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► **To cite this version:**

Gabriel Turinici. Calibration of local volatility using the local and implied instantaneous variance. 2008. hal-00338114v1

**HAL Id: hal-00338114**

**<https://hal.science/hal-00338114v1>**

Preprint submitted on 11 Nov 2008 (v1), last revised 21 Dec 2008 (v2)

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Calibration of local volatility using the local  
and implied instantaneous variance

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Nov 11, 2008



# Calibration of local volatility using the local and implied instantaneous variance

## Abstract

We document the calibration of the local volatility through a functional to be optimized; our calibration variables are the local and implied instantaneous variances whose theoretical properties are fully explored within a particular class of volatilities. We confirm the theoretical results through a numerical procedure where we separate the parametric optimization (performed with any suitable optimization algorithm) from the computation of the functional by the use of an adjoint to obtain an approximation. The procedure performs well on benchmarks from the literature and on FOREX data.

Keywords: calibration, local volatility, implied volatility, Dupire formula, adjoint, instantaneous local variance, instantaneous implied variance, implied variance.

## 1 Motivation: the local volatility surface

Let us consider a security  $S_t$  (e.g. a stock, a FOREX rate, etc.) whose price, under the risk-neutral [Musiela and Rutkowski, 2005, Hull, 2006] measure,

follows the stochastic differential equation

$$dS_t/S_t = (r(t) - q(t))dt + \sigma dW_t \quad (1)$$

with  $r(t)$  being the time dependent risk-free rate,  $q(t)$  the continuous dividend rate,  $\sigma$  the volatility (we will make explicit its dependence latter) and  $W_t$  a Brownian motion.

Let us consider (for now) plain vanilla call options contingent on  $S_t$  and recall that when the volatility (and the discount rate  $r$ ) are known the Black-Scholes model [Black and Scholes, 1973] gives a formula for the price  $C(S, t)$  of such claims. It is standard to note that the reverse is also true, i.e., provided  $r$  is known, from the observed market prices denoted  $C_{K_l, T_l}^{market}$  (with strikes  $K_l$  and maturities  $T_l$ ,  $l = 1, \dots, L$ ) one can find (i.e. calibrate) the unique *implied volatilities*  $\sigma_{K_l, T_l}^I$  that, when introduced in the Black-Scholes formulae, match the observed market prices  $C_{K_l, T_l}^{market}$ . However the implied volatilities  $\sigma_{K_l, T_l}^I$  thus obtained are not the same for all  $K_l$  and  $T_l$  (the *smile* effect) which is inconsistent with the initial model. To address this issue it was independently proposed by Rubinstein [Rubinstein, 1994], Dupire [Dupire, 1994] and Derman and Kani [Derman and Kani, 1994] to take the volatility  $\sigma$  as depending on the time and the security price  $S$  :  $\sigma = \sigma(S, t)$ ; the model is named *local volatility*. Historically the proposals in [Rubinstein, 1994, Derman and Kani, 1994] build on the Cox-Ross-Rubinstein binomial tree [Cox et al., 1979] and

are described as *implied trees*.

Let us make clear that we do not discuss here the local volatility model itself nor its dynamics. We only see the local volatility as a way to express the non-arbitrage relationships between the set of derivatives contracts contingent on the same (set of) underlying instruments (much similar to the way one uses the *risk neutral* probability measure as a tool to compute prices but does not necessarily want to assign it to any real world probabilities).

Although efficient computation of the map  $\sigma \rightarrow C$  can, e.g. be performed with the Dupire formula [Dupire, 1994, Hull, 2006, Achdou and Pironneau, 2005] the inverse map i.e. matching observed prices is often unstable and the problem becomes now an inverse problem [Bouchouev and Isakov, 1997, 1999].

When the number of quoted market prices  $C_{K_l, T_l}^{market}$  is large/dense enough (i.e.  $K_l, T_l$  cover well the range of  $S$  and  $t$ ) the local volatility can be expressed using the Dupire formula [Dupire, 1994, Hull, 2006, Achdou and Pironneau, 2005]. However, when only a few prices are known, the Dupire formula is less effective and other methods have to be used [Avellaneda et al., 1997, Bodurtha and Jermakyan, 1999]. Among those, Coleman, Li & Verma [Coleman et al., 2001] introduced a parametric procedure which we refine in this contribution. Further, L. Jiang, and co-authors established a mathematical grounding for formulating this problem as a control problem [Jiang et al.,

2003]; we will retain in this paper the adjoint state technique that we adapt to take into account the constraints (see [Lagnado and Osher, 1997, 1998] for related endeavors). Our procedure combines the approaches above and is accelerated by the use of an approximation of the functional through the use of the adjoint (11).

Motivated by considerations concerning the convexity of the quality functional, the specificity of our approach is to work directly with the local and implied instantaneous variances as primal variables; we show that in some particular cases our choice of variables render the optimization strictly convex, that it converges to the correct local volatility when the number of observations is dense enough and that Tychonoff regularization provides a stable way to converge to the expected solution; moreover the convexity does not only speed the convergence but also eliminates local minima and operates a coherent selection of the adequate local surface, as confirmed in numerical experiments.

This approach (rather natural since option traders often only quote the implied volatility and not the price) is especially useful in markets that heavily rely on Greeks: e.g. in the FOREX market one quotes *risk reversals* which involve Deltas and the implied volatility.

## 1.1 Mathematical preliminaries

From a given local volatility  $\sigma$  the Dupire formula [Dupire, 1994, Hull, 2006, Achdou and Pironneau, 2005] allows to recover a continuum of option prices for all strikes  $K$  and time to maturities  $t < T$ , denoted  $C(K, t, \sigma)$ ;

Let us denote by  $\mathcal{P}$  the map from the local volatility  $\sigma$  to prices i.e.  $\mathcal{P}(\sigma)(K, t) = C(K, t, \sigma)$ . We also introduce the map  $\mathcal{V}$  from the square  $v = \sigma^2$  of the local volatility, named hereafter *instantaneous local variance* to the square of the implied volatility, named hereafter *instantaneous implied variance* (both as functions of  $K$  and  $t$ ) i.e., with our notations  $\mathcal{V}(v)(K, t)$  is the square  $w = (\sigma^I(K, t))^2$  of the implied volatility of the option with price  $C(K, t, \sigma)$ .

