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Rate of convergence to self-similarity for Smoluchowski's coagulation equation with constant coefficients

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Abstract

We show that solutions to Smoluchowski's equation with a constant coagulation kernel and an initial datum with some regularity and exponentially decaying tail converge exponentially fast to a self-similar profile. This convergence holds in a weighted Sobolev norm which implies the L^2 convergence of derivatives up to a certain order k depending on the regularity of the initial condition. We prove these results through the study of the linearized coagulation equation in self-similar variables, for which we show a spectral gap in a scale of weighted Sobolev spaces. We also take advantage of the fact that the Laplace or Fourier transforms of this equation can be explicitly solved in this case.

Keywords. Smoluchowski's equation; coagulation equation; constant coagulation kernel; self-similar variables; spectral gap; exponential relaxation rate; explicit.

AMS Subject Classification. 82C21, 45K05, 82C05.

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1 Introduction

1.1 Smoluchowski's equation with a constant kernel

Smoluchowski's equation is a well-known model for the time evolution of irreversible aggregation processes [3, 2, 13, 11]. The object of our study is the continuous version of this equation, with the additional assumption that the rate at which any two nuclei coalesce (the *coagulation rate* or *kernel*) is independent of their sizes. If the number of nuclei of size $y > 0$ at a given time t is given by $f(t, y) dy$, then Smoluchowski's equation reads:

$$\partial_t f(t, y) = C(f(t, \cdot), f(t, \cdot))(y), \quad (1)$$

where

$$C(f, f)(y) := \frac{1}{2} \int_0^y f(x) f(y-x) dx - f(y) \int_0^\infty f(x) dx. \quad (2)$$

When other coagulation rates are considered, a coagulation coefficient $a(x, y)$ appears in the equation; here we have set $a(x, y) = 1$ for all

$x, y > 0$. The integral $\int_0^\infty f(t, y) y dy$, called *the mass of f* at time t , represents the total mass of all nuclei at time t , and is a conserved quantity for this equation.

One of the more remarkable properties of Smoluchowski's equation is the self-similar behavior that its solutions exhibit at large times, which is expected by formal arguments but has been rigorously proved only in very particular cases. This behavior is our main concern in this paper, and among our main results we highlight the following one:

Theorem 1.1. *Take a locally absolutely continuous function $f_0 : (0, \infty) \rightarrow [0, \infty)$ such that, for some $\nu > 0$,*

$$\int_0^\infty (f_0'(y))^2 y^4 e^{\nu y} dy < \infty.$$

Let f be the solution of Smoluchowski's equation (1) with initial condition f_0 (this problem is known to be well-posed; see [16, 14]). Then there is a number $\mu > 0$ such that for any $0 < \delta < 1$,

$$\int_0^\infty (t^2 \partial_y(f(t, ty)) - \partial_y g_\rho)^2 y^4 e^{\mu y} dy \leq K^2 t^{-2\delta} \quad \text{for all } t > 0, \quad (3)$$

for some $K > 0$ which depends only on the initial condition f_0 . In particular,

$$\sup_{y>0} \{y^2 |t^2 f(t, ty) - g_\rho(y)|\} \leq K t^{-\delta} \quad \text{for all } t > 0,$$

where ρ is the mass of f_0 and $g_\rho(y) := \frac{4}{\rho} e^{-\frac{2}{\rho}y}$ for $y > 0$.

This theorem is a direct consequence of Theorem 1.6 below, proved in Section 6. Results along these lines were already obtained in [14] and [15], where it was proved that with some regularity of the initial condition it holds that

$$\sup_{y>0} \{y |t^2 f(t, ty) - g_\rho(y)|\} \rightarrow 0 \quad \text{when } t \rightarrow \infty. \quad (4)$$

A result similar to the latter, but giving uniform convergence only in compact sets, can be found in [9]; weak convergence was proved in [12] using entropy arguments. Hence, the main improvement of Theorem 1.1 is that we give an explicit rate of convergence, which is to our knowledge a new result, and that the convergence in eq. (3) is stronger than uniform convergence. In fact, we give corresponding results for the rate of convergence of higher derivatives of f as long as the initial condition is regular enough. We note that this rate is optimal in general, as can be seen from the evolution of the moment of order 0 of f , $\int f$. These results, and the methods we use to arrive at them, are best described in terms of a rescaled form of eq. (1), which we introduce next.

1.2 The self-similar equation

A *scaling solution* or *self similar solution* of Smoluchowski's equation is a solution f to eq. (1) which is a rescaling of some fixed function g for any time t :

$$f(t, y) = p(t)g(q(t)y) \quad (t, y > 0), \quad (5)$$

where $p, q > 0$ are functions of time. When such a solution exists, g is called a *scaling profile* or *self-similar profile*.

Let us briefly describe the problem of the scaling behavior of Smoluchowski's equation for a general coagulation rate, and then we will focus on the problem for a constant rate. It is expected that for a general class of homogeneous coagulation kernels, there is a unique self-similar profile with given mass $\rho > 0$, and that for a very wide class of initial conditions solutions to Smoluchowski's equation approximate, for large times, the self-similar solution with the same mass, in a sense to be precised. For general homogeneous coagulation rates there are no rigorous proofs of this behavior, except for recent results that have shown the existence of self-similar profiles [7, 6] and given some of their properties [8, 4]. Nevertheless, for the specific coagulation rates given by $a(x, y) = x + y$ and $a(x, y) = 1$ (our case), it is known that the self-similar profile is unique for each given finite mass, and the convergence to it has been proved in [9, 14, 15, 12]. For $a(x, y) = 1$, the profile corresponding to a mass ρ is the function g_ρ mentioned in Theorem 1.1, and the convergence to it holds at least in the sense of eq. (4) under quite general conditions on the initial data.

In our case, if f is a self-similar solution of eq. (1), then it is known that the scaling in eq. (5) is determined: it must happen that for some $\tau \in \mathbb{R}$,

$$f(t + \tau, y) = \frac{1}{(1 + t)^2} g\left(\frac{y}{1 + t}\right) \quad \text{for } t \geq 0, y > 0.$$

Hence, up to time translations, we can consider that all self-similar solutions are of this form. The following change of variables is then useful: if f is a solution of eq. (1) for $t \in [0, \infty)$, we set

$$g(t, y) := e^{2t} f(e^t - 1, e^t y) \quad \text{for } t \geq 0, y > 0, \quad (6)$$

whose inverse is

$$f(t, y) = \frac{1}{(1 + t)^2} g\left(\log(1 + t), \frac{y}{1 + t}\right) \quad \text{for } t \geq 0, y > 0. \quad (7)$$

The fact that f is a solution of eq. (1) is equivalent to g satisfying the following *self-similar coagulation equation*:

$$\partial_t g = 2g + y\partial_y g + C(g, g) \quad (8)$$

$$=: D(g) + C(g, g). \quad (9)$$

Through this change of variables, self-similar solutions f of eq. (1) correspond, up to a time translation, to stationary solutions of eq. (8). Given this equivalence, we study eq. (8), which is more convenient when considering the scaling behavior of solutions.

Let us be more precise as to what we mean by a solution to eq. (8). Below we denote our functional spaces by

$$L^1 := L^1(0, \infty) \quad L_1^1 := L^1((0, \infty); y \, dy).$$

Definition 1.2. For any $g, h \in L^1$, we define $C(g, h)$ by

$$\begin{aligned} C(g, h)(y) := & \frac{1}{2} \int_0^y g(y')h(y - y') \, dy \, dy' \\ & - \frac{1}{2} \int_0^\infty g(y)h(y') \, dy' \\ & - \frac{1}{2} \int_0^\infty h(y)g(y') \, dy'. \end{aligned} \quad (10)$$

Notice that this is in agreement with the expression in eq. (2), and that this operator is symmetric: $C(g, h) = C(h, g)$ for any g, h .

Definition 1.3. A *solution* of eq. (8) on $[0, \infty)$ is a nonnegative function $g \in \mathcal{C}([0, \infty); L^1 \cap L_1^1)$ such that eq. (8) holds in the following sense: for all $\phi \in \mathcal{C}_0^\infty$, g satisfies

$$\begin{aligned} \int_0^\infty g(t, y)\phi(y) \, dy = & \int_0^\infty g(0, y)\phi(y) \, dy + \int_0^t \int_0^\infty g(s, y)\phi(y) \, dy \, ds \\ & - \int_0^t \int_0^\infty y g(s, y)\phi'(y) \, dy \, ds \\ & + \int_0^t \int_0^\infty C(g(s), g(s))(y)\phi(y) \, dy \, ds \quad \text{for } t > 0. \end{aligned} \quad (11)$$

The results in [4] or [14] prove that for any nonnegative $g^0 \in L^1 \cap L_1^1$ there exists a unique solution to eq. (8) in the above sense with the initial condition $g(0, y) = g^0(y)$ for $y > 0$.

Eq. (8) has a one-parameter family of stationary solutions $\{g_\rho\}_{\rho>0}$ which is explicitly given by

$$g_\rho(y) := \frac{4}{\rho} e^{-\frac{2y}{\rho}}. \quad (12)$$

The solution g_ρ has mass (first moment) ρ and number of particles (zeroth moment) equal to 2. If we write the result in eq. (4) in terms of a solution g to eq. (8), then it means that

$$\limsup_{t \rightarrow \infty} \sup_{y > 0} \{y |g(t, y) - g_\rho(y)|\} = 0.$$

Some regularity conditions need to be imposed on the initial data for the latter convergence to hold; we refer the reader to [15], where the result is proved, for details.

