap A_3). \quad (5.68)$$

We will see that (5.68) is consistent with the multiplication formulae of next section.

(iv) Let $n = 6$, and $\sigma = \{\{1,2\},\{3\},\{4\},\{5,6\}\}$, so that

$$\mathbf{B}_2(\sigma) = \{\{1,2\},\{5,6\}\}$$

and the class $\mathbf{PB}_2(\sigma)$ contains the two pairs

$$(\{1,2\};\{5,6\}) \quad \text{and} \quad (\{5,6\};\{1,2\}).$$

First, (5.57) gives

$$\mathrm{St}_{\geq\sigma}^{\hat{N},[6]}(A_1 \times ... \times A_6) = N(A_1 \cap A_2)\,N(A_5 \cap A_6)\,\hat{N}(A_3)\,\hat{N}(A_4).$$

Moreover, we deduce from (5.58) that

$$\begin{aligned}
\mathrm{St}_{\sigma}^{\hat{N},[6]}(A_1 \times ... \times A_6) &= \nu(A_1 \cap A_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[3]}((A_5 \cap A_6) \times A_3 \times A_4) \\
&\quad + \nu(A_5 \cap A_6)\,\mathrm{St}_{\hat{0}}^{\hat{N},[3]}((A_1 \cap A_2) \times A_3 \times A_4) \\
&\quad + \mathrm{St}_{\hat{0}}^{\hat{N},[4]}((A_1 \cap A_2) \times (A_5 \cap A_6) \times A_3 \times A_4) \\
&\quad + \nu(A_1 \cap A_2)\,\nu(A_5 \cap A_6)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(A_3 \times A_4).
\end{aligned}$$

The last displayed equation becomes in integral form

$$\begin{aligned}
&\mathrm{St}_{\sigma}^{\hat{N},[6]}(f) \\
&= \int_{\substack{z_1=z_2,\ z_5=z_6,\ z_3\neq z_4 \\ z_3\neq z_1,\ z_4\neq z_1 \\ z_3\neq z_5,\ z_4\neq z_5}} f(z_1,...,z_6)\,\hat{N}(dz_1)\cdots\hat{N}(dz_6) \\
&= \int_{w,x\neq y\neq z} f(w,w,x,y,z,z)\,\nu(dw)\,\hat{N}(dx)\,\hat{N}(dy)\,\hat{N}(dz) \\
&\quad + \int_{w\neq x\neq y,z} f(w,w,x,y,z,z)\,\hat{N}(dw)\,\hat{N}(dx)\,\hat{N}(dy)\,\nu(dz) \\
&\quad + \int_{w\neq x\neq y\neq z} f(w,w,x,y,z,z)\,\hat{N}(dw)\,\hat{N}(dx)\,\hat{N}(dy)\,\hat{N}(dz) \\
&\quad + \int_{w,x\neq y,z} f(w,w,x,y,z,z)\,\nu(dw)\,\hat{N}(dx)\,\hat{N}(dy)\,\nu(dz).
\end{aligned}$$

Indeed, let us denote the RHS of the last expression as $(I)+(II)+(III)+(IV)$. For (I) and (II), we use (5.59)–(5.60) with $R_1=\{\{1,2\}\}$ and $R_2=\{\{5,6\}\}$ and $k=2$, which corresponds to the number of singletons $\{3\}$, $\{4\}$. For (III), we use (5.61), with $k+|\mathbf{B}_2(\sigma)|=2+2=4$, and $\mathbf{B}_2(\sigma)=\{\{1,2\},\{5,6\}\}$. For (5.62) we use $k=2$ and our $\mathbf{B}_2(\sigma)$.

(v) Let $n=6$, and $\sigma=\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$. Here, $\mathbf{B}_2(\sigma)=\{\{1,2\}\}$ and the class $\mathbf{PB}_2(\sigma)$ is empty. Then,

$$\mathrm{St}_{\geq\sigma}^{\hat{N},[6]}(A_1\times...\times A_6)=N(A_1\cap A_2)\,\hat{N}(A_3)\,\hat{N}(A_4)\,\hat{N}(A_5)\,\hat{N}(A_6),$$

and also

$$\begin{aligned}\mathrm{St}_{\sigma}^{\hat{N},[6]}(A_1\times...\times A_6) &= \nu(A_1\cap A_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[4]}(A_3\times A_4\times A_5\times A_6)\\ &\quad+\mathrm{St}_{\hat{0}}^{\hat{N},[4]}((A_1\cap A_2)\times A_3\times A_4\times A_5\times A_6).\end{aligned}$$

In integral form,

$$\begin{aligned}&\mathrm{St}_{\sigma}^{\hat{N},[6]}(f)\\ =&\int_{\substack{z_1=z_2,\\ z_1\neq z_j,\ j=3,...,6\\ z_i\neq z_j,\ 3\leq i\neq j\leq 6}} f(z_1,z_2,z_3,z_4,z_5,z_6)\prod_{j=1}^{6}\hat{N}(dz_j)\\ =&\int_{\substack{z_1\neq z_j,\ j=3,...,6\\ z_i\neq z_j,\ 3\leq i\neq j\leq 6}} f(z_1,z_1,z_3,z_4,z_5,z_6)\,\nu(dz_1)\prod_{j=3}^{6}\hat{N}(dz_j)\\ &+\int_{\substack{z_1\neq z_j,\ j=3,...,6\\ z_i\neq z_j,\ 3\leq i\neq j\leq 6}} f(z_1,z_1,z_3,z_4,z_5,z_6)\,\hat{N}(dz_1)\prod_{j=3}^{6}\hat{N}(dz_j).\end{aligned}$$

Corollary 5.1 *Suppose that the Assumptions of Theorem 5.2 hold. Fix $\sigma\in\mathcal{P}([n])$ and assume that $|b|\geq 2$, for every $b\in\sigma$. Then,*

$$\mathbb{E}\left[\mathrm{St}_{\sigma}^{\hat{N},[n]}(A_1\times\cdots\times A_n)\right]=\prod_{m=2}^{n}\prod_{b\in\{j_1,...,j_m\}\in\sigma}\nu(A_{j_1}\cap\cdots\cap A_{j_m}). \tag{5.69}$$

Proof. Use (5.58)–(5.62) and note that, by assumption, $k=0$ (the partition σ does not contain any singleton $\{j\}$). It follows that the sum in (5.59) vanishes, and one is left with

$$\begin{aligned}&\mathbb{E}\left[\mathrm{St}_{\sigma}^{\hat{N},[n]}(A_1\times\cdots\times A_n)\right]\\ =&\ \mathbb{E}\left[\mathrm{St}_{\hat{0}}^{\hat{N},[k+|\mathbf{B}_2(\sigma)|]}\left(\mathop{\mathrm{X}}_{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}(A_{i_1}\cap\cdots\cap A_{i_\ell})\times A_{j_1}\times\cdots\times A_{j_k}\right)\right]\\ &+\prod_{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}\nu(A_{i_1}\cap\cdots\cap A_{i_\ell})\\ =&\prod_{b=\{i_1,...,i_\ell\}\in\mathbf{B}_2(\sigma)}\nu(A_{i_1}\cap\cdots\cap A_{i_\ell}),\end{aligned}$$

which is equal to the RHS of (5.69). ■

6 Multiplication formulae

6.1 The general case

The forthcoming Theorem 6.1 applies to every good completely random measure φ. It gives a universal combinatorial rule, according to which every product of multiple stochastic integrals can be represented as a sum over diagonal measures that are indexed by non-flat diagrams (as defined in Section 4.1). We will see that product formulae are crucial in order to deduce explicit expressions for the cumulants and the moments of multiple integrals. As discussed later in this section, Theorem 6.1 contains (as special cases) two celebrated *product formulae* for integrals with respect to Gaussian and Poisson random measures. We provide two proofs of Theorem 6.1: the first one is new and it is based on a decomposition of partially diagonal sets; the second consists in a slight variation of the combinatorial arguments displayed in the proofs of [106, Th. 3 and Th. 4], and is included for the sake of completeness. The theorem is formulated for simple kernels to ensure that the integrals are always defined, in particular the quantity $\mathrm{St}_{\sigma}^{\varphi,[n]}$, which appears on the RHS of (6.2).

Theorem 6.1 (Rota et Wallstrom) *Let φ be a good completely random measure with non-atomic control ν. For $n_1, n_2, ..., n_k \geq 1$, we write $n = n_1 + \cdots + n_k$, and we denote by π^* the partition of $[n]$ given by*

$$\pi^* = \{\{1, ..., n_1\}, \{n_1 + 1, ..., n_1 + n_2\}, ..., \{n_1 + ... + n_{k-1} + 1, ..., n\}\}. \tag{6.1}$$

Then, if the kernels $f_1, ..., f_k$ are such that $f_j \in \mathcal{E}_{s,0}(\nu^{n_j})$ $(j = 1, ..., k)$, one has that

$$\boxed{\prod_{j=1}^{k} I_{n_j}^{\varphi}(f_j) = \prod_{j=1}^{k} \mathrm{St}_{\hat{0}}^{\varphi,[n_j]}(f_j) = \sum_{\sigma \in \mathcal{P}([n]):\sigma \wedge \pi^* = \hat{0}} \mathrm{St}_{\sigma}^{\varphi,[n]}(f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k),} \tag{6.2}$$

where, by definition, the function in n variables $f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k \in \mathcal{E}(\nu^n)$ is defined as

$$f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k(x_1, x_2, ..., x_n) = \prod_{j=1}^{k} f_j\left(x_{n_1+...+n_{j-1}+1}, ..., x_{n_1+...+n_j}\right), \quad (n_0 = 0). \tag{6.3}$$

Proof. (*First proof*) From the discussion of the previous section, one deduces that

$$\prod_{j=1}^{k} \mathrm{St}_{\hat{0}}^{\varphi,[n_j]}(f_j) = \mathrm{St}_{\geq \hat{0}}^{\varphi,[n]}\left[(f_1 \otimes_0 \cdots \otimes_0 f_k)\, \mathbf{1}_{A^*}\right],$$

where

$$A^* = \{(z_1, ..., z_n) \in \mathcal{Z}^n : z_i \neq z_j, \forall i \neq j \text{ such that } i \sim_{\pi^*} j\},$$

that is, A^* is obtained by excluding all diagonals within each block of π^*. We shall prove that

$$A^* = \bigcup_{\sigma \in \mathcal{P}([n]):\sigma \wedge \pi^* = \hat{0}} Z_{\sigma}^{n}. \tag{6.4}$$

Suppose first σ is such that $\sigma \wedge \pi^* = \hat{0}$, that is, the meet of σ and π^* is given by the singletons. For every $(z_1, ..., z_n) \in Z_{\sigma}^{n}$ the following implication holds: if $i \neq j$ and $i \sim_{\pi^*} j$, then i and j are

in two different blocks of σ, and therefore $z_i \neq z_j$. This implies that $Z_\sigma^n \subset A^*$. For the converse, take $(z_1, ..., z_n) \in A^*$, and construct a partition $\sigma \in \mathcal{P}([n])$ by the following rule: $i \sim_\sigma j$ if and only if $z_i = z_j$. For every pair $i \neq j$ such that $i \sim_{\pi^*} j$, one has (by definition of A^*) $z_i \neq z_j$, so that $\sigma \wedge \pi^* = \hat{0}$, and hence (6.4). To conclude the proof of the theorem, just use the additivity of $\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}$ to write

$$\begin{aligned}\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}\left[(f_1 \otimes_0 \cdots \otimes_0 f_k)\,\mathbf{1}_{A^*}\right] &= \sum_{\sigma\wedge\pi^*=\hat{0}} \mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}\left[(f_1 \otimes_0 \cdots \otimes_0 f_k)\,\mathbf{1}_{Z_\sigma^n}\right] \\ &= \sum_{\sigma\wedge\pi^*=\hat{0}} \mathrm{St}_{\sigma}^{\varphi,[n]}\left(f_1 \otimes_0 \cdots \otimes_0 f_k\right),\end{aligned}$$

by using the relation $\mathrm{St}_{\geq\hat{0}}^{\varphi,[n]}\left[(\cdot)\,\mathbf{1}_{Z_\sigma^n}\right] = \mathrm{St}_{\sigma}^{\varphi,[n]}[\cdot]$.

(*Second proof – see [106]*) This proof uses Proposition 2.1 and Proposition 2.2. To simplify the discussion (and without loss of generality) we can assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. For $j = 1, ..., k$ we have that

$$\mathrm{St}_{\hat{0}}^{\varphi,[n_j]}(f_j) = \sum_{\sigma\in\mathcal{P}([n_j])} \mu\left(\hat{0},\sigma\right) \mathrm{St}_{\geq\sigma}^{\varphi,[n_j]}(f_j),$$

where we have used (5.34) with $\pi = \hat{0}$. From this relation one obtains

$$\begin{aligned}\prod_{j=1}^{k} \mathrm{St}_{\hat{0}}^{\varphi,[n_j]}(f_j) &= \sum_{\sigma_1\in\mathcal{P}([n_1])} \cdots \sum_{\sigma_k\in\mathcal{P}([n_k])} \prod_{j=1}^{k} \mu\left(\hat{0},\sigma_j\right) \mathrm{St}_{\geq\sigma_j}^{\varphi,[n_j]}(f_j) \\ &= \sum_{\rho\in\mathcal{P}([n]):\rho\leq\pi^*} \mu\left(\hat{0},\rho\right) \mathrm{St}_{\geq\rho}^{\varphi,[n]}\left(f_1 \otimes_0 \cdots \otimes_0 f_k\right), \qquad (6.5)\end{aligned}$$

where π^* is defined in (6.1). To prove equality (6.5), recall the definition of "class" in Section 2.3, as well as the last example in that section. Observe that the segment $\left[\hat{0},\pi^*\right]$ has class $(n_1, ..., n_k)$, thus yielding (thanks to Proposition 2.2) that $\left[\hat{0},\pi^*\right]$ is isomorphic to the lattice product of the $\mathcal{P}([n_j])$'s. This implies that each vector

$$(\sigma_1, ..., \sigma_k) \in \mathcal{P}([n_1]) \times \cdots \times \mathcal{P}([n_k])$$

has indeed the form

$$(\sigma_1, ..., \sigma_k) = \psi^{-1}(\rho)$$

for a unique $\rho \in \left[\hat{0},\pi^*\right]$, where ψ is a bijection defined as in (2.16). Now use, in order, Part 2 and Part 1 of Proposition 2.1 to deduce that

$$\prod_{j=1}^{k} \mu\left(\hat{0},\sigma_j\right) = \mu\left(\hat{0},(\sigma_1, ..., \sigma_k)\right) = \mu\left(\hat{0},\psi^{-1}(\rho)\right) = \mu\left(\hat{0},\rho\right). \qquad (6.6)$$

Observe that

$$\hat{0} = \{\{1\}, ..., \{n_j\}\} \quad \text{in} \quad \mu\left(\hat{0},\sigma_j\right),$$

whereas

$$\hat{0} = \{\{1\}, ..., \{n\}\} \quad \text{in} \quad \mu(\hat{0},\rho).$$

Also, one has the relation

$$\prod_{j=1}^{k} \mathrm{St}_{\geq\sigma_j}^{\varphi,[n_j]}(f_j) = \mathrm{St}_{\geq\rho}^{\varphi,[n]}(f_1 \otimes_0 \cdots \otimes_0 f_k), \tag{6.7}$$

(by the definition of $\mathrm{St}_{\geq\rho}^{\varphi,[n]}$ as the measure charging all the diagonals contained in the diagonals associated with the blocks of the σ_j $(j = 1, ..., k)$). Then, (6.6) and (6.7) yield immediately (6.5). To conclude the proof, write

$$\begin{aligned}
&\sum_{\rho\in\mathcal{P}([n]):\rho\leq\pi^*} \mu\left(\hat{0},\rho\right) \mathrm{St}_{\geq\rho}^{\varphi,[n]}(f_1 \otimes_0 \cdots \otimes_0 f_k) \\
= &\sum_{\rho\in\mathcal{P}([n]):\rho\leq\pi^*} \mu\left(\hat{0},\rho\right) \sum_{\gamma\geq\rho} \mathrm{St}_{\gamma}^{\varphi,[n]}(f_1 \otimes_0 \cdots \otimes_0 f_k) \\
= &\sum_{\gamma\in\mathcal{P}([n])} \mathrm{St}_{\gamma}^{\varphi,[n]}(f_1 \otimes_0 \cdots \otimes_0 f_k) \sum_{\hat{0}\leq\rho\leq\pi^*\wedge\gamma} \mu\left(\hat{0},\rho\right).
\end{aligned}$$

Since, by (2.14),

$$\sum_{\hat{0}\leq\rho\leq\pi^*\wedge\gamma} \mu\left(\hat{0},\rho\right) = \begin{cases} 0 & \text{if } \pi^*\wedge\gamma \neq \hat{0} \\ 1 & \text{if } \pi^*\wedge\gamma = \hat{0}. \end{cases},$$

relation (6.2) is obtained. ■

Remark. The RHS of (6.2) can also be reformulated in terms of diagrams and in terms of graphs, as follows:

$$\sum_{\sigma\in\mathcal{P}([n]):\Gamma(\pi^*,\sigma) \text{ is non-flat}} \mathrm{St}_{\sigma}^{\varphi,[n]}(f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k),$$

where $\Gamma(\pi^*,\sigma)$ is the diagram of (π^*,σ), as defined in Section 4.1, or, whenever every $\Gamma(\pi^*,\sigma)$ involved in the previous sum is Gaussian,

$$\sum_{\sigma\in\mathcal{P}([n]):\hat{\Gamma}(\pi^*,\sigma) \text{ has no loops}} \mathrm{St}_{\sigma}^{\varphi,[n]}(f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k).$$

where $\hat{\Gamma}(\pi^*,\sigma)$ is the graph of (π^*,σ) defined in Section 4.3. This is because, thanks to Proposition 4.1, the relation $\pi^*\wedge\sigma = \hat{0}$ indicates that $\Gamma(\pi^*,\sigma)$ is non-flat or, equivalently in the case of Gaussian diagrams, that $\hat{\Gamma}(\pi^*,\sigma)$ has no loops.

Examples. (i) Set $k = 2$ and $n_1 = n_2 = 1$ in Theorem 6.1. Then, $n = 2$, $\mathcal{P}([2]) = \left\{\hat{0},\hat{1}\right\}$ and $\pi^* = \{\{1\},\{2\}\} = \hat{0}$. Since $\hat{0}\wedge\hat{1} = \hat{0}$, (6.2) gives immediately that, for every pair of elementary functions f_1, f_2,

$$\begin{aligned}
I_1^{\varphi}(f_1)\times I_1^{\varphi}(f_2) &= \mathrm{St}_{\hat{0}}^{\varphi,[2]}(f_1\otimes_0 f_2) + \mathrm{St}_{\hat{1}}^{\varphi,[2]}(f_1\otimes_0 f_2) \\
&= I_2^{\varphi}(f_1\otimes_0 f_2) + \mathrm{St}_{\hat{1}}^{\varphi,[2]}(f_1\otimes_0 f_2),
\end{aligned} \tag{6.8}$$

Note that, if $\varphi = G$ is Gaussian, then relation (5.51) yields that

$$\mathrm{St}_{\hat{1}}^{G,[2]}(f_1\otimes_0 f_2) = \int_Z f_1(z)\, f_2(z)\, \nu(dz),$$

so that, in integral notation,

$$I_1^G(f_1)\times I_1^G(f_2)=\int\int_{z_1\neq z_2} f_1(z_1)\,f_2(z_2)\,G(dz_1)\,G(dz_2)+\int_Z f_1(z)\,f_2(z)\,\nu(dz).$$

On the other hand, if φ is compensated Poisson, then $\mathrm{St}_{\hat{1}}^{\varphi,[2]}(f_1\otimes_0 f_2)=\int_Z f_1(z)\,f_2(z)\,N(dz)$, so that, by using the relation $N=\hat{N}+\nu$, (6.8) reads

$$\begin{aligned}
I_1^{\hat{N}}(f_1)\times I_1^{\hat{N}}(f_2) &= I_2^{\hat{N}}(f_1\otimes_0 f_2)+\int_z f_1(z)\,f_2(z)\,\hat{N}(dz)+\int_z f_1(z)\,f_2(z)\,\nu(dz)\\
&= \int\int_{z_1\neq z_2} f_1(z_1)\,f_2(z_2)\,\hat{N}(dz_1)\,\hat{N}(dz_2)\\
&\quad+\int_z f_1(z)\,f_2(z)\,\hat{N}(dz)+\int_z f_1(z)\,f_2(z)\,\nu(dz)\\
&= I_2^{\hat{N}}(f_1\otimes_0 f_2)+I_1^{\hat{N}}(f_1f_2)+\int_z f_1(z)\,f_2(z)\,\nu(dz)
\end{aligned}$$

(ii) Consider the case $k=2$, $n_1=2$ and $n_2=1$. Then, $n=3$, and $\pi^*=\{\{1,2\},\{3\}\}$. There are three elements $\sigma_1,\sigma_2,\sigma_3\in\mathcal{P}([3])$ such that $\sigma_i\wedge\pi^*=\hat{0}$, namely $\sigma_1=\hat{0}$, $\sigma_2=\{\{1,3\},\{2\}\}$ and $\sigma_3=\{\{1\},\{2,3\}\}$. Then, (6.2) gives that, for every pair $f_1\in\mathcal{E}_{s,0}\left(\nu^2\right)$, $f_2\in\mathcal{E}(\nu)$,

$$\begin{aligned}
I_2^{\varphi}(f_1)\times I_1^{\varphi}(f_2) &= \mathrm{St}_{\hat{0}}^{\varphi,[3]}(f_1\otimes_0 f_2)+\mathrm{St}_{\sigma_2}^{\varphi,[3]}(f_1\otimes_0 f_2)+\mathrm{St}_{\sigma_3}^{\varphi,[3]}(f_1\otimes_0 f_2).\\
&= \mathrm{St}_{\hat{0}}^{\varphi,[3]}(f_1\otimes_0 f_2)+2\mathrm{St}_{\sigma_2}^{\varphi,[3]}(f_1\otimes_0 f_2),
\end{aligned}$$

where we have used the fact that, by the symmetry of f_1, $\mathrm{St}_{\sigma_2}^{\varphi,[3]}(f_1\otimes_0 f_2)=\mathrm{St}_{\sigma_3}^{\varphi,[3]}(f_1\otimes_0 f_2)$.

When $\varphi=G$ is a Gaussian measure, one can use (5.53) applied to σ_2 to deduce that

$$\mathrm{St}_{\sigma_2}^{G,[3]}(f_1\otimes_0 f_2)=I_1^G\left[\int_Z f_1(\cdot,z)\,f_2(z)\,\nu(dz)\right],$$

or, more informally,

$$\mathrm{St}_{\sigma_2}^{G,[3]}(f_1\otimes_0 f_2)=\int_Z\int_Z f_1\left(z',z\right)f_2\left(z'\right)\nu(dz)\,G\left(dz'\right),$$

so that one gets

$$\begin{aligned}
I_2^G(f_1)\times I_1^G(f_2) &= \mathrm{St}_{\hat{0}}^{G,[3]}(f_1\otimes_0 f_2)+2I_1^G\left[\int_Z f_1(\cdot,z)\,f_2(z)\,\nu(dz)\right]\\
&= \int\int\int_{z_1\neq z_2\neq z_3} f_1(z_1,z_2)\,f_2(z_3)\,G(dz_1)\,G(dz_2)\,G(dz_3)\\
&\quad+2\int_Z\int_Z f_1\left(z',z\right)f_2(z)\,\nu(dz)\,G\left(dz'\right).
\end{aligned}$$

When $\varphi=\hat{N}$ is compensated Poisson, as shown in (5.67), formula (5.58), applied to σ_2, yields

$$\mathrm{St}_{\sigma_2}^{\hat{N},[3]}(f_1\otimes_0 f_2)=I_1^{\hat{N}}\left[\int_Z f_1(\cdot,z)\,f_2(z)\,\nu(dz)\right]+I_2^{\hat{N}}\left[f_1\otimes_1^0 f_2\right],$$

where $f_1 \otimes_1^0 f_2(z', z) = f_1(z', z) f_2(z)$.