Calibration can be recast as an inverse problem of the form  $T(u) = f$  with  $T$  being one of the operators  $\mathcal{P}$ ,  $\mathcal{V}$  and possibly any similar mappings (see [Berestycki et al., 2002] that use  $\mathcal{B}(\xi) = \frac{1}{\sqrt{\mathcal{V}(\sigma^2)}}$  where  $\xi = \frac{1}{\sigma}$ ) ; in practice this problem is solved by optimization of some quality functional  $J(u) = \|Tu - f\|^2$  (to which one may add some Tychonoff regularization terms, see [Achdou and Pironneau, 2005, Bonnans et al., 2006] and latter in this paper); most often used are the mappings  $\sigma \rightarrow \mathcal{P}$  and  $\sigma = \sqrt{v} \rightarrow \sigma^I = \sqrt{\mathcal{V}(v)}$ .

To make our notations easier we will suppose from now on that the prob-



lem has a solution  $\sigma_0$ ; we denote  $v_0 = \sigma_0^2$ .

Among the properties of the optimization functional  $J(u)$  a distinguished property is the convexity. If the convexity is not complete, we would like to have it at least on some distinguished classes of local volatilities. To the best of our knowledge, no systematic studies exist to motivate the choice of minimization variables. In particular none of the mappings  $\sigma \rightarrow \mathcal{P}$  or  $\sigma \rightarrow \sqrt{\mathcal{V}(\sigma^2)}$  has been shown to be convex on any particular classed of functions. On the contrary, for  $\sigma^2 = v \rightarrow \mathcal{V}(v)$  we give below a result for the situation when the local surface is strike independent but has (arbitrary) time dependence.

Our primary variable is from now on the local instantaneous variance  $v = \sigma^2$  and the quantity to fit the implied instantaneous variance  $w = \mathcal{V}(\sigma^2)$ .

## **2 Mathematical properties : strike independent local volatility**

The dependence  $v \rightarrow w$  (see previous section for notations) is not straightforward to analyze, we refer to [Berestycki et al., 2002, Gatheral, 2006] for equations relating the two. However when the local volatility depends only on time  $\sigma = \sigma(t)$  we can understand much of its basic properties. Accord-

ingly we will suppose throughout this section that the volatility is strike independent.

**Lemma 1** *The mapping  $v \in L^2(0, T) \rightarrow \|\mathcal{V}(v) - \mathcal{V}(v_0)\|^2$  is strictly convex.*

*In particular equation  $\mathcal{V}(v) = \mathcal{V}(v_0)$  has an unique solution  $(v_0)$ .*

**Proof** When  $\sigma(S, t) = \sigma(t)$  we know (see [Gatheral, 2006]) that the implied instantaneous variance  $w$  is the average of the local instantaneous variance  $v$ :  $\mathcal{V}(v)(t) = \frac{1}{t} \int_0^t v(s) ds$  and  $\mathcal{V}(v)(0) = v(0)$ . Thus  $\mathcal{V}$  is a linear mapping **in these variables**. All that remains to be proved is that  $V$  is non degenerate.

We will prove more, namely that  $V$  is a strictly positive operator. From the above formula we have  $v = (tw)'$ ; then  $\langle \mathcal{V}(v), v \rangle_{L^2(0, T)} = \int_0^T (tw)' w dt = \frac{T}{2} w^2(T) + \int_0^T w^2 dt$  which is strictly positive except when  $w$  is identically null.

As a consequence we have that  $\mathcal{V}$  is a continuous map from  $L^2(0, T)$  to itself.

The existence and uniqueness of the solution is a direct consequence of the positivity of the map.  $\square$

We know now, by the lemma above, that the solution to the calibration problem  $\mathcal{V}(v) = \mathcal{V}(v_0)$  is unique; however the inverse mapping is not stable i.e. if we modify slightly  $v_0$  the solution can change dramatically. In order to lift the ill-posedness of the problem it is classical to add a regularization term [Achdou and Pironneau, 2005] which in this variable reads  $\|v'\|_{L^2}^2$ . We will prove latter (Thm. 2) that adding this term gives indeed a stable way to

invert the mapping.

However stability with respect to small variations in the overall volatility surface is not enough; in practice data does not cover all possible maturities but only a discrete set of times  $T_i$ ,  $i = 1, \dots, N$ ; a desirable property of the calibration process is the stability also respect to the amount of data available. The following result suggests a procedure that converges in a stable way:

**Theorem 1** 1/ For any division  $\mathcal{T} = T_0 = 0 \leq T_1 \leq \dots \leq T_i \leq \dots \leq T_n = T$  the minimum

$$\min_{v \in H^1; \mathcal{V}(v)(T_i) = \mathcal{V}(v_0)(T_i)} \|v'\|_{L^2(0,T)}^2 \quad (2)$$

is attained in an unique point  $v^{\mathcal{T}}$ .

2/ Let  $d(\mathcal{T}) = \max_i(T_{i+1} - T_i)$  be the size of the division  $\mathcal{T}$ ; then if  $v_0 \in H^s$  with  $s > 1$  the sequence  $v^{\mathcal{T}}$  converges to  $v_0$  :  $\|v^{\mathcal{T}} - v_0\|_{H^1} \rightarrow 0$  as  $d(\mathcal{T}) \rightarrow 0$ .

**Proof** 1/ The problem can also be written as

$$\min_{v \in H^1; v(0) = v_0(0); \int_{T_i}^{T_{i+1}} v(t) dt = \int_{T_i}^{T_{i+1}} v_0(t) dt, \forall i \leq n-1} \|v'\|_{L^2(0,T)}^2 \quad (3)$$

Obviously, the space of  $H^1$  functions  $\{v \in H^1; \mathcal{V}(v)(T_i) = \mathcal{V}(v_0)(T_i)\}$  that fits the data at the points  $T_i$  is convex and closed  $L^2$  and  $H^1$ . By taking a minimizing sequence  $v_n$  we obtain by classical arguments that  $v'_n$  is  $L^2$

bounded which, together with the weak lower semi-continuity of the norm gives the existence of a minimizer. The uniqueness follows from the strict convexity of the norm on the convex space of constraints. Let us note that the minimizer satisfies the following Euler equation  $-v'' = cst$  on any  $[T_i, T_{i+1}]$ ; in addition it is also continuous (as element of  $H^1([0, T])$ ) and satisfies the other constraints  $v(0) = v_0(0)$ ,  $\int_{T_i}^{T_{i+1}} v(t)dt = \int_{T_i}^{T_{i+1}} v_0(t)dt$ ,  $\forall i \leq n - 1$ .