1.3 Main results

Our main objective is to give an explicit exponential rate of convergence to equilibrium for solutions to the above equation in various norms. Explicitly, for an integrable function $h : (0, \infty) \rightarrow \mathbb{R}$, a positive number μ and an integer $k \geq -1$ we consider the norms

$$\|h\|_{k,\mu}^2 := \int_0^\infty y^{2(k+1)} (D^k h(y))^2 e^{\mu y} dy, \quad (13)$$

where $D^k h$ represents the k -th derivative of h and $D^{-1}h$ denotes the primitive of h given by

$$D^{-1}h(y) \equiv -H(y) := - \int_y^\infty h(x) dx. \quad (14)$$

Note that the subindex μ indicates the exponential weight, and not the power of the function as is usual when considering L^p spaces. The latter norm with $\mu = 2/\rho$ is directly suggested by the quadratic approximation to the following *relative entropy* functional near a stationary solution, and is the initial motivation for it:

$$F[g|g_\rho] := \int_0^\infty \left(g(y) \left(\log \frac{g(y)}{g_\rho(y)} - 1 \right) + g_\rho(y) \right) dy. \quad (15)$$

We prove both local and global convergence results; among the local ones we have the following, proved in Section 5:

Theorem 1.4. *Let g be a solution to eq. (8) with mass ρ and non-negative initial condition g^0 at $t = 0$. Take $0 < \mu < \nu < 2/\rho$ and an integer $k \geq -1$. Choose $\epsilon > 0$, and assume that*

1. *The exponential moment of order $\nu/2$ is bounded by a constant $E > 0$ for all $t \geq 0$; this is,*

$$\int_0^\infty |g(t, y)| e^{\frac{\nu}{2}y} dy \leq E \quad \text{for all } t \geq 0.$$

2. *The relative entropy $F[g^0|g_\rho]$ is less than ϵ .*

Then, for any $0 < \delta < 1$ there is an $\epsilon_0 = \epsilon_0(\rho, E, \delta, k, \mu, \nu)$ such that if $\epsilon \leq \epsilon_0$ then

$$\|g(t) - g_\rho\|_{k,\mu} \leq K \|g^0 - g_\rho\|_{k,\nu} e^{-\delta t} \quad \text{for } t \geq 0, \quad (16)$$

for some constant $K = K(\rho, E, \delta, k, \mu, \nu)$. When $k = -1$, the same is true with $\|g^0 - g_\rho\|_{-1, \mu}$ instead of $\|g^0 - g_\rho\|_{-1, \nu}$:

$$\|g(t) - g_\rho\|_{-1, \mu} \leq K \|g^0 - g_\rho\|_{-1, \mu} e^{-\delta t} \quad \text{for } t \geq 0.$$

Observe that these results are meaningful when $\|g^0\|_{k, \nu}$, or $\|g^0\|_{k, \mu}$ in this last case, are finite.

Remark 1.5. Below we give intrinsic conditions on the initial data which are equivalent to point 1 in the above theorem (see Section 4, and concretely eq. (92)), so it is really a condition on g^0 . In particular, it is satisfied if $g^0(y) \leq \nu e^{-\frac{\nu}{2}y}$ for all $y > 0$.

Our main global convergence result can be stated as follows:

Theorem 1.6. *Let g be a solution to eq. (8) with mass ρ and non-negative initial condition g^0 such that $(g^0)' \in L^1$. Assume that, for some $\nu > 0$ and $k \geq -1$,*

$$\int_0^\infty g^0(y) e^{\nu y} dy < \infty \quad \text{and} \quad \|g^0\|_{k, \nu} < \infty.$$

Then there exists $0 < \mu < 2/\rho$ such that for any $0 < \delta < 1$,

$$\|g(t) - g_\rho\|_{k, \mu} \leq K e^{-\delta t} \quad \text{for } t \geq 0 \quad (17)$$

for some number K which depends in an explicit way on k, ν, δ and g^0 .

For $k = 1$, this implies Theorem 1.1 through the self-similar change of variables in eq. (6). When $k = -1$ or 0 , the condition $\|g^0\|_{k, \nu} < \infty$ is redundant, as it is already implied by $\int g^0 e^{\nu y} < \infty$ and $g_0' \in L^1$. We also remark that all the constants in the previous theorems are constructive and can be explicitly given by following their proofs.

We point out that all the arguments in the paper can be in fact carried out with the norms

$$\|h\|_{k, \mu}^2 := \int_0^\infty y^{2k} (D^k h(y))^2 e^{\mu y} dy \quad (18)$$

for $k \geq 0$ and $\mu > 0$, and

$$\|h\|_{-1, \mu}^2 = \|h\|_{-1, \mu}^2 := \int_0^\infty H(y)^2 e^{\mu y} dy, \quad (19)$$

as before. Notice that here the power weight is lower in all norms with $k \geq 0$. In this case, Theorems 1.4 and 1.6 are still true with the sole difference that one obtains an exponential rate of convergence

with $\delta < 1/2$ instead of $\delta < 1$. In particular, one obtains a statement as that in Theorem 1.1 with a modified power weight and rate of convergence: in the conditions of Theorem 1.1, and for $0 < \delta < 1/2$,

$$\int_0^\infty (t^2 \partial_y (f(t, ty)) - \partial_y g_\rho)^2 y^2 e^{\mu y} dy \leq K^2 t^{-2\delta} \quad \text{for all } t > 0,$$

and in particular,

$$\sup_{y>0} \{y |t^2 f(t, ty) - g_\rho(y)|\} \leq K t^{-\delta} \quad \text{for all } t > 0.$$

We have not included the calculations for the norms (18) with different power weights in order to make the paper more readable, but there are no difficulties in rederiving our results in this case.

We also note that the result on exponential convergence in the L^2 norm, with an exponential rate of convergence of $1/2$, can be arrived at by explicit computations on the solution of the Fourier transform of eq. (1). We do this in Lemma 6.1.

These results strongly suggest that the behavior near $y = 0$ of solutions to eq. (1) is decisive when studying the speed of convergence of solutions to a self-similar one: it seems likely that one cannot obtain, for example, the convergence of derivatives at $y = 0$, so some weight near $y = 0$ is probably unavoidable.

The strategy of proof for our results stems from the relative entropy functional defined in eq. (15), which was first introduced in the context of eq. (8) in [12], where it was proved that

$$\frac{d}{dt} \frac{F[g(t, \cdot) | g_\rho]}{\int g(t, y) dy} \leq 0, \quad \text{and} \quad (20)$$

$$\frac{d}{dt} F[G(t, \cdot) | G_\rho] \leq 0, \quad (21)$$

for any solution g of eq. (8), where G and G_ρ are the primitives of g and g_ρ defined as in eq. (14). That is: the above expressions are Liapunov functionals for this equation. If one wants to carry out the classical method of studying the convergence to equilibrium in a neighborhood of a stationary solution by looking at the spectral properties of the linearization of the equation near such a solution, the above functionals suggest that we consider the linearized operator in the norm which appears in their quadratic approximation. For eq. (20), this norm is precisely $\|\cdot\|_{0, \frac{2}{\rho}}$; for the functional in eq. (21), it is $\|\cdot\|_{-1, \frac{2}{\rho}}$. We follow this path and study the equation

$$\partial_t h = Lh := 2h + y \partial_y h + 2C(h, g_\rho), \quad (22)$$

where L is the linearization of the operator $D(g) + C(g, g)$ from eq. (9) near the stationary solution g_ρ . We find that L , when restricted to functions h such that $\int_0^\infty yh(y) dy = 0$, has a spectral gap in the norm $\|\cdot\|_{k,\mu}$ for $0 < \mu \leq 2/\rho$ and integer $k \geq -1$; see Sections 3.2–3.4 for a precise statement (see also [5] for a spectral study of L in an L^1 framework). This shows exponential convergence to 0 for the linear equation (22) in these norms, and can also be used to prove exponential convergence to equilibrium for the nonlinear equation (8) in a close-to-equilibrium regime.

In order to treat solutions far from the equilibrium, the results of [15] were already available. However, as we are interested in the rate of convergence, we prove a similar result which gives an explicit rate of convergence in the L^2 norm, using estimates on the explicit solution for the Fourier transform of (8) which resemble those in the proof of uniform convergence in [15]. With further estimates on the derivatives of the explicit solution one could probably reproduce “by hand” our results on the rate of convergence for the derivatives. Nevertheless, the calculations soon become very involved, and in our opinion do not add much to the understanding of the equation, while a linear study proves to be less technical, and possibly better suited for generalization to other coagulation kernels.

The paper is organized as follows: in Section 2 we derive our main functional inequality, obtained from a linearization of an inequality of Aizenman and Bak [1], which is then used to prove that the linear operator L has a spectral gap in the norm $\|\cdot\|_{-1,2/\rho}$. In Section 3 we define precisely the linear operator L in the spaces associated to the norms mentioned above, and prove that it has a spectral gap in them. Section 4 can be read independently from the rest of the paper; in it, we find an explicit expression for the evolution of the exponential moments of solutions to eq. (8). This is a remarkable piece of information, which can be obtained because of the widely known fact that the Laplace transform of eq. (1) is explicitly solvable. In turn, this information on the exponential moments is needed in the arguments of later sections. In Section 5 we give local exponential convergence results for the nonlinear equation (8) with the help of the linear study from Section 3, and in particular we prove Theorem 1.4. Finally, in Section 6 we state and prove our global convergence results, and in particular prove Theorem 1.6.