(iii) Consider the case $k = 3$, $n_1 = n_2 = n_3 = 1$. Then, $n = 3$, and $\pi^* = \{\{1\}, \{2\}, \{3\}\} = \hat{0}$. For every $\sigma \in \mathcal{P}([3])$ one has that $\sigma \wedge \pi^* = \hat{0}$. Note also that $\mathcal{P}([3]) = \left\{\hat{0}, \rho_1, \rho_2, \rho_3, \hat{1}\right\}$, where

$$\rho_1 = \{\{1,2\}, \{3\}\}, \ \ \rho_2 = \{\{1,3\}, \{2\}\}, \ \ \rho_3 = \{\{1\}, \{2,3\}\},$$

so that (6.2) gives that, for every $f_1, f_2, f_3 \in \mathcal{E}(\nu)$,

$$\begin{aligned} I_1^{\varphi}(f_1) I_1^{\varphi}(f_2) I_1^{\varphi}(f_3) &= \mathrm{St}_{\hat{0}}^{\varphi,[3]}(f_1 \otimes_0 f_2 \otimes_0 f_3) + \mathrm{St}_{\rho_1}^{\varphi,[3]}(f_1 \otimes_0 f_2 \otimes_0 f_3) \\ &\quad + \mathrm{St}_{\rho_2}^{\varphi,[3]}(f_1 \otimes_0 f_2 \otimes_0 f_3) + \mathrm{St}_{\rho_3}^{\varphi,[3]}(f_1 \otimes_0 f_2 \otimes_0 f_3) \\ &\quad + \mathrm{St}_{\hat{1}}^{\varphi,[3]}(f_1 \otimes_0 f_2 \otimes_0 f_3). \end{aligned}$$

In particular, by taking $f_1 = f_2 = f_3 = f$ and by symmetry,

$$I_1^{\varphi}(f)^3 = \mathrm{St}_{\hat{0}}^{\varphi,[3]}(f \otimes_0 f \otimes_0 f) + \mathrm{St}_{\hat{1}}^{\varphi,[3]}(f \otimes_0 f \otimes_0 f) + 3\mathrm{St}_{\rho_1}^{\varphi,[3]}(f \otimes_0 f \otimes_0 f). \tag{6.9}$$

When $\varphi = G$ is Gaussian, then $\mathrm{St}_{\hat{1}}^{G,[3]} = 0$ by (5.51) and $\mathrm{St}_{\rho_1}^{\varphi,[3]}(f \otimes_0 f \otimes_0 f) = \|f\|^2 I_1^G(f)$, so that (6.9) becomes

$$\begin{aligned} I_1^G(f)^3 &= I_3^G(f \otimes_0 f \otimes_0 f) + 3\|f\|^2 I_1^G(f) \\ &= \int\int\int_{z_1 \neq z_2 \neq z_3} f(z_1) f(z_2) f(z_3) G(dz_1) G(dz_2) G(dz_3) \\ &\quad + 3\int_Z f(z)^2 \nu(dz) \times \int_Z f(z) G(dz). \end{aligned}$$

When $\varphi = \hat{N}$ is compensated Poisson, from (5.55), then

$$\mathrm{St}_{\hat{1}}^{\hat{N},[3]}(f \otimes_0 f \otimes_0 f) = \int f(z)^3 N(dz) = I_1^{\hat{N}}\left(f^3\right) + \int_Z f^3(z) \nu(dz)$$

by (5.67), and also

$$\mathrm{St}_{\rho_1}^{\hat{N},[3]}(f \otimes_0 f \otimes_0 f) = \|f\|^2 I_1^{\hat{N}}(f) + I_2^{\hat{N}}\left(f^2 \otimes_0 f\right),$$

where $f^2 \otimes_0 f(z, z') = f^2(z) f(z')$, so that (6.9) becomes

$$\begin{aligned} I_1^{\hat{N}}(f)^3 &= I_3^{\hat{N}}(f \otimes_0 f \otimes_0 f) + I_1^{\hat{N}}\left(f^3\right) + \int_Z f^3(z) \nu(dz) \\ &\quad + 3\|f\|^2 I_1^{\hat{N}}(f) + I_2^{\hat{N}}\left(f^2 \otimes_0 f\right) \\ &= \int\int\int_{z_1 \neq z_2 \neq z_3} f(z_1) f(z_2) f(z_3) \hat{N}(dz_1) \hat{N}(dz_2) \hat{N}(dz_3) \\ &\quad + \int_Z f(z)^3 \left(\hat{N} + \nu\right)(dz) + 3\int_Z f(z)^2 \nu(dz) \times \int_Z f(z) \hat{N}(dz) \\ &\quad + \int\int_{z_1 \neq z_2} f^2(z_1) f(z_2) \hat{N}(dz_1) \hat{N}(dz_2). \end{aligned}$$

General applications to the Gaussian and Poisson cases are discussed, respectively, in Subsection 6.4 and Subsection 6.5.

6.2 Contractions

As anticipated, the statement of Theorem 6.1 contains two well-known *multiplication formulae*, associated with the Gaussian and Poisson cases. In order to state these results, we shall start with a standard definition of the *contraction kernels* associated with two symmetric functions f and g. Roughly speaking, given $f \in L^2_s(\nu^p)$ and $g \in L^2_s(\nu^q)$, the contraction of f and g on $Z^{p+q-r-l}$ ($r = 0, ..., q \wedge p$ and $l = 1, ..., r$), noted $f \star^l_r g$, is obtained by reducing the number of variables in the tensor product $f(x_1, ..., x_p)\, g(x_{p+1}, ..., x_{p+q})$ as follows: r variables are identified, and of these, l are integrated out with respect to ν. The formal definition of $f \star^l_r g$ is given below.

Definition 7 *Let ν be a σ-finite measure on $(Z, \mathcal{Z})$. For every $q, p \geq 1$, $f \in L^2(\nu^p)$, $g \in L^2(\nu^q)$ (not necessarily symmetric), $r = 0, ..., q \wedge p$ and $l = 1, ..., r$, the contraction (of index (r, l)) of f and g on $Z^{p+q-r-l}$, is the function $f \star^l_r g$ of $p + q - r - l$ variables defined as follows:*

$$\begin{aligned} & f \star^l_r g(\gamma_1, \ldots, \gamma_{r-l}, t_1, \ldots, t_{p-r}, s_1, \ldots, s_{q-r}) \\ = & \int_{Z^l} f(z_1, \ldots, z_l, \gamma_1, \ldots, \gamma_{r-l}, t_1, \ldots, t_{p-r}) \times \\ & \qquad \times g(z_1, \ldots, z_l, \gamma_1, \ldots, \gamma_{r-l}, s_1, \ldots, s_{q-r}) \nu^l(dz_1 ... dz_l)\,. \end{aligned} \tag{6.10}$$

and, for $l = 0$,

$$\begin{aligned} & f \star^0_r g(\gamma_1, \ldots, \gamma_r, t_1, \ldots, t_{p-r}, s_1, \ldots, s_{q-r}) \\ = & f(\gamma_1, \ldots, \gamma_r, t_1, \ldots, t_{p-r}) g(\gamma_1, \ldots, \gamma_r, s_1, \ldots, s_{q-r}), \end{aligned} \tag{6.11}$$

so that

$$f \star^0_0 g(t_1, \ldots, t_p, s_1, \ldots, s_q) = f(t_1, \ldots, t_p) g(s_1, \ldots, s_q).$$

For instance, if $p = q = 2$, one gets

$$f \star^0_1 g(\gamma, t, s) = f(\gamma, t)\, g(\gamma, s)\,, \quad f \star^1_1 g(t, s) = \int_Z f(z, t)\, g(z, s)\, \nu(dz) \tag{6.12}$$

$$f \star^1_2 g(\gamma) = \int_Z f(z, \gamma)\, g(z, \gamma)\, \nu(dz)\,, \tag{6.13}$$

$$f \star^2_2 g = \int_Z \int_Z f(z_1, z_2)\, g(z_1, z_2)\, \nu(dz_1)\, \nu(dz_2)\,. \tag{6.14}$$

One also has

$$\begin{aligned} & f \star^r_r g(x_1, ..., x_{p+q-2r}) \\ = & \int_{Z^r} f(z_1, ..., z_r, x_1, ..., x_{p-r})\, g(z_1, ..., z_r, x_{p-r+1}, ..., x_{p+q-2r})\, \nu(dz_1) \cdots \nu(dz_r)\,, \end{aligned} \tag{6.15}$$

but, in analogy with (6.3), we set $\star^r_r = \otimes_r$, and consequently write

$$f \star^r_r g(x_1, ..., x_{p+q-2r}) = f \otimes_r g(x_1, ..., x_{p+q-2r})\,, \tag{6.16}$$

so that, in particular,

$$f \star^0_0 g = f \otimes_0 g.$$

The following elementary result is proved by using the Cauchy-Schwarz inequality. It ensures that the contractions of the type (6.16) are still square-integrable kernels.

Lemma 6.1 *Let $f \in L^2(\nu^p)$ and $g \in L^2(\nu^q)$. Then, for every $r = 0, ..., p \wedge q$, one has that $f \otimes_r g \in L^2\left(\nu^{p+q-2r}\right)$.*

Proof. Just write

$$\begin{aligned} & \int_{Z^{p+q-2r}} (f \otimes_r g)^2 \, d\nu^{p+q-2r} \\ = & \int_{Z^{p+q-2r}} \left(\int_{Z^r} f(a_1, ..., a_r, z_1, ..., z_{p-r}) \right. \\ & \left. g(a_1, ..., a_r, z_{p-r+1}, ..., z_{p+q-r}) \, \nu^r(da_1, ...da_r) \right)^2 \nu^{p+q-2r}(dz_1, ..., dz_{p+q-r}) \\ \leq & \|f\|^2_{L^2(\nu^p)} \times \|g\|^2_{L^2(\nu^q)} . \end{aligned}$$

■

6.3 Symmetrization of contractions

Suppose that $f \in L^2(\nu^p)$ and $g \in L^2(\nu^q)$, and let " $\sim$ " denote symmetrization. Then $f = \widetilde{f}$ and $g = \widetilde{g}$. However, in general, the fact that f and g are symmetric *does not* imply that the contraction $f \otimes_r g$ is symmetric. For instance, if $p = q = 1$,

$$\widetilde{f \otimes_0 g}(s,t) = \frac{1}{2}\left[f(s) g(t) + g(s) f(t)\right];$$

if $p = q = 2$

$$\widetilde{f \otimes_1 g}(s,t) = \frac{1}{2}\int_Z \left[f(x,s) g(x,t) + g(x,s) f(x,t)\right] \nu(dx) .$$

In general, due to the symmetry of f and g, for every $p, q \geq 1$ and every $r = 0, ..., p \wedge q$ one has the relation

$$\begin{aligned} \widetilde{f \otimes_r g}(t_1, ..., t_{p+q-2r}) = & \frac{1}{\binom{p+q-2r}{p-r}} \times \\ & \times \sum_{1 \leq i_1 < \cdots < i_{p-r} \leq p+q-2r} \int_{Z^r} f\left(\mathbf{t}_{(i_1,...,i_{p-r})}, \mathbf{a}_r\right) g\left(\mathbf{t}_{(i_1,...,i_{p-r})^c}, \mathbf{a}_r\right) \nu^r(d\mathbf{a}_r) , \end{aligned}$$

where we used the shorthand notation

$$\begin{aligned} \mathbf{t}_{(i_1,...,i_{p-r})} &= \left(t_{i_1}, ..., t_{i_{p-r}}\right) \\ \mathbf{t}_{(i_1,...,i_{p-r})^c} &= (t_1, ..., t_{p+q-2r}) \setminus \left(t_{i_1}, ..., t_{i_{p-r}}\right) \\ \mathbf{a}_r &= (a_1, ..., a_r) \\ \nu^r(d\mathbf{a}_r) &= \nu^r(da_1, ..., da_r) . \end{aligned}$$

Using the definition (6.10), one has also that $\widetilde{f \star_r^l g}$ indicates the symmetrization of $f \star_r^l g$, where $l < r$. For instance, if $p = 3$, $q = 2$, $r = 2$ and $l = 1$, one has that

$$f \star_r^l g(s,t) = f \star_2^1 g(s,t) = \int_Z f(z,s,t) \, g(z,s) \, \nu(dz) ,$$

and consequently, since f is symmetric,

$$\widetilde{f \star_r^l g}(s,t) = \widetilde{f \star_2^1 g}(s,t) = \frac{1}{2}\int_Z \left[f(z,s,t)\, g(z,s) + f(z,s,t)\, g(z,t)\right] \nu(dz).$$

6.4 The product of two integrals in the Gaussian case

The main result of this section is the following general formula for products of Gaussian multiple integrals.

Proposition 6.1 *Let $\varphi = G$ be a centered Gaussian measure with σ-finite and non-atomic control measure ν. Then, for every $q, p \geq 1$, $f \in L_s^2(\nu^p)$ and $g \in L_s^2(\nu^q)$,*

$$I_p^G(f)\, I_q^G(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}^G\left(\widetilde{f \otimes_r g}\right), \tag{6.17}$$

where the symbol $(\widetilde{\ })$ indicates a symmetrization, the contraction $f \otimes_r g$ is defined in (6.16), and for $p = q = r$, we write

$$\begin{aligned} & I_0^G\left(\widetilde{f \otimes_p g}\right) \\ = \;& f \otimes_r g = \int_{Z^p} f(z_1,...,z_p)\, g(z_1,...,z_p)\, \nu(dz_1)\cdots\nu(dz_p) \\ = \;& (f,g)_{L^2(\nu^p)}. \end{aligned}$$

Remark. Since, in general, one has that $I_n^G(\tilde{h}) = I_n^G(h)$ (see formula (5.43)), one could dispense with the symmetrization " $\widetilde{\ }$ " in formula (6.17).

Proof of Proposition 6.1. We start by assuming that $f \in \mathcal{E}_{s,0}(\nu^p)$ and $g \in \mathcal{E}_{s,0}(\nu^q)$, and we denote by π^* the partition of the set $[p+q] = \{1,...,p+q\}$ given by

$$\pi^* = \{\{1,...,p\}, \{p+1,...,p+q\}\}.$$

According to formula (6.2)

$$I_p^G(f)\, I_q^G(g) = \sum_{\sigma\in\mathcal{P}([p+q]):\sigma\wedge\pi^*=\hat{0}} \mathrm{St}_\sigma^{G,[n]}(f\otimes_0 g).$$

Every partition $\sigma \in \mathcal{P}([p+q])$ such that $\sigma \wedge \pi^* = \hat{0}$ is necessarily composed of r $(0 \leq r \leq p\wedge q)$ two-elements blocks of the type $\{i,j\}$ where $i \in \{1,...,p\}$ and $j \in \{p+1,...,p+q\}$, and $p+q-2r$ singletons. Moreover, for every fixed $r \in \{0,...,p\wedge q\}$, there are exactly $r!\binom{p}{r}\binom{q}{r}$ partitions of this type. To see this, observe that, to build such a partition, one should first select one of the $\binom{p}{r}$ subsets of size r of $\{1,...,p\}$, say A_r, and a one of the $\binom{q}{r}$ subset of size r of $\{p+1,...,p+q\}$, say B_r, and then choose one of the $r!$ bijections between A_r and B_r. When $r = 0$, and therefore $\sigma = \hat{0}$, one obtains immediately

$$\mathrm{St}_\sigma^{G,[p+q]}(f\otimes_0 g) = \mathrm{St}_{\hat{0}}^{G,[p+q]}\left(\widetilde{f\otimes_0 g}\right) = I_{p+q}^G\left(\widetilde{f\otimes_0 g}\right),$$

where the first equality is a consequence of the symmetry of $\mathrm{St}_{\hat{0}}^{G,[n]}$ (see Proposition 5.3). On the other hand, we claim that every partition $\sigma \in \mathcal{P}([p+q])$ such that $\sigma \wedge \pi^* = \hat{0}$ and σ contains $r \geq 1$ two-elements blocks of the type $b = \{i,j\}$ (with $i \in \{1,...,p\}$ and $j \in \{p+1,...,p+q\}$) and $p+q-2r$ singletons, satisfies also

$$\mathrm{St}_{\sigma}^{G,[p+q]}(f \otimes_0 g) = \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}(f \otimes_r g) = I_{p+q-2r}^{G}\left(\widetilde{f \otimes_r g}\right). \tag{6.18}$$

We give a proof of (6.18). Consider first the (not necessarily symmetric) functions

$$f^{\circ} = \mathbf{1}_{A_1 \times \cdots \times A_p} \quad \text{and} \quad g^{\circ} = \mathbf{1}_{A_{p+1} \times \cdots \times A_{p+q}},$$

where $A_l \in \mathcal{Z}_{\nu}$, $l = 1,...,p+q$. Then, one may use (5.53), in order to obtain

$$\begin{aligned}
\mathrm{St}_{\sigma}^{G,[p+q]}(f^{\circ} \otimes_0 g^{\circ}) &= \prod_{b=\{i,j\}\in\sigma} \nu(A_i \cap A_j)\, \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}\left(A_{j_1} \times \cdots \times A_{j_{p+q-2r}}\right) \\
&= \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}\left(\prod_{b=\{i,j\}\in\sigma} \nu(A_i \cap A_j)\, \mathbf{1}_{A_{j_1} \times \cdots \times A_{j_{p+q-2r}}}\right),
\end{aligned}$$

where $\{\{j_1\},...,\{j_{p+q-2r}\}\}$ are the singletons of σ (by the symmetry of $\mathrm{St}_{\hat{0}}^{G,[p+q-2r]}$ we can always suppose, here and in the following, that the singletons $\{j_1\},...,\{j_{p-r}\}$ are contained in $\{1,...,p\}$ and that the singletons $\{j_{p-r+1}\},...,\{j_{p+q-2r}\}$ are in $\{p+1,...,p+q\}$). For every pair of permutations $w \in \mathfrak{S}_{[p]}$ and $w' \in \mathfrak{S}_{[p+1,p+q]}$ (the group of permutations of the set $[p+1,p+q] = \{p+1,...,p+q\}$), we define $f^{\circ,w} = \mathbf{1}_{A_1^w \times \cdots \times A_p^w}$ and $g^{\circ,w'} = \mathbf{1}_{A_{p+1}^{w'} \times \cdots \times A_{p+q}^{w'}}$, where $A_j^w = A_{w(j)}$, $j = 1,...,p$ (and analogously for w'). In this way,

$$\begin{aligned}
&\mathrm{St}_{\sigma}^{G,[p+q]}\left(f^{\circ,w} \otimes_0 g^{\circ,w'}\right) \qquad (6.19)\\
&= \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}\left(\prod_{b=\{i,j\}\in\sigma} \nu\left(A_i^w \cap A_j^{w'}\right) \mathbf{1}_{A_{j_1}^w \times \cdots \times A_{j_{p-r}}^w \times A_{j_{p-r+1}}^{w'} \times \cdots \times A_{j_{p+q-2r}}^{w'}}\right) \\
&= \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}\left(\prod_{b=\{i,j\}\in\sigma} \nu\left(A_i^w \cap A_j^{w'}\right) \mathbf{1}_{A_{j_1}^w \times \cdots \times A_{j_{p-r}}^w \times \widetilde{A_{j_{p-r+1}}^{w'}} \times \cdots \times A_{j_{p+q-2r}}^{w'}}\right).
\end{aligned}$$

Now write

$$f = \sum_{w \in \mathfrak{S}_{[p]}} f^{\circ,w} \quad \text{and} \quad g = \sum_{w \in \mathfrak{S}_{[p+1,p+q]}} g^{\circ,w},$$

and observe that (by using (6.16))

$$\sum_{w \in \mathfrak{S}_{[p]}} \sum_{w' \in \mathfrak{S}_{[p+1,p+q]}} \prod_{b=\{i,j\}\in\sigma} \nu\left(A_i^w \cap A_j^{w'}\right) \mathbf{1}_{A_{j_1}^w \times \cdots \times A_{j_{p-r}}^w \times \widetilde{A_{j_{p-r+1}}^{w'}} \times \cdots \times A_{j_{p+q-2r}}^{w'}} = \widetilde{f \otimes_r g}.$$

Since (6.19) gives

$$\begin{aligned}
&\mathrm{St}_{\sigma}^{G,[p+q]}(f \otimes_0 g) \\
&= \mathrm{St}_{\hat{0}}^{G,[p+q-2r]}\left(\sum_{w \in \mathfrak{S}_{[p]}} \sum_{w' \in \mathfrak{S}_{[p+1,p+q]}} \prod_{b=\{i,j\}\in\sigma} \nu\left(A_i^w \cap A_j^{w'}\right) \mathbf{1}_{A_{j_1}^w \times \cdots \times A_{j_{p-r}}^w \times \widetilde{A_{j_{p-r+1}}^{w'}} \times \cdots \times A_{j_{p+q-2r}}^{w'}}\right),
\end{aligned}$$

we obtain (6.18), so that, in particular, (6.17) is proved for symmetric simple functions vanishing on diagonals. The general result is obtained by using the fact that the linear spaces $\mathcal{E}_{s,0}(\nu^p)$ and $\mathcal{E}_{s,0}(\nu^q)$ are dense, respectively, in $L^2_s(\nu^p)$ and $L^2_s(\nu^q)$. Indeed, to conclude the proof it is sufficient to observe that, if $\{f_k\} \subset \mathcal{E}_{s,0}(\nu^p)$ and $\{g_k\} \subset \mathcal{E}_{s,0}(\nu^q)$ are such that $f_k \to f$ in $L^2_s(\nu^p)$ and $g_k \to g$ in $L^2_s(\nu^q)$, then, for instance by Cauchy-Schwarz, $I^G_p(f_k) I^G_q(g_k) \to I^G_p(f) I^G_q(g)$ in any norm $L^s(\mathbb{P})$, $s \geq 1$ (use e.g. (5.46)), and also

$$\widetilde{f_k \otimes_r g_k} \to \widetilde{f \otimes_r g}$$

in $L^2_s\left(\nu^{p+q-2r}\right)$, so that $I^G_{p+q-2r}\left(\widetilde{f_k \otimes_r g_k}\right) \to I^G_{p+q-2r}\left(\widetilde{f \otimes_r g}\right)$ in $L^2(\mathbb{P})$. ■

Other proofs of Proposition 6.1 can be found e.g. in [49], [16] or [75, Proposition 1.1.3].

Examples. (i) When $p = q = 1$, one obtains

$$I^G_1(f) I^G_1(g) = I^G_2\left(\widetilde{f \otimes_0 g}\right) + I^G_0\left(\widetilde{f \otimes_1 g}\right) = I^G_2\left(\widetilde{f \otimes_0 g}\right) + \langle f, g\rangle_{L^2(\nu)},$$

which is consistent with (6.8).

(ii) When $p = q = 2$, one obtains

$$I^G_2(f) I^G_2(g) = I^G_4\left(\widetilde{f \otimes_0 g}\right) + 4I^G_2\left(\widetilde{f \otimes_1 g}\right) + \langle f, g\rangle_{L^2(\nu)}.$$

(iii) When $p = 3$ and $q = 2$, one obtains

$$I^G_3(f) I^G_2(g) = I^G_5\left(\widetilde{f \otimes_0 g}\right) + 6I^G_3\left(\widetilde{f \otimes_1 g}\right) + 6I^G_1(f \otimes_2 g),$$

where $f \otimes_2 g(z) = \int_{Z^2} f(z,x,y)\, g(x,y)\, \nu(dx)\, \nu(dy)$.

6.5 The product of two integrals in the Poisson case

We now focus on the product of two Poisson integrals.

Proposition 6.2 *Let $\varphi = \hat{N}$ be a compensated Poisson measure, with σ-finite and non-atomic control measure ν. Then, for every $q, p \geq 1$, $f \in \mathcal{E}_{s,0}(\nu^p)$ and $g \in \mathcal{E}_{s,0}(\nu^q)$,*

$$I^{\widehat{N}}_p(f) I^{\widehat{N}}_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} \sum_{l=0}^{r} \binom{r}{l} I^{\widehat{N}}_{p+q-r-l}\left(\widetilde{f \star^l_r g}\right). \tag{6.20}$$

Formula (6.20) continues to hold for functions $f \in L^2_s(\nu^p)$ and $g \in L^2_s(\nu^q)$ such that $f \star^l_r g \in L^2\left(\nu^{q+p-r-l}\right)$, $\forall r = 0, ..., p \wedge q$, $\forall l = 0, ..., r$.