2/ For the second part we need a more detailed description of the solution. Let us denote  $W^{\mathcal{T}}(t) = t\mathcal{V}(v^{\mathcal{T}})(t)$ ,  $W_0(t) = t\mathcal{V}(v_0)(t)$  and recall that  $v^{\mathcal{T}}(t) = (W^{\mathcal{T}})'(t)$ . The minimization problem of which  $v^{\mathcal{T}}$  is solution can be rephrased in terms of  $W^{\mathcal{T}}$  :

$$W^{\mathcal{T}} = \arg \min_{W(T_i)=W_0(T_i), i=0, \dots, n} \|W''\|_{L^2(0, T)}^2 \quad (4)$$

We know that the solution  $W^{\mathcal{T}}$  is the cubic spline interpolant of  $W_0$  at points  $T_i$ . The conclusion follows from the approximation properties of the cubic splines, more precisely the convergence of the derivatives of  $H^2$  functions, cf. de Boor [2001].  $\square$

At this point we have a stable procedure involving, at each moment, only a finite number of information on option prices. But we still do not know how to compute in a stable manner the solutions  $v^{\mathcal{T}}$ . The answer is given in the next result; for simplicity we will omit to mark explicitly the dependence in  $\mathcal{T}$ .

**Theorem 2** *Let us consider*

$$J^\epsilon(v) = \epsilon \|v'\|_{L^2}^2 + \sum_i (\mathcal{V}(v)(T_i) - (\mathcal{V}(v_0)(T_i)))^2. \quad (5)$$

*The problem  $\min_{v \in H^1} J^\epsilon(v)$  has an unique solution  $v^\epsilon$ . Moreover  $v^\epsilon$  converges in  $H^1$  to  $v^{\mathcal{T}}$  as  $\epsilon \rightarrow 0$ .*

**Proof** The existence of  $v^\epsilon$  follows from arguments of convexity; we refer the reader to the proof of Thm.3 in [Berestycki et al., 2002] where they use a very similar functional  $\mathcal{V}$  (albeit over whole  $\mathbb{R}$  and with other variables). They also prove that  $v^\epsilon$  converges to some limit  $v_l$  with  $\mathcal{V}(v_l)(T_i) = \mathcal{V}(v_0)(T_i)$ ,  $\forall i \leq n$ . We will now identify this limit  $v_l$ . Let  $v$  be such that  $\mathcal{V}(v)(T_i) = \mathcal{V}(v_0)(T_i)$   $\forall i \leq n$ ; from  $J^\epsilon(v^\epsilon) \leq J^\epsilon(v)$  one obtains in particular  $\|(v^\epsilon)'\| \leq \|v'\|$ . Thus  $\limsup_{\epsilon \rightarrow 0} \|(v^\epsilon)'\| \leq \|v'\|$ . Recalling the weak lower semi-continuity of the norm and passing to infimum over all  $v$  with  $\mathcal{V}(v_l)(T_i) = \mathcal{V}(v_0)(T_i)$  one obtains

$$\|(v_l)'\| \leq \liminf_{\epsilon \rightarrow 0} \|(v^\epsilon)'\| \leq \limsup_{\epsilon \rightarrow 0} \|(v^\epsilon)'\| \leq \|(v^{\mathcal{T}})'\|. \quad (6)$$

which, together with definition of  $v^{\mathcal{T}}$  gives  $v_l = v^{\mathcal{T}}$  and the convergence will be in  $H^1$ .  $\square$

The reader interested in having the corresponding results for the situation of a parametric optimization is referred to the Appendix A.

### 3 Numerical implementation : adjoint and the cost functional

We consider now the full generality of the  $S$  and  $t$  dependence of  $\sigma$  :  $\sigma = \sigma(S, t)$  and use the functional

$$J^\epsilon(v) = \epsilon \|v'\|_{L^2}^2 + \sum_{i=1}^L (\mathcal{V}(v)(K_i, T_i) - (\mathcal{V}(v_0)(K_i, T_i)))^2. \quad (7)$$

where now  $v_0$  is the unknown market instantaneous local variance. In order to minimize the functional  $J^\epsilon$  one needs to compute the gradient  $\frac{\partial \mathcal{V}(v)(K_i, T_i)}{\partial v}$ .

Recall that the price  $C(S, t)$  of a European call on  $S_t$  and maturity  $t = T$  satisfies the (Black-Scholes) equation [Hull, 2006] for all  $S \geq 0$  and  $t \in [0, T]$ :

$$\partial_t C + (r - q)S \partial_S C + \frac{\sigma^2 S^2}{2} \partial_{SS} C - rC = 0 \quad (8)$$

$$C(S, t = T) = (S - K)_+ \quad (9)$$

**Remark 1** For a stock  $q(t)$  is the (known) dividend rate while for a FOREX spot  $q(t)$  is the foreign interest rate (with  $r(t)$  being the domestic rate).

The price at  $t = 0$  of the contract is  $C(S_{t=0}, t = 0)$ ; note the retrograde nature of the equation (8)-(9). We will use the technique of the adjoint state [Achdou and Pironneau, 2005, Jiang et al., 2003] and view the price as a implicit functional of  $\sigma$  (here  $\delta$  is the Dirac operator):  $C(t = 0; S = S_0) = \langle \delta_{t=0, S=S_0}, C(S, t) \rangle$ . Then the variation  $\frac{\delta C}{\delta(\sigma^2)}$  of  $C$  with respect to  $\sigma^2$

will be

$$\frac{\delta C}{\delta(\sigma^2)} = \frac{S^2}{2}(\partial_{SS}C)\chi. \quad (10)$$

Here the adjoint state  $\chi$  is the solution of:

$$\partial_t\chi + \partial_S((r - q)S\chi) - \partial_{SS}\left(\frac{\sigma^2 S^2}{2}\chi\right) + r\chi = 0 \quad (11)$$

$$\chi(S, t = 0) = \delta_{S=S_0}. \quad (12)$$

**Remark 2** *Both problems (8) and (11) can be solved e.g. through a Crank-Nicholson finite-difference scheme (in time and space) [Hull, 2006, Andersen and Brotherton-Ratcliffe, 1998] that we explain in Appendix B.*

**Remark 3** *In the numerical resolution of the adjoint state we replace  $\delta_{S=S_0}$  by a regularized version  $\delta_{S=S_0}^a = \frac{1}{a\sqrt{\pi}}e^{-(S-S_0)^2/a^2}$  with  $a$  of the order of one percent of  $S_0$ ; this choice (and in fact many other) was seen to give consistently good numerical results.*

To illustrate the nature of this gradient we display an example in Figure 1 where we note two singularities appearing in  $(t = 0, S = S_0)$  (from eqn (12)) and  $(t = 1, S = K)$  (from  $\partial_{SS}(S - K)_+$ ) (see also [Avellaneda et al., 1997] for similar conclusions).