2 Functional inequalities

The study of the linearization of eq. (8) in the norm $\|\cdot\|_{-1,2/\rho}$ defined in eq. (13) is directly suggested by the Liapunov functional in eq. (21).

Here we prove the necessary inequality to show that the linearized operator L (which we define more precisely later) has a spectral gap in the norm $\|\cdot\|_{-1,2/\rho}$; more precisely,

$$\langle h, Lh \rangle_{-1, \frac{2}{\rho}} \leq -\|h\|_{-1, \frac{2}{\rho}}^2 \quad (23)$$

for any sufficiently regular function h with $\int_0^\infty yh(y) dy = 0$, where $\langle \cdot, \cdot \rangle_{-1, 2/\rho}$ denotes the scalar product associated to the norm $\|\cdot\|_{-1, 2/\rho}$; see Proposition 2.6 and Lemma 3.12 for a more precise statement.

In order to prove this we use an inequality of Aizenman and Bak, which was first introduced in their paper [1] to study the convergence to equilibrium for eq. (1) with an additional fragmentation term. In [12], Aizenman-Bak's inequality is also needed in the proof that the relative entropy $F[G|G_\rho]$ is nonincreasing, and here we use a quadratic approximation of it to show eq. (23).

We will first prove a needed auxiliary inequality; then, we prove eq. (23).

2.1 Some notation

Here we explain some of our notation conventions, and in particular that for weighted L^p spaces; for the specific spaces used in our study of the linearized coagulation operator, see Section 3.1.

As most of our functions will be defined on $(0, \infty)$, we omit the limits in integrals when this does not lead to confusion; thus, when we omit the limits of integration, it is understood that integration is on $(0, \infty)$; when we omit the variable, it is understood that integration is on the y variable. For example, $\int_0^\infty g(y) dy$ is understood to be an integral over $y \in (0, \infty)$, and for a solution $g = g(t, y)$ depending on time t and size y , the expression $\int g$ denotes $\int_0^\infty g(t, y) dy$. In the same way, derivatives are with respect to the size variable unless explicitly noted.

We use several spaces of functions on $(0, \infty)$, and we similarly omit the explicit mention of the interval. Thus, we use:

$$L^1 := L^1(0, \infty) \quad L_k^1 := L^1((0, \infty); y^k dy) \quad (k \in \mathbb{R}). \quad (24)$$

In general, for a function $y \mapsto m(y)$ and $1 \leq p < \infty$, we write

$$L^p(m(y)) := L^p((0, \infty); m(y) dy), \quad (25)$$

to denote the space of functions f such that $y \mapsto |f(y)|^p m(y)$ is integrable. Likewise, \mathcal{C} and \mathcal{C}^k denote the spaces of continuous and k -times continuously differentiable functions on $(0, \infty)$, respectively;

$\mathcal{C}_0, \mathcal{C}_0^k$ are the corresponding spaces of compactly supported functions on $(0, \infty)$.

For any function which is denoted g or h , and which is integrable on (ϵ, ∞) for any $\epsilon > 0$, we always write G, H to mean the primitives given by

$$G(y) := \int_y^\infty g(x) dx, \quad H(y) := \int_y^\infty h(x) dx \quad (y > 0). \quad (26)$$

The same is done for the pair g_ρ, G_ρ , where g_ρ is the stationary solution from eq. (12). Hence,

$$G_\rho(y) := \int_y^\infty g_\rho(x) dx = 2e^{-\frac{2y}{\rho}} \quad (y > 0). \quad (27)$$

2.2 Aizenman-Bak's inequality and its linearization

The following lemma can be found in [1, Proposition 4.3].

Lemma 2.1 (Aizenman-Bak inequality). *Take $f : (0, \infty) \rightarrow \mathbb{R}$ with $f \geq 0$, $f \in L^1$, and $f \log f \in L^1$. Then,*

$$\begin{aligned} & \int_0^\infty \int_0^\infty f(x)f(y) \log f(x+y) dx dy \\ & \leq \int_0^\infty \int_0^\infty f(x)f(y) \log f(y) dx dy - \left(\int_0^\infty f(x) dx \right)^2, \end{aligned} \quad (28)$$

and the equality is attained only for the exponential functions $f(y) = e^{-\mu y}$, with $\mu > 0$.

Now we expand this inequality to second order near an exponential function to find the following:

Lemma 2.2 (Linearized Aizenman-Bak inequality). *For any function $h \in L^2((1+y)e^{\mu y})$ and any $\mu > 0$, it holds that*

$$4 \int_0^\infty h H e^{\mu y} \leq 2\mu \left(\int h \right) \left(\int y h \right) + \int_0^\infty h^2 y e^{\mu y} + \frac{1}{\mu} \int_0^\infty h^2 e^{\mu y}, \quad (29)$$

where H was given in (26).

Proof. Fix $\mu > 0$, and take a continuous function h with compact support on $(0, \infty)$. For small enough $\epsilon > 0$, the function

$$f_\epsilon(y) := e^{-\mu y} + \epsilon h(y) \quad (y > 0) \quad (30)$$

is in the conditions of Lemma 2.1, so Aizenman-Bak's inequality holds for the function f_ϵ , for all small enough ϵ . One can easily check that both sides of the inequality are differentiable in ϵ . As the equality is attained at $\epsilon = 0$, it must hold that

$$\left. \frac{d^2}{d\epsilon^2} \right|_{\epsilon=0} I_1 \leq \left. \frac{d^2}{d\epsilon^2} \right|_{\epsilon=0} I_2, \quad (31)$$

where I_1 and I_2 are the left and right hand sides of Aizenman-Bak's inequality for f_ϵ , respectively. This will give the inequality we are interested in, so let us calculate these terms.

Step 1: First order derivative. Let us calculate the first order derivative of all terms in the inequality. The regularity of f_ϵ justifies the derivation under the integral sign, and we obtain the following for small enough ϵ :

$$\begin{aligned} \frac{d}{d\epsilon} I_1 &= \frac{d}{d\epsilon} \int_0^\infty \int_0^\infty f_\epsilon(x) f_\epsilon(y) \log f_\epsilon(x+y) dx dy \\ &= 2 \int_0^\infty \int_0^\infty h(x) f_\epsilon(y) \log f_\epsilon(x+y) dx dy \\ &\quad + \int_0^\infty \int_0^\infty f_\epsilon(x) f_\epsilon(y) \frac{h(x+y)}{f_\epsilon(x+y)} dx dy =: I_{11} + I_{12}. \end{aligned} \quad (32)$$

For the first term in I_2 ,

$$\begin{aligned} \frac{d}{d\epsilon} \int_0^\infty \int_0^\infty f_\epsilon(x) f_\epsilon(y) \log f_\epsilon(y) dx dy &= \int_0^\infty h(x) \int_0^\infty f_\epsilon(y) \log f_\epsilon(y) dy dx \\ &\quad + \int_0^\infty f_\epsilon(x) \int_0^\infty h(y) \log f_\epsilon(y) dy dx \\ &\quad + \left(\int h \right) \left(\int f_\epsilon \right) =: I_{21} + I_{22} + I_{23}. \end{aligned} \quad (33)$$

And for the last term in I_2 we have

$$-\frac{d}{d\epsilon} \left(\int_0^\infty f_\epsilon(x) dx \right)^2 = -2 \left(\int f_\epsilon \right) \left(\int h \right) =: I_{24}. \quad (34)$$

Step 2: Second order derivative. We have:

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{11} &= -2\mu \int_0^\infty \int_0^\infty h(x)h(y)(x+y) dx dy \\ &\quad + 2 \int_0^\infty \int_0^\infty h(x)e^{\mu x} h(x+y) dx dy \\ &= -4\mu \left(\int h \right) \left(\int yh \right) + 2 \int hHe^{\mu y} \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{12} &= 2 \int_0^\infty \int_0^\infty h(x)e^{\mu x} h(x+y) dx dy \\ &\quad - \int_0^\infty \int_0^\infty e^{\mu(x+y)} h(x+y)^2 dx dy \\ &= 2 \int hHe^{\mu y} - \int h^2 ye^{\mu y}. \end{aligned} \quad (36)$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{21} = -\mu \left(\int h \right) \left(\int yh \right) + \left(\int h \right)^2. \quad (37)$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{22} = -\mu \left(\int h \right) \left(\int yh \right) + \frac{1}{\mu} \int h^2 e^{\mu y}. \quad (38)$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{23} = \left(\int h \right)^2. \quad (39)$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} I_{24} = -2 \left(\int h \right)^2. \quad (40)$$

Step 3: Final inequality. Putting together eqs. (35) and (36) we have

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} I_1 = 4 \int hHe^{\mu y} - 4\mu \left(\int h \right) \left(\int yh \right) - \int h^2 ye^{\mu y}, \quad (41)$$

and eqs. (37)–(40) show that

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} I_2 = -2\mu \left(\int h \right) \left(\int yh \right) + \frac{1}{\mu} \int h^2 e^{\mu y}. \quad (42)$$

Finally, writing out the inequality in eq. (31) in view of eqs. (41)–(42), we obtain precisely eq. (29). This shows our inequality for any $h \in \mathcal{C}_0(0, \infty)$, and taking a suitable limit of such functions shows the inequality for any $h \in L^2((1+y)e^{\mu y})$ (note that for such an h all terms in the inequality are finite). \square