Sketch of the proof. We shall only prove formula (6.20) in the simple case where $p = q = 2$. The generalization to general indices $p, q \geq 1$ (left to the reader) does not present any particular

additional difficulty, except for the need of a rather heavy notation. We shall therefore prove that

$$
\begin{aligned}
I_2^{\widehat{N}}(f)I_2^{\widehat{N}}(g) &= \sum_{r=0}^{2} r!\binom{2}{r}\binom{2}{r}\sum_{l=0}^{r}\binom{r}{l} I_{4-r-l}^{\widehat{N}}(\widetilde{f\star_r^l g}) && (6.21)\\
&= I_4^{\widehat{N}}(\widetilde{f\star_0^0 g}) && (6.22)\\
&\quad +4\left[I_3^{\widehat{N}}(\widetilde{f\star_1^0 g}) + I_2^{\widehat{N}}(\widetilde{f\star_1^1 g})\right] && (6.23)\\
&\quad +2\left[I_2^{\widehat{N}}(\widetilde{f\star_2^0 g}) + 2I_1^{\widehat{N}}(\widetilde{f\star_2^1 g}) + \langle f,g\rangle_{L^2(\nu^2)}\right]. && (6.24)
\end{aligned}
$$

Moreover, by linearity, we can also assume that

$$
f = \mathbf{1}_{A_1\times A_2} + \mathbf{1}_{A_2\times A_1} \quad \text{and} \quad g = \mathbf{1}_{B_1\times B_2} + \mathbf{1}_{B_2\times B_1},
$$

where $A_1 \cap A_2 = B_1 \cap B_2 = \varnothing$. Denote by π^* the partition of $[4] = \{1,...,4\}$ given by

$$
\pi^* = \{\{1,2\},\{3,4\}\},
$$

and apply the general result (6.2) to deduce that

$$
I_2^{\widehat{N}}(f)I_2^{\widehat{N}}(g) = \sum_{\sigma\in\mathcal{P}([4]):\sigma\wedge\pi^*=\hat{0}} \mathrm{St}_\sigma^{\hat{N},[n]}(f\otimes_0 g).
$$

We shall prove that

$$
\sum_{\sigma\in\mathcal{P}([4]):\sigma\wedge\pi^*=\hat{0}} \mathrm{St}_\sigma^{\hat{N},[n]}(f\otimes_0 g) = (6.22)+(6.23)+(6.24).
$$

To see this, observe that the class

$$
\left\{\sigma\in\mathcal{P}([4]) : \sigma\wedge\pi^* = \hat{0}\right\}
$$

contains exactly 7 elements, that is:

(I) the trivial partition $\hat{0}$, containing only singletons;

(II) four partitions $\sigma_1,...,\sigma_4$ containing one block of two elements and two singletons, namely

$$
\begin{aligned}
\sigma_1 &= \{\{1,3\},\{2\},\{4\}\}, \quad \sigma_2 = \{\{1,4\},\{2\},\{3\}\}\\
\sigma_3 &= \{\{1\},\{2,3\},\{4\}\} \quad \text{and} \quad \sigma_4 = \{\{1\},\{2,4\},\{3\}\};
\end{aligned}
$$

(III) two partitions σ_5,σ_6 composed of two blocks of two elements, namely

$$
\sigma_5 = \{\{1,3\},\{2,4\}\} \quad \text{and} \quad \sigma_6 = \{\{1,4\},\{2,3\}\}.
$$

By definition, one has that

$$\mathrm{St}_{\hat{0}}^{\hat{N},[n]}(f\otimes_0 g)=I_4^{\widehat{N}}(\widetilde{f\star_0^0 g}),$$

giving (6.22). Now consider the partition σ_1, as defined in Point **(II)** above. By using the notation (5.54), one has that $\mathbf{B}_2(\sigma_1)=\{\{1,3\}\}$, $|\mathbf{B}_2(\sigma_1)|=1$ and $\mathbf{PB}_2(\sigma_1)=\varnothing$. It follows from formula (5.58) that

$$\begin{aligned}
&\mathrm{St}_{\sigma_1}^{\hat{N},[4]}(f\otimes_0 g)=\mathrm{St}_{\sigma_1}^{\hat{N},[4]}\left((\mathbf{1}_{A_1\times A_2}+\mathbf{1}_{A_2\times A_1})\otimes_0(\mathbf{1}_{B_1\times B_2}+\mathbf{1}_{B_2\times B_1})\right)\\
=\;&\mathrm{St}_{\hat{0}}^{\hat{N},[3]}\left(\mathbf{1}_{(A_1\cap B_1)\times A_2\times B_2}+\mathbf{1}_{(A_1\cap B_2)\times A_2\times B_1}+\mathbf{1}_{(A_2\cap B_1)\times A_1\times B_2}+\mathbf{1}_{(A_2\cap B_2)\times A_1\times B_1}\right)\\
&+\nu(A_1\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_2\times B_2})+\nu(A_1\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_2\times B_1})\\
&+\nu(A_2\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_1\times B_2})+\nu(A_2\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_1\times B_1}).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\mathrm{St}_{\hat{0}}^{\hat{N},[3]}\left(\mathbf{1}_{(A_1\cap B_1)\times A_2\times B_2}+\mathbf{1}_{(A_1\cap B_2)\times A_2\times B_1}+\mathbf{1}_{(A_2\cap B_1)\times A_1\times B_2}+\mathbf{1}_{(A_2\cap B_2)\times A_1\times B_1}\right)\\
=\;&I_3^{\widehat{N}}(\widetilde{f\star_1^0 g}),
\end{aligned}$$

and moreover,

$$\begin{aligned}
&\nu(A_1\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_2\times B_2})+\nu(A_1\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_2\times B_1})+\\
&\nu(A_2\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_1\times B_2})+\nu(A_2\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[2]}(\mathbf{1}_{A_1\times B_1})\\
=\;&I_2^{\widehat{N}}(f\star_1^1 g)=I_2^{\widehat{N}}(\widetilde{f\star_1^1 g}).
\end{aligned}$$

By repeating exactly same argument, one sees immediately that

$$\mathrm{St}_{\sigma_1}^{\hat{N},[4]}(f\otimes_0 g)=\mathrm{St}_{\sigma_i}^{\hat{N},[4]}(f\otimes_0 g),$$

for every for $i=2,3,4$ (the partitions σ_i being defined as in Point (II) above) so that the quantity

$$\sum_{i=1,\ldots,4}\mathrm{St}_{\sigma_i}^{\hat{N},[n]}(f\otimes_0 g)$$

equals necessarily the expression appearing in (6.23). Now we focus on the partition σ_5 appearing in Point (III). Plainly (by using once again the notation introduced in (5.54)), $\mathbf{B}_2(\sigma_5)=\{\{1,3\},\{2,4\}\}$, $|\mathbf{B}_2(\sigma_5)|=2$, and the set $\mathbf{PB}_2(\sigma_5)$ contains two elements, namely

$$(\{\{1,3\}\};\{\{2,4\}\})\quad\text{and}\quad(\{\{2,4\}\};\{\{1,3\}\})$$

(note that we write $\{\{1,3\}\}$ (with two accolades), since the elements of $\mathbf{PB}_2(\sigma_5)$ are pairs of collections of blocks of σ_2, so that $\{\{1,3\}\}$ is indeed the singleton whose only element is $\{1,3\}$).

We can now apply formula (5.58) to deduce that

$$\begin{aligned}
&\mathrm{St}_{\sigma_5}^{\hat{N},[4]}(f\otimes_0 g)=\mathrm{St}_{\sigma_5}^{\hat{N},[4]}\left((\mathbf{1}_{A_1\times A_2}+\mathbf{1}_{A_2\times A_1})\otimes_0(\mathbf{1}_{B_1\times B_2}+\mathbf{1}_{B_2\times B_1})\right) \qquad (6.25)\\
=\;& 2\Big[\nu(A_1\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_2\cap B_2})+\nu(A_1\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_2\cap B_1})\\
&+\nu(A_2\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_1\cap B_2})+\nu(A_2\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_1\cap B_1})\Big]\\
&+\mathrm{St}_{\hat{0}}^{\hat{N},[2]}\left(\mathbf{1}_{(A_1\cap B_1)\times(A_2\cap B_2)}+\mathbf{1}_{(A_1\cap B_2)\times(A_2\cap B_1)}+\mathbf{1}_{(A_2\cap B_1)\times(A_1\cap B_2)}+\mathbf{1}_{(A_2\cap B_2)\times(A_1\cap B_1)}\right)\\
&+\nu(A_1\cap B_1)\,\nu(A_2\cap B_2)+\nu(A_1\cap B_2)\,\nu(A_2\cap B_1)\\
&+\nu(A_2\cap B_1)\,\nu(A_1\cap B_2)+\nu(A_2\cap B_2)\,\nu(A_1\cap B_1).
\end{aligned}$$

One easily verifies that

$$\begin{aligned}
&2\Big[\nu(A_1\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_2\cap B_2})+\nu(A_1\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_2\cap B_1})\\
&+\nu(A_2\cap B_1)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_1\cap B_2})+\nu(A_2\cap B_2)\,\mathrm{St}_{\hat{0}}^{\hat{N},[1]}(\mathbf{1}_{A_1\cap B_1})\Big]\\
=\;& 2I_1^{\hat{N}}(f\star_2^1 g)=2I_1^{\hat{N}}\widetilde{(f\star_2^1 g)}, \qquad (6.26)
\end{aligned}$$

and moreover

$$\begin{aligned}
&\mathbf{1}_{(A_1\cap B_1)\times(A_2\cap B_2)}+\mathbf{1}_{(A_1\cap B_2)\times(A_2\cap B_1)}+\mathbf{1}_{(A_2\cap B_1)\times(A_1\cap B_2)}+\mathbf{1}_{(A_2\cap B_2)\times(A_1\cap B_1)}\\
=\;&\widetilde{f\star_2^0 g} \qquad (6.27)
\end{aligned}$$

and

$$\begin{aligned}
\langle f,g\rangle_{L^2(\nu^2)} =\;& \nu(A_1\cap B_1)\,\nu(A_2\cap B_2)+\nu(A_1\cap B_2)\,\nu(A_2\cap B_1) \qquad (6.28)\\
&+\nu(A_2\cap B_1)\,\nu(A_1\cap B_2)+\nu(A_2\cap B_2)\,\nu(A_1\cap B_1).
\end{aligned}$$

Since, trivially, $\mathrm{St}_{\sigma_5}^{\hat{N},[4]}(f\otimes_0 g)=\mathrm{St}_{\sigma_6}^{\hat{N},[4]}(f\otimes_0 g)$, we deduce immediately from (6.25)–(6.28) that the sum $\mathrm{St}_{\sigma_5}^{\hat{N},[4]}(f\otimes_0 g)+\mathrm{St}_{\sigma_6}^{\hat{N},[4]}(f\otimes_0 g)$ equals the expression appearing in (6.24). This proves the first part of the Proposition. The last assertion in the statement can be proved by a density argument, similar to the one used in order to conclude the proof of Proposition 6.1. ■

Other proofs of Proposition 6.2 can be found for instance in [38, 121, 126].

Examples. (i) When $p=q=1$, one obtains

$$I_1^{\hat{N}}(f)\,I_1^{\hat{N}}(g)=I_2^{\hat{N}}\left(\widetilde{f\otimes_0 g}\right)+I_1^{\hat{N}}\left(\widetilde{f\star_1^0 g}\right)+\langle f,g\rangle_{L^2(\nu)}.$$

(ii) When $p=2$, and $q=1$, one has

$$\begin{aligned}
I_2^{\hat{N}}(f)\,I_1^{\hat{N}}(g) =\;& I_3^{\hat{N}}\left(\widetilde{f\otimes_0 g}\right)+2I_2^{\hat{N}}\left(\widetilde{f\star_1^0 g}\right)+2I_1^{\hat{N}}\left(\widetilde{f\star_1^1 g}\right)\\
=\;&\int\int\int_{z_1\neq z_2\neq z_3} f(z_1,z_2)\,g(z_3)\,\hat{N}(dz_1)\,\hat{N}(dz_2)\,\hat{N}(dz_3)\\
&+2\int\int_{z_1\neq z_2} f(z_1,z_2)\,g(z_1)\,\hat{N}(dz_1)\,\hat{N}(dz_2)\\
&+2\int\left(\int f(z_1,x)\,g(z_1)\,\nu(dx)\right)\hat{N}(dz_1).
\end{aligned}$$

7 Diagram formulae

We now want a general formula for computing cumulants and expectations of products of multiple integrals, that is, formulae for objects of the type

$$\mathbb{E}\left[I_{n_1}^{\varphi}(f_1)\times\cdots\times I_{n_k}^{\varphi}(f_k)\right] \quad \text{and} \quad \chi\left(I_{n_1}^{\varphi}(f_1),...,I_{n_k}^{\varphi}(f_k)\right).$$

7.1 Formulae for moments and cumulants

As in the previous sections, we shall focus on completely random measures φ that are also *good* (in the sense of Definition 3), so that moments and cumulants are well-defined. As usual, we shall assume that the control measure ν is non-atomic. This last assumption is not enough, however, because while the measure $\nu(A)=\mathbb{E}\varphi(A)^2$ may be non-atomic, for some $n\geq 2$ the mean measure (concentrated on the "full diagonal")

$$\langle\Delta_n^{\varphi}(A)\rangle \triangleq \mathbb{E}\left[\mathrm{St}_{\hat{1}}^{\varphi,[n]}(A)\right]=\mathbb{E}\left[\varphi^{\otimes n}\{(z_1,...,z_n)\in A: z_1=\cdots=z_n\}\right]$$

may be atomic. We shall therefore assume that φ is "multiplicative", that is, that this phenomenon does not take place for any $n\geq 2$.

Proceeding formally, let φ be a good random measure on Z, and fix $n\geq 2$. Recall that $\mathcal{Z}_{\nu}^n$ denotes the collection of all sets B in $\mathcal{Z}^{\otimes n}$ such that $\nu^{\otimes n}(B)=\nu^n(B)<\infty$ (see (5.3)). As before, for every partition $\pi\in\mathcal{P}([n])$, the class $\mathcal{Z}_{\pi}^n$ is the collection of all π-diagonal elements of $\mathcal{Z}^n$ (see (5.1)). Recall also that $\mathrm{St}_{\pi}^{\varphi,[n]}$ is the restriction of the measure $\varphi^{\otimes n}=\varphi^{[n]}$ on Z_{π}^n (see (5.28)). Now let

$$\left\langle\mathrm{St}_{\pi}^{\varphi,[n]}\right\rangle(C) = \mathbb{E}\left[\mathrm{St}_{\pi}^{\varphi,[n]}(C)\right], \quad C\in\mathcal{Z}_{\nu}^n, \tag{7.1}$$

$$\Delta_1^{\varphi}(A) = \varphi(A), \tag{7.2}$$

$$\Delta_n^{\varphi}(A) = \mathrm{St}_{\hat{1}}^{\varphi,[n]}(\underbrace{A\times\cdots\times A}_{n\text{ times}}), \quad A\in\mathcal{Z}_{\nu}, \tag{7.3}$$

$$\langle\Delta_n^{\varphi}\rangle(A) = \mathbb{E}\left[\Delta_n^{\varphi}(A)\right], \quad A\in\mathcal{Z}_{\nu}. \tag{7.4}$$

Thus, $\Delta_n^{\varphi}(A)$ denotes the random measure concentrated on the full diagonal $z_1=...=z_n$ of the $n$tuple product $A\times\cdots\times A$, and $\langle\cdot\rangle$ denotes expectation.

Definition 8 *We say that the good completely random measure φ is **multiplicative** if the deterministic measure $A\mapsto\langle\Delta_n^{\varphi}\rangle(A)$ is non-atomic for every $n\geq 2$. We show in the examples below that a Gaussian or compensated Poisson measure, with non-atomic control measure ν, is always multiplicative.*

The term "multiplicative" (which we take from [106]) originates from the fact that φ is multiplicative (in the sense of the previous definition) if and only if for every partition π the non-random measure $\left\langle\mathrm{St}_{\pi}^{\varphi,[n]}\right\rangle(\cdot)$ can be written as a product measure. In particular (see Proposition 8 in [106]), the completely random measure φ is multiplicative if and only if for every $\pi\in\mathcal{P}([n])$ and every $A_1,...,A_n\in\mathcal{Z}_{\nu}$,

$$\left\langle\mathrm{St}_{\pi}^{\varphi,[n]}\right\rangle(A_1\times\cdots\times A_n)=\prod_{b\in\pi}\left\langle\mathrm{St}_{\hat{1}}^{\varphi,[|b|]}\right\rangle\left(\underset{j\in b}{\mathrm{X}}A_j\right), \tag{7.5}$$

where, for every $b = \{j_1, ..., j_k\} \in \pi$, we used once again the notation $\underset{j\in b}{\mathrm{X}} A_j \triangleq A_{j_1} \times \cdots \times A_{j_k}$. Note that the RHS of (7.5) involves products over blocks of the partition π, in which there is concentration over the diagonals associated with the blocks. Thus, in view of (5.2) and (7.3), one has that

$$\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle (A_1 \times \cdots \times A_n) = \prod_{b\in\pi} \left\langle \Delta_{|b|}^{\varphi} \right\rangle (\cap_{j\in b} A_j) \,, \tag{7.6}$$

that is, we can express the LHS of (7.5) as a product of measures involving sets in $\mathcal{Z}_\nu$. Observe that one can rewrite relation (7.5) in the following (compact) way:

$$\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle = \bigotimes_{b\in\pi} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle . \tag{7.7}$$

Examples. (i) When φ is Gaussian with non-atomic control measure ν, relation (7.5) implies that $\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle$ is 0 if π contains at least one block b such that $|b| \neq 2$. If, on the other hand, every block of π contains exactly two elements, we deduce from (5.51) and (5.53) that

$$\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle (A_1 \times \cdots \times A_n) = \prod_{b=\{i,j\}\in\pi} \nu (A_i \cap A_j) \,, \tag{7.8}$$

which is not atomic.

(ii) If φ is a compensated Poisson measure with non-atomic control measure ν, then $\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle$ is 0 whenever π contains at least one block b such that $|b| = 1$ (indeed, that block would have measure 0, since φ is centered). If, on the other hand, every block of π has more than two elements, then, by Corollary 5.1

$$\left\langle \mathrm{St}_{\pi}^{\varphi,[n]} \right\rangle (A_1 \times \cdots \times A_n) = \prod_{k=2}^{n} \prod_{b=\{j_1,...,j_k\}\in\pi} \nu (A_{j_1} \cap \cdots \cap A_{j_k}) \,, \tag{7.9}$$

which is non-atomic. See [106] for (quite pathological) examples of non-multiplicative measures.

Notation. In what follows, the notation

$$\int_{Z^n} f(z_1, ..., z_n) \bigotimes_{b\in\pi} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (dz_1, ..., dz_n) \triangleq \bigotimes_{b\in\pi} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (f) \tag{7.10}$$

will be used for every function $f \in \mathcal{E}(\nu^n)$. [7]

The next result gives a new universal combinatorial formula for the computation of the cumulants and the moments associated with the multiple Wiener-Itô integrals with respect to a completely random *multiplicative* measure.

[7] The integral $\int_{Z^n} f\{d \bigotimes_{b\in\pi} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle\}$ in (7.10) is well defined, since the set function $\bigotimes_{b\in\pi} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (\cdot)$ is a σ-additive signed measure (thanks to (5.8)) on the algebra generated by the products of the type $A_1 \times \cdots \times A_n$, where each A_j is in $\mathcal{Z}_\nu$.

Theorem 7.1 (Diagram formulae) *Let φ be a good completely random measure, with non-atomic control measure ν, and suppose that φ is also multiplicative in the sense of Definition 8. For every $n_1, ..., n_k \geq 1$, we write $n = n_1 + \cdots + n_k$, and we denote by π^* the partition of $[n] = \{1, ..., n\}$ given by (6.1). Then, for every collection of kernels $f_1, ..., f_k$ such that $f_j \in \mathcal{E}_{s,0}(\nu^{n_j})$, one has that*

$$\mathbb{E}\left[I^{\varphi}_{n_1}(f_1)\cdots I^{\varphi}_{n_k}(f_k)\right] = \sum_{\left\{\sigma\in\mathcal{P}([n]):\sigma\wedge\pi^*=\hat{0}\right\}} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}^{\varphi,[|b|]}_{\hat{1}} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k), \tag{7.11}$$

and

$$\chi\left(I^{\varphi}_{n_1}(f_1), \cdots, I^{\varphi}_{n_k}(f_k)\right) = \sum_{\substack{\sigma\wedge\pi^*=\hat{0}\\ \sigma\vee\pi^*=\hat{1}}} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}^{\varphi,[|b|]}_{\hat{1}} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k), \tag{7.12}$$

Proof. Formula (7.11) is a consequence of Theorem 6.1 and (7.5). In order to prove (7.12), we shall first show that the following two equalities hold

$$\begin{aligned} &\chi\left(I^{\varphi}_{n_1}(f_1), \cdots, I^{\varphi}_{n_k}(f_k)\right) \\ &= \sum_{\pi^*\leq\rho=(r_1,...,r_l)\in\mathcal{P}([n])} \mu(\rho,\hat{1}) \prod_{j=1}^{l} \mathbb{E}\left[\prod_{a:\{n_1+\cdots+n_{a-1}+1,...,n_1+\cdots+n_a\}\subseteq r_j} I^{\varphi}_{n_a}(f_a)\right] \qquad (7.13)\\ &= \sum_{\pi^*\leq\rho=(r_1,...,r_l)\in\mathcal{P}([n])} \mu(\rho,\hat{1}) \sum_{\substack{\gamma\leq\rho\\ \gamma\wedge\pi^*=\hat{0}}} \bigotimes_{b\in\gamma} \left\langle \mathrm{St}^{\varphi,[|b|]}_{\hat{1}} \right\rangle (f_1 \otimes_0 \cdots \otimes_0 f_k), \qquad (7.14) \end{aligned}$$

where $n_1 + n_0 = 0$ by convention.

The proof of (7.13) uses arguments analogous to those in the proof of Malyshev's formula (4.17). Indeed, one can use relation (3.5) to deduce that

$$\chi\left(I^{\varphi}_{n_1}(f_1), \cdots, I^{\varphi}_{n_k}(f_k)\right) = \sum_{\sigma=\{x_1,...,x_l\}\in\mathcal{P}([k])} (-1)^{l-1}(l-1)! \prod_{j=1}^{l} \mathbb{E}\left[\prod_{a\in x_j} I^{\varphi}_{n_a}(f_a)\right]. \tag{7.15}$$

Now observe that there exists a *bijection*

$$\mathcal{P}([k]) \to \left[\pi^*, \hat{1}\right] : \sigma \mapsto \rho^{(\sigma)},$$

between $\mathcal{P}([k])$ and the segment $\left[\pi^*, \hat{1}\right]$, which is defined as the set of those $\rho \in \mathcal{P}([n])$ such that $\pi^* \leq \rho$, where π^* is given by (6.1). Such a bijection is realized as follows: for every $\sigma = \{x_1, ..., x_l\} \in \mathcal{P}([k])$, define $\rho^{(\sigma)} \in \left[\pi^*, \hat{1}\right]$ by merging two blocks

$$\{n_1 + \cdots + n_{a-1} + 1, ..., n_1 + \cdots + n_a\} \quad \text{and} \quad \{n_1 + \cdots + n_{b-1} + 1, ..., n_1 + \cdots + n_b\}$$

of π^* $(1 \leq a \neq b \leq k)$ if and only if $a \sim_\sigma b$. Note that this construction implies that $|\sigma| = \left|\rho^{(\sigma)}\right| = l$, so that (2.9) yields

$$(-1)^{l-1}(l-1)! = \mu(\sigma,\hat{1}) = \mu\left(\rho^{(\sigma)},\hat{1}\right) \tag{7.16}$$

(observe that the two Möbius functions appearing in (7.16) refer to two different lattices of partitions). Now use the notation $\rho^{(\sigma)} = \left\{r_1^{(\sigma)}, ..., r_l^{(\sigma)}\right\}$ to indicate the blocks of $\rho^{(\sigma)}$: since, by construction,

$$\prod_{j=1}^{l} \mathbb{E}\left[\prod_{a\in x_j} I_{n_a}^{\varphi}(f_a)\right] = \prod_{j=1}^{l} \mathbb{E}\left[\prod_{a:\{n_1+\cdots+n_{a-1}+1,\ldots,n_1+\cdots+n_a\}\subseteq r_j^{(\sigma)}} I_{n_a}^{\varphi}(f_a)\right], \tag{7.17}$$

we immediately obtain (7.13) by plugging (7.16) and (7.17) into (7.15).