Same technique works for any other quantity dependent on the price, e.g. the instantaneous implied variance  $\mathcal{V}(v)(K_i, T_i)$ . Recall that an explicit formula links the price to the implied volatility  $C = C(\sigma^I)$  and as such

$\frac{\partial \mathcal{V}(v)(K_i, T_i)}{\partial v} = \frac{\partial (\sigma^I)^2}{\partial \sigma^2} = 2\sigma^I \frac{\partial \sigma^I}{\partial C} \frac{\partial C}{\partial \sigma^2}$ . We recognize in the term  $\frac{\partial C}{\partial \sigma^I}$  the Black-Scholes vega, that we will denote  $\nu_{BS}^I$ . We obtain

$$\frac{\partial \mathcal{V}(v)(K_i, T_i)}{\partial v} = 2\sqrt{\mathcal{V}(v)(K_i, T_i)} \frac{1}{\nu_{BS}^I} \frac{\partial C}{\partial \sigma^2}. \quad (13)$$

Repeated application of the formula (13) allows to compute the gradient of the second part of  $J^\epsilon$  in Eqn (7) with respect to  $v$ . Note that for each term  $\mathcal{V}(v)(K_i, T_i)$  one needs to solve a PDE for the price and a corresponding PDE for the adjoint and use them as in (10) (i.e.  $2L$  PDEs).

**Remark 4** *An alternative solution could be to fit directly the whole surface  $\mathcal{V}(v)$  instead of an ensemble of fitting points  $K_i, T_i$ . The computation of the full surface  $\mathcal{P}(\sigma)$  of option prices (then of  $\mathcal{V}(v)$ ) can be performed directly from the Dupire equation at the price of one PDE (or from the equation relating local and implied volatilities cf [Berestycki et al., 2002, Gatheral, 2006]). But then we need a different adjoint PDE each time we want to compute the gradient for a new descent direction. The computational speed-up (or not) will depend on the relative number of gradient computations (for Dupire) and  $2L$  times the number of macro iterations (cf. Section 4.1) for the procedure we propose. As it will be seen in Section 5 for our target applications (FOREX) we need between 5 and 10 macro iterations to converge and the data is not so abundant (less than 30 known implied volatilities to fit). We plan to document in a future work a comparison between the two*



*methods.*

*An additional argument (but there are ways to circumvent this) is that we want to specifically emphasize the points  $K_i, T_i$  where we have market information and not fit the entire (necessarily interpolated) implied surface.*

## 4 Surface space and the optimization procedure

A traditional choice to avoid singularities and address the possible non-uniqueness of the solution (for  $\epsilon = 0$   $J^\epsilon$  has an infinity of minima) is to parametrize the surface  $\sigma(S, t)$  [Achdou and Pironneau, 2005, Coleman et al., 2001]; the result will be the optimal surface in the class.

We give here a possible choice to describe the space of available surface shapes. We consider continuous affine functions with degrees of freedom being the values on some grid ( $S_i = S_0 + i\Delta S, t_j = t_0 + j\Delta t$ ),  $i \leq I, j \leq J$ . We denote by  $f_{ij}(S, t)$  the unique piecewise linear and continuous function that has value of 1 at  $(t_i, S_j)$ , and is zero everywhere else. The surfaces are linear combinations of the shapes  $f_{ij}(S, t)$ :

$$v(S, t) = \sigma^2(S, t) = \sum \alpha_{ij} f_{ij}(S, t). \quad (14)$$

The advantage of linear interpolation is that the shape functions have

nice localization properties: the scalar product of two such functions (or their gradient) is zero except if they are neighbors i.e. matrices (18)-(19) are sparse. Also setting constraints e.g.  $v(S, t) \in [v_{min}, v_{max}]$  for all  $S, t$  is equivalent to asking that all  $\alpha_{ij}$  are in  $[v_{min}, v_{max}]$ .

**Remark 5** *We also tested cubic splines interpolation and it performed equally satisfactory. Imposing the constraints  $v(S, t) \in [v_{min}, v_{max}]$  in this case is done pointwise: we ask that the local volatility  $v(S_j, t_i)$  be between  $[v_{min}, v_{max}]$  in any interpolation node  $(S_j, t_i)$ ; this reduces to a set of linear constraints in the coefficients  $\alpha$ :  $\sum_{kl} \alpha_{kl} f_{kl}(S_j, t_i) \in [v_{min}, v_{max}]$ .*

**Remark 6** *A possible procedure would be to optimize the cost functional (7) expressed as a function of the coefficients  $\alpha_{ij}$  of  $v$  in (14). But, depending on the interpolation function, this dependence is highly nonlinear and the resulting optimization can have many unwanted local extrema.*

Chain rule gives the gradient of any  $\mathcal{V}(v)(K_i, T_i)$  with respect to variations of the instantaneous local variance  $v = \sigma^2$  inside the admissible surface space. This is in fact just a matter of projecting the exact gradient (10) onto each shape  $f_{ij}$ . We obtain a first order approximation formula around the current local instantaneous variance  $v$ :

$$\mathcal{V}\left(v + \sum_{ij} \alpha_{ij} f_{ij}(S, t)\right)(K_i, T_i) = \mathcal{V}(v)(K_i, T_i) + \sum_{ij} \left\langle \frac{\partial \mathcal{V}(v)}{\partial v}, f_{ij} \right\rangle_{L^2_{S,t}} \alpha_{ij} + o(\alpha). \quad (15)$$

In discrete formulation the second part of the cost functional  $J^\epsilon$  will employ the matrix

$$M_{ab;rs} = \sum_{i=1}^L \left\langle \frac{\partial \mathcal{V}(v)}{\partial v}(K_i, T_i), f_{ab} \right\rangle_{L_{S,t}^2} \left\langle \frac{\partial \mathcal{V}(v)}{\partial v}(K_i, T_i), f_{rs} \right\rangle_{L_{S,t}^2}. \quad (16)$$

Note that (15) already provides (some) second order information; also note that around  $v = \sum_{ij} \beta_{ij} f_{ij}(S, t)$  the Tychonoff regularization term  $\epsilon \|\nabla v\|^2$  [Achdou and Pironneau, 2005, Bonnans et al., 2006] can be written

$$\epsilon \langle \beta + \alpha, Q_S(\beta + \alpha) \rangle + \epsilon \langle \beta + \alpha, Q_t(\beta + \alpha) \rangle \quad (17)$$

where

$$(Q_S)_{ij;kl} = \int \int \frac{\partial f_{ij}(S, t)}{\partial S} \frac{\partial f_{kl}(S, t)}{\partial S} dS dt \quad (18)$$

and

$$(Q_t)_{ij;kl} = \int \int \frac{\partial f_{ij}(S, t)}{\partial t} \frac{\partial f_{kl}(S, t)}{\partial t} dS dt. \quad (19)$$