2.3 Spectral gap inequality

Take $\rho > 0$, which will be kept fixed in the rest of this section. One can linearize the self-similar eq. (8) around the profile g_ρ with mass ρ (given in eq. (12)) to obtain $\partial_t h = Lh$, with L given by the following definition:

Definition 2.3. For a function $h \in \mathcal{C}_0^1$, we set

$$Lh(y) := 2h(y) + yh'(y) + 2C(g_\rho, h)(y). \quad (43)$$

Using the expression of $C(g_\rho, h)$ from eq. (10) and noticing that $h * g_\rho = -\frac{4}{\rho}H + g_\rho \int h + \frac{2}{\rho}H * g_\rho$ (an integration by parts), one has

$$2C(g_\rho, h) = h * g_\rho - g_\rho \int h - 2h = -2h - \frac{4}{\rho}H + \frac{2}{\rho}H * g_\rho, \quad (44)$$

so (43) may be rewritten as

$$Lh = y\partial_y h - \frac{4}{\rho}H + \frac{2}{\rho}H * g_\rho. \quad (45)$$

Observe that this expression can be generalized to functions h which are not necessarily integrable at $y = 0$, as it is H which appears in the convolution; it is for this reason that it will be useful later. The following expression for the primitive of the operator C will also be needed below:

Lemma 2.4. For $g, h \in L^1$, and $y > 0$, it holds that

$$\int_y^\infty C(g, h)(x) dx = \frac{1}{2} \int_0^y g(x)H(y-x) dx - \frac{1}{2}H(y) \int_0^\infty g(x) dx,$$

or, written more compactly,

$$2 \int_y^\infty C(g, h) = g * H - H \int g,$$

where H is given in eq. (26).

A proof of the above result can be obtained by direct integration of eq. (10), and we omit it here. For an explicit proof, see for example that of Proposition 9 in [12]. A direct consequence of this is the next expression for the primitive of the operator L :

Lemma 2.5. For $h \in \mathcal{C}_0^1$ and $y > 0$, the primitive of Lh is given by

$$\mathcal{L}h(y) := \int_y^\infty Lh(x) dx = -H(y) - yh(y) + g_\rho * H(y), \quad (46)$$

where H, G_ρ are the primitives of h, g_ρ given in eqs. (26) and (27), respectively.

Proof. Using Lemma 2.4 and integrating by parts,

$$\begin{aligned}\mathcal{L}h(y) &= 2H(y) + \int_y^\infty x h'(x) dx + 2 \int_y^\infty C(h, g_\rho)(x) dx \\ &= H(y) - yh(y) + g_\rho * H - H \int g_\rho = -H(y) - yh(y) + g_\rho * H,\end{aligned}$$

as the integral of g_ρ is equal to 2. \square

Proposition 2.6 (Spectral gap inequality). *For any $h \in \mathcal{C}_0^1$ such that*

$$\int_0^\infty H(y)^2 e^{\frac{2}{\rho}y} dy < \infty \quad \text{and} \quad \int_0^\infty y h(y) dy = 0,$$

it holds that

$$\langle h, Lh \rangle_{-1, \frac{2}{\rho}} \leq -\|h\|_{-1, \frac{2}{\rho}}^2, \quad (47)$$

where $\langle \cdot, \cdot \rangle_{-1, \frac{2}{\rho}}$ is the scalar product associated to the norm $\|\cdot\|_{-1, 2/\rho}$,

$$\langle h, Lh \rangle_{-1, \frac{2}{\rho}} := \int_0^\infty H(y) \mathcal{L}h(y) e^{-\frac{2}{\rho}y} dy.$$

Proof. Set $\mu := 2/\rho$ as a shorthand. We explicitly calculate this expression by using eq. (46):

$$\begin{aligned}\langle h, Lh \rangle_{-1, \mu} &= \int_0^\infty H(y) \int_y^\infty Lh(x) dx e^{\mu y} dy \\ &= - \int H^2 e^{\mu y} - \int Hh y e^{\mu y} + \int H(g_\rho * H) e^{\mu y}.\end{aligned} \quad (48)$$

We rewrite these terms using integration by parts:

$$\int Hh y e^{\mu y} = - \int Hh y e^{\mu y} + \int H^2 e^{\mu y} + \mu \int H^2 y e^{\mu y}, \quad (49)$$

and hence

$$\int Hh y e^{\mu y} = \frac{1}{2} \int H^2 e^{\mu y} + \frac{\mu}{2} \int H^2 y e^{\mu y}. \quad (50)$$

As for the third term in eq. (48),

$$\int_0^\infty H(g_\rho * H) e^{\mu y} = \frac{4}{\rho} \int_0^\infty H(y) \int_0^y H(x) e^{\mu x} dx dy = \frac{4}{\rho} \int H \mathcal{H} e^{\mu y}, \quad (51)$$

where \mathcal{H} is the primitive of H :

$$\mathcal{H}(y) := \int_y^\infty H(x) dx.$$

Putting eqs. (50) and (51) into (48) one obtains

$$\langle h, Lh \rangle_{-1, \mu} = -\frac{3}{2} \int H^2 e^{\frac{2}{\rho}y} - \frac{1}{\rho} \int H^2 y e^{\frac{2}{\rho}y} + \frac{4}{\rho} \int \mathcal{H} H e^{\frac{2}{\rho}y}.$$

Using the linearized Aizenman-Bak inequality (Lemma 2.2) with H instead of h and $\mu := 2/\rho$ gives

$$\begin{aligned} \langle h, Lh \rangle_{-1, \mu} &\leq -\frac{3}{2} \int H^2 e^{\frac{2}{\rho}y} - \frac{1}{\rho} \int H^2 y e^{\frac{2}{\rho}y} \\ &\quad + \frac{1}{\rho} \int H^2 y e^{\frac{2}{\rho}y} + \frac{1}{2} \int H^2 e^{\frac{2}{\rho}y} = - \int H^2 e^{\frac{2}{\rho}y} = - \|h\|_{-1, \mu}^2, \end{aligned}$$

where we have taken into account that $\int y h = 0$ when omitting one of the terms in the inequality. \square

3 Spectral gaps of the linearized operator

3.1 Functional spaces and definition of L

For $\mu > 0$ and integer $k \geq -1$, we consider the following norms for a function $h : (0, \infty) \rightarrow \mathbb{R}$, defined when the expression below make sense:

$$\|h\|_{k, \mu}^2 := \int_0^\infty y^{2(k+1)} (D^k h(y))^2 e^{\mu y} dy. \quad (52)$$

Here, $D^k h$ denotes the k -th derivative of h ; for $D^1 h$ we often use the more common h' , and when $k = -1$ we set $D^{-1} h := -H$ (cf. eq. (14)). We also consider the following spaces:

$$X_{-1, \mu} := \left\{ H' \mid H \in L^2(e^{\mu y}) \quad \text{and} \quad \int H = 0 \right\} \quad (53)$$

$$X_{k, \mu} := \left\{ h \mid y^{k+1} D^k h \in L^2(e^{\mu y}) \quad \text{and} \quad \int y h = 0 \right\}, \quad (54)$$

where it is understood that the members of these spaces are at least distributions on $(0, \infty)$, and the derivatives are meant also in the sense of distributions. In each $X_{k, \mu}$ for integer $k \geq -1$ and $\mu > 0$, the above defined $\|\cdot\|_{k, \mu}$ gives a Hilbert norm, and their associated scalar products are denoted by $\langle \cdot, \cdot \rangle_{k, \mu}$. We will prove later that the linearized coagulation operator has a spectral gap in these spaces, and considering only functions h with $\int y h = 0$ is tied to the fact that the coagulation

equation conserves the mass, so its linearization cannot have a spectral gap in a space without a restriction of this kind. Observe that the restriction $\int H = 0$ in $X_{-1,\mu}$ is the same as $\int y H' = -\int y h = 0$ when H is regular enough, so it is the same as in the other spaces.

For given $\mu > 0$, the spaces $X_{k,\mu}$ form a scale where each space is dense in the next one, as we show in the next two lemmas.