To prove (7.14), fix $\rho = \{r_1, ..., r_l\}$ such that $\pi^* \leq \rho$. For $j = 1, ..., l$, we write $\pi^*(j)$ to indicate the partition of the block r_j whose blocks are the sets $\{n_1 + \cdots + n_{a-1} + 1, ..., n_1 + \cdots + n_a\}$ such that

$$\{n_1 + \cdots + n_{a-1} + 1, ..., n_1 + \cdots + n_a\} \subseteq r_j. \tag{7.18}$$

According to (7.11),

$$\mathbb{E}\left[\prod_{a:\{n_1+\cdots+n_{a-1}+1,\ldots,n_1+\cdots+n_a\}\subseteq r_j} I_{n_a}^{\varphi}(f_a)\right] = \sum_{\left\{\sigma\in\mathcal{P}(r_j):\sigma\wedge\pi^*(j)=\hat{0}\right\}} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle \left(\left\{\otimes_{r_j,0} f\right\}\right),$$

where the function $\left\{\otimes_{r_j,0} f\right\}$ is obtained by juxtaposing the $|\pi^*(j)|$ functions f_a such that the index a verifies (7.18). Now observe that $\gamma \in \mathcal{P}([n])$ satisfies

$$\gamma \leq \rho \quad \text{and} \quad \gamma \wedge \pi^* = \hat{0},$$

if and only if γ admits a (unique) representation as a union of the type

$$\gamma = \bigcup_{j=1}^{l} \sigma(j),$$

where each $\sigma(j)$ is an element of $\mathcal{P}(r_j)$ such that $\sigma(j) \wedge \pi^*(j) = \hat{0}$. This yields

$$\begin{aligned} &\prod_{j=1}^{l} \sum_{\left\{\sigma\in\mathcal{P}(r_j):\sigma\wedge\pi^*(j)=\hat{0}\right\}} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle \left(\left\{\otimes_{r_j,0} f\right\}\right) \\ = &\sum_{\substack{\gamma\leq\rho \\ \gamma\wedge\pi^*=\hat{0}}} \bigotimes_{b\in\gamma} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (f_1 \otimes_0 \cdots \otimes_0 f_k). \end{aligned}$$

This relation, together with (7.16) and (7.17), shows that (7.13) implies (7.14). To conclude the proof, just observe that, by inverting the order of summation in (7.14), one obtains that

$$\begin{aligned} \chi\left(I_{n_1}^{\varphi}(f_1), \cdots, I_{n_k}^{\varphi}(f_k)\right) &= \sum_{\gamma\wedge\pi^*=\hat{0}} \bigotimes_{b\in\gamma} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (f_1 \otimes_0 \cdots \otimes_0 f_k) \sum_{\pi^*\vee\gamma\leq\rho\leq\hat{1}} \mu(\rho, \hat{1}) \\ &= \sum_{\substack{\gamma\wedge\pi^*=\hat{0} \\ \pi^*\vee\gamma=\hat{1}}} \bigotimes_{b\in\gamma} \left\langle \mathrm{St}_{\hat{1}}^{\varphi,[|b|]} \right\rangle (f_1 \otimes_0 \cdots \otimes_0 f_k), \end{aligned}$$

where the last equality is a consequence of the relation

$$\sum_{\pi^* \vee \gamma \leq \rho \leq \hat{1}} \mu\left(\rho, \hat{1}\right) = \begin{cases} 1 & \text{if } \pi^* \vee \gamma = \hat{1} \\ 0 & \text{otherwise,} \end{cases}$$

which is in turn a special case of (2.14). ■

Remark. Observe that the only difference between the moment formula (7.11) and the cumulant formula (7.12) is that in the first case the sum is over all σ such that $\sigma \wedge \pi^* = \hat{0} = \{\{1\}, ..., \{n\}\}$, and that in the second case σ must satisfy in addition that $\sigma \vee \pi^* = \hat{1} = \{[n]\}$. Moreover, the relations (7.11) and (7.12) can be restated in terms of diagrams by rewriting the sums as

$$\sum_{\left\{\sigma \in \mathcal{P}([n]) : \sigma \wedge \pi^* = \hat{0}\right\}} = \sum_{\sigma \in \mathcal{P}([n]) : \Gamma(\pi^*, \sigma) \text{ is non-flat}} \quad ; \quad \sum_{\substack{\sigma \wedge \pi^* = \hat{0} \\ \sigma \vee \pi^* = \hat{1}}} = \sum_{\substack{\sigma \in \mathcal{P}([n]) : \Gamma(\pi^*, \sigma) \text{ is non-flat} \\ \text{and connected}}},$$

where $\Gamma\left(\pi^*, \sigma\right)$ is the diagram of (π^*, σ), as defined in Section 4.1.

7.2 The Gaussian case

We shall now provide a version of Theorem 7.1 in the case where φ is, respectively, Gaussian and Poisson. For convenience, using the same notation as that in Theorem 7.1, let

$$\mathcal{M}\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{P}\left([n]\right) : \sigma \vee \pi^* = \hat{1} \text{ and } \sigma \wedge \pi^* = \hat{0}\right\} \tag{7.19}$$

$$\mathcal{M}^0\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{P}\left([n]\right) : \sigma \wedge \pi^* = \hat{0}\right\} \tag{7.20}$$

and

$$\mathcal{M}_2\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{M}\left([n], \pi^*\right) : |b| = 2, \ \forall b \in \sigma\right\} \tag{7.21}$$

$$\mathcal{M}_2^0\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{M}^0\left([n], \pi^*\right) : |b| = 2, \ \forall b \in \sigma\right\} \tag{7.22}$$

$$\mathcal{M}_{\geq 2}\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{M}\left([n], \pi^*\right) : |b| \geq 2, \ \forall b \in \sigma\right\} \tag{7.23}$$

$$\mathcal{M}_{\geq 2}^0\left([n], \pi^*\right) \triangleq \left\{\sigma \in \mathcal{M}^0\left([n], \pi^*\right) : |b| \geq 2, \ \forall b \in \sigma\right\} \tag{7.24}$$

where the partition $\pi^* \in \mathcal{P}\left([n]\right)$ is defined in (6.1). The sets $\mathcal{M}_2\left([n], \pi^*\right)$ and $\mathcal{M}_2^0\left([n], \pi^*\right)$ appear in the case where φ is Gaussian. Note that, by using the formalism of diagrams Γ and multigraphs $\hat{\Gamma}$ introduced in Section 4, one has that

$$\mathcal{M}\left([n], \pi^*\right) = \left\{\sigma \in \mathcal{P}\left([n]\right) : \Gamma\left(\pi^*, \sigma\right) \text{ is non-flat and connected}\right\} \tag{7.25}$$

$$\mathcal{M}^0\left([n], \pi^*\right) = \left\{\sigma \in \mathcal{P}\left([n]\right) : \Gamma\left(\pi^*, \sigma\right) \text{ is connected}\right\} \tag{7.26}$$

$$\mathcal{M}_2\left([n], \pi^*\right) = \left\{\sigma \in \mathcal{P}\left([n]\right) : \Gamma\left(\pi^*, \sigma\right) \text{ is Gaussian, non-flat and connected}\right\} \tag{7.27}$$

$$= \left\{\sigma \in \mathcal{P}\left([n]\right) : \hat{\Gamma}\left(\pi^*, \sigma\right) \text{ has no loops and is connected}\right\}$$

$$\mathcal{M}_2^0\left([n], \pi^*\right) = \left\{\sigma \in \mathcal{P}\left([n]\right) : \Gamma\left(\pi^*, \sigma\right) \text{ is Gaussian and non-flat}\right\} \tag{7.28}$$

$$= \left\{\sigma \in \mathcal{P}\left([n]\right) : \hat{\Gamma}\left(\pi^*, \sigma\right) \text{ has no loops}\right\}.$$

Clearly, $\mathcal{M}_2\left([n], \pi^*\right) \subset \mathcal{M}_2^0\left([n], \pi^*\right)$, $\mathcal{M}_2\left([n], \pi^*\right) \subset \mathcal{M}_{\geq 2}\left([n], \pi^*\right)$ and

$$\mathcal{M}_2^0\left([n], \pi^*\right) \subset \mathcal{M}_{\geq 2}^0\left([n], \pi^*\right).$$

The sets $\mathcal{M}_{\geq 2}([n],\pi^*)$ and $\mathcal{M}^0_{\geq 2}([n],\pi^*)$ appear when φ is a compensated Poisson measure, namely $\varphi = \hat{N}$.

Corollary 7.1 (Gaussian measures) *Suppose $\varphi = G$ is a centered Gaussian measure with non-atomic control measure ν, fix integers $n_1,...,n_k \geq 1$ and let $n = n_1 + \cdots + n_k$. Write π^* for the partition of $[n]$ appearing in (6.1). Then, for any vector of functions $(f_1,...,f_k)$ such that $f_j \in L^2_s(\nu^{n_j})$, $j = 1,...,k$, the following relations hold:*

1. *If $\mathcal{M}_2([n],\pi^*) = \varnothing$ (in particular, if n is odd), then $\chi\left(I^G_{n_1}(f_1),\cdots,I^G_{n_k}(f_k)\right) = 0$;*
2. *If $\mathcal{M}_2([n],\pi^*) \neq \varnothing$, then*

$$\chi\left(I^G_{n_1}(f_1),\cdots,I^G_{n_k}(f_k)\right) = \sum_{\sigma\in\mathcal{M}_2([n],\pi^*)} \int_{Z^{n/2}} f_{\sigma,k}\, d\nu^{n/2}, \tag{7.29}$$

   where, for every $\sigma \in \mathcal{M}_2([n],\pi^*)$, the function $f_{\sigma,k}$, of $n/2$ variables, is obtained by identifying the variables x_i and x_j in the argument of $f_1 \otimes_0 \cdots \otimes_0 f_{n_k}$ (as given in (6.3)) if and only if $i \sim_\sigma j$;
3. *If $\mathcal{M}^0_2([n],\pi^*) = \varnothing$ (in particular, if n is odd), then $\mathbb{E}\left(I^G_{n_1}(f_1)\cdots I^G_{n_k}(f_k)\right) = 0$;*
4. *If $\mathcal{M}^0_2([n],\pi^*) \neq \varnothing$,*

$$\mathbb{E}\left(I^G_{n_1}(f_1)\cdots I^G_{n_k}(f_k)\right) = \sum_{\sigma\in\mathcal{M}^0_2([n],\pi^*)} \int_{Z^{n/2}} f_{\sigma,k}\, d\nu^{n/2} \tag{7.30}$$

Proof. First observe that, since $\varphi = G$ is Gaussian, then $\left\langle \mathrm{St}^{G,[|b|]}_{\hat{1}} \right\rangle \equiv 0$ whenever $|b| \neq 2$. Assume for the moment that $f_j \in \mathcal{E}_{s,0}(\nu^{n_j})$, $j = 1,...,k$. In this case, we can apply formula (7.12) and obtain that

$$\begin{aligned}
&\chi\left(I^G_{n_1}(f_1),\cdots,I^G_{n_k}(f_k)\right)\\
=&\sum_{\substack{\{\sigma:\sigma\wedge\pi^*=\hat{0}\ ;\\ \sigma\vee\pi^*=\hat{1}\ ;\ |b|=2\ \forall b\in\sigma\}}} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}^{G,[|b|]}_{\hat{1}} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k)\\
=&\sum_{\sigma\in\mathcal{M}_2([n],\pi^*)} \bigotimes_{b\in\sigma} \left\langle \mathrm{St}^{G,[|b|]}_{\hat{1}} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k),
\end{aligned}$$

where we have used (7.21). The last relation trivially implies Point 1 in the statement. Moreover, since, for every $B, C \in \mathcal{Z}_\nu$,

$$\left\langle \mathrm{St}^{G,[2]}_{\hat{1}} \right\rangle (B\times C) = \left\langle \mathrm{St}^{G,[2]}_{\hat{1}} \right\rangle ((B\cap C)\times(B\cap C)) = \left\langle \Delta^G_2 \right\rangle (B\cap C), \tag{7.31}$$

one deduces immediately that the support of the deterministic measure $\bigotimes_{b\in\sigma} \left\langle \mathrm{St}^{G,[|b|]}_{\hat{1}} \right\rangle$ is contained in the set

$$Z^n_{\geq\sigma} = \{(z_1,...,z_n) : z_i = z_j \text{ for every } i,j \text{ such that } i \sim_\sigma j\}.$$

Since, by (5.51) and (7.31),

$$\left\langle \mathrm{St}_{\hat{1}}^{G,[|b|]} \right\rangle (B \times C) = \nu (B \cap C), \tag{7.32}$$

for every $B, C \in \mathcal{Z}_\nu$, we infer that

$$\bigotimes_{b \in \sigma} \left\langle \mathrm{St}_{\hat{1}}^{G,[|b|]} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k) = \bigotimes_{b \in \sigma} \left\langle \mathrm{St}_{\hat{1}}^{G,[|b|]} \right\rangle (f_\sigma) = \int_{Z^{n/2}} f_{\sigma,k} d\nu^{n/2}. \tag{7.33}$$

where the function $f_{\sigma,k}$ is defined in the statement. To obtain the last equality in (7.33), one should start with functions f_j of type $f_j\left(z_1, ..., z_{n_j}\right) = \mathbf{1}_{C_1^{(j)} \times \cdots \times C_{n_j}^{(j)}}\left(z_1, ..., z_{n_j}\right)$, where the $C_\ell^{(j)} \in \mathcal{Z}_\nu$ are disjoint, and then apply formula (7.8), so that the extension to general functions $f_j \in \mathcal{E}_{s,0}\left(\nu^{n_j}\right)$ is obtained by the multilinearity of the application

$$(f_1, ..., f_k) \mapsto \int_{Z^{n/2}} f_{\sigma,k} d\nu^{n/2}.$$

To obtain (7.29) for general functions $f_1, ..., f_k$ such that $f_j \in L_s^2\left(\nu^{n_j}\right)$, start by observing that $\mathcal{E}_{s,0}\left(\nu^{n_j}\right)$ is dense in $L_s^2\left(\nu^{n_j}\right)$, and then use the fact that, if a sequence $f_1^{(r)}, ..., f_k^{(r)}$, $r \geq 1$, is such that $f_j^{(r)} \in \mathcal{E}_{s,0}\left(\nu^{n_j}\right)$ and $f_j^{(r)} \to f_j$ in $L_s^2\left(\nu^{n_j}\right)$ $(j = 1, ..., k)$, then

$$\chi\left(I_{n_1}^G\left(f_1^{(r)}\right), \cdots, I_{n_k}^G\left(f_k^{(r)}\right)\right) \to \chi\left(I_{n_1}^G(f_1), \cdots, I_{n_k}^G(f_k)\right),$$

by (5.46), and moreover

$$\int_{Z^{n/2}} f_{\sigma,k}^{(r)} d\nu^{n/2} \to \int_{Z^{n/2}} f_{\sigma,k} d\nu^{n/2},$$

where $f_{\sigma,k}^{(r)}$ is constructed from $f_1^{(r)}, ..., f_k^{(r)}$, as specified in the statement (a similar argument was needed in the proof of Proposition 6.1). Points 3 and 4 in the statement are obtained analogously, by using the relations

$$\begin{aligned} & \mathbb{E}\left(I_{n_1}^G(f_1), \cdots, I_{n_k}^G(f_k)\right) \\ = & \sum_{\substack{\{\sigma : \sigma \wedge \pi^* = \hat{0}\ ; \\ |b| = 2\ \forall b \in \sigma\}}} \bigotimes_{b \in \sigma} \left\langle \mathrm{St}_{\hat{1}}^{G,[|b|]} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k) \\ = & \sum_{\sigma \in \mathcal{M}_2^0([n], \pi^*)} \bigotimes_{b \in \sigma} \left\langle \mathrm{St}_{\hat{1}}^{G,[|b|]} \right\rangle (f_1 \otimes_0 f_2 \otimes_0 \cdots \otimes_0 f_k), \end{aligned}$$

and then by applying the same line of reasoning as above. ■

Examples. (i) We want to use Corollary 7.1 to compute the cumulant of the two integrals

$$\begin{aligned} I_{n_1}^G(f_1) &= \int_{Z_{\hat{0}}^{n_1}} f_1(z_1, ..., z_{n_1})\, G(dz_1) \cdots G(dz_{n_1}) \\ I_{n_2}^G(f_2) &= \int_{Z_{\hat{0}}^{n_2}} f_2(z_1, ..., z_{n_2})\, G(dz_1) \cdots G(dz_{n_2}), \end{aligned}$$

that is, the quantity

$$\chi\left(I_{n_1}^G(f_1), I_{n_2}^G(f_2)\right) = \mathbb{E}\left(I_{n_1}^G(f_1) I_{n_2}^G(f_2)\right).$$

Here, $\pi^* \in \mathcal{P}([n_1+n_2])$ is given by $\pi^* = \{\{1,...,n_1\}, \{n_1+1,...,n_1+n_2\}\}$. It is easily seen that $\mathcal{M}_2([n_1+n_2], \pi^*) \neq \varnothing$ if and only if $n_1 = n_2$. Indeed, each partition $\mathcal{M}_2([n_1+n_2], \pi^*)$ is of the form

$$\sigma = \{\{i_1, i_2\} : i_1 \in \{1,...,n_1\}, i_2 \in \{n_1+1,...,n_1+n_2\}\} \tag{7.34}$$

(this is the case because σ must have blocks of size $|b| = 2$ only, and no blocks can be constructed using only the indices $\{1,...,n_1\}$ or $\{n_1+1,...,n_1+n_2\}$, since the corresponding diagram must be non-flat). In the case where $n_1 = n_2$, there are exactly $n_1!$ partitions as in (7.34), since to each element in $\{1,...,n_1\}$ one attaches one element of $\{n_1+1,...,n_1+n_2\}$. Moreover, for any such σ one has that

$$\int_{Z^{n/2}} f_{\sigma,2} d\nu^{n/2} = \int_{Z^{n_1}} f_1 f_2 d\nu^{n_1}, \tag{7.35}$$

where $n = n_1 + n_2$ and we have used the symmetry of f_1 and f_2 to obtain that

$$f_{\sigma,2}\left(z_1,...,z_{\frac{n}{2}}\right) = f_{\sigma,2}(z_1,...,z_{n_1}) = f_1(z_1,...,z_{n_1}) f_2(z_1,...,z_{n_1}).$$

From (7.30) and (7.35), we deduce that

$$\mathbb{E}\left(I_{n_1}^G(f_1) I_{n_2}^G(f_2)\right) = \mathbf{1}_{n_1=n_2} \times n_1! \int_{Z^{n_1}} f_1 f_2 d\nu^{n_1},$$

as expected (see (5.42)). Note also that, since every diagram associated with π^* has two rows, one also has

$$\mathcal{M}_2([n_1+n_2], \pi^*) = \mathcal{M}_2^0([n_1+n_2], \pi^*),$$

that is, every non-flat diagram is also connected, thus yielding (thanks to (7.29) and (7.30))

$$\chi\left(I_{n_1}^G(f_1), I_{n_2}^G(f_2)\right) = \mathbb{E}\left(I_{n_1}^G(f_1) I_{n_2}^G(f_2)\right).$$

(ii) We fix an integer $k \geq 3$ and set $n_1 = ... = n_k = 1$, that is, we focus on functions f_j, $j = 1,...,k$, of one variable, so that the integral $I_1^G(f_j)$ is Gaussian for every j, and we consider $\chi\left(I_1^G(f_1),...,I_1^G(f_k)\right)$ and $\mathbb{E}\left[I_1^G(f_1),...,I_1^G(f_k)\right]$. In this case, $n_1 + \cdots + n_k = k$, and $\pi^* = \{\{1\},...,\{k\}\} = \hat{0}$. For instance, for $k = 6$, π^* is represented in Fig. 19.

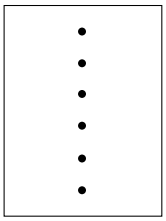

Figure 19: *A representation of the partition* $\hat{0}$

In that case $\mathcal{M}_2([k],\pi^*)=\varnothing$, because all diagrams will be disconnected. One of such diagrams is represented in Fig. 20.

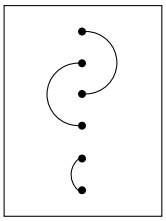

Figure 20: *A disconnected Gaussian diagram*

(**Exercise**: give an algebraic proof of the fact that $\mathcal{M}_2([k],\pi^*)=\varnothing$). It follows from Point 1 of Corollary 7.1 that

$$\chi\left(I_1^G(f_1),...,I_1^G(f_k)\right)=0$$

(this is consistent with the properties of cumulants of Gaussian vectors noted in Section 3). Now focus on $\mathcal{M}_2^0([k],\pi^*)$. If k is odd the class $\mathcal{M}_2^0([k],\pi^*)$ is empty, and, for k even, $\mathcal{M}_2^0([k],\pi^*)$ coincides with the collection of all partitions

$$\sigma=\left\{\{i_1,j_1\},...,\left\{i_{\frac{k}{2}},j_{\frac{k}{2}}\right\}\right\}\in\mathcal{P}([k]) \tag{7.36}$$

whose blocks have size two (that is, $\mathcal{M}_2^0([k],\pi^*)$ is the class of all perfect matchings of the first k integers). For σ as in (7.36), we have

$$f_{\sigma,k}\left(z_1,...,z_{\frac{k}{2}}\right)=\prod_{\substack{\{i_l,j_l\}\in\sigma\\ l=1,...,k/2}}f_{i_l}(z_l)\,f_{j_l}(z_l).$$

Points 3 and 4 of Corollary 7.1 yield therefore

$$\begin{aligned}&\mathbb{E}\left(I_1^G(f_1)\cdots I_1^G(f_k)\right)\\ =\;&\begin{cases}\sum_{\sigma=\left\{\{i_1,j_1\},...,\left\{i_{k/2},j_{k/2}\right\}\right\}\in\mathcal{P}([k])}\int_Z f_{i_1}f_{j_1}d\nu\cdots\int_Z f_{i_{k/2}}f_{j_{k/2}}d\nu, & k\text{ even}\\ 0, & k\text{ odd.}\end{cases},\end{aligned}$$

which is just a special case of (3.16), since $\mathbb{E}\left(I_1^G(f_i)\,I_1^G(f_j)\right)=\int_Z f_{i_1}f_{j_1}d\nu$. For instance, if $k=4$, one has that

$$\begin{aligned}\mathbb{E}\left(I_1^G(f_1)\cdots I_1^G(f_4)\right)\;=\;&\int_Z f_1f_2d\nu\times\int_Z f_3f_4d\nu+\int_Z f_1f_3d\nu\times\int_Z f_2f_4d\nu\\ &\int_Z f_1f_4d\nu\times\int_Z f_2f_3d\nu.\end{aligned}$$

(iii) Consider the case $k=3$, $n_1=2$, $n_2=n_3=1$. Here, $n=n_1+n_2+n_3=4$, and $\pi^*=\{\{1,2\},\{3\},\{4\}\}$. The partition π^* is represented in Fig. 21.

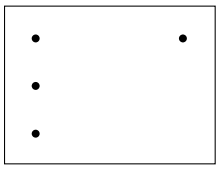

Figure 21: *A three-block partition*

The class $\mathcal{M}_2([4],\pi^*)$ contains only two elements, namely

$$\sigma_1 = \{\{1,3\},\{2,4\}\} \quad \text{and} \quad \sigma_2 = \{\{1,4\},\{2,3\}\},$$

whose diagrams are given in Fig. 22.