A last ingredient involves bounds on  $v(S, t)$ ; indeed,  $v(S, t)$  cannot be negative. Even when this is the case, local volatilities with very low values (e.g. 3% !) are obviously not realistic. Enforcing constraints on the local volatilities is a very important step towards selecting meaningful candidates. A choice that is consistent with other observations in the literature [Rubin-

stein, 1994, Derman and Kani, 1994] is to ask

$$\begin{aligned}
v(t, S) &= \sigma(t, S)^2 \in [v_{min}, v_{max}] \text{ with} \\
v_{min} &= \left( \frac{1}{2} \min\{\sigma_{K_l, T_l}^{I;market}; l = 1, \dots, L\} \right)^2, \\
v_{max} &= \left( 2 \max\{\sigma_{K_l, T_l}^{I;market}; l = 1, \dots, L\} \right)^2.
\end{aligned} \tag{20}$$

## 4.1 Optimization procedure

The algorithm operates as follows: first we choose as initial guess  $v^0$  to be the (projection on the space  $Vect\{f_{ij}\}$ ) of the known implied instantaneous variance surface  $\mathcal{V}(v_0)$ , eventually corrected to be between bounds  $\sigma_{min}$  and  $\sigma_{max}$ .

The iterative procedure operates in macro-iteration cycles by using the approximation formula (15) and solving a quadratic minimization problem around each point:

1/ Around  $v^k = \sum_{ij} \beta_{ij} f_{ij}(S, t)$  compute the gradient in formula (15) and obtains the functional  $J^k(\alpha)$  that approximates  $J^\epsilon(v^k + \sum_{ij} \alpha_{ij} f_{ij}(S, t))$ :

$$J^k(\alpha) = \frac{1}{2} \langle \alpha^T, (M + \epsilon Q_S + \epsilon Q_t) \alpha \rangle + w^t \alpha + J^\epsilon(v). \tag{21}$$

2/ solve the (quadratic) optimization problem

$$\begin{aligned}
\min \quad & J^k(\alpha) \\
v_{min} & \leq \beta + \alpha \leq v_{max}
\end{aligned} \tag{22}$$

3/ update the local instantaneous variance  $v^{k+1} = v^k + \alpha$ ; if the replication error is small then exit, otherwise set  $k = k + 1$  and return in 1/ for a new macro-iteration.

In practice very few macro-iteration cycles 1/-3/ are necessary. We tested on several indices and in the FOREX market and the numbers varied between 5 and 10 macro-iterations.

**Remark 7** *In order to remain in a region where the approximation (15) holds we have also imposed constraints on the maximum change in  $\alpha$  in step 2/. The bounds that proved satisfactory were of the order  $\frac{v_{max}-v_{min}}{10}$  although we did not try to optimize this bounds.*

**Remark 8** *The quadratic problem (22) can be solved by any suitable algorithm. The advantage of the approach is precisely to separate the optimization itself from the formulation of the problem. For instance Matlab uses by default a subspace trust-region method based on the interior-reflective Newton method described in [Coleman and Li, 1996].*

*We also tested a simple projected gradient: at each step we advanced a fixed step size in the direction of the gradient; then, points that do not satisfy the constraints  $[v_{min}, v_{max}]$  are projected to either  $v_{min}$  or  $v_{max}$ . We were surprised to see that in all cases we tested the projected gradient performs as well as a general quadratic optimization algorithm. We expect that the*

*reason lies with the advantageous choice of variables in which the problem was expressed i.e. variables  $v = \sigma^2$  and  $w = \mathcal{V}(v)$ . This numerical observation may indicate further convexity of the functional  $J^\epsilon(v)$  than what we were able to prove in Section 2.*

## 5 Results and conclusions

A specificity of the approach is that instead of a unique optimization in the parametric space we perform one optimization around each current point; this reduces the number of computations of the PDE (8). But, equally importantly, the separation between the optimization and the approximation of the functional provides flexibility in the information that can be fitted, e.g. we can ignore some prices should them not be available or if one wants to arbitrage against them (in contrast with the pioneering approaches [Rubinstein, 1994, Dupire, 1994, Derman and Kani, 1994] that need a uniform set of data to perform the inversion); in particular no interpolation is required to fill this information when missing.

The use of the gradient not in an optimization procedure but to obtain an approximation of the functional around the current point is a acknowledgement of the fact that the main difficulty is not finding a solution but choosing one among all compatible surfaces (i.e. ill-posedness).

Numerical results (not shown here) show a clear improvement when using the formulation involving volatilities instead of prices.

We used throughout a grid with  $I = 24$  values of  $S$  and  $J = 13$  values of  $t$  i.e. 312 shapes  $f_{ij}$ , cf. eqn. (14). The regularization parameter  $\epsilon$  was in the range  $[1.e - 3, 1.e - 2]$ . The finite differences used 100 time steps and 200 spacial steps.

Let us now iterate through several benchmarks from the literature; we begin with the European call data on the S& P index from [Andersen and Brotherton-Ratcliffe, 1998, Coleman et al., 2001]. Similar to [Andersen and Brotherton-Ratcliffe, 1998, Coleman et al., 2001], we use in our computation only the options with maturities of less than two years. The initial index, interest rate and dividend rate are the same (see Figure 3). We first checked (not shown) that for  $L = 1$  the problem recovers the implied volatility; it did so with only one cycle. When we took all the  $L = 70$  data the resulting local volatility surface is given Figure 3.

We next moved to a FOREX example (from [Avellaneda et al., 1997]) where synchronous option prices (based on bid- ask volatilities and risk-reversals) are provided for the USD/DEM 20,25 and 50 delta risk-reversals quoted on August 23rd 1995. The results in Figure 4 show a very good fit quality with only five cycles 1/-3/.

We remain in the FOREX market and take as the next example 10,25 and

50-Delta risk-reversal and strangles for EUR/USD dated March 18th 2008 (courtesy of Reuters Financial Software). We recall that e.g. a 25 Delta risk reversal contract consists in a long position in a call option with  $\text{delta}=0.25$  and a short position in a put option with  $\text{delta} = -0.25$ ; the contract is quoted in terms of the difference of the implied volatilities of these two options. Note that the price of the options never appears in the quotes. In order to set the implied surface we used 10 and 25 Delta strangles which are quoted as the arithmetic mean of the implied volatilities of the two options above. Of course, from this data one can next recover the implied volatilities of each option, then all other characteristics. We give in Tables 1,2 and 3 the resulting data which is to be fitted; the data is to be understood in the following way: for a given Delta and maturity, e.g.  $\text{Delta}=0.25$  and  $T_i = 31$  days to expiry one finds its strike  $K_i = 1.6165$  in Table 1, the implied volatility  $\sigma^I(K_i, T_i) = 12.95\%$  from Table 2 and the premium (consistency check)  $C(K_i, T_i) = 0.0088\%$  from Table 3, all this with spot price  $S_0 = 1.5755$ ,  $r = r_{USD} = 0.2485$  and 'dividend' rate  $q = 0.0455$ . We present in Figure 5 the implied and the calibrated local volatility. The procedure was also tested (not shown here) on other currencies pairs (GBP, CHF, JPY, KRW, THB, ZAR all with respect to USD as domestic currency) and performed well.