Lemma 3.1. *For any $\mu \geq 0$, any integer $n \geq 0$ and $h \in L^2(y^{2n+2}e^{\mu y})$ it holds that*

$$\int_0^\infty H^2 y^{2n} e^{\mu y} dy \leq 4 \int_0^\infty h^2 y^{2(n+1)} e^{\mu y} dy. \quad (55)$$

Proof. Notice that h is integrable on (ϵ, ∞) for any $\epsilon > 0$, so H is well-defined and $H(y) \rightarrow 0$ as $y \rightarrow +\infty$. We use Cauchy-Schwarz's and Hardy's inequality as follows:

$$\begin{aligned} \int_0^\infty H^2 y^{2n} e^{\mu y} dy &= \int_0^\infty \int_y^\infty H(y) h(x) y^{2n} e^{\mu y} dx dy \\ &= \int_0^\infty h(x) \int_0^x H(y) y^{2n} e^{\mu y} dy dx \\ &\leq \int_0^\infty |h(x)| x^{n+1} e^{\frac{\mu}{2}x} \frac{1}{x} \int_0^x |H(y)| y^n e^{\frac{\mu}{2}y} dy dx \\ &\leq \left(\int_0^\infty h^2 x^{2n+2} e^{\mu x} dx \right)^{\frac{1}{2}} \left(\int_0^\infty \left(\frac{1}{x} \int_0^x H y^n e^{\frac{\mu}{2}y} dy \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq 2 \left(\int_0^\infty h^2 x^{2n+2} e^{\mu x} dx \right)^{\frac{1}{2}} \left(\int_0^\infty H^2 y^{2n} e^{\mu y} dy \right)^{\frac{1}{2}}. \end{aligned}$$

This proves the inequality in the lemma. \square

Lemma 3.2. *For $\mu > 0$ and integer $k \geq 0$, $X_{k,\mu}$ is contained and dense in $X_{k-1,\mu}$.*

Proof. Lemma 3.1 proves the inclusion. The claim that they are dense inclusions is a consequence of the fact that the set $\{f \in \mathcal{C}_0^\infty([0, \infty)) \mid \int y f = 0\}$ is dense in all of them. \square

Let us also give other useful inequalities:

Lemma 3.3. *For $\mu > 0$ and any $h \in L^2(e^{\mu y})$,*

$$\int H^2 e^{\mu y} \leq \frac{4}{\mu^2} \int h^2 e^{\mu y}. \quad (56)$$

Proof. The inequality $(2h - \mu H)^2 \geq 0$ and an integration by parts gives

$$\begin{aligned} 4 \int h^2 e^{\mu y} &\geq 4\mu \int h H e^{\mu y} - \mu^2 \int H^2 e^{\mu y} \\ &= 2\mu \left(\int h \right)^2 + \mu^2 \int H^2 e^{\mu y} \geq \mu^2 \int H^2 e^{\mu y}. \end{aligned}$$

□

A consequence of Lemma (56) applied to the function $y^{k+1} D^k H$ in the place of H , and Lemma 3.1, is the following inequality:

Lemma 3.4. *For integer $k \geq 0$ and $\mu > 0$, there is a number $C = C(k, \mu) > 0$ such that*

$$\|H\|_{k, \mu} \leq C \|h\|_{k, \mu}$$

for all $h \in X_{k, \mu}$.

Another useful property of these spaces is that exchanging derivatives and powers of y gives other equivalent norms. More precisely:

Lemma 3.5. *Take $\mu > 0$, and nonnegative integers k, a, b, n, m such that $a + b = n + m = k + 1$. Assume that $h \in \mathcal{C}^{k-1}$ is such that*

$$\lim_{y \rightarrow 0} y^{i+2} D^i h(y) = 0 \quad (i = 0, \dots, k-1). \quad (57)$$

Then there is a (constructive) constant $K = K(k, a, b, n, m) > 0$ such that

$$\left\| y^a D^k (y^b h) \right\|_{L^2(e^{\mu y})} \leq K \left\| y^n D^k (y^m h) \right\|_{L^2(e^{\mu y})}$$

for any $h \in L^2(e^{\mu y})$, in the sense that the left hand side is finite whenever the right hand side is, and then the inequality holds.

One can prove this by repeated application of weighted Hardy inequalities (cf. [10, Example 0.3]), and we omit the details.

Remark 3.6. From this it is easy to see that the $L^2(e^{\mu y})$ -norm of any combination of products by positive powers of y and derivatives of h gives an equivalent norm as long as the total sum of all the powers and that of the orders of all derivatives is fixed. It is important to restrict these considerations to functions h satisfying (57) and having some integrability property at $+\infty$, (e.g., $h \in L^2(e^{\mu y})$ in our case), as otherwise the finiteness of the right hand term does not imply that of the left one.

The above equivalence of norms can be used in our $X_{k,\mu}$ spaces for $\mu, k > 0$, as the condition in the previous lemma holds:

Lemma 3.7. *If $h \in X_{k,\mu}$ with $\mu > 0$ and integer $k > 0$, then*

$$\lim_{y \rightarrow 0} y^{i+2} D^i h(y) = 0 \quad (i = 0, \dots, k-1). \quad (58)$$

Proof. Take an integer $0 \leq i < k$. As h is in $X_{i+1,\mu} \subset X_{i,\mu}$, one directly sees that the first derivative of $y^{i+2} D^i h(y)$ is integrable near $y = 0$. Hence, $y^{i+2} D^i h(y)$ has a limit at $y = 0$. As $y^{i+1} D^i h(y)$ is integrable, it must be that this limit is 0. \square

One can define the operator L in the spaces $X_{k,\mu}$, of course agreeing with our previous one in Definition 2.3 whenever both are applicable. For the rest of Section 3 we fix $\rho > 0$, and consider the linearization of the nonlinear operator in eq. (8) around the self-similar profile g_ρ :

Definition 3.8. Take $\mu > 0$. For $h \in X_{0,\mu}$ (in particular, for $h \in X_{k,\mu}$ with $k \geq 0$) we define Lh by eq. (45), where the derivative of h is taken in the sense of distributions.

Our aim in this section is to show that the operator L is defined as $L : X_{k+1,\mu} \rightarrow X_{k,\mu}$ for $k \geq -1$, and that it is a closed operator in the spaces $X_{k,\mu}$ for $\mu > 0$ and any integer $k \geq -1$. Before proving this we will need two lemmas in which we study the operator $C(g, h)$ which appears in the definition of L .

Lemma 3.9. *For $\mu > 0$, integer $k \geq 1$ and $g, h \in X_{k,\mu}$, it holds that*

$$D^k (y^{k+1} (g * h)) = \sum_{i=0}^{k+1} \binom{k+1}{i} (D^{k+1-i} (y^{k+1-i} g)) * (D^{i-1} (y^i h)).$$

In fact, this holds when $y^k D^{k+1} g, y^{k+1} D^k h \in L^2(e^{\mu y})$; this is, when g, h satisfy the conditions for being in $X_{k,\mu}$, but without imposing that $\int y h = \int y g = 0$. In particular, this applies to $g = g_\rho$ and $h \in X_{k,\mu}$.

Proof. One has

$$y^{k+1} (g * h)(y) = \int_0^y y^{k+1} g(x) h(y-x) dx,$$

and the identity in the lemma can be obtained by writing the binomial expansion for $y^{k+1} = (x + (y-x))^{k+1}$ and differentiating the resulting convolutions. The border terms which appear when differentiating the integrals between 0 and y vanish due to (58). \square

Lemma 3.10. For $0 < \mu < 4/\rho$ and integer $k \geq -1$ there is a constant $K = K(\rho, k, \mu)$ such that

$$\|C(h, g_\rho)\|_{k, \mu} \leq K \|h\|_{k, \mu} \quad \text{for all } h \in X_{k, \mu}.$$

Proof. Here, $C(h, g_\rho)$ is understood to be defined by eq. (44), which makes sense for $h \in X_{k, \mu}$ (which implies $H \in L^2(e^{\mu y})$). Numbers K_1, K_2, \dots below are assumed to depend only on ρ, k and μ .

Let us first prove the lemma for $k \geq 0$. From

$$2C(g_\rho, h) = -2h - \frac{4}{\rho}H + \frac{2}{\rho}H * g_\rho$$

and our previous lemma, we deduce that

$$2\|C(g_\rho, h)\|_{k, \mu} \leq 2\|h\|_{k, \mu} + \frac{4}{\rho}\|H\|_{k, \mu} + \frac{2}{\rho}\|H * g_\rho\|_{k, \mu}. \quad (59)$$

The second term in eq. (59) is bounded by $K_1\|h\|_{k, \mu}$ thanks to Lemma 3.4, for some $K_1 > 0$. For the third term in (59) we have:

$$\begin{aligned} \|H * g_\rho\|_{k, \mu} &\leq K_2 \left\| D^k \left(y^{k+1} (H * g_\rho) \right) \right\|_{L^2(e^{\mu y})} \\ &\leq K_3 \sum_{i=0}^{k+1} \binom{k+1}{i} \|h\|_{i-1, \mu} \int \left| D^{k-i} (y^{k-i} g_\rho) \right| y^{k-i} e^{\frac{\mu}{2}y} \\ &\leq K_4 \|h\|_{k, \mu} \end{aligned}$$

for some constants $K_2, K_3, K_4 > 0$, where we have used Lemma 3.5 in the first inequality (observe that the conditions at 0 are met), Lemma 3.9 in the second, and Lemma 3.1 for the third one. As the integrals bounded in the last inequality are finite as long as $0 \leq \mu < 4/\rho$, the lemma is proved for $k \geq 0$.

Now, for $k = -1$, by density it is enough to prove the inequality when h is $\mathcal{C}_0^\infty([0, \infty))$. We use the expression of the primitive of $C(g, h)$ from Lemma 2.4 to obtain, using Young's inequality, that

$$\begin{aligned} 2\|C(g_\rho, h)\|_{-1, \mu} &\leq \|g_\rho * H\|_{L^2(e^{\mu y})} + \|h\|_{-1, \mu} \int g_\rho \\ &\leq \|H\|_{L^2(e^{\mu y})} \int |g_\rho| e^{\frac{\mu}{2}y} + \|h\|_{-1, \mu} \int g_\rho \leq 2\|h\|_{-1, \mu} \int |g_\rho| e^{\frac{\mu}{2}y}. \quad (60) \end{aligned}$$

□

We finally have the following:

Proposition 3.11. *For any integer $k \geq -1$ and $0 < \mu < 4/\rho$, the operator L is defined between $X_{k+1,\mu}$ and $X_{k,\mu}$:*

$$L : X_{k+1,\mu} \rightarrow X_{k,\mu}.$$

Seen as an unbounded operator on $X_{k,\mu}$ for integer $k \geq -1$ and real $\mu > 0$, L is a closed operator with dense domain. In addition, for every $\mu > 0$ and integer $k \geq -1$ there is a constant $K = K(\rho, k, \mu)$ such that

$$\langle Lh, h \rangle_{k,\mu} \leq K \|h\|_{k,\mu}^2 \quad (h \in X_{k+1,\mu}), \quad (61)$$

and consequently L generates an evolution semigroup in each of the spaces $X_{k,\mu}$.