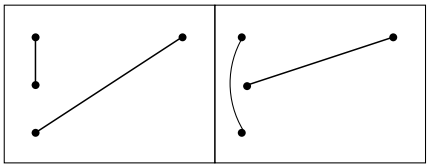

Figure 22: *The two elements of* $\mathcal{M}_2([4],\pi^*)$

Since the rows of these diagrams cannot be divided into two subsets (see Section 4.1), they are connected, and one has $\mathcal{M}_2([4],\pi^*) = \mathcal{M}_2^0([4],\pi^*)$, that is, cumulants equal moments by (7.29) and (7.30). Moreover,

$$\begin{aligned} f_{\sigma_1,3}(z_1,z_2) &= f_1(z_1,z_2) f_2(z_1) f_3(z_2) \\ f_{\sigma_2,3}(z_1,z_2) &= f_1(z_1,z_2) f_2(z_2) f_3(z_1). \end{aligned}$$

It follows that

$$\begin{aligned} \chi\left(I_2^G(f_1), I_1^G(f_2), I_1^G(f_3)\right) &= \mathbb{E}\left(I_2^G(f_1) I_1^G(f_2) I_1^G(f_3)\right) \\ &= \int_{Z^2} \{f_{\sigma_1,3}(z_1,z_2) + f_{\sigma_2,3}(z_1,z_2)\} \nu^2(dz_1,dz_2) \\ &= 2\int_{Z^2} f_1(z_1,z_2) f_2(z_1) f_3(z_2) \nu^2(dz_1,dz_2), \end{aligned}$$

where in the last equality we have used the symmetry of f_1.

(iv) We want to use Point 1 and 2 of Corollary 7.1 to compute the kth cumulant

$$\chi_k\left(I_2^G(f)\right) = \chi(\underbrace{I_2^G(f), ..., I_2^G(f)}_{k \text{ times.}}),$$

for every $k \geq 3$. This can be done by specializing formula (7.29) to the case: $k \geq 3$ and $n_1 = n_2 = ... = n_k = 2$. Here, $n = 2k$ and $\pi^* = \{\{1,2\},\{3,4\},...,\{2k-1,2k\}\}$; for instance, for $k = 4$ the partition π^* can be represented as in Fig. 23.

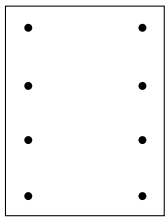

Figure 23: *A four-block partition*

One element of the set $\mathcal{M}_2([2k], \pi^*)$ is given by the partition

$$\sigma^* = \{\{1, 2k\}, \{2, 3\}, \{4, 5\}, ..., \{2k-2, 2k\}\} \in \mathcal{P}([2k]),$$

whose diagram (for $k = 4$) appears in Fig. 24.

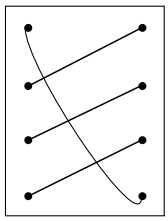

Figure 24: *A circular diagram with four rows*

Note that such a diagram is circular, and that the corresponding multigraph looks like the one in Fig. 16. Therefore,

$$f_{\sigma^*,k}(z_1, ..., z_k) = f(z_1, z_2) f(z_2, z_3) \cdots f(z_{k-1}, z_k) f(z_k, z_1). \tag{7.37}$$

It is not difficult to see that $\mathcal{M}_2([2k], \pi^*)$ contains exactly $2^{k-1}(k-1)!$ elements and that the diagram $\Gamma(\pi^*, \sigma)$ associated to each $\sigma \in \mathcal{M}_2([2k], \pi^*)$ is equivalent (up to a permutation, or equivalently to a renumbering, of the rows) to a circular diagram (see Section 4.1). It follows that, for every $\sigma \in \mathcal{M}_2([2k], \pi^*)$, one has that

$$f_{\sigma,k}(z_1, ..., z_k) = f_{\sigma^*,k}(z_1, ..., z_k),$$

where $f_{\sigma^*,2k}$ is given in (7.37). This yields the classic formula (see e.g. [22])

$$\begin{aligned} & \chi_k\left(I_2^G(f)\right) \\ = \; & 2^{k-1}(k-1)! \int_{Z^k} f(z_1, z_2) f(z_2, z_3) \cdots f(z_{k-1}, z_k) f(z_k, z_1) \nu(dz_1) \cdots \nu(dz_k). \end{aligned} \tag{7.38}$$

For a non-combinatorial proof of (7.38) see e.g. [67, Section 2.2].

7.3 The Poisson case

The following result provides diagram formulae for Wiener-Itô integrals with respect to compensated Poisson measures. It is stated only for elementary functions, so as not to have to deal with convergence issues. The proof is similar to the one of Corollary 7.1, and it is only sketched. We let $\mathcal{M}_{\geq 2}([n],\pi^*)$ and $\mathcal{M}^0_{\geq 2}([n],\pi^*)$ be defined as in Section 7.2.

Corollary 7.2 (Poisson measures) *Suppose $\varphi = \hat{N}$ is a centered Poisson measure with non-atomic control measure ν, fix integers $n_1,...,n_k \geq 1$ and let $n = n_1 + \cdots + n_k$. Write π^* for the partition of $[n]$ appearing in (6.1). Then, for any vector of functions $(f_1,...,f_k)$ such that $f_j \in \mathcal{E}_{s,0}(\nu^{n_j})$, $j = 1,...,k$, the following relations hold:*

1. *If $\mathcal{M}_{\geq 2}([n],\pi^*) = \varnothing$, then $\chi\left(I^{\hat{N}}_{n_1}(f_1),\cdots,I^{\hat{N}}_{n_k}(f_k)\right) = 0$;*

2. *If $\mathcal{M}_{\geq 2}([n],\pi^*) \neq \varnothing$, then*

$$\chi\left(I^{\hat{N}}_{n_1}(f_1),\cdots,I^{\hat{N}}_{n_k}(f_k)\right) = \sum_{\sigma\in\mathcal{M}_{\geq 2}([n],\pi^*)} \int_{Z^{|\sigma|}} f_{\sigma,k}\, d\nu^{|\sigma|}, \tag{7.39}$$

   where, for every $\sigma \in \mathcal{M}_{\geq 2}([n],\pi^*)$, the function $f_{\sigma,k}$, in $|\sigma|$ variables, is obtained by identifying the variables x_i and x_j in the argument of $f_1 \otimes_0 \cdots \otimes_0 f_{n_k}$ (as defined in (6.3)) if and only if $i \sim_\sigma j$;

3. *If $\mathcal{M}^0_{\geq 2}([n],\pi^*) = \varnothing$, then $\mathbb{E}\left(I^{\hat{N}}_{n_1}(f_1)\cdots I^{\hat{N}}_{n_k}(f_k)\right) = 0$;*

4. *If $\mathcal{M}^0_{\geq 2}([n],\pi^*) \neq \varnothing$,*

$$\mathbb{E}\left(I^{\hat{N}}_{n_1}(f_1),\cdots,I^{\hat{N}}_{n_k}(f_k)\right) = \sum_{\sigma\in\mathcal{M}^0_{\geq 2}([n],\pi^*)} \int_{Z^{|\sigma|}} f_{\sigma,k}\, d\nu^{|\sigma|}. \tag{7.40}$$

Sketch of the Proof. The proof follows closely that of Corollary 7.1. The only difference is in evaluating (7.6). Instead of having (7.8) which requires considering $\mathcal{M}_2$ and $\mathcal{M}^0_2$, one has (7.9), which implies that one must use $\mathcal{M}_{\geq 2}$ and $\mathcal{M}^0_{\geq 2}$. ■

Remark. Corollaries 7.1 and 7.2 are quite similar. In the Poisson case, however, $f_{\sigma,k}$ depends on $|\sigma|$ variables, whereas in the Gaussian case it depends on $n/2$ variables.

Examples. All kernels appearing in the following examples are symmetric, elementary and vanishing on diagonals (this ensures that multiple integrals have moments of all orders).

(i) We apply Corollary 7.2 in order to compute the cumulant

$$\chi\left(I^{\hat{N}}_{n_1}(f_1), I^{\hat{N}}_{n_2}(f_2)\right) = \mathbb{E}\left(I^{\hat{N}}_{n_1}(f_1)\, I^{\hat{N}}_{n_2}(f_2)\right),$$

where $n_1, n_2 \geq 1$ are arbitrary. In this case, $\pi^* \in \mathcal{P}([n_1+n_2])$ is given by

$$\pi^* = \{\{1,...,n_1\},\{n_1+1,...,n_1+n_2\}\}.$$

Moreover,

$$\begin{aligned}\mathcal{M}_2^0\left([n_1+n_2],\pi^*\right) &= \mathcal{M}_2\left([n_1+n_2],\pi^*\right)\\ &= \mathcal{M}_{\geq 2}\left([n_1+n_2],\pi^*\right)=\mathcal{M}_{\geq 2}^0\left([n_1+n_2],\pi^*\right)\end{aligned}$$

(indeed, since any diagram of π^* is composed of two rows, every non-flat diagram must be necessarily connected and Gaussian). This gives, in particular, $\mathcal{M}_{\geq 2}\left([n_1+n_2],\pi^*\right)\neq\varnothing$ if and only if $n_1=n_2$. The computations performed in the Gaussian case thus apply and therefore yield

$$\chi\left(I_{n_1}^{\hat{N}}(f_1),I_{n_2}^{\hat{N}}(f_2)\right)=\mathbb{E}\left(I_{n_1}^{\hat{N}}(f_1)\,I_{n_2}^{\hat{N}}(f_2)\right)=\mathbf{1}_{n_1=n_2}\times n_1!\int_{Z^{n_1}}f_1f_2d\nu^{n_1},$$

which is once again consistent with (5.42).

(ii) Consider the case $k=3$, $n_1=n_2=2$, $n_3=1$. Here, $n=n_1+n_2+n_3=5$, and $\pi^*=\{\{1,2\},\{3,4\},\{5\}\}$. The partition π^* can be represented as in Fig. 25.

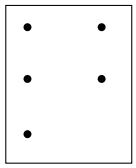

Figure 25: *A three-row partition*

The class $\mathcal{M}_{\geq 2}^0\left([5],\pi^*\right)$, of σ's such that $\sigma\wedge\pi^*=\hat{0}$, contains four elements, that is,

$$\begin{aligned}\sigma_1 &= \{\{1,3,5\},\{2,4\}\},\ \ \sigma_2=\{\{1,4\},\{2,3,5\}\}\\ \sigma_3 &= \{\{1,3\},\{2,4,5\}\}\ \ \text{and}\ \ \sigma_4=\{\{1,4,5\},\{2,3\}\},\end{aligned}$$

whose diagrams are given in Fig. 26.

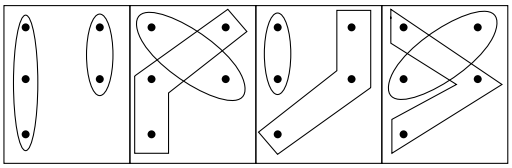

Figure 26: *The four elements of* $\mathcal{M}_{\geq 2}([5],\pi^*)$

Since all these diagrams are connected, the class $\mathcal{M}_{\geq 2}\left([5],\pi^*\right)$ coincides with $\mathcal{M}_{\geq 2}^0\left([5],\pi^*\right)$. Note also that, since the above diagrams have an odd number of vertices, $\mathcal{M}_{\geq 2}\left([5],\pi^*\right)$ does not contain partitions σ whose diagram is Gaussian. Thus,

$$\begin{aligned}f_{\sigma_1,3}(z_1,z_2) &= f_1(z_1,z_2)\,f_2(z_1,z_2)\,f_3(z_1)\\ f_{\sigma_2,3}(z_1,z_2) &= f_1(z_1,z_2)\,f_2(z_2,z_1)\,f_3(z_2)\\ f_{\sigma_3,3}(z_1,z_2) &= f_1(z_1,z_2)\,f_2(z_1,z_2)\,f_3(z_2)\\ f_{\sigma_4,3}(z_1,z_2) &= f_1(z_1,z_2)\,f_2(z_2,z_1)\,f_3(z_1).\end{aligned}$$

For instance, $f_{\sigma_1,3}(z_1,z_2)$ has been obtained by identifying the variables of $f_1(x_1,x_2)\,f_2(x_3,x_4)\,f_3(x_5)$ as $x_1=x_3=x_5=z_1$ and $x_2=x_4=z_2$. By exploiting the symmetry of f_1 and f_2, one deduces that the four quantities

$$\int_{Z^2} f_{\sigma_i,3}(z_1,z_2)\,\nu^2(dz_1,dz_2), \quad i=1,...,4,$$

are equal. It follows from (7.39) and (7.40) that

$$\begin{aligned}\chi\left(I_2^{\hat{N}}(f_1), I_2^{\hat{N}}(f_2), I_1^{\hat{N}}(f_3)\right) &= \mathbb{E}\left(I_2^{\hat{N}}(f_1)\,I_2^{\hat{N}}(f_2)\,I_1^{\hat{N}}(f_3)\right)\\ &= 4\int_{Z^2}\{f_1(z_1,z_2)\,f_2(z_1,z_2)\,f_3(z_1)\}\,\nu^2(dz_1,dz_2).\end{aligned}$$

(iii) Consider the case $k=4$ and $n_i=1$, $i=1,...,4$. Here, $\pi^*=\hat{0}=\{\{1\},\{2\},\{3\},\{4\}\}$, and consequently π^* can be represented as a single column of four vertices. The class $\mathcal{M}_{\geq 2}([4],\pi^*)$ contains only the maximal partition $\hat{1}=\{\{1,2,3,4\}\}$, whereas $\mathcal{M}^0_{\geq 2}([5],\pi^*)$ contains $\hat{1}$ and the three elements

$$\begin{aligned}\sigma_1 &= \{\{1,2\},\{3,4\}\},\ \sigma_2=\{\{1,3\},\{2,4\}\},\ \text{and}\\ \sigma_3 &= \{\{1,4\},\{2,3\}\}.\end{aligned}$$

The diagrams associated with the class $\mathcal{M}^0_{\geq 2}([5],\pi^*)=\left\{\hat{1},\sigma_1,\sigma_2,\sigma_3\right\}$ are represented in Fig. 27.

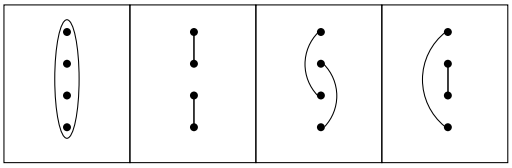

Figure 27: *The elements of* $\mathcal{M}^0_{\geq 2}([5],\pi^*)$

Now take $f_i=f$ for $i=1,...,4$, where f is an elementary kernel. One has that

$$\begin{aligned}f_{\hat{1},4}(z) &= f(z)^4\\ f_{\sigma_j,4}(z_1,z_2) &= f(z_1)^2\,f(z_2)^2, \quad j=1,2,3.\end{aligned}$$

It follows from (7.39) and (7.40) that

$$\begin{aligned}\chi\left(I_1^{\hat{N}}(f), I_1^{\hat{N}}(f), I_1^{\hat{N}}(f), I_1^{\hat{N}}(f)\right) &= \chi_4\left(I_1^{\hat{N}}(f)\right)=\int_Z f(z)^4\,\nu(dz)\\ \mathbb{E}\left(I_1^{\hat{N}}(f)^4\right) &= \int_Z f(z)^4\,\nu(dz)\\ &\quad +3\int_{Z^2} f(z_1)^2\,f(z_2)^2\,\nu^2(dz_1,dz_2).\end{aligned}$$

(iv) Let Y be a centered random variable with finite moments of all orders, and suppose that Y is infinitely divisible and such that

$$\mathbb{E}\left[\exp(i\lambda Y)\right]=\exp\left[\int_{\mathbb{R}}\left(e^{i\lambda u}-1-i\lambda u\right)\rho(du)\right], \tag{7.41}$$

where the measure ρ is such that $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} |u|^k \rho(du) < \infty$ for every $k \geq 1$. Then, combining (7.41) and (3.2), one deduces that

$$\chi_k(Y) = \int_{\mathbb{R}} u^k \rho(du), \quad k \geq 2, \tag{7.42}$$

(note that $\chi_1(Y) = \mathbb{E}(Y) = 0$). We shall prove that (7.42) is consistent with (7.39). Indeed, according to the discussion contained in Section 5.2, one has that

$$Y \overset{law}{=} \int_{\mathbb{R}} \int_0^1 u\hat{N}(du, dx) = I_1^{\hat{N}}(f),$$

where $f(u,x) = u\mathbf{1}_{[0,1]}(x)$, and $\hat{N}$ is a centered Poisson measure on $[0,1] \times \mathbb{R}$, with control $\rho(du)\,dx$. It follows that

$$\chi_k(Y) = \chi_k(I_1^{\hat{N}}(f)) = \chi(\underbrace{I_1^{\hat{N}}(f), ..., I_1^{\hat{N}}(f)}_{k \text{ times}}). \tag{7.43}$$

The RHS of (7.43) can be computed by means of Corollary 7.2 in the special case where $n_j = 1$ $(\forall j = 1, ..., k)$, $n = \Sigma_j n_j = k$, and $\pi^* = \hat{0} = \{\{1\}, ..., \{k\}\}$. One has clearly that $\hat{1}$ is the only partition such that the diagram $\Gamma(\hat{0}, \hat{1})$ is connected, so that

$$\mathcal{M}_{\geq 2}([k], \pi^*) = \{\hat{1}\} = \{\{1, ..., k\}\}.$$

Since

$$f_{\hat{1},k}(u,x) = u^k \mathbf{1}_{[0,1]}(x),$$

we can now use (7.39) to deduce that

$$\chi_k(I_1^{\hat{N}}(f)) = \int_{\mathbb{R}} \int_0^1 f_{\hat{1},k}(u,x)\,\rho(du)\,dx = \int_0^1 dx \int_{\mathbb{R}} u^k \rho(du) = \int_{\mathbb{R}} u^k \rho(du).$$

(v) As an explicit example of (7.42), consider the case where Y is a centered Gamma random variable with shape parameter $a > 0$ and unitary scale parameter, that is,

$$\mathbb{E}[\exp(i\lambda Y)] = \frac{e^{-i\lambda a}}{(1-i\lambda)^a} = \exp\left[a \int_0^\infty \left(e^{i\lambda u} - 1 - i\lambda u\right) e^{-u} \frac{du}{u}\right].$$

Thus, $\rho(du) = a\mathbf{1}_{\{u>0\}} u^{-1} e^{-u} du$. It follows that, $\chi_1(Y) = \mathbb{E}(Y) = 0$ and, for $k \geq 2$,

$$\chi_k(Y) = a \int_{\mathbb{R}} u^k e^{-u} \frac{du}{u} = a\Gamma(k) = a(k-1)!.$$

8 From Gaussian measures to isonormal Gaussian processes

For the sake of completeness, in this section we show how to generalize part of the previous results to the case of an *isonormal Gaussian process*. These objects have been introduced by R.M. Dudley in [17], and are a natural generalization of the Gaussian measures introduced in Section 5.1. In particular, the concept of isonormal Gaussian process can be very useful in the study of fractional fields. See e.g. Pipiras and Taqqu [94, 95, 96], or the second edition of Nualart's book [75]. For a general approach to Gaussian analysis by means of Hilbert space techniques, and for further details on the subjects discussed in this section, the reader is referred to Janson [36].

8.1 General definitions and examples

Let $\mathfrak{H}$ be a real separable Hilbert space with inner product $(\cdot,\cdot)_{\mathfrak{H}}$. In what follows, we will denote by

$$X = X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$$

an *isonormal Gaussian process* over $\mathfrak{H}$. This means that X is a centered real-valued Gaussian family, indexed by the elements of $\mathfrak{H}$ and such that

$$\mathbb{E}\left[X(h)\, X\left(h'\right)\right] = \left(h, h'\right)_{\mathfrak{H}}, \quad \forall h, h' \in \mathfrak{H}. \tag{8.1}$$

In other words, relation (8.1) means that X is a centered Gaussian Hilbert space (with respect to the inner product canonically induced by the covariance) isomorphic to $\mathfrak{H}$.

Example (*Euclidean spaces*). Fix an integer $d \geq 1$, set $\mathfrak{H} = \mathbb{R}^d$ and let $(e_1, ..., e_d)$ be an orthonormal basis of $\mathbb{R}^d$ (with respect to the usual Euclidean inner product). Let $(Z_1, ..., Z_d)$ be a Gaussian vector whose components are i.i.d. $N(0,1)$. For every $h = \sum_{j=1}^{d} c_j e_j$ (where the c_j are real and uniquely defined), set $X(h) = \sum_{j=1}^{d} c_j Z_j$ and define $X = \left\{X(h) : h \in \mathbb{R}^d\right\}$. Then, X is an isonormal Gaussian process over $\mathbb{R}^d$.

Example (*Gaussian measures*). Let $(Z, \mathcal{Z}, \nu)$ be a measure space, where ν is positive, σ-finite and non atomic. Consider a completely random Gaussian measure $G = \{G(A) : A \in \mathcal{Z}_\nu\}$ (as defined in Section 5.1), where the class $\mathcal{Z}_\nu$ is given by (5.3). Set $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$ (thus, for every $h, h' \in \mathfrak{H}$, $(h, h')_{\mathfrak{H}} = \int_Z h(z)h'(z)\nu(dz)$) and, for every $h \in \mathfrak{H}$, define $X(h) = I_1^G(h)$ to be the Wiener-Itô integral of h with respect to G, as defined in (5.14). Recall that $X(h)$ is a centered Gaussian random variable with variance given by $\|h\|_{\mathfrak{H}}^2$. Then, relation (5.15) implies that the collection $X = \left\{X(h) : h \in L^2(Z, \mathcal{Z}, \nu)\right\}$ is an isonormal Gaussian process over $L^2(Z, \mathcal{Z}, \nu)$.

Example (*Isonormal spaces built from covariances*). Let $Y = \{Y_t : t \geq 0\}$ be a real-valued centered Gaussian process indexed by the positive axis, and set $R(s,t) = \mathbb{E}[Y_s Y_t]$ to be the covariance function of Y. Then, one can embed Y into some isonormal Gaussian process as follows: (i) define $\mathcal{E}$ as the collection of all finite linear combinations of indicator functions of the type $\mathbf{1}_{[0,t]}$, $t \geq 0$; (ii) define $\mathfrak{H} = \mathfrak{H}_R$ to be the Hilbert space given by the closure of $\mathcal{E}$ with respect to the inner product

$$(f, h)_R := \sum_{i,j} a_i c_j R(s_i, t_j),$$

where $f = \sum_i a_i \mathbf{1}_{[0,s_i]}$ and $h = \sum_j c_j \mathbf{1}_{[0,t_j]}$ are two generic elements of $\mathcal{E}$; (iii) for $h = \sum_j c_j \mathbf{1}_{[0,t_j]} \in \mathcal{E}$, set $X(h) = \sum_j c_j Y_{t_j}$; (iv) for $h \in \mathfrak{H}_R$, set $X(h)$ to be the $L^2(\mathbb{P})$ limit of any sequence of the type $X(h_n)$, where $\{h_n\} \subset \mathcal{E}$ converges to h in $\mathfrak{H}_R$. Note that such a sequence $\{h_n\}$ necessarily exists and may not be unique (however, the definition of $X(h)$ does not depend on the choice of the sequence $\{h_n\}$). Then, by construction, the Gaussian space $\{X(h) : h \in \mathfrak{H}\}$ is an isonormal Gaussian process over $\mathfrak{H}_R$. See Janson [36, Ch. 1] or Nualart [75] for more details on this construction.

Example (*Even functions and symmetric measures*). Other classic examples of isonormal Gaussian processes (see e.g., [11, 24, 49, 122]) are given by objects of the type $X_\beta =$

$\{X_\beta(\psi) : \psi \in \mathfrak{H}_{E,\beta}\}$, where β is a real non-atomic symmetric measure on $(-\pi, \pi]$ (that is, $\beta(dx) = \beta(-dx)$), and

$$\mathfrak{H}_{E,\beta} = L^2_E((-\pi, \pi], d\beta) \tag{8.2}$$

stands for the collection of *real* linear combinations of complex-valued *even* functions that are square-integrable with respect to β (recall that a function ψ is even if $\overline{\psi(x)} = \psi(-x)$). The class $\mathfrak{H}_{E,\beta}$ is indeed a real Hilbert space, endowed with the inner product

$$(\psi_1, \psi_2)_\beta = \int_{-\pi}^{\pi} \psi_1(x)\, \psi_2(-x)\, \beta(dx) \in \mathbb{R}. \tag{8.3}$$

This type of construction is used in the spectral theory of time series.