## Acknowledgements

We gratefully thank two anonymous referees for their thoughtful feedback and relevant remarks that largely contributed to improve the paper; we also thank Mohammed Aissaoui, Marc Lailat and Kalide Brassier from the Adfin Analytics team of Thomson Reuters for providing the data in Tables 1,2 and 3 and for helpful discussions.

## Appendix A: parametric space versions of theoretical results of Section 2

In this section we would also like to know what happens when we look for the solution in a parametric space  $K$ . A standard example of such a set is a bounded part of some finite dimensional linear (parametric) space of surfaces. The requirement that at least one element exists that fits the data is a requirement on the diversity of elements of the set  $K$ .

**Theorem 3** *Let  $\mathcal{T} = T_0 = 0 \leq T_1 \leq \dots \leq T_i \leq \dots \leq T_n = T$  be any division of  $[0, T]$  and  $K_{\mathcal{T}}$  be a compact closed subset of  $H^1$  such that there exists at least one  $v_{0,K} \in K$  with  $\mathcal{V}(v_{0,K})(T_i) = \mathcal{V}(v_0)(T_i)$  for all  $i \leq N$ . Then the optimization problem  $\min\{\|v'\|_{L^2(0,T)}^2 | v \in K; \mathcal{V}(v)(T_i) = \mathcal{V}(v_0)(T_i)\}$  has an unique solution  $v_K^{\mathcal{T}}$  which attains thus the minimum.*

**Proof** The existence is straightforward due to the compactness of  $K$ . The uniqueness uses the convexity of  $K$  and of the norm.  $\square$

**Theorem 4** *Let  $K$  be a compact convex set and suppose that at least one element  $v_K^* \in K$  exists that will fit the observed implied volatilities:  $\mathcal{V}(v_K^*)(T_i) = w_0(T_i) \forall i \leq N$ . Then the problem  $\min_{v \in K} J^\epsilon(v)$  has an unique solution  $v_{\epsilon,K} \in K$ ; in addition the sequence  $(v_{\epsilon,K})_{\epsilon > 0}$  converges to  $v_{0,K}$  uniformly over  $[0, T]$ .*

**Proof** Same arguments as the proof in the whole space apply (plus the compactness of  $K$ ).  $\square$

## Appendix B: the numerical discretization scheme

We briefly explain in this section the resolution of equations (8) and (11). We use a standard finite differences scheme [Hull, 2006, Andersen and Brotherton-Ratcliffe, 1998]: denote by  $C_k^n$  the approximation of the value  $C(S_k, t_n)$  where  $S_k = S_{min} + k \times dS$  is the  $k$ -th spatial point and  $t^n = n \times dt$  the  $n$ -th time step; the equation for  $C$  is backward, i.e. we know  $C_k^{m+1}$  and want to compute  $C_k^n$ ; the numerical scheme is then (we use  $\theta = 1/2$ ):

$$\begin{aligned}
& \frac{C_k^n - \frac{C_{k+1}^{n+1} + C_{k-1}^{n+1}}{2}}{-dt} + (r - q)S_k \frac{C_{k+1}^n - C_{k-1}^n}{2dS} - rC_k^n \\
& + \theta \frac{S_k^2 v(S_k, t^n)}{2} \left\{ \frac{C_{k+1}^n + C_{k-1}^n - 2C_k^n}{dS^2} \right\} \\
& + (1 - \theta) \frac{S_k^2 v(S_k, t^{n+1})}{2} \left\{ \frac{C_{k+1}^{n+1} + C_{k-1}^{n+1} - 2C_k^{n+1}}{dS^2} \right\} = 0. \quad (23)
\end{aligned}$$

It is best to use for (11) the numerical adjoint of (8) (cf. also [Achdou and Pironneau, 2005] Section 8.3 for an example of numerical adjoint, albeit for the Dupire equation). This means that if the vector  $C^n = (C_k^n)_{k \geq 1}$  solves the equation  $\mathcal{A}^n C^n = \mathcal{D}^n C^{n+1}$  then the vector  $\chi^n$  that approximates  $\chi(\cdot, n \times dt)$  solves  $(\mathcal{A}^n)^T \chi^n = (\mathcal{D}^{n-1})^T \chi^{n-1}$ .

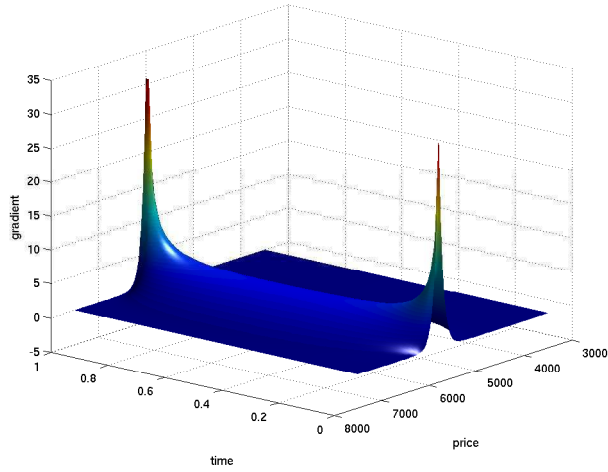


Figure 1: Gradient  $\frac{\delta C}{\delta(\sigma^2)}$  (see eqn. (10)) of the price  $C$  of a plain vanilla call with respect to the local instantaneous variance  $v = \sigma^2$ . Note the two singularities at the initial time (around the spot price) and at the expiration around the strike. These singularities prevent the direct use of a gradient method in these variables otherwise the resulting surface will be singular.