Proof. To see that L is defined between $X_{k+1,\mu}$ and $X_{k,\mu}$ for any $k \geq -1$, take $h \in X_{k+1,\mu}$ and notice that every term in (43) is in $X_{k,\mu}$, as can be seen from Lemma 3.10 and the fact that $h \in X_{k+1,\mu}$ implies that $yh' \in X_{k,\mu}$ (see Lemma 3.5 and the remark that follows).

For $k \geq -1$, let us prove that $L : X_{k+1,\mu} \rightarrow X_{k,\mu}$ is a closed operator in the space $X_{k,\mu}$. Take a sequence h_n in $X_{k+1,\mu}$ which converges to some $h \in X_{k,\mu}$ in the norm of $X_{k,\mu}$, and such that $Lh_n \rightarrow \tilde{h}$ in $X_{k,\mu}$ for some $\tilde{h} \in X_{k,\mu}$. We need to show that $\tilde{h} \in X_{k+1,\mu}$ and $Lh = \tilde{h}$. For such a sequence we have

$$Lh_n = 2h_n + yh'_n + 2C(g_\rho, h_n)$$

As the sequence $\{h_n\}$ converges to h , it is clear from Lemma 3.10 that the first and last terms in the expression of Lh_n converge to $2h$ and $2C(g_\rho, h)$, respectively, in $X_{k,\mu}$. As Lh_n converges to \tilde{h} , it follows that the second term, yh'_n , converges to something in $X_{k,\mu}$. From the equivalence of norms in Lemma 3.5, this implies that in fact h_n converges to something in the norm of $X_{k+1,\mu}$. As this is a stronger norm than $\|\cdot\|_{k,\mu}$, we see that the limit must be h , which implies that $h \in X_{k+1,\mu}$ and $Lh_n \rightarrow Lh$ in $X_{k,\mu}$. Hence, $\tilde{h} = Lh$.

Finally, let us prove the inequality (61) for $k \geq 0$ (for $k = -1$ a similar argument proves it). From expression (43) we have

$$\begin{aligned} \langle Lh, h \rangle_{k,\mu} &= 2 \|h\|_{k,\mu}^2 + \langle yh', h \rangle_{k,\mu} + 2 \langle C(g_\rho, h), h \rangle_{k,\mu} \\ &\leq 2 \|h\|_{k,\mu}^2 + \langle yh', h \rangle_{k,\mu} + 2 \|C(g_\rho, h)\|_{k,\mu} \|h\|_{k,\mu}. \end{aligned} \quad (62)$$

For the second term, as $D^k(yh') = kD^k h + yD^{k+1}h$,

$$\langle yh', h \rangle_{k,\mu} = k \|h\|_{k,\mu}^2 + \int y^{2k+3} (D^{k+1}h)(D^k h) e^{\mu y} \leq k \|h\|_{k,\mu}^2, \quad (63)$$

as an integration by parts shows that the term we omitted is negative:

$$\begin{aligned} & \int y^{2k+3} (D^{k+1}h)(D^k h) e^{\mu y} \\ &= -\frac{2k+3}{2} \int y^{2k+2} (D^k h)^2 e^{\mu y} - \frac{\mu}{2} \int y^{2k+3} (D^k h)^2 e^{\mu y} \\ &= -\frac{2k+3}{2} \|h\|_{k,\mu}^2 - \frac{\mu}{2} \int y^{2k+3} (D^k h)^2 e^{\mu y}. \end{aligned} \quad (64)$$

For the last term in (62), using Lemma 3.10,

$$\|C(g_\rho, h)\|_{k,\mu} \leq K \|h\|_{k,\mu} \quad (65)$$

Hence, we finally obtain

$$\langle Lh, h \rangle_{k,\mu} \leq K_1 \|h\|_{k,\mu}^2$$

for some K_1 depending on ρ and μ . \square

3.2 Spectral gap in $X_{-1,2/\rho}$

A direct consequence of Proposition 2.6 and a limit argument is the exponential decay of the evolution semigroup defined by L :

Lemma 3.12. *For $h \in X_{0,2/\rho}$,*

$$\langle h, Lh \rangle_{-1,2/\rho} \leq -\|h\|_{-1,2/\rho}^2.$$

As a consequence, for $h^0 \in X_{-1,2/\rho}$ we have

$$\|e^{tL} h^0\|_{-1,2/\rho} \leq \|h^0\|_{-1,2/\rho} e^{-t} \quad (t \geq 0). \quad (66)$$

3.3 Extension of spectral gaps to lower exponential weights

The evolution semigroup generated by L in the spaces $X_{k,\mu}$ has the remarkable property of creating exponential moments of $(D^k h)^2$ in finite time. This will allow us to prove that, if L has a spectral gap in $X_{k,\nu}$ for some $k \geq -1$, then it also has a spectral gap in $X_{k,\mu}$ for any $0 < \mu \leq \nu$.

Lemma 3.13. *Take $0 < \mu < \nu < 4/\rho$ and an integer $k \geq -1$. If h is a solution of the linear self-similar equation $\partial_t h = Lh$ with $h(0) \in X_{k,\mu}$, then the norm $\|h(t)\|_{k,\nu}$ is finite at time $t = t_0 := \log(\nu/\mu)$, and*

$$\|h(t_0)\|_{k,\nu} \leq \left(\frac{\nu}{\mu}\right)^K \|h(0)\|_{k,\mu}$$

for some positive constant $K = K(\rho, k, \mu, \nu)$.

Proof. Let us first prove the lemma for $k \geq 0$. From eqs. (62)–(65) in the proof of Proposition 3.11, for any $0 < \gamma < 4/\rho$ we have

$$\langle h, Lh \rangle_{k,\gamma} \leq K \|h\|_{k,\gamma}^2 - \frac{\gamma}{2} \int y^{2k+3} (D^k h)^2 e^{\gamma y}, \quad (67)$$

where the constant $K = K(\rho, k, \gamma) > 0$ is the same as that of Proposition 3.11. With this, take $\phi(t) := \mu e^t$ and carry out the following computation:

$$\begin{aligned} \frac{d}{dt} \|h\|_{k,\phi(t)}^2 &= 2 \langle h, Lh \rangle_{k,\phi(t)} + \phi'(t) \int y^{2k+3} (D^k h)^2 e^{\mu y} \\ &\leq 2K(\phi(t)) \|h\|_{\phi(t)}^2 + (\phi'(t) - \phi(t)) \int y^{2k+3} (D^k h)^2 e^{\mu y} \\ &= 2K(\phi(t)) \|h\|_{\phi(t)}^2, \end{aligned} \quad (68)$$

where $K(\phi(t))$ is the constant obtained from (67), for which we write the dependence on $\phi(t)$ explicitly. Now, for $t \leq t_0$,

$$\frac{d}{dt} \|h\|_{\phi(t)}^2 \leq 2K(\phi(t)) \|h\|_{\phi(t)}^2 \leq K_1 \|h\|_{\phi(t)}^2 \quad (69)$$

for some $K_1 > 0$, as the constant $K(\phi(t))$ is bounded for $0 \leq t \leq t_0$, which can be checked from the proof of Proposition 3.11. Hence, as $\phi(0) = \mu$,

$$\|h\|_{\phi(t_0)}^2 \leq e^{K_1 t_0} \|h^0\|_{\mu}^2 = \left(\frac{\nu}{\mu}\right)^{K_1} \|h^0\|_{\mu}^2,$$

which proves the lemma for $k \geq 0$, given that $\phi(t_0) = \mu e^{t_0} = \nu$.

For $k = -1$ the same argument can be carried out by using the inequality

$$\langle h, Lh \rangle_{-1,\gamma} \leq K \|h\|_{-1,\gamma}^2 - \frac{\gamma}{2} \int y H^2 e^{\gamma y}$$

instead of (67), and the following instead of (68):

$$\frac{d}{dt} \|h\|_{-1,\phi(t)}^2 = 2 \langle h, Lh \rangle_{-1,\phi(t)} + \phi'(t) \int y e^{\phi(t)y} H^2.$$

□

Lemma 3.14. *Take $0 < \nu < 4/\rho$ and an integer $k \geq -1$. Assume that the operator L has a spectral gap in $X_{k,\nu}$ of size $\delta > 0$; this is, there exists $C \geq 1$ such that*

$$\|e^{tL} h^0\|_{k,\nu} \leq C \|h^0\|_{k,\nu} e^{-\delta t}.$$

Then, L has a spectral gap of the same size in $X_{k,\mu}$ for any $0 < \mu < \nu$; this is, there is $C' = C'(\mu, C) \geq 1$ such that

$$\|e^{tL} h^0\|_{k,\mu} \leq C' \|h^0\|_{k,\mu} e^{-\delta t}.$$

Proof. Take h^0 in the domain of L as an operator on $X_{k,\mu}$ (i.e., $X_{k+1,\mu}$ if $k \geq 0$, and $L^2(y e^{\mu y})$ if $k = -1$). Consider the solution $t \mapsto h(t)$ to eq. (8) with initial condition h^0 . The idea is to estimate $\|h(t)\|_{k,\mu}$ using eq. (61) until a time t_0 for which $\|h(t_0)\|_{-1,\nu}$ is finite, and after that time use the fact that the latter norm is exponentially decreasing thanks to the spectral gap we are assuming in the norm of $X_{k,\nu}$.