8.2 Hermite polynomials and Wiener chaos

We shall now show how to extend the notion of *Wiener chaos* (as defined in Section 5.6) to the case of an isonormal Gaussian process. The reader is referred to [75, Ch. 1] for a complete discussion of this subject. We need some further (standard) definitions.

Definition 9 *The sequence of* ***Hermite polynomials*** $\{H_q : q \geq 0\}$ *on* $\mathbb{R}$*, is defined via the following relations:* $H_0 \equiv 1$ *and, for* $q \geq 1$*,*

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

For instance, $H_1(x) = 1$*,* $H_2(x) = x^2 - 1$ *and* $H_3(x) = x^3 - 3x$*.*

Recall that the sequence $\{(q!)^{-1/2} H_q : q \geq 0\}$ is an orthonormal basis of $L^2(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2} dx)$.

Definition 10 *From now on, the symbol* $\mathcal{A}_\infty$ *will denote the class of those sequences* $\alpha = \{\alpha_i : i \geq 1\}$ *such that: (i) each* α_i *is a nonnegative integer, (ii)* α_i *is different from zero only for a finite number of indices* i*. A sequence of this type is called a* ***multiindex.*** *For* $\alpha \in \mathcal{A}_\infty$*, we use the notation* $|\alpha| = \sum_i \alpha_i$*. For* $q \geq 1$*, we also write*

$$\mathcal{A}_{\infty,q} = \{\alpha \in \mathcal{A}_\infty : |\alpha| = q\}.$$

Remark on notation. Fix $q \geq 2$. Given a real separable Hilbert space $\mathfrak{H}$, we denote by $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$, respectively, the qth *tensor power* of $\mathfrak{H}$ and the qth *symmetric tensor power* of $\mathfrak{H}$ (see e.g. [36]). We conventionally set $\mathfrak{H}^{\otimes 1} = \mathfrak{H}^{\odot 1} = \mathfrak{H}$.

We recall four classic facts concerning tensors powers of Hilbert spaces (see e.g. [36]).

(I) The spaces $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ are real separable Hilbert spaces, such that $\mathfrak{H}^{\odot q} \subset \mathfrak{H}^{\otimes q}$.

(**II**) Let $\{e_j : j \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$; then, an orthonormal basis of $\mathfrak{H}^{\otimes q}$ is given by the collection of all tensors of the type

$$e_{j_1} \otimes \cdots \otimes e_{j_q}, \quad j_1, ..., j_d \geq 1.$$

(**III**) Let $\{e_j : j \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$ and endow $\mathfrak{H}^{\odot q}$ with the inner product $(\cdot, \cdot)_{\mathfrak{H}^{\otimes q}}$; then, an orthogonal (and, in general, *not* orthonormal) basis of $\mathfrak{H}^{\odot q}$ is given by all elements of the type

$$\mathbf{e}(j_1, ..., j_q) = \mathbf{sym}\left\{e_{j_1} \otimes \cdots \otimes e_{j_q}\right\}, \quad 1 \leq j_1 \leq ... \leq j_q < \infty, \tag{8.4}$$

where $\mathbf{sym}\{\cdot\}$ stands for a canonical symmetrization. **Exercise**: find an orthonormal basis of $\mathfrak{H}^{\odot q}$.

(**IV**) If $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, where ν is σ-finite and non-atomic, then $\mathfrak{H}^{\otimes q}$ can be identified with $L^2(Z^q, \mathcal{Z}^q, \nu^q)$ and $\mathfrak{H}^{\odot q}$ can be identified with $L^2_s(Z^q, \mathcal{Z}^q, \nu^q)$, where $L^2_s(Z^q, \mathcal{Z}^q, \nu^q)$ is the subspace of $L^2(Z^q, \mathcal{Z}^q, \nu^q)$ composed of symmetric functions.

Now observe that, once an orthonormal basis of $\mathfrak{H}$ is fixed and due to the symmetrization, each element $\mathbf{e}(j_1, ..., j_q)$ in (8.4) can be completely described in terms of a unique multiindex $\alpha \in \mathcal{A}_{\infty,q}$, as follows: (i) set $\alpha_i = 0$ if $i \neq j_r$ for every $r = 1, ..., q$, (ii) set $\alpha_j = k$ for every $j \in \{j_1, ..., j_q\}$ such that j is repeated exactly k times in the vector $(j_1, ..., j_q)$ $(k \geq 1)$.

Examples. (i) The multiindex $(1, 0, 0, ...)$ is associated with the element of $\mathfrak{H}$ given by e_1.

(ii) Consider the element $\mathbf{e}(1, 7, 7)$. In $(1, 7, 7)$ the number 1 is not repeated and 7 is repeated twice, hence $\mathbf{e}(1, 7, 7)$ is associated with the multiindex $\alpha \in \mathcal{A}_{\infty,3}$ such that $\alpha_1 = 1$, $\alpha_7 = 2$ and $\alpha_j = 0$ for every $j \neq 1, 7$, that is, $\alpha = (1, 0, 0, 0, 0, 0, 2, 0, 0, ...)$.

(iii) The multindex $\alpha = (1, 2, 2, 0, 5, 0, 0, 0, ...)$ is associated with the element of $\mathfrak{H}^{\odot 10}$ given by $\mathbf{e}(1, 2, 2, 3, 3, 5, 5, 5, 5, 5)$.

In what follows, given $\alpha \in \mathcal{A}_{\infty,q}$ $(q \geq 1)$, we shall write $\mathbf{e}(\alpha)$ in order to indicate the element of $\mathfrak{H}^{\odot q}$ uniquely associated with α.

Definition 11 *For every $h \in \mathfrak{H}$, we set $I_1^X(h) = X(h)$. Now fix an orthonormal basis $\{e_j : j \geq 1\}$ of $\mathfrak{H}$: for every $q \geq 2$ and every $h \in \mathfrak{H}^{\odot q}$ such that*

$$h = \sum_{\alpha \in \mathcal{A}_{\infty,q}} c_\alpha e(\alpha)$$

(with convergence in $\mathfrak{H}^{\odot q}$, endowed with the inner product $(\cdot, \cdot)_{\mathfrak{H}^{\otimes q}}$), we set

$$I_q^X(h) = \sum_{\alpha \in \mathcal{A}_{\infty,q}} c_\alpha \prod_j H_{\alpha_j}(X(e_j)), \tag{8.5}$$

*where the products only involve the non-zero terms of each multiindex α, and H_m indicates the mth Hermite polynomial . For $q \geq 1$, the collection of all random variables of the type $I_q^X(h)$, $h \in \mathfrak{H}^{\odot q}$, is called the **$q$th Wiener chaos associated with** X and is denoted by $C_q(X)$. One sets conventionally $C_0(X) = \mathbb{R}$.*

Examples. (i) If $h = e(\alpha)$, where $\alpha = (1,1,0,0,0,...) \in \mathcal{A}_{\infty,2}$, then

$$I_2^X(h) = H_1(X(e_1)) H_1(X(e_2)) = X(e_1) X(e_2).$$

(ii) If $\alpha = (1,0,1,2,0,...) \in \mathcal{A}_{\infty,4}$, then

$$\begin{aligned} I_4^X(h) &= H_1(X(e_1)) H_1(X(e_3)) H_2(X(e_4)) \\ &= X(e_1) X(e_3) \left(X(e_4)^2 - 1\right) \\ &= X(e_1) X(e_3) X(e_4)^2 - X(e_1) X(e_3). \end{aligned}$$

(iii) If $\alpha = (3,1,1,0,0,...) \in \mathcal{A}_{\infty,5}$, then

$$\begin{aligned} I_5^X(h) &= H_3(X(e_1)) H_1(X(e_2)) H_1(X(e_3)) \\ &= \left(X(e_1)^3 - 3X(e_1)\right) X(e_2) X(e_3) \\ &= X(e_1)^3 X(e_2) X(e_3) - 3X(e_1) X(e_2) X(e_3). \end{aligned}$$

The following result collects some well-known facts concerning Wiener chaos and isonormal Gaussian processes. In particular: the first point characterizes the operators I_q^X as isomorphisms; the second point is an equivalent of the chaotic representation property for Gaussian measures, as stated in formula (5.45); the third point establishes a formal relation between random variables of the type $I_q^X(h)$ and the multiple Wiener-Itô integrals introduced in Section 5.4 (see [75, Ch. 1] for proofs and further discussions of all these facts).

Proposition 8.1 *1. For every $q \geq 1$, the qth Wiener chaos $C_q(X)$ is a Hilbert subspace of $L^2(\mathbb{P})$, and the application*

$$h \mapsto I_q^X(h), \quad h \in \mathfrak{H}^{\odot q},$$

defines a Hilbert space isomorphism between $\mathfrak{H}^{\odot q}$, endowed with the inner product $q!(\cdot,\cdot)_{\mathfrak{H}^{\otimes q}}$, and $C_q(X)$.

2. For every $q, q' \geq 0$ such that $q \neq q'$, the spaces $C_q(X)$ and $C_{q'}(X)$ are orthogonal in $L^2(\mathbb{P})$.

3. Let F be a functional of the isonormal Gaussian process X satisfying $\mathbb{E}[F(X)^2] < \infty$: then, there exists a unique sequence $\{f_q : q \geq 1\}$ such that $f_q \in \mathfrak{H}^{\odot q}$, and

$$F = \mathbb{E}(F) + \sum_{q=1}^{\infty} I_q^X(f_q),$$

where the series converges in $L^2(\mathbb{P})$.

4. Suppose that $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, where ν is σ-finite and non-atomic. Then, for $q \geq 2$, the symmetric power $\mathfrak{H}^{\odot q}$ can be identified with $L_s^2(Z^q, \mathcal{Z}^q, \nu^q)$ and, for every $f \in \mathfrak{H}^{\odot q}$, the random variable $I_q^X(f)$ coincides with the Wiener-Itô integral (see Definition 5) of f with respect to the Gaussian measure given by $A \to X(\mathbf{1}_A)$, $A \in \mathcal{Z}_\nu$.

Remark. The combination of Point 1. anf Point 2. in the statement of Proposition 8.1 implies that, for every $q, q' \geq 1$,

$$\mathbb{E}\left[I_q^X(f) I_{q'}^X(f')\right] = \mathbf{1}_{q=q'} q! \left(f, f'\right)_{\mathfrak{H}^{\otimes q}}$$

(compare with (5.42)).

8.3 Contractions, products and some explicit formulae

We start by introducing the notion of *contraction* in the context of powers of Hilbert spaces.

Definition 12 *Consider a real separable Hilbert space $\mathfrak{H}$, and let $\{e_i : i \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$. For every $n, m \geq 1$, every $r = 0, ..., n \wedge m$ and every $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, we define the contraction of order r, of f and g, as the element of $\mathfrak{H}^{\otimes n+m-2r}$ given by*

$$f \otimes_r g = \sum_{i_1,...,i_r=1}^{\infty} (f, e_{i_1} \otimes \cdots \otimes e_{i_r})_{\mathfrak{H}^{\otimes r}} \otimes (g, e_{i_1} \otimes \cdots \otimes e_{i_r})_{\mathfrak{H}^{\otimes r}}, \tag{8.6}$$

and we denote by $\widetilde{f \otimes_r g}$ its symmetrization.

Remark. One can prove (**Exercise!**) the following result: if $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, $f \in \mathfrak{H}^{\odot n} = L^2_s(Z^n, \mathcal{Z}^n, \nu^n)$ and $g \in \mathfrak{H}^{\odot m} = L^2_s(Z^m, \mathcal{Z}^m, \nu^m)$, then the definition of the contraction $f \otimes_r g$ given in (8.6) and the one given in (6.16) coincide.

The following result extends the product formula (6.17) to the case of isonormal Gaussian processes. The proof (which is left to the reader) can be obtained from Proposition 6.1, by using the fact that every real separable Hilbert space is isomorphic to a space of the type $L^2(Z, \mathcal{Z}, \nu)$, where ν is σ-finite and non-atomic. An alternative proof (by induction) can be found in [75, Ch. 1].

Proposition 8.2 *Let X be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$. Then, for every $n, m \geq 1$, $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$,*

$$I^X_n(f) I^X_m(g) = \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I^X_{n+m-2r}\left(\widetilde{f \otimes_r g}\right), \tag{8.7}$$

where the symbol $\widetilde{\ }$ indicates a symmetrization, the contraction $f \otimes_r g$ is defined in (8.6), and for $m = n = r$, we write

$$I^X_0\left(\widetilde{f \otimes_n g}\right) = (f, g)_{\mathfrak{H}^{\otimes n}}.$$

We stress that one can obtain a generalization of the cumulant formulae (7.29) in the framework of isonormal Gaussian processes. To do this, one should represent each integral of the type $\int_{Z^{n/2}} f_\sigma d\nu^{n/2}$, appearing in (7.29), as the inner product between two iterated contractions of the kernels $\{f_{n_j}\}$, and then use the canonical isomorphism between $\mathfrak{H}$ and a space of the form $L^2(Z, \mathcal{Z}, \nu)$. However, the formalism associated with this extension is rather heavy (and not really useful for the discussion to follow), and is left to the reader. Here, we will only state the following formula (proved in [79]) giving an explicit expression for the fourth cumulant of a random variable of the type $I^X_d(f)$, $f \in \mathfrak{H}^{\odot d}$, $d \geq 2$:

$$\chi_4\left(I^X_d(f)\right) = \mathbb{E}\left[I^X_d(f)^4\right] - 3(d!)^2 \|f\|^4_{\mathfrak{H}^{\otimes d}} \tag{8.8}$$

$$= \sum_{p=1}^{d-1} \frac{(d!)^4}{(p!(d-p)!)^2} \left\{ \|f \otimes_p f\|^2_{\mathfrak{H}^{\otimes 2(d-p)}} + \binom{2d-2p}{d-p} \left\| \widetilde{f \otimes_p f} \right\|^2_{\mathfrak{H}^{\otimes 2(d-p)}} \right\}. \tag{8.9}$$

As pointed out in [79, Corollary 2], formula (8.9) can be used in order to prove that, for every isonormal Gaussian process X, every $d \geq 2$ and every $f \in \mathfrak{H}^{\odot d}$, the random variable $I_d^X(f)$ *cannot be Gaussian* (see also [36, Ch. 6]).

Example. We focus once again on isonormal Gaussian processes of the type $X_\beta = \{X_\beta(\psi) : \psi \in \mathfrak{H}_{E,\beta}\}$, where the Hilbert space $\mathfrak{H}_{E,\beta}$ is given in (8.2). In this case, for $d \geq 2$, the symmetric power $\mathfrak{H}_{E,\beta}^{\odot d}$ can be identified with the real Hilbert space of those functions ψ_d that are symmetric on $(-\pi,\pi]^d$, square integrable with respect to β^d, and such that $\overline{\psi_d(x_1,...,x_d)} = \psi_d(-x_1,...,-x_d)$. For every $n_1,...,n_k \geq 1$, one can write explicitly a diagram formula as follows:

$$\chi\left(I_{n_1}^{X_\beta}(\psi_1),\cdots,I_{n_k}^{X_\beta}(\psi_k)\right) = \sum_{\sigma\in\mathcal{M}_2([n],\pi^*)} \int_{Z^{n/2}} \psi_\sigma d\beta^{n/2},$$

where $\mathcal{M}_2([n],\pi^*)$ is defined in (7.21) and ψ_σ is the function in $(n_1+\cdots+n_k)/2$ variables obtained by setting $x_i = -x_j$ in $\psi_1 \otimes_0 \cdots \otimes_0 \psi_d$ if and only if $i \sim_\sigma j$. The field X_β is often defined in terms of a complex Gaussian measure (see [11, 24, 49]).

9 Simplified CLTs, contractions and circular diagrams

In a recent series of papers (see [52, 65, 69, 70, 71, 76, 79, 83, 90, 91] for the Gaussian case, and [86, 87, 88, 89] for the Poisson case) a set of new results has been established, allowing to obtain neat Central Limit Theorems (CLTs) for sequences of random variables belonging to a fixed Wiener chaos of some Gaussian or Poisson field. The techniques adopted in the above references are quite varied, as they involve stochastic calculus ([79, 83, 91]), Malliavin calculus ([65, 76, 90]), Stein's method ([69, 70, 71, 86]) and decoupling ([88, 87, 89]). However, all these contributions may be described as "drastic simplifications" of the method of *moments and cumulants* (see e.g. [11, 49], as well as the discussion below) which is a common tool for proving weak convergence results for non linear functionals of random fields.

The aim of this section is to draw the connection between the above quoted CLTs and the method of moments and cumulants into further light, by providing a detailed discussion of the combinatorial implications of the former. This discussion will involve the algebraic formalism introduced in Section 2–4, as well as the diagram formulae proved in Section 7.

9.1 A general problem

In what follows, we will be interested in several variations of the following problem.

Problem A . *Let φ be a completely random Gaussian or Poisson measure over some space $(Z,\mathcal{Z},\nu)$, where ν is σ-finite and non-atomic. For $m \geq 1$ and $d_1,...,d_m \geq 1$, let $\{f_j^{(k)} : j = 1,...,m,\ \ k \geq 1\}$ be a collection of kernels such that $f_j^{(k)} \in L_s^2\left(Z^{d_j},\mathcal{Z}^{d_j},\nu^{d_j}\right)$ (the vector $(d_1,...,d_m)$ does not depend on k), and*

$$\lim_{k\to\infty} \mathbb{E}\left[I_{d_i}^{\varphi}\left(f_i^{(k)}\right) I_{d_j}^{\varphi}\left(f_j^{(k)}\right)\right] = C(i,j), \quad 1 \leq i,j \leq m, \tag{9.1}$$

where the integrals $I_{d_i}^{\varphi}\left(f_i^{(k)}\right)$ are defined via (5.41) and $\mathbf{C} = \{C(i,j)\}$ is a $m \times m$ positive definite matrix. We denote by $\mathbf{N}_m(0,\mathbf{C})$ a m-dimensional centered Gaussian vector with covariance matrix $\mathbf{C}$. Find conditions on the sequence $\left(f_1^{(k)},...,f_m^{(k)}\right)$, $k \geq 1$, in order to have the

CLT

$$\mathbf{F}_k \triangleq \left(I^{\varphi}_{d_1}\left(f_1^{(k)}\right), ..., I^{\varphi}_{d_m}\left(f_m^{(k)}\right)\right) \xrightarrow{law} \mathbf{N}_m\left(0, \mathbf{C}\right), \quad k \to \infty. \tag{9.2}$$

We observe that, if $d_i \neq d_j$ in (9.1), then necessarily $C(i,j) = 0$ by Point 2 in Proposition 8.1. The relevance of Problem A comes from the chaotic representation (5.45), implying that a result such as (9.2) may be a key tool in order to establish CLTs for more general functionals of the random measure φ. We recall (see e.g. [36, Ch. 6]) that, if φ is Gaussian and $d \geq 2$, then a random variable of the type $I^{\varphi}_d(f)$ cannot be Gaussian.

Plainly, when φ is Gaussian, a solution of Problem A can be immediately deduced from the results discussed in Section 7.2. Indeed, if the normalization condition (9.1) is satisfied, then the moments of the sequence $\{\mathbf{F}_k\}$ are uniformly bounded (to see this, one can use (5.46)), and the CLT (9.2) takes place if and only if every cumulant of order ≥ 3 associated with $\mathbf{F}_k$ converges to zero when $k \to \infty$. Moreover, an explicit expression for the cumulants can be deduced from (7.29). This method of proving the CLT (9.2) (which is known as the *method of cumulants*) has been used e.g. in the references [9, 11, 24, 51, 54, 55], where the authors proved CLTs for non-linear functionals of Gaussian fields with a non trivial covariance structure (for instance, sequences with long memory or isotropic spherical fields). However, such an approach (e.g. in the study of fractional Gaussian processes) may be technically quite demanding, since it involves an infinity of asymptotic relations (one for every cumulant of order ≥ 3). If one uses the diagram formulae (7.29), the method of cumulants requires that one explicitly computes and controls an infinity of expressions of the type $\int_{Z^{n/2}} f_{\sigma,k} d\nu^{n/2}$, where the partition σ is associated with a non-flat, Gaussian and connected diagram (see Section 4.1).

Remarks. (i) We recall that (except for trivial cases), when φ is Gaussian the explicit expression of the characteristic function of a random variable of the type $I^{\varphi}_d(f)$, $d \geq 3$, is unknown (for $d = 2$ see e.g. [79, p. 185]).

(ii) Thanks to the results discussed in Section 7.3 (in particular, formula (7.39)), the method of cumulants and diagrams can be also used when φ is a completely random Poisson measure. Clearly, since (7.39) also involves non-Gaussian diagrams, the use of this approach in the Poisson case is even more technically demanding.

In the forthcoming sections we will show how one can successfully bypass the method of moments and cumulants when dealing with CLTs on a fixed Wiener chaos.

9.2 One-dimensional CLTs in the Gaussian case

We now consider an isonormal Gaussian process $X = \{X(h) : h \in \mathfrak{H}\}$ over some real separable Hilbert space $\mathfrak{H}$. Recall (see Section 8) that the notion of isonormal Gaussian process is more general than the one of Gaussian measure. The following result involves one-dimensional sequences of multiple stochastic integrals, and collects the main findings of [79] and [69]. We recall that the *total variation distance*, between the law of two general real-valued random variables Y and Z, is given by

$$d_{TV}(Y, Z) = \sup \left|\mathbb{P}(Y \in B) - \mathbb{P}(Z \in B)\right|,$$

where the supremum is taken over all Borel sets $B \in \mathcal{B}(\mathbb{R})$. Observe that the topology induced by d_{TV}, on the class of probability measures on $\mathbb{R}$, is strictly stronger than the topology of

weak convergence and thus $\lim_{n\to\infty} d_{TV}(B_n, B) = 0$ is a stronger result than $B_n \overset{law}{\to} B$ (see e.g. Dudley [17, Ch. 11] for a discussion of other relevant properties of d_{TV}).

Theorem 9.1 (See [79] and [69]) *Fix an integer $d \geq 2$, and define the operator I_d^X according to (8.5). Then,for every sequence $\left\{f^{(k)} : k \geq 1\right\}$ such that $f^{(k)} \in \mathfrak{H}^{\odot d}$ for every k, and*

$$\lim_{k\to\infty} d! \left\|f^{(k)}\right\|^2_{\mathfrak{H}^{\otimes d}} = \lim_{k\to\infty} \mathbb{E}\left[I_d^X\left(f^{(k)}\right)^2\right] = 1, \tag{9.3}$$

the following three conditions are equivalent

1. $\lim_{k\to\infty} \chi_4\left(I_d^X\left(f^{(k)}\right)\right) = 0$;
2. *for every $r = 1, ..., d-1$,*

$$\lim_{k\to\infty} \left\|f^{(k)} \otimes_r f^{(k)}\right\|^2_{\mathfrak{H}^{\otimes 2(d-r)}} = 0 \ , \tag{9.4}$$

   where the contraction $f^{(k)} \otimes_r f^{(k)}$ is defined according to (8.6);

3. *as $k \to \infty$, the sequence $\left\{I_d^X\left(f^{(k)}\right) : k \geq 1\right\}$ converges towards a centered standard Gaussian random variable $Z \sim N(0,1)$.*

Moreover, the following bound holds for every fixed k:

$$\begin{aligned} d_{TV}\left(I_d^X\left(f^{(k)}\right), Z\right)^2 &\leq \left(1 - d!\left\|f^{(k)}\right\|^2_{\mathfrak{H}^{\otimes d}}\right)^2 \\ &+ d^2 \sum_{r=1}^{d-1} (2q-2r)!\,(r-1)!^2 \binom{q-1}{r-1}^2 \left\|f^{(k)} \otimes_r f^{(k)}\right\|^2_{\mathfrak{H}^{\otimes 2(d-r)}} \end{aligned} \tag{9.5}$$

Observe that condition (1.) in the previous statement holds if and only if

$$\lim_{k\to\infty} \mathbb{E}\left[I_d^X\left(f^{(k)}\right)^4\right] = 3.$$

The equivalence of (1.), (2.) and (3.) has been first proved in [79] by means of stochastic calculus techniques. The paper [76] contains an alternate proof with additional necessary and sufficient conditions, as well as several crucial connections with Malliavin calculus operators (see e.g. [75]). The upper bound (9.5) is proved in [69], by means of Malliavin calculus and the so-called *Stein's method* for normal approximation (see e.g. [119]).