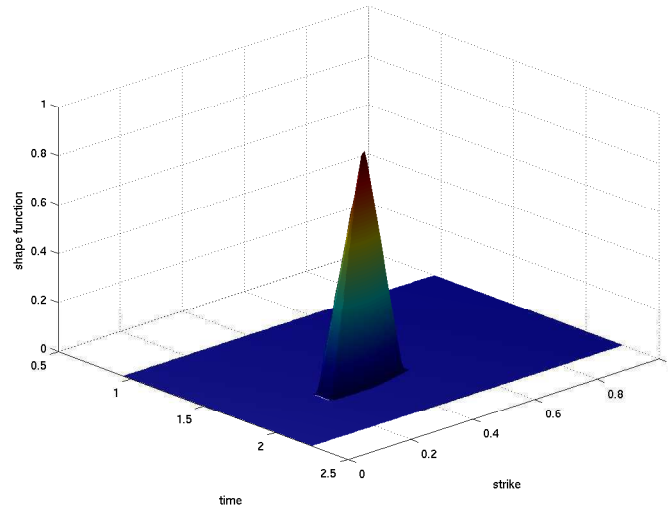


Figure 2: The local instantaneous variance  $v = \sigma^2(S, t)$  is sought after as a linear combination of basic shapes  $f_{ij}(S, t)$ :  $v(S, t) = \sum_{ij} \alpha_{ij} f_{ij}$ . A possible choice is to take  $f_{ij}(S, t)$  as the (unique) linear interpolation which is zero except in some point  $(S_i, t_j)$  (part of a grid in  $S$  and  $t$ ). We display here such a shape.

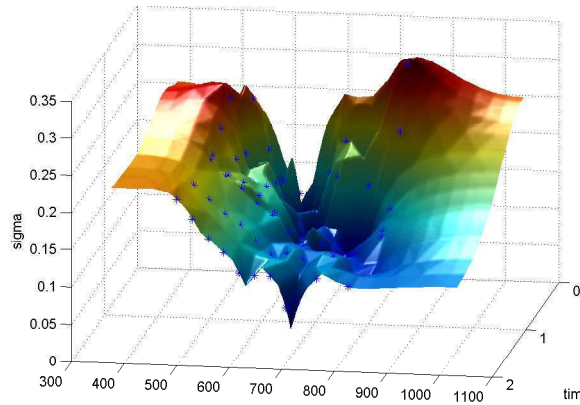


Figure 3: Local volatility surface of the S&P 500 index as recovered from the published European call options data [Andersen and Brotherton-Ratcliffe, 1998, Coleman et al., 2001]; spot price is \$590; discount rate  $r = 6\%$ , dividend rate 2.62%. The blue marks on the surface indicate the option prices that were used to invert i.e. the  $K_l$  and  $T_l$  ( $L = 70$ ). After 10 iterations the prices are recovered up to 0.04% (relative to spot) and the implied volatility up to 0.28%. Setting regularization parameter  $\epsilon$  to smaller values gives better fit but less smooth surfaces.

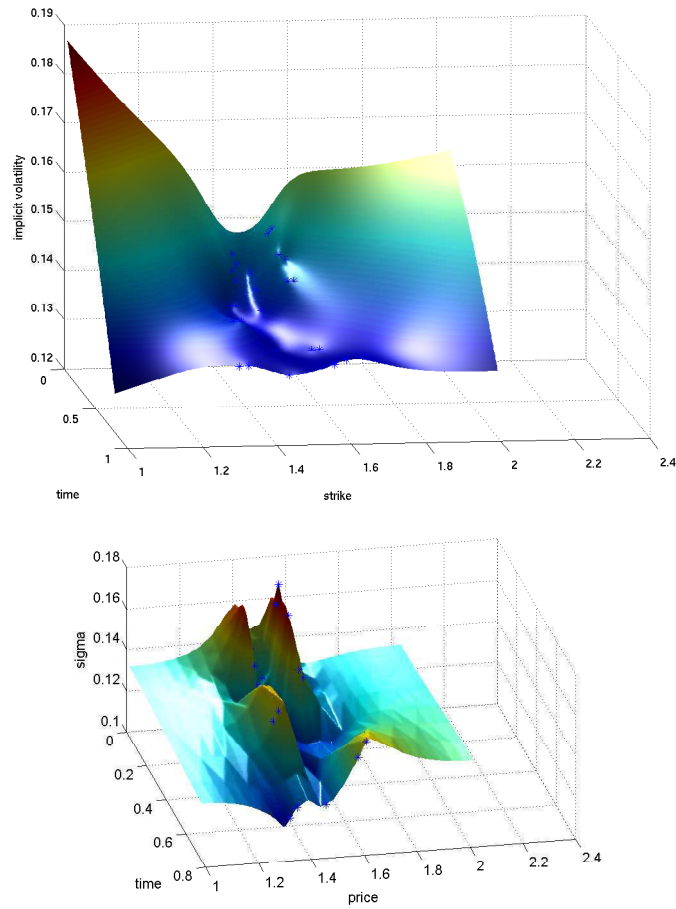


Figure 4: Top: implied volatility surface of the USD/DEM rate from [Avellaneda et al., 1997]; blue marks on the surface represent the available prices (to be matched). Bottom: local volatility surface as recovered from quoted 20,25 and 50-delta risk-reversals [Avellaneda et al., 1997]; (mid) spot price is 1.48875; *USD* discount rate  $r = 5.91\%$ , and *DEM* rate 4.27%. The blue marks on the surface indicate the option prices that were used to invert i.e. the  $K_l$  and  $T_l$  ( $L = 30$ ). After 5 iterations the prices are recovered up to 0.005% (relative to spot), below the PDE resolution, and the implied volatility up to 0.03% (below the bid/ask spread).

Delta	0,1	0,25	0,5	0,75	0,9
Days to Expiry					
7	1,6177	1,5965	1,5753	1,5544	1,5341
31	1,6564	1,6150	1,5740	1,5335	1,4935
59	1,6804	1,6253	1,5720	1,5191	1,4653
92	1,7013	1,6333	1,5686	1,5042	1,4368
184	1,7449	1,6474	1,5592	1,4711	1,3728
365	1,8030	1,6611	1,5391	1,4164	1,2665

Table 1: Strikes of the EUR/USD data derived from March 18th 2008 10,25 and 50 Delta risk-reversals and straddles corresponding to results in Figure 5.



Delta	0,1	0,25	0,5	0,75	0,9
Days to Expiry					
7	14,8250%	14,1750%	13,9250%	14,1750%	14,8250%
31	13,5250%	12,9500%	12,7750%	13,1000%	13,8250%
59	12,7750%	12,1500%	12,0250%	12,4000%	13,2750%
92	12,4250%	11,7500%	11,6250%	12,1000%	13,1250%
184	12,0875%	11,2125%	11,0500%	11,6375%	12,9625%
365	11,9125%	10,8625%	10,7000%	11,3375%	12,8875%

Table 2: Implied volatilities of the EUR/USD data derived from March 18th 2008 10,25 and 50 Delta risk-reversals and straddles corresponding to results in Figure 5.