Choose $t_0 := \log \frac{\nu}{\mu}$. Then, for $t \leq t_0$, the estimate in eq. (61) gives

$$\frac{d}{dt} \|h(t)\|_{k,\mu}^2 = 2 \langle h(t), Lh(t) \rangle_{k,\mu} \leq K_1 \|h(t)\|_{k,\mu}^2.$$

So, for $0 \leq t \leq t_0$ we have

$$\|h(t)\|_{k,\mu}^2 \leq \|h^0\|_{k,\mu}^2 e^{K_1 t} \leq \|h^0\|_{k,\mu}^2 e^{K_1 t_0} = \|h^0\|_{k,\mu}^2 \left(\frac{\nu}{\mu}\right)^{K_1} \quad (70)$$

At $t = t_0$ we can use Lemma 3.13 with $\nu := 2/\rho$ to get that

$$\|h(t_0)\|_{k,\nu}^2 \leq \|h^0\|_{k,\mu}^2 \left(\frac{\nu}{\mu}\right)^K,$$

where K is the one from the lemma. Now, as we are assuming a spectral gap of a certain size δ in $X_{k,\nu}$, we have for $t \geq t_0$

$$\begin{aligned} \|h(t)\|_{k,\mu}^2 &\leq \|h(t)\|_{k,\nu}^2 \leq \|h(t_0)\|_{k,\nu}^2 e^{-2\delta(t-t_0)} \\ &\leq \|h^0\|_{k,\mu}^2 \left(\frac{\nu}{\mu}\right)^K e^{-2\delta(t-t_0)} = \|h^0\|_{k,\mu}^2 \left(\frac{\nu}{\mu}\right)^{K+2\delta} e^{-2\delta t}. \end{aligned}$$

Together with eq. (70), this shows that

$$\|h(t)\|_{k,\mu} \leq C \|h^0\|_{k,\mu} e^{-\delta t} \quad \text{for all } t \geq 0, \quad (71)$$

with

$$C^2 := \max \left\{ \left(\frac{\nu}{\mu}\right)^{K+2\delta}, \left(\frac{\nu}{\mu}\right)^{K_1} \right\}.$$

The result is extended to all $h^0 \in X_{k,\mu}$ by density. \square

Then, from the spectral gap we showed in Lemma 3.12 we obtain the following:

Corollary 3.15. *Take $0 < \mu \leq 2/\rho$, and consider the semigroup e^{tL} generated by L in the space $X_{-1,\mu}$. Then there is a number $K = K(\mu) \geq 1$ such that*

$$\|e^{tL} h^0\|_{-1,\mu} \leq K \|h^0\|_{-1,\mu} e^{-t} \quad \text{for any } h^0 \in X_{-1,\mu}.$$

3.4 Extension of the spectral gap to $X_{k,\mu}$ with $k \geq 0$

Let us shortly discuss the strategy of the proofs in this section. One could try the following argument in order to obtain a spectral gap for a norm involving a certain derivative $D^k h$ of order $k \geq 0$. We could hope that, when calculating

$$\frac{d}{dt} \int_0^\infty (D^k h(y))^2 e^{\frac{2}{\rho} y} dy,$$

one obtains a negative multiple of this same norm, plus some terms which involve L^2 norms of h with some weight, and which can already be controlled by the semigroup decay results in previous sections; this idea can be arrived at by trying it for $k = 0$, in which case it works perfectly. But for $k > 0$, the term which involves the same norm is actually a *positive* multiple of it, so our tentative argument does not directly give a spectral gap in this case. It is for this reason that we consider instead norms with a power weight y^m for a certain m : one is forced to raise the power m if one wants to obtain a spectral gap involving L^2 norms of higher derivatives of h . Then, another problem appears: if one calculates, say, the time derivative of $\int y^m (h')^2 e^{\mu y}$, then some terms which involve L^2 norms of h will appear *with the same weight* y^m as we were using in the first place: this is, a term involving $\int h^2 y^m e^{\mu y}$ will appear, for which we have no previous spectral gap results. Hence, to close the estimates we use a spectral gap in a norm with an exponential weight of order slightly higher than the one we started with.

Lemma 3.16. *Take $0 < \mu < 2/\rho$ and an integer $k \geq 0$, and consider the operator L in the space $X_{k,\mu}$, with domain $X_{k+1,\mu}$. Then, L has a spectral gap in $X_{k,\mu}$; more precisely, there is some constant $K = K(k, \mu, \rho) > 0$ such that*

$$\|e^{tL} h^0\|_{k,\mu} \leq K \|h^0\|_{k,\mu} e^{-t} \quad (72)$$

for any $h^0 \in X_{k,\mu}$.

Proof. Take $k \geq 0$, assume the result true up to $k-1$ (the case $k = -1$ was proved in Corollary 3.15). Denote by h the solution of the linear equation with initial condition h^0 (i.e., $h(t, \cdot) = e^{tL} h^0$), and in order to simplify the notation write $\|\cdot\|$ instead of $\|\cdot\|_{k,\mu}$ and $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{k,\mu}$. The numbers K, K_1, K_2, \dots which appear in this proof are understood to be positive and depend only on k, μ, ρ and δ .

Pick $\mu < \nu < 2/\rho$. Then, after the time t_0 given in Lemma 3.13, the norm $\|h(t)\|_{k,\nu}$ is finite and we have, from the expression for L in

eq. (45):

$$\frac{1}{2} \frac{d}{dt} \|h\|^2 = \langle h, Lh \rangle = \langle h, yh' \rangle - \frac{4}{\rho} \langle h, H \rangle + \frac{2}{\rho} \langle h, H * g_\rho \rangle. \quad (73)$$

For the first term in (73), eqs. (63)–(64) show that

$$\langle h, yh' \rangle \leq -\frac{3}{2} \|h\|^2. \quad (74)$$

For the middle term in eq. (73),

$$\begin{aligned} \langle h, H \rangle &= \int (D^k h)(D^k H) y^{2k+2} e^{\mu y} dy \\ &= - \int (D^k h)(D^{k-1} h) y^{2k+2} e^{\mu y} dy \\ &= (k+1) \int (D^{k-1} h)^2 y^{2k+1} e^{\mu y} dy \\ &\quad + \frac{\mu}{2} \int (D^{k-1} h)^2 y^{2k+2} e^{\mu y} dy \leq K_1 \|h\|_{k-1, \nu}^2, \end{aligned} \quad (75)$$

where in the case $k = 0$ we denote $D^{-1}h := -H$. Observe that here the exponential weight has been changed from μ to ν , as the power weights in the norms which appear are not the right ones for the $\|\cdot\|_{k-1, \mu}$ norm, but higher. We overcome this difficulty by using a norm with a slightly higher exponential weight.

To bound the middle term in (73), we will need the following expression for $D^k(h * g_\rho)$, which can easily be proved by induction:

$$D^k(H * g_\rho) = H * (D^k g_\rho) + \sum_{i=0}^{k-1} (D^{i-1} h)(D^{k-1-i} g_\rho)(0). \quad (76)$$

From this,

$$\begin{aligned} \|H * g_\rho\| &= \left\| y^{k+1} D^k(H * g_\rho) \right\|_{L^2(e^{\mu y})} \\ &\leq \left\| y^{k+1} (H * (D^k g_\rho)) \right\|_{L^2(e^{\mu y})} + \sum_{i=0}^{k-1} \left| (D^{k-1-i} g_\rho)(0) \right| \left\| y^k D^{i-1} h \right\|_{L^2(e^{\mu y})} \\ &\leq \left\| y^{k+1} (H * (D^k g_\rho)) \right\|_{L^2(e^{\mu y})} + K_2 \|h\|_{k-1, \nu}, \end{aligned} \quad (77)$$

where the last bound is possible because all the norms inside the sum are bounded by a constant times $\|h\|_{k, \nu}$, using a higher exponential weight as before. For the remaining term in (77),

$$y^{k+1} (H * (D^k g_\rho)) = \sum_{i=0}^{k+1} \binom{k+1}{i} (y^i H) * (y^{k+1-i} D^k g_\rho)$$

and hence

$$\begin{aligned}
& \left\| y^{k+1} (H * (D^k g_\rho)) \right\|_{L^2(e^{\mu y})} \leq \\
& \sum_{i=0}^{k+1} \binom{k+1}{i} \left\| (y^i H) * (y^{k+1-i} D^k g_\rho) \right\|_{L^2(e^{\mu y})} \\
& \leq \sum_{i=0}^{k+1} \binom{k+1}{i} \|y^i H\|_{L^2(e^{\mu y})} \int y^{k+1-i} |D^k g_\rho| e^{\frac{\mu}{2} y} \\
& \leq K_3 \|h\|_{-1, \nu} \leq K_4 \|h\|_{k-1, \nu}.
\end{aligned}$$