Remark. Theorem 9.1, as well as its multidimensional generalizations (see Section 9.4 below), has been applied to a variery of frameworks, such as: quadratic functionals of bivariate Gaussian processes (see [15]), quadratic functionals of fractional processes (see [79]), high-frequency limit theorems on homogeneous spaces (see [52, 53]), self-intersection local times of fractional Brownian motion (see [29, 76]), Berry-Esséen bounds in CLTs for Gaussian subordinated sequences (see [69, 70, 71]), needleets analysis on the sphere (see [5]), power variations of iterated processes (see [68]), weighted variations of fractional processes (see [66, 72]) and of related random functions (see [6, 13]).

9.3 Combinatorial implications of Theorem 9.1

The implication (1.) $\Longrightarrow$ (3.) in Theorem 9.1 provides the announced "drastic simplification" of the methods of moments and cumulants. However, as demonstrated by the applications of Theorem 9.1 listed above, condition (9.4) is often much easier to verify. Indeed, it turns out that the implication (2.) $\Longrightarrow$ (3.) has an interesting combinatorial interpretation.

To see this, we shall fix $d \geq 2$ and suppose that $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, with ν a σ-finite and non-atomic measure. According to Proposition 8.1, in this case the random variable $I_d^X(f)$, where $f \in L_s^2\left(Z^d, \mathcal{Z}^d, \nu^d\right) = L_s^2\left(\nu^d\right)$, is the multiple Wiener-Itô integral of f with respect to the Gaussian measure $A \to X(\mathbf{1}_A)$, as defined in Definition 5. We shall also use some notation from Sections 2–7, in particular:

- For every $n \geq 2$, the symbol $\pi^*([nd]) \in \mathcal{P}([nd])$ stands for the partition of $[nd] = \{1, 2, ..., nd\}$ obtained by taking n consecutive blocks of size d, that is:

$$\pi^*([nd]) = \{\{1, ..., d\}, \{d+1, ..., 2d\}, ..., \{(n-1)d+1, ..., nd\}\}.$$

- The class of partitions $\mathcal{M}_2([nd], \pi^*([nd]))$ is defined according to formula (7.21). Recall that, according to (7.27), a partition $\sigma \in \mathcal{P}([nd])$ is an element of $\mathcal{M}_2([nd], \pi^*([nd]))$ if and only if the diagram $\Gamma(\pi^*([nd]), \sigma)$ (see Section 4.1) is Gaussian, non-flat and connected, which is equivalent to saying that the graph $\hat{\Gamma}(\pi^*([nd]), \sigma)$ (see Section 4.3) is connected and has no loops.

- As in formula (7.29), for every $f \in L_s^2\left(\nu^d\right)$, every n such that nd is even, and every $\sigma \in \mathcal{M}_2([nd], \pi^*([nd]))$, we denote by $f_{\sigma,n}$ the function in $dn/2$ variables, obtained by identifying two variables x_i and x_j in the argument of

$$\underbrace{f \otimes_0 \cdots \otimes_0 f}_{n \text{ times}} \tag{9.6}$$

  if and only if $i \sim_\sigma j$.

We will also denote by

$$\mathcal{M}_2^c([nd], \pi^*([nd]))$$

the subset of $\mathcal{M}_2([nd], \pi^*([nd]))$ composed of those partitions σ such that the diagram

$$\Gamma(\pi^*([nd]), \sigma)$$

is *circular* (see Section 4.1). We also say that a partition $\sigma \in \mathcal{M}_2^c([nd], \pi^*([nd]))$ has *rank* r ($r = 1, ..., d-1$) if the diagram $\Gamma(\pi^*([nd]), \sigma)$ has exactly r edges linking the first and the second row.

Examples. (i) The partition whose diagram is given in Fig. 24 (Section 7.2) is an element of

$$\mathcal{M}_2^c([8], \pi^*([9]))$$

and has rank $r = 1$.

(ii) Consider the case $d = 3$ and $n = 4$, as well as the partition $\sigma \in \mathcal{M}_2([12], \pi^*([12]))$ given by

$$\sigma = \{\{1,4\},\{2,5\},\{3,12\},\{6,9\},\{7,10\},\{8,11\}\}.$$

Then, the diagram $\Gamma(\pi^*([12]), \sigma)$ is the one in Fig. 28, and therefore $\sigma \in \mathcal{M}_2^c([12], \pi^*([12]))$ and σ has rank $r = 2$.

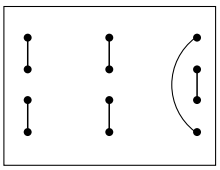

Figure 28: *A circular diagram*

The following technical result links the notions of circular diagram, rank and contraction. For $d \geq 2$ and $\sigma \in \mathcal{M}_2^c([4d], \pi^*([4d]))$, let $f_{\sigma,4}$ be the function in $2d$ variables obtained by identifying x_i and x_j in the argument of the tensor product (9.6) (with $n = 4$) if and only if $i \sim_\sigma j$. For instance, if $d = 3$ and $\sigma \in \mathcal{M}_2^c([12], \pi^*([12]))$ is associated with the diagram in Fig. 28, then

$$f_{\sigma,4}(x_1,x_2,x_3,x_4,x_5,x_6) = f(x_1,x_2,x_3)f(x_1,x_2,x_4)f(x_5,x_6,x_4)f(x_5,x_6,x_3).$$

Lemma 9.1 *Fix $f \in L_s^2(\nu^d)$, $d \geq 2$, and, for $r = 1,...,d-1$, define the contraction $f \otimes_r f$ according to (6.16). Then, for every $\sigma \in \mathcal{M}_2^c([4d], \pi^*([4d]))$ with rank $r \in \{1,...,d-1\}$,*

$$\int_{Z^{2d}} f_{\sigma,4} d\nu^{2d} = \|f \otimes_r f\|^2_{L^2(\nu^{2(d-r)})} = \|f \otimes_{d-r} f\|^2_{L^2(\nu^{2r})} \tag{9.7}$$

Proof. It is sufficient to observe that f is symmetric by definition, and then to use the relation

$$\begin{aligned} & \|f \otimes_r f\|^2_{L^2(\nu^{2(d-r)})} \\ = & \int_{Z^{d-r}} \int_{Z^{d-r}} \int_{Z^r} \int_{Z^r} f(\mathbf{a}_{d-r}, \mathbf{b}_r) f(\mathbf{b}_r, \mathbf{a}'_{d-r}) \times \\ & \qquad \times f(\mathbf{a}'_{d-r}, \mathbf{b}'_r) f(\mathbf{b}'_r, \mathbf{a}_{d-r}) \nu^{d-r}(d\mathbf{a}_{d-r}) \nu^{d-r}(d\mathbf{a}'_{d-r}) \nu^r(d\mathbf{b}_r) \nu^r(d\mathbf{b}'_r). \end{aligned}$$

■

Remark. Formula (9.7) implies that, for a fixed f and for every $\sigma \in \mathcal{M}_2^c([4d], \pi^*([4d]))$, the value of the integral $\int_{Z^{2d}} f_\sigma d\nu^{2d}$ *depends on σ uniquely through r* (or $d-r$), where r is the rank of σ.

By using Lemma 9.1, one obtains immediately the following result, which provides a combinatorial description of the implication (2.) $\Longrightarrow$ (3.) in Theorem 9.1.

Proposition 9.1 *For every $d \geq 2$ and every sequence $\{f^{(k)} : k \geq 1\} \subset L_s^2(\nu^d)$ such that $d! \|f^{(k)}\|^2_{L^2(\nu^d)} \to 1$ ($k \to \infty$), the following relations are equivalent:*

1. as $k \to \infty$

$$\sum_{\sigma \in \mathcal{M}_2([nd],\pi^*([nd]))} \int_{Z^{nd/2}} f_\sigma^{(k)} d\nu^{nd/2} \to 0, \quad \forall n \geq 3; \tag{9.8}$$

2. for every partition $\sigma \in \mathcal{M}_2^c([4d], \pi^*([4d]))$*, as* $k \to \infty$*,*

$$\int_{Z^{2d}} f_\sigma^{(k)} d\nu^{2d} \to 0. \tag{9.9}$$

Proof. Thanks to formula 7.29, one deduces that

$$\sum_{\sigma \in \mathcal{M}_2([nd],\pi^*([nd]))} \int_{Z^{nd/2}} f_\sigma^{(k)} d\nu^{nd/2} = \chi_n \left(I_d^X \left(f^{(k)} \right) \right),$$

where $I_d^X\left(f^{(k)}\right)$ is the multiple Wiener-Itô integral of $f^{(k)}$ with respect to the Gaussian measure induced by X, and χ_n indicates the nth cumulant. It follows that, since $d! \left\| f^{(k)} \right\|^2_{L^2(\nu^d)} = \mathbb{E}\left[I_d^X \left(f^{(k)} \right) \right] \to 1$, relation (9.8) is equivalent to $I_d^X \left(f^{(k)} \right) \overset{law}{\to} Z \sim N(0,1)$. On the other hand, one deduces from Lemma 9.1 that (9.9) takes place if and only if (9.4) holds. Since, according to Theorem 9.1, condition (9.4) is necessary and sufficient in order to have $I_d^X \left(f^{(k)} \right) \overset{law}{\to} Z$, we immediately obtain the desired conclusion. ■

Corollary 9.1 *Fix* $d \geq 2$ *and suppose that the sequence* $\left\{ f^{(k)} : k \geq 1 \right\} \subset L_s^2\left(\nu^d\right)$ *is such that* $d! \left\| f^{(k)} \right\|^2_{L^2(\nu^d)} \to 1$ *(*$k \to \infty$*). Then, (9.9) takes places if and only if* $I_d^X(f^{(k)}) \overset{law}{\to} Z \sim N(0,1)$.

Proof. As pointed out in the proof of Proposition 9.1, since the normalization condition $d! \left\| f^{(k)} \right\|^2_{L^2(\nu^d)} \to 1$ is in order, relation (9.8) is equivalent to the fact that the sequence $I_d^X \left(f^{(k)} \right)$, $k \geq 1$, converges in law to a standard Gaussian random variables. The implication (2.) $\Longrightarrow$ (1.) in the statement of Proposition 9.1 yields the desired result. ■

Remarks. (1) Corollary 9.1 implies that, in order to prove a CLT on a fixed Wiener chaos, *it is sufficient to compute and control a finite number of expressions of the type* $\int_{Z^{2d}} f_\sigma^{(k)} d\nu^{2d}$*, where* σ *is associated with a connected Gaussian circular diagram with four rows.* Moreover, these expressions determine the speed of convergence in total variation, via the upper bound given in (9.5).

(2) Relation (9.7) also implies that: (i) for d even, (9.9) takes place for every

$$\sigma \in \mathcal{M}_2^c([4d], \pi_d^*([4d]))$$

if and only if for every $r = 1, ..., d/2$, there exists a partition $\sigma \in \mathcal{M}_2^c([4d], \pi_d^*([4d]))$ with rank r and such that (9.9) holds; (ii) for d odd, (9.9) takes place for every $\sigma \in \mathcal{M}_2^c([4d], \pi_d^*([4d]))$ if and only if for every $r = 1, ..., (d+1)/2$, there exists a partition $\sigma \in \mathcal{M}_2^c([4d], \pi_d^*([4d]))$ with rank r and such that (9.9) holds.

(3) When $d = 2$, the implication (9.9) $\Rightarrow$ (9.8) is a consequence of the fact that, for every $n \geq 3$ and up to a permutation of the rows, the diagram associated with any element of $\mathcal{M}_2([2n], \pi_2^*([2n]))$ is equivalent to a circular diagram (this fact has been already pointed out

at the end of Section 7.2). For instance, it is always possible to permute the blocks of $\pi_2^*([10])$ in such a way that the diagram $\Gamma(\pi_2^*([10]), \sigma)$, associated with some $\sigma \in \mathcal{M}_2([10], \pi_2^*([10]))$, has the form of the diagram in Fig. 29. By using this fact, one can prove that (9.9) $\Rightarrow$ (9.8) by means of the Cauchy-Schwarz inequality and of a recurrence argument (for another proof of Theorem 9.1 in the case $d = 2$, by means of an explicit expression of the Fourier transform of $I_2^X\left(f^{(k)}\right)$, see [79, p. 185]).

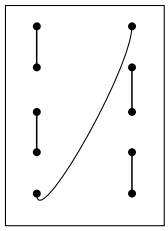

Figure 29: *A circular diagram with five rows*

9.4 A multidimensional CLT

The paper [91] (but see also [71, 77, 83]) contains a complete solution of Problem A in the Gaussian case, for every $m \geq 2$. For such an index m, we denote by V_m the set of all vectors $(i_1, i_2, i_3, i_4) \in (1, ..., m)^4$ such that at least one of the following three properties is verified: (a) $i_1 \neq i_2 = i_3 = i_4$, (b) $i_1 \neq i_2 = i_3 \neq i_4$ and $i_4 \neq i_1$, (c) the elements of $(i_1, ..., i_4)$ are all different. In what follows, $X = \{X(h) : h \in \mathfrak{H}\}$ indicates an isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$.

Theorem 9.2 *Let $m \geq 2$ and $d_1, ..., d_m \geq 1$ be fixed and let*

$$\left\{f_j^{(k)} : j = 1, ..., m, \;\; k \geq 1\right\}$$

be a collection of kernels such that $f_j^{(k)} \in \mathfrak{H}^{\odot d_j}$ and the normalization condition (9.1) is verified. Then, the following conditions are equivalent:

1. *as $k \to \infty$, the vector $\mathbf{F}_k = \left(I_{d_1}^X\left(f_1^{(k)}\right), ..., I_{d_m}^X\left(f_m^{(k)}\right)\right)$ converges in law towards a m-dimensionnel Gaussian vector $\mathbf{N}_m(0, \mathbf{C}) = (N_1, ..., N_m)$ with covariance matrix $\mathbf{C} = \{C(i,j)\}$;*

2. $$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}\left[\left(\sum_{i=1,...,m} I_{d_i}^X\left(f_i^{(k)}\right)\right)^4\right] \\ = \;& 3\left(\sum_{i=1}^m C(i,i) + 2\sum_{1\leq i<j\leq m} C(i,j)\right)^2 = \mathbb{E}\left[\left(\sum_{i=1}^m N_i\right)^4\right], \end{aligned}$$

and

$$\lim_{k\to\infty} \mathbb{E}\left[\prod_{l=1}^{4} I_{d_{i_l}}^{X}\left(f_{i_l}^{(k)}\right)\right] = \mathbb{E}\left[\prod_{l=1}^{4} N_{i_l}\right]$$

$\forall\,(i_1,i_2,i_3,i_4)\in V_m$;

3. *for every* $j=1,...,m$, *the sequence* $I_{d_j}^{X}\left(f_j^{(k)}\right)$ $k\geq 1$, *converges in law towards* N_j, *that is, towards a centered Gaussian variable with variance* $C(j,j)$;

4. $\forall j=1,...,m$, $\lim_{k\to\infty}\chi_4\left(I_{d_j}^{X}\left(f_j^{(k)}\right)\right)=0$;

5. $\forall j=1,...,m$

$$\lim_{k\to\infty}\left\|f_j^{(k)}\otimes_r f_j^{(k)}\right\|_{\mathfrak{H}^{\otimes 2\left(d_j-r\right)}}=0, \tag{9.10}$$

$\forall r=1,...,d_j-1$.

The original proof of Theorem 9.2 uses arguments from stochastic calculus. See [71] and [77], respectively, for alternate proofs based on Malliavin calculus and Stein's method. In particular, in [71] one can find bounds analogous to (9.5), concerning the multidimensional Gaussian approximation of $\mathbf{F}_k$ in the Wasserstein distance. The crucial element in the statement of Theorem 9.2 is the implication (3.) $\Rightarrow$ (1.), which yields the following result.

Corollary 9.2 *Let the vectors* $\mathbf{F}_k$, $k\geq 1$, *be as in the statement of Theorem 9.2, and suppose that (9.1) is satisfied. Then, the convergence in law of each component of the vectors* $\mathbf{F}_k$, *towards a Gaussian random variable, always implies the joint convergence of* $\mathbf{F}_k$ *towards a Gaussian vector with covariance* $\mathbf{C}$.

Thanks to Theorem 9.1, it follows that a CLT such as (9.2) can be uniquely deduced from (9.1) and from the relations (9.10), involving the contractions of the kernels $f_j^{(k)}$.

When $\mathfrak{H}=L^2(Z,\mathcal{Z},\nu)$ (with ν non atomic), the combinatorial implications of Theorem 9.2 are similar to those of Theorem 9.1. Indeed, thanks to the implication (5.) $\Rightarrow$ (1.), one deduces that, for a sequence $\left(f_1^{(k)},...,f_m^{(k)}\right)$, $k\geq 1$, as in (9.1), if

$$\int_{Z^{2d}}\left(f_j^{(k)}\right)_{\sigma}d\nu^{2d}\to 0,\quad \forall\sigma\in\mathcal{M}_2^{c}\left([4d],\pi^{*}([4d])\right),$$

then

$$\sum_{\sigma\in\mathcal{M}_2([n],\pi^{*})}\int_{Z^{n/2}} f_{\sigma,\ell}^{(k)}d\nu^{n/2}\to 0,$$

for every integer n which is the sum of $\ell\geq 3$ components $(d_{i_1},d_{i_2},...,d_{i_\ell})$ of the vector $(d_1,...,d_m)$ (with possible repetitions of the indices $i_1,...,i_\ell$), with

$$\pi^{*}=\left\{\{1,...,d_{i_1}\},...,\left\{d_1+...+d_{i_{\ell-1}}+1,...,n\right\}\right\}\in\mathcal{P}([n]),$$

and every function $f_{\sigma,\ell}^{(k)}$, in $n/2$ variables, is obtained by identifying two variables x_k and x_j in the argument of $f_{i_1}\otimes_0\cdots\otimes_0 f_{i_\ell}$ if and only if $k\sim_\sigma j$.

As already pointed out, the chaotic representation property (5.45) allows to use Theorem 9.2 in order to obtain CLTs for general functionals of an isonormal Gaussian process X. We now present a result in this direction, obtained in [29], whose proof can be deduced from Theorem 9.2.

Theorem 9.3 (See [29]) *We consider a sequence $\{F_k : k \geq 1\}$ of centered and square-integrable functionals of an isonormal Gaussian process X, admitting the chaotic decomposition*

$$F_k = \sum_{d=1}^{\infty} I_d^X\left(f_d^{(k)}\right), \quad k \geq 1.$$

Assume that

- $\lim_{N\to\infty} \limsup_{k\to\infty} \sum_{d\geq N+1} d! \left\|f_d^{(k)}\right\|^2_{\mathfrak{H}^{\otimes d}} \to 0$,
- *for every $d \geq 1$,* $\lim_{k\to\infty} d! \left\|f_d^{(k)}\right\|^2_{\mathfrak{H}^{\otimes d}} = \sigma_d^2$,
- $\sum_{d=1}^{\infty} \sigma_d^2 \triangleq \sigma^2 < \infty$,
- *for every $d \geq 1$,* $\lim_{k\to\infty} \left\|f_d^{(k)} \otimes_r f_d^{(k)}\right\|_{\mathfrak{H}^{\otimes 2(d-r)}} = 0$, $\forall r = 1, ..., d-1$.

Then, as $k \to \infty$, $F_k \overset{law}{\to} N\left(0, \sigma^2\right)$, where $N\left(0, \sigma^2\right)$ is a centered Gaussian random variable with variance σ^2.

9.5 Simplified CLTs in the Poisson case: the case of double integrals

We conclude this survey by discussing a simplified CLT for sequences of double integrals with respect to a Poisson random measure. Note that this result (originally obtained in [89]) has been generalized in [86], where one can find CLTs for sequences of multiple integrals of arbitrary orders – with explicit Berry-Esséen bounds in the Wasserstein distance obtained once again via Stein's method.

In this section, $(Z, \mathcal{Z}, \nu)$ is a measure space, with ν σ-finite and non-atomic. Also, $\hat{N} = \{\hat{N}(B) : B \in \mathcal{Z}_\nu\}$ is a compensated Poisson measure with control measure given by ν. In [89], we have used some decoupling techniques developed in [87] in order to prove CLTs for sequences of random variables of the type:

$$F_k = I_2^{\hat{N}}\left(f^{(k)}\right), \quad k \geq 1, \tag{9.11}$$

where $f^{(k)} \in L_s^2\left(\nu^2\right)$. In particular, we focus on sequences $\{F_k\}$ satisfying the following assumption

Assumption N. The sequence $f^{(k)}$, $k \geq 1$, in (9.11) verifies :

N.i (*integrability*) $\forall k \geq 1$,

$$\int_Z f^{(k)}(z, \cdot)^2 \nu(dz) \in L^2(\nu) \quad \text{and} \quad \left\{\int_Z f^{(k)}(z, \cdot)^4 \nu(dz)\right\}^{\frac{1}{2}} \in L^1(\nu); \tag{9.12}$$

N.ii (*normalization*) As $k \to \infty$,

$$2 \int_Z \int_Z f^{(k)}\left(z, z'\right)^2 \nu\left(dz\right) \nu\left(dz'\right) \to 1; \tag{9.13}$$

N.iii (*fourth power*) As $k \to \infty$,

$$\int_Z \int_Z f^{(k)}\left(z, z'\right)^4 \nu\left(dz\right) \nu\left(dz'\right) \to 0 \tag{9.14}$$

(in particular, this implies that $f^{(k)} \in L^4\left(\nu^2\right)$).

Remarks. (1) The conditions in (9.12) are technical : the first ensures the existence of the stochastic integral of $\int_Z f^{(k)}\left(z, \cdot\right)^2 \nu\left(dz\right)$ with respect to $\hat{N}$; the second allows to use some Fubini arguments in the proof of the results to follow.

(2) Suppose that there exists a set B, independent of n, such that $\nu\left(B\right) < \infty$ and $f^{(k)} = f^{(k)}\mathbf{1}_B$, a.e.–$d\nu^2$, $\forall k \geq 1$ (this holds, in particular, when ν is finite). Then, by the Cauchy-Schwarz inequality, if (9.14) is true, then $\left(f^{(k)}\right)$ converges necessarily to zero. Therefore, in order to study more general sequences $\left(f^{(k)}\right)$, we must assume that $\nu\left(Z\right) = +\infty$.

The next theorem is the main result of [89].

Theorem 9.4 *Let $F_k = I_2^{\hat{N}}(f^{(k)})$ with $f^{(k)} \in L_s^2(\nu^2)$, $k \geq 1$, and suppose that Assumption N is verified. Then, $f^{(k)} \star_1^0 f^{(k)} \in L^2(\nu^3)$ and $f^{(k)} \star_1^1 f^{(k)} \in L_s^2(\nu^2)$, $\forall k \geq 1$, and also :*

1. *if*
$$\left\| f^{(k)} \star_2^1 f^{(k)} \right\|_{L^2(\nu)} \to 0 \text{ and } \left\| f^{(k)} \star_1^1 f^{(k)} \right\|_{L^2(\nu^2)} \to 0 \ , \tag{9.15}$$
then
$$F_k \overset{law}{\to} N\left(0,1\right), \tag{9.16}$$
where $N\left(0,1\right)$ is a centered Gaussian random variable with unitary variance.

2. *if $F_k \in L^4\left(\mathbb{P}\right)$, $\forall k$, a sufficient condition in order to have (9.15) is*
$$\chi_4\left(F_n\right) \to 0; \tag{9.17}$$

3. *if the sequence $\left\{F_k^4 : k \geq 1\right\}$ is uniformly integrable, then the three conditions (9.15), (9.16) and (9.17) are equivalent.*

Remark. See [14] and [85] for several applications of Theorem 9.4 to Bayesian non-parametric survival analysis.