Delta	0,1	0,25	0,5	0,75	0,9
Days to Expiry					
7	0,0015	0,0046	0,0121	0,0253	0,0425
31	0,0029	0,0088	0,0231	0,0486	0,0824
59	0,0038	0,0113	0,0298	0,0632	0,1088
92	0,0046	0,0136	0,0358	0,0767	0,1341
184	0,0063	0,0182	0,0479	0,1039	0,1883
365	0,0086	0,0247	0,0650	0,1430	0,2724

Table 3: Premiums of the EUR/USD data derived from March 18th 2008 10,25 and 50 Delta risk-reversals and straddles corresponding to results in Figure 5.

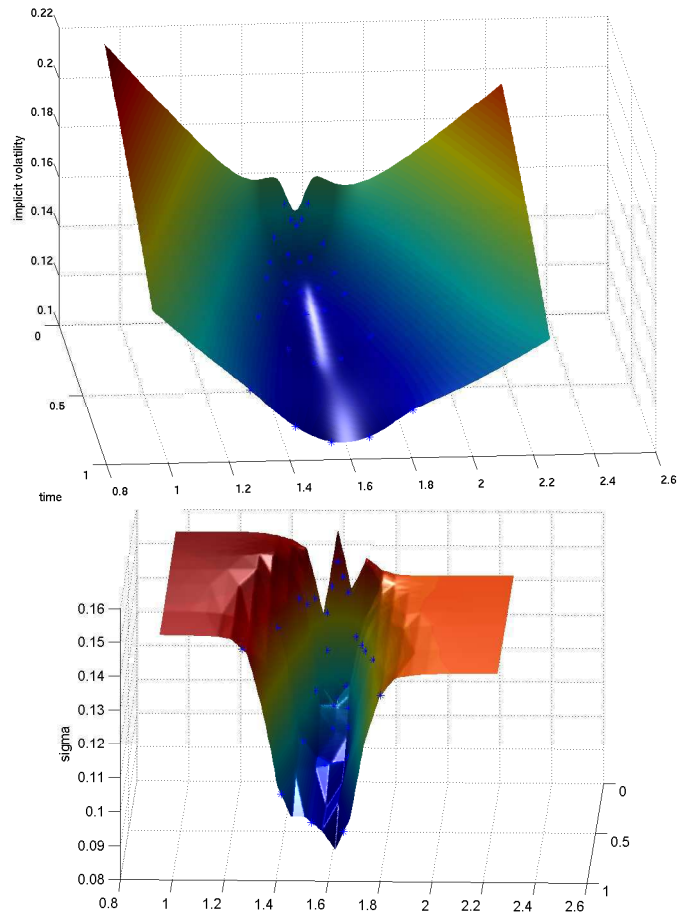


Figure 5: Top: implied volatility surface of the EUR/USD rate from Tables 1,2 and 3); marks on the surface represent the available prices (to be matched). Bottom: local volatility surface as recovered from quoted 10,25 and 50-delta risk-reversals and straddles; (mid) spot price is 1.5755; *USD* discount rate was set to  $r_{USD} = 2.485\%$ , and  $r_{EUR} = 4.550\%$ . The blue marks on the surface indicate the option prices that were used to invert i.e. the  $K_l$  and  $T_l$  ( $L = 30$ ). After 5 iterations the prices are recovered up to 0.001% (relative to spot) and the implied volatility up to 0.02% (below the bid/ask spread).

## References

- Yves Achdou and Olivier Pironneau. *Computational methods for option pricing*, volume 30 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005. ISBN 0-89871-573-3.
- L. Andersen and R. Brotherton-Ratcliffe. The equity option volatility smile: an implicit finite difference approach. *The Journal of Computational Finance*, 1:532, 1998.
- M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi. Calibrating volatility surfaces via relative-entropy minimization. *Applied Mathematical Finance*, 4(1):37–64, 1997. URL <http://www.informaworld.com/10.1080/135048697334827>.
- H. Berestycki, J. Busca, and I. Florent. Asymptotics and calibration of local volatility models. *Quantitative Finance*, 2, 2002. URL <http://www.informaworld.com/10.1088/1469-7688/2/1/305>.
- F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81, May-June 1973.
- J. Jr. Bodurtha and M. Jermakyan. Non-parametric estimation of an implied volatility surface. *J. Comput. Finance*, 2:2961, 1999.

- J. Frédéric Bonnans, J. Charles Gilbert, Claude Lemaréchal, and Claudia A. Sagastizábal. *Numerical optimization*. Universitext. Springer-Verlag, Berlin, second edition, 2006. ISBN 3-540-35445-X. Theoretical and practical aspects.
- I. Bouchouev and V. Isakov. The inverse problem of option pricing. *Inverse Problems*, 13:L117, 1997.
- I. Bouchouev and V. Isakov. Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets. *Inverse Problems*, 15:R95116, 1999.
- Thomas F. Coleman and Yuying Li. A reflective Newton method for minimizing a quadratic function subject to bounds on some of the variables. *SIAM J. Optim.*, 6(4):1040–1058, 1996. ISSN 1052-6234.
- Thomas F. Coleman, Yuying Li, and Arun Verma. Reconstructing the unknown local volatility function [J. Comput. Finance **2** (1999), no. 3, 77–100]. In *Quantitative analysis in financial markets*, pages 192–215. World Sci. Publ., River Edge, NJ, 2001.
- J. Cox, S. Ross, and M. Rubinstein. Option pricing: a simplified approach. *J. Financial Economics*, 7:22963, 1979.

- Carl de Boor. *A Practical Guide to Splines*. Applied mathematical sciences. Springer Verlag, Berlin, 2001.
- E. Derman and I. Kani. Riding on a Smile. *Risk*, 7(2):329, February 1994.
- B. Dupire. Pricing with a Smile. *RISK*, 7(1):18–20, 1994.
- Jim Gatheral. *The volatility surface : a practitioner's guide*. John Wiley & Sons, 2006.
- J. Hull. *Options, Futures, and Other Derivatives*. Prentice Hall, sixth edition, 2006.
- Lishang Jiang, Qihong Chen, Lijun Wang, and Jin E. Zhang. A new well-posed algorithm to recover implied local volatility. *Quantitative Finance*, 3:451–457, 2003. URL <http://www.informaworld.com/10.1088/1469-7688/3/6/304>.
- R. Lagnado and S. Osher. Reconciling differences. *Risk*, 10(4):7983, 1997.
- R. Lagnado and S. Osher. A technique for calibrating derivative security pricing models: numerical solution of the inverse problem. *J. Comput. Finance*, 1:1325, 1998.
- Marek Musiela and Marek Rutkowski. *Martingale methods in financial mod-*

*elling*, volume 36 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, second edition, 2005. ISBN 3-540-20966-2.

M. Rubinstein. Implied binomial trees. *J. Finance*, 49:771–818, 1994.