Notice that the integrals involving g_ρ are numbers that depend only on k, μ and ρ . From (77),

$$\|H * g_\rho\| \leq K_5 \|h\|_{k-1, \nu},$$

so

$$|\langle h, H * g_\rho \rangle| \leq \|h\| \|H * g_\rho\| \leq K_5 \|h\| \|h\|_{k-1, \nu}. \quad (78)$$

Finally, all three terms from (73) are bounded in (74), (75) and (78), and together give

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|h\|^2 & \leq -\frac{3}{2} \|h\|^2 + \frac{4}{\rho} K_1 \|h\|_{k-1, \nu}^2 + \frac{2}{\rho} K_5 \|h\| \|h\|_{k-1, \nu} \\
& \leq -\frac{5}{4} \|h\|^2 + K_6 \|h\|_{k-1, \nu}^2,
\end{aligned}$$

where we used the elementary inequality $\|h\| \|h\|_{k-1, \nu} \leq \epsilon^2 \|h\|^2 + (1/\epsilon) \|h\|_{k-1, \nu}^2$ for an arbitrary $\epsilon > 0$. Thanks to the spectral gap in $\|\cdot\|_{k-1, \nu}$ and Lemma 3.13, we know that for any $\delta < 1$ there is some constant $K > 0$ such that $\|h(t)\|_{k-1, \nu} \leq K \|h(t_0)\|_{k-1, \nu} e^{-\delta(t-t_0)}$ for $t \geq t_0$. Hence, for $t \geq t_0$,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|h\|^2 & \leq -\frac{5}{4} \|h\|^2 + K_7 \|h^0\|_{k-1, \nu}^2 e^{-2\delta(t-t_0)} \\
& \leq -\frac{5}{4} \|h\|^2 + K_8 \|h^0\|_{k-1, \nu}^2 e^{-2\delta t}
\end{aligned}$$

A Gronwall lemma, together with inequality (61) for $t \leq t_0$, finishes the proof. \square

4 Behavior of moments

For later use we will need to have some information on the time evolution of exponential moments of solutions to the coagulation equation

(8), which is also interesting by itself, as this equation is one of the few particular cases where one can find an explicit expression for the evolution of moments. For a function $g : (0, \infty) \rightarrow \mathbb{R}$ we denote

$$M_k[g] := \int_0^\infty y^k |g(y)| dy \quad \text{for } k \in \mathbb{R}, \quad (79)$$

$$E_\mu[g] := \int_0^\infty e^{\mu y} |g(y)| dy \quad \text{for } \mu \in \mathbb{R}. \quad (80)$$

In general it is slightly simpler to solve explicitly the evolution of moments in eq. (1) than in eq. (8); but of course, one of them being a rescaling of the other, there is a simple relationship between the moments of their solutions: if f and g are related by the change of variables in eqs. (6)–(7), then the relationship between moments of f and moments of g is

$$M_k[g(t)] = e^{t(1-k)} M_k[f(e^t - 1)] \quad (81)$$

$$M_k[f(t)] = (t + 1)^{k-1} M_k[g(\log(1 + t))]. \quad (82)$$

That between exponential moments of f and g is

$$E_\mu[g(t)] = e^t E_{\mu e^{-t}}[f(e^t - 1)], \quad (83)$$

$$E_\mu[f(t)] = \frac{1}{t + 1} E_{\mu(t+1)}[g(\log(t + 1))]. \quad (84)$$

The evolution of moments of order 0, 1 and 2 for a nonnegative solution f to eq. (1) (or the corresponding solution g to eq. (8)) is easily obtained and well-known: the first moment is a constant, called its *mass*. For the moment of order 0,

$$\frac{d}{dt} M_0[f] = -\frac{1}{2} M_0[f]^2,$$

so

$$M_0[f] = \frac{2}{t + 2/M_0[f^0]}. \quad (85)$$

Using the relation in eq. (81) (with $\lambda = 0$) we have

$$M_0[g] = \frac{2}{1 - e^{-t} + 2e^{-t}/M_0[g^0]}. \quad (86)$$

The second moment of f in eq. (1) with $a \equiv 1$ satisfies the equation

$$\frac{d}{dt} M_2[f] = \frac{1}{2} M_1[f]^2 = \frac{1}{2} \rho^2,$$

where ρ is the mass of the solution. Hence,

$$M_2[f] = M_2[f^0] + \frac{1}{2} \rho^2 t, \quad (87)$$

and the second moment for the self-similar equation is

$$M_2[g] = e^{-t}M_2[g^0] + \frac{1}{2}\rho^2(1 - e^{-t}). \quad (88)$$

For exponential moments one can also find an explicit expression: again from eq. (1), and using eq. (85),

$$\begin{aligned} \frac{d}{dt}E_\mu[f] &= \frac{1}{2}E_\mu[f]^2 - M_0[f]E_\mu[f] \\ &= \frac{1}{2}E_\mu[f]^2 - \frac{2}{t+K}E_\mu[f], \end{aligned}$$

with $K := 2/M_0[f^0]$. This has an explicit solution:

$$E_\mu[f(t)] = \frac{2}{t + \frac{2}{M_0^0}} + \frac{2}{\frac{2}{E_\mu^0 - M_0^0} - t}, \quad (89)$$

where M_0^0 and E_μ^0 denote $M_0[f(0)]$ and $E_\mu[f(0)]$, resp.

Using eq. (83) we obtain the evolution of exponential moments for the self-similar equation with $\lambda = 0$:

$$E_\mu[g(\log(s))] = \frac{2s}{s-1 + \frac{2}{M_0^0}} + \frac{2s}{\frac{2}{E_{\mu/s}^0 - M_0^0} - s + 1}. \quad (90)$$

This directly implies the following lemma:

Lemma 4.1. *If g is a solution of eq. (8) with initial data g^0 and $\mu > 0$, then $E_\mu[g]$ is finite for all $t > 0$ if and only if the initial data satisfies*

$$E_\theta^0 - M_0^0 < \frac{2}{\frac{\mu}{\theta} - 1} \quad \text{for all } 0 < \theta < \mu. \quad (91)$$

$E_\mu[g]$ is uniformly bounded for all $t \geq 0$ if and only if the initial data satisfies the following for some $\nu > \mu$:

$$E_\theta^0 - M_0^0 \leq \frac{2}{\frac{\nu}{\theta} - 1} \quad \text{for all } 0 < \theta < \mu. \quad (92)$$

In particular, if $E_\nu[g]$ is finite for all $t > 0$, then for each $\mu < \nu$ the moment $E_\mu[g]$ is uniformly bounded for all $t \geq 0$.

Next we prove that if *some* positive exponential moment is initially finite, then every exponential moment less than $\frac{2}{\rho}$ becomes bounded after some time. The precise result is the following:

Lemma 4.2. *Let g be a solution of eq. (8) with initial data g^0 . Assume that there exists $\mu > 0$ such that $E_\mu[g^0] < \infty$. Then, for each $0 < \nu < \frac{2}{\rho}$ there is a time $T_\nu \geq 0$ (which depends on the initial condition g^0) such that*

$$E_\nu[g(t)] < \infty \quad \text{for all } t > T_\nu. \quad (93)$$

As a consequence, for each $0 < \nu < \frac{2}{\rho}$ there is a time $T_\nu^ \geq 0$ and a constant $K_\nu > 0$ such that*

$$E_\nu[g(t)] \leq K_\nu \quad \text{for all } t \geq T_\nu^*. \quad (94)$$

Proof. The uniform bound eq. (94) is a consequence of eq. (93) in view of Lemma 4.1, so we will only prove eq. (93). Take $0 < \nu < \frac{2}{\rho}$. In view of eq. (90), it is enough to show that, for some $\mu > \epsilon > 0$,

$$E_\theta[g^0] - M_0[g^0] < \frac{2}{\frac{\nu}{\theta} - 1} \quad \text{for all } \theta \leq \epsilon, \quad (95)$$

and then eq. (93) holds with $T_\nu := \log \frac{\nu}{\epsilon}$. As eq. (95) is a local inequality, it is enough to study both terms near $\theta = 0$: they are 0 at $\theta = 0$, and their derivatives are:

$$\begin{aligned} \frac{d}{d\theta} \int_0^\infty g(y) e^{\theta y} dy \Big|_{\theta=0} &= \int_0^\infty yg(y) dy = \rho \\ \frac{d}{d\theta} \frac{2}{\frac{\nu}{\theta} - 1} \Big|_{\theta=0} &= \frac{2\nu}{(\nu - \theta)^2} \Big|_{\theta=0} = \frac{2}{\nu}. \end{aligned}$$

Hence, the inequality

$$\nu < \frac{2}{\rho}$$

shows that eq. (95) holds for some $\epsilon > 0$, and the result is proved. \square

5 Local exponential convergence

In this section we will prove Theorem 1.4, which is a *local* exponential convergence result of the following kind: we show that if a solution g to eq. (8) is initially *close enough* to the stationary solution, then it converges exponentially fast to it with a given rate in the norm $\|\cdot\|_{k,\mu}$ for $0 < \mu < 2/\rho$ and integer $k \geq -1$, as long as this norm is finite at time $t = 0$.

Here, the concept of *close enough* is measured in the sense that the initial relative entropy to the equilibrium is small. The reason for this is that this closeness must be enough to guarantee that the nonlinear

term in the evolution equation is small when compared to the linear part, which we know is well-behaved. However, if g is a solution, the smallness of the nonlinear part essentially involves norms of the kind $\int |g - g_\rho|$, which cannot be controlled by our norm $\|g - g_\rho\|_{-1,2/\rho}