We now give a combinatorial interpretation (in terms of diagrams) of the three asymptotic conditions appearing in formulae (9.14) and (9.15). To do this, consider the set $[8] = \{1, ..., 8\}$, as well as the partition $\pi^* = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}\} \in \mathcal{P}\left([8]\right)$. We define the set of partitions $\mathcal{M}_{\geq 2}\left([8], \pi^*\right) \subset \mathcal{P}\left([8]\right)$ according to (7.23). Given an element $\sigma \in \mathcal{M}_{\geq 2}\left([8], \pi^*\right)$ and

given $f \in L^2_s\left(\nu^2\right)$, the function $f_{\sigma,4}$, in $|\sigma|$ variables, is obtained by identifying the variables x_i and x_j in the argument of $f \otimes_0 f \otimes_0 f \otimes_0 f$ (as defined in (6.3)) if and only if $i \sim_\sigma j$. We define three partitions $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{M}_{\geq 2}\left([8], \pi^*\right)$ as follows:

$$\begin{aligned}\sigma_1 &= \{\{1,3,5,7\},\{2,4,6,8\}\}\\ \sigma_2 &= \{\{1,3,5,7\},\{2,4\},\{6,8\}\}\\ \sigma_3 &= \{\{1,3\},\{4,6\},\{5,7\},\{2,8\}\}.\end{aligned}$$

The diagrams $\Gamma\left(\pi^*, \sigma_1\right)$, $\Gamma\left(\pi^*, \sigma_2\right)$ and $\Gamma\left(\pi^*, \sigma_3\right)$ are represented (in order) in Fig. 30.

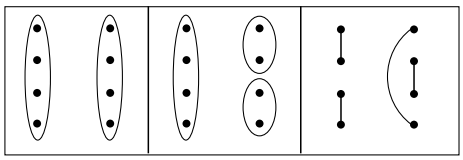

Figure 30: *Three diagrams associated with contractions*

One has therefore the following combinatorial representation of the three norms appearing in formulae (9.14) and (9.15) (the proof is elementary, and left to the reader).

Proposition 9.2 *For every $f \in L^2_s\left(\nu^2\right)$, one has that*

$$\begin{aligned}\int_Z \int_Z f\left(z,z'\right)^4 \nu\left(dz\right)\nu\left(dz'\right) &= \|f\|^4_{L^4(\nu^2)} = \int_{Z^2} f_{\sigma_1,4}\left(z,z'\right)\nu\left(dz\right)\nu\left(dz'\right)\\ \int_Z \left[\int_Z f\left(z,z'\right)^2 \nu\left(dz\right)\right]^2 \nu\left(dz'\right) &= \left\| f \star_2^1 f\right\|^2_{L^2(\nu)} = \int_Z f_{\sigma_2,4}\left(z\right)\nu\left(dz\right)\\ \int_{Z^2}\left[\int_Z f\left(z,z'\right) f\left(z,z''\right)\nu\left(dz\right)\right]^2 \nu\left(dz'\right)\nu\left(dz''\right) &= \left\| f \star_1^1 f\right\|^2_{L^2(\nu)} = \int_Z f_{\sigma_3,4}\left(z\right)\nu\left(dz\right).\end{aligned}$$

In particular, Proposition 9.2 implies that, on the second Wiener chaos of a Poisson measure, one can establish CLTs by focusing uniquely on expressions related to three connected diagrams with four rows. Similar characterizations for sequences belonging to chaoses of higher orders can be deduced from the main findings of [86].

Acknowledgements. Part of this survey has been written while the authors were visiting the Departement of Mathematics and Applied Statistics of Turin University, in june 2007. The authors heartily thank Massimo Marinacci and Igor Prünster for their kind hospitality and support. Giovanni Peccati acknowledges support from ISI Foundation–Lagrange Project. Murad S. Taqqu acknowledges support by NSF under grant DMS-0706786 at Boston University. We thank Florent Benaych-Georges, Domenico Marinucci and Marc Yor for a careful reading of an earlier draft of this manuscript, as well as for valuable suggestions.

References

[1] R.J. Adler (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*. Lecture Notes-Monograph Series **12**, Institut of Mathematical Statistics, Hayward, California.

[2] M. Aigner (1979). *Combinatorial theory.* Springer-Verlag, Berlin Heidelberg New York.

[3] M. Anshelevich (2001). Partition-dependent stochastic measures and q-deformed cumulants. *Documenta Mathematica* **6**, 343-384.

[4] M. Anshelevich (2005). Linearization coefficients for orthogonal polynomials using stochastic processes. *The Annals of Probabability* **33**(1), 114-136.

[5] P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard (2006). High-frequency asymptotics for wavelet-based tests for Gaussianity and isotropy on the torus. Preprint.

[6] O. Barndorff-Nielsen, J. Corcuera, M. Podolskij and J. Woerner (2008). Bipower variations for Gaussian processes with stationary increments. Preprint.

[7] P. Billingsley (1995). *Probability and Measure*, 3rd Edition. Wiley, New York.

[8] K. Bitcheler (1980). Stochastic integration and L^p theory of semimartingales. *The Annals of Probability* **9**(1), 49-89.

[9] P. Breuer et P. Major (1983). Central limit theorems for non-linear functionals of Gaussian fields. *Journal of Multivariate Analysis* **13**, 425-441.

[10] M.S. Brodskii (1971), *Triangular and Jordan Representations of Linear Operators*. Transl. Math. Monographs **32**, AMS, Providence.

[11] D. Chambers et E. Slud (1989). Central limit theorems for nonlinear functionals of stationary Gaussian processes. *Probability Theory and Related Fields* **80**, 323-349.

[12] S. Cohen et M.S. Taqqu (2004). Small and large scale behavior of the Poissonized Telecom process. *Methodology and Computing in Applied Probability* **6**, 363-379.

[13] J.M. Corcuera, D. Nualart et J.H.C. Woerner (2006). Power variation of some integral long memory process. *Bernoulli* **12**(4), 713-735.

[14] P. de Blasi, G. Peccati and I. Prünster (2008). Asymptotics for posterior hazards. To appear in: *The Annals of Statistics.*

[15] P. Deheuvels, G. Peccati et M. Yor (2006) On quadratic functionals of the Brownian sheet and related processes. *Stochastic Processes and their Applications* **116**, 493-538.

[16] C. Dellacherie, B. Maisonneuve et P.-A. Meyer (1992). *Probabilités et Potentiel (Chapitres XVII à XXIV).* Hermann, Paris.

[17] R.M. Dudley (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *Journal of Functional Analysis* **1**, 290-330.

[18] D.D. Engel (1982). The multiple stochastic integral. *Memoirs of the AMS* **38**, 1-82.

[19] M. Farré, M. Jolis and F. Utzet (2008). Multiple Stratonovich integral and Hu-Meyer formula for Lévy processes. Preprint.

[20] P.D. Feigin (1985). Stable convergence of semimartingales. *Stochastic Processes and their Applications* **19**, 125-134.T.S.

[21] Ferguson (1973). A Bayesian analysis of some non-parametric problems. *The Annals of Statistics* **1** (2), 209-230.

[22] R. Fox et M.S. Taqqu (1987). Multiple stochastic integrals with dependent integrators. *Journal of Multivariate Analysis* **21**(1), 105-127.

[23] E. Giné and V.H. de la Peña (1999). *Decoupling.* Springer-Verlag. Berlin Heidelberg New York.

[24] L. Giraitis and D. Surgailis (1985). CLT and other limit theorems for functionals of Gaussian processes. *Zeitschrift für Wahrsch. verw. Gebiete* **70**, 191-212.

[25] J. Goldberg and M.S. Taqqu (1982). Regular multigraphs and their applications to the Monte Carlo evaluation of moments of non-linear functions of Gaussian processes. *Stochastic Processes and their Applications* **13**, 121-138.

[26] G. Gripenberg and I. Norros (1996). On the prediction of fractional Brownian motion. *Journal of Applied Probability* **33**, 400-410.

[27] K. Handa (2005). Sampling formulae for symmetric selection. *Electronic Communications in Probabability* **10**, 223-234 (Electronic).

[28] K. Handa (2007). The two-parameter Poisson-Dirichlet point process. Preprint.

[29] Y. Hu and D. Nualart (2005). Renormalized self-intersection local time for fractional Brownian motion. *The Annals of Probabability* **33**(3), 948-983.

[30] K. Itô (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3**, 157–169

[31] J. Jacod, A. Klopotowski et J. Mémin (1982). Théorème de la limite centrale et convergence fonctionnelle vers un processus à accroissements indépendants : la méthode des martingales. *Annales de l'Institut H. Poincaré* (*PR*) **1**, 1-45.

[32] J. Jacod et A.N. Shiryaev (1987). *Limit Theorems for Stochastic Processes.* Springer, Berlin Heidelberg New York.

[33] A. Jakubowski (1986). Principle of conditioning in limit theorems for sums of random variables. *The Annals of Probability* **11**(3), 902-915.

[34] L.F. James, A. Lijoi and I. Prünster (2005). Conjugacy as a distinctive feature of the Dirichlet process. *Scandinavian Journal of Statistics* **33**, 105-120.

[35] L. James, B. Roynette and M. Yor (2008). Generalized Gamma Convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probability Surveys* **5**, 346-415.

[36] S. Janson (1997). *Gaussian Hilbert Spaces.* Cambridge University Press, Cambridge.

[37] O. Julià et D. Nualart (1988). The distribution of a double stochastic integral with respect to two independent Brownian Sheets. *Stochastics* **25**, 171-182.

[38] Y. Kabanov (1975). On extended stochastic integrals. *Theory of Probability and its applications* **20**, 710-722.

[39] O. Kallenberg et J. Szulga (1991). Multiple integration with respect to Poisson and Lévy processes. *Probability Theory and Related Fields* **83**, 101-134.

[40] J.F.C. Kingman (1967). Completely random measures. *Pacific Journal of Mathematics* **21**, 59-78.

[41] H.-H. Kuo (1975). *Gaussian measures in Banach spaces.* LNM **463**. Springer-Verlag, Berlin Heidelberg New-York.

[42] A.U. Kussmaul (1977). *Stochastic integration and generalized martingales.* Pitman research notes in mathematic, **11**. London.

[43] S. Kwapień and W.A. Woyczyński (1991). Semimartingale integrals via decoupling inequalities and tangent processes. *Probability and Mathematical Statisitics* **12**(2), 165-200.

[44] S. Kwapień and W.A. Woyczyński (1992). *Random Series and Stochastic Integrals: Single and Multiple.* Birkhäuser, Basel.

[45] V.P. Leonov and A.N. Shiryaev (1959). On a method of calculations of semi-invariants. *Theory of Probability and its Applications* **4**, 319-329.

[46] M.A. Lifshits (1995). *Gaussian Random Functions.* Kluwer, Dordrecht.

[47] W. Linde (1986). *Probability in Banach spaces: stable and infinitely divisible ditributions.* Wiley, New York.

[48] J. Ma, Ph. Protter and J. San Martin (1998) Anticipating integrals for a class of martingales. *Bernoulli*, **4**, 81-114.

[49] P. Major (1981). *Multiple Wiener-Itô integrals.* LNM **849**. Springer-Verlag, Berlin Heidelberg New York.

[50] V.A. Malyshev (1980). Cluster expansion in lattice models of statistical physics and quantum fields theory. *Uspehi Mat. Nauk* **35**, 3-53.

[51] D. Marinucci (2006). High resolution asymptotics for the angular bispectrum of spherical random fields. *The Annals of Statistics* **34**, 1-41.

[52] D. Marinucci and G. Peccati (2007). High-frequency asymptotics for subordinated stationary fields on an Abelian compact group. *Stochastic Processes and their Applications* **118**(4), 585-613.

[53] D. Marinucci and G. Peccati (2007). Group representations and high-frequency central limit theorems for subordinated random fields on a sphere. Preprint.

[54] G. Maruyama (1982). Applications of the multiplication of the Itô-Wiener expansions to limit theorems. *Proc. Japan Acad.* **58**, 388-390.

[55] G. Maruyama (1985). Wiener functionals and probability limit theorems, I: the central limit theorem. *Osaka Journal of Mathematics* **22**, 697-732.

[56] P.R. Masani (1995) The homogeneous chaos from the standpoint of vector measures, *Phil. Trans. R. Soc. Lond, A* **355**, 1099-1258

[57] P. McCullagh (1987). *Tensor Methods in Statistics.* Chapman and Hall. London.

[58] R.D. Mauldin, W.D. Sudderth and S.C. Williams (1992). Pólya trees and random distributions. *The Annals of Statistics*, **20** (3), 1203-1221

[59] P.-A. Meyer (1976). Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X*, LNM **511**. Springer-Verlag, Berlin Heidelberg New York, pp. 245-400.

[60] P.-A. Meyer (1992). *Quantum probability for probabilists.* LNM **1538**. Springer-Verlag, Berlin Heidelberg New York.

[61] A. Neuenkirch and I. Nourdin (2006). Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. Prépublication.

[62] J. Neveu (1968). *Processus Aléatoires Gaussiens.* Presses de l'Université de Montréal.

[63] A. Nica and R. Speciher (2006). *Lectures on the combinatorics of free probability.* London Mathematical Society Lecture Notes Series **335.** Cambridge University Press, Cambridge.

[64] I. Nourdin (2005). Schémas d'approximation associés à une équation différentielle dirigée par une fonction höldérienne; cas du mouvement Brownien fractionnaire. *C.R.A.S.* **340**(8), 611-614.

[65] I. Nourdin and D. Nualart (2007). Central limit theorems for multiple stable integrals. Preprint.

[66] I. Nourdin, D. Nualart and C.A. Tudor (2007). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. Preprint.

[67] I. Nourdin and G. Peccati (2007). Non-central convergence of multiple integrals. To appear in: *The Annals of Probability.*

[68] I. Nourdin and G. Peccati (2008). Weighted power variations of iterated Brownian motion. *The Electronic Journal of Probability.* **13**, n. 43, 1229-1256 (Electronic).

[69] I. Nourdin and G. Peccati (2008). Stein's method on Wiener chaos. To appear in: *Probability Theory and Related Fields.*

[70] I. Nourdin and G. Peccati (2008). Stein's method and exact Berry-Esséen asymptotics for functionals of Gaussian fields. Preprint.

[71] I. Nourdin, G. Peccati and A. Réveillac (2008). Multivariate normal approximation using Stein's method and Malliavin calculus. To appear in: *Annales de l'Institut H. Poincaré.*

[72] I. Nourdin and A. Réveillac (2008). Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. Preprint.

[73] D. Nualart (1983), On the distribution of a double stochastic integral. *Z. Wahrscheinlichkeit verw. Gebiete* **65**, 49-60

[74] D. Nualart (1998). Analysis on Wiener space and anticipating stochastic calculus. *Lectures on Probability Theory and Statistics. École de probabilités de St. Flour XXV (1995)*, LNM **1690**. Springer-Verlag, Berlin Heidelberg New York, pp. 123-227.

[75] D. Nualart (2006). *The Malliavin Calculus and related topics* ($2^{\text{ème}}$ édition). Springer-Verlag, Berlin Heidelberg New York.

[76] D. Nualart and S. Ortiz-Latorre (2007). Intersection local times for two independent fractional Brownian motions. *Journal of Theoretical Probabability* **20**(4), 759-767.

[77] D. Nualart and S. Ortiz-Latorre (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Processes and their Applications* **118**(4), 614-628

[78] D. Nualart and E. Pardoux (1988). Stochastic calculus with anticipating integrands. *Probabability Theory Related Fields*, **78**, 535-581.

[79] D. Nualart and G. Peccati (2005). Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability*, **33**(1), 177-193.

[80] D. Nualart and W. Schoutens (2000). Chaotic and predictable representation for Lévy processes. *Stochastic Processes and their Applications* **90**, 109-122.

[81] D. Nualart and J. Vives (1990). Anticipative calculus for the Poisson space based on the Fock space. *Séminaire de Probabilités XXIV*, LNM **1426**. Springer-Verlag, Berlin Heidelberg New York, pp. 154-165.

[82] G. Peccati (2001). On the convergence of multiple random integrals. *Studia Sc. Mat. Hungarica*, **37**, 429-470.

[83] G. Peccati (2007). Gaussian approximations of multiple integrals. *Electronic Communications in Probability* **12**, 350-364 (electronic).

[84] G. Peccati (2008). Multiple integral representation for functionals of Dirichlet processes. *Bernoulli* **14**(1), 91-124

[85] G. Peccati and I. Prünster (2008). Linear and quadratic functionals of random hazard rates: an asymptotic analysis. *The Annals of Applied Probability* **18**(5), 1910-1943

[86] G. Peccati, J.-L. Solé, F. Utzet and M.S. Taqqu (2008). Stein's method and Gaussian approximation of Poisson functionals. Preprint.

[87] G. Peccati and M.S. Taqqu (2007). Stable convergence of generalized L^2 stochastic integrals and the principle of conditioning. *The Electronic Journal of Probability*, **12**, 447-480, n. 15 (electronic).

[88] G. Peccati and M.S. Taqqu (2008). Limit theorems for multiple stochastic integrals. To appear in: *ALEA*.

[89] G. Peccati and M.S. Taqqu (2008). Central limit theorems for double Poisson integrals. *Bernoulli* **14**(3), 791-821.

[90] G. Peccati and M.S. Taqqu (2007). Stable convergence of multiple Wiener-Itô integrals. To appear in: *The Journal of Theoretical Probability.*

[91] G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. *Séminaire de Probabilités XXXVIII*, LNM **1857**. Springer-Verlag, Berlin Heidelberg New York, pp. 247-262.

[92] G. Peccati and M. Yor (2004). Hardy's inequality in $L^2([0,1])$ and principal values of Brownian local times. *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communications Series, 49-74.

[93] G. Peccati and M. Yor (2004). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communication Series, 75-87.

[94] V. Pipiras and M.S. Taqqu (2000). Integration questions related to fractional Brownian motion. *Probability Theory and Related Fields* **118**(2), 251-291.

[95] V. Pipiras and M.S. Taqqu (2001). Are classes of deterministic integrands for fractional Brownian motion complete? *Bernoulli* **7**(6), 873-897

[96] V. Pipiras and M.S. Taqqu (2003). Fractional calculus and its connection to fractional Brownian motion. In: *Long Range Dependence*, 166-201, Birkhäuser, Basel.

[97] J. Pitman (2006). *Combinatorial Stochastic Processes.* LNM **1875**. Springer-Verlag, Berlin Heidelberg New York.

[98] N. Privault (1994). Chaotic and variational calculus in discrete and continuous time for the Poisson process. *Stochastics and Stochastics Reports* **51**, 83-109.

[99] N. Privault (1994). Inégalités de Meyer sur l'espace de Poisson. *C.R.A.S.* **318**, 559-562.

[100] N. Privault, J.L. Solé and J. Vives (2001). Chaotic Kabanov formula for the Azéma martingales. *Bernoulli* **6**(4), 633-651.

[101] N. Privault et J.-L. Wu (1998). Poisson stochastic integration in Hilbert spaces. *Ann. Math. Blaise Pascal* **6**(2), 41-61.

[102] P. Protter (2005). *Stochastic Integration and Differential Equations* ($2^{\text{ème}}$ édition). Springer-Verlag, Berlin Heidelberg New York.

[103] B.S. Rajput and J. Rosinski (1989). Spectral representation of infinitely divisible processes. *Probability Theory and Related Fields* **82**, 451-487.

[104] D. Revuz and M. Yor (1999). *Continuous martingales and Brownian motion.* Springer-Verlag, Berlin Heidelberg New York.

[105] G.-C. Rota and J. Shen (2000). On the combinatorics of cumulants. *Journal of Combinatorial Theory Series A* **91**, 283-304.

[106] G.-C. Rota and C. Wallstrom (1997). Stochastic integrals: a combinatorial approach. *The Annals of Probability* **25**(3), 1257-1283.

[107] J. Rosińsky and W.A. Woyczyński (1984). Products of random measures, multilinear random forms and multiple stochastic integrals. *Proc. Conference of Measure Theory*, Oberwolfach 1983, LNM 1089. Springer-Verlag, Berlin Heidelberg New York, pp. 294-315.

[108] F. Russo and P. Vallois (1998). Product of two multiple stochastic integrals with respect to a normal martingale. *Stochastic Processes and their Applications* **73**(1), 47-68.

[109] G. Samorodnitsky and M.S. Taqqu (1994). *Stable Non-Gaussian Random Processes.* Chapman and Hall, New York.

[110] K.-I. Sato (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics **68**. Cambridge University Press.

[111] W. Schoutens (2000). Stochastic processes and orthogonal polynomials. Lecture Notes in Staistics 146. Springer-Verlag, Berlin Heidelberg New York.

[112] M. Schreiber (1969). Fermeture en probabilité de certains sous-espaces d'un espace L^2. *Zeitschrift Warsch. verw. Gebiete* **14**, 36-48.

[113] A.N. Shyryayev (1984). *Probability.* Springer-Verlag. Berlin Heidelberg New York.

[114] E.V. Slud (1993). The moment problem for polynomial forms in normal random variables. *The Annals of Probabability* **21**(4), 2200-2214.

[115] J.-L. Solé and F. Utzet (2008). Time-space harmonic polynomials associated with a Lévy process. *Bernoulli* **14**(1), 1-13.

[116] J.-L. Solé and F. Utzet (2008). On the orthogonal polynomials associated to a Lévy process. To appear in: *The Annals of Probability.*

[117] T. Speed (1983). Cumulants and partitions lattices. *Australian Journal of Statistics* **25**(2), 378-388.

[118] R. Stanley (1997). *Enumerative combinatorics, Vol. 1.* Cambridge University Press.

[119] Ch. Stein (1986). *Approximate computation of expectations.* Institute of Mathematical Statistics Lecture Notes - Monograph Series, **7**. Institute of Mathematical Statistics, Hayward, CA.

[120] D.W. Stroock (1987). Homogeneous chaos revisited. *Séminaire de Probabilités XXI*, LNM **1247**, Springer-Verlag, Berlin Heidelberg New York, pp. 1-8.

[121] D. Surgailis (1984). On multiple Poisson stochastic integrals and associated Markov semigroups. *Probab. Math. Statist.* **3**(2), 217-239.

[122] D. Surgailis (2000). CLTs for Polynomials of Linear Sequences: Diagram Formulae with Applications. Dans : *Long Range Dependence.* Birkhäuser, Basel, pp. 111-128.

[123] D. Surgailis (2000). Non-CLT's: U-Statistics, Multinomial Formula and Approximations of Multiple Wiener-Itô integrals. Dans : *Long Range Dependence*. Birkhäuser, Basel, pp. 129-142.

[124] N.V. Tsilevich and A.M. Vershik (2003). Fock factorizations and decompositions of the L^2 spaces over general Lévy processes. *Russian Math. Surveys* **58**(3), 427-472.

[125] N. Tsilevich, A.M. Vershik and M. Yor (2001). An infinite-dimensional analogue of the Lebesgue measure and distinguished properties of the gamma process. *J. Funct. Anal.* **185**(1), 274-296.

[126] C. Tudor (1997). Product formula for multiple Poisson-Itô integrals. *Revue Roumaine de Math. Pures et Appliquées* **42**(3-4), 339-345.

[127] C.A. Tudor and J. Vives (2002) The indefinite Skorohod integral as integrator on the Poisson space. *Random Operators and Stochastic Equations* **10**, 29-46.

[128] A.S. Üstünel and M. Zakai (1997). The construction of filtrations on Abstract Wiener Space. *Journal of Functional Analysis* **143**, 10-32.

[129] R.A. Vitale (1990). Covariances of symmetric statistics. *Journal of Multivariate Analysis* **41**, 14-26.

[130] N. Wiener (1938). The homogeneous chaos. *Amer. J. Math.* **60**, 879-936.

[131] R.L. Wolpert and M.S. Taqqu (2005). Fractional Ornstein-Uhlenbeck Lévy Processes and the Telecom Process: Upstairs and Downstairs. *Signal Processing* **85**(8), 1523-1545.

[132] L.M. Wu (1990). Un traitement unifié de la représentation des fonctionnelles de Wiener. *Séminaire de Probabilités XXIV*, LNM **1426**, Springer-Verlag, Berlin Heidelberg New York, pp. 166-187.

[133] X.-H. Xue (1991). On the principle of conditioning and convergence to mixtures of distributions for sums of dependent random variables. *Stochastic Processes and their Applications* **37**(2), 175-186.

[134] K. Yosida (1980). *Functional analysis*. Springer-Verlag, Berlin Heidelberg New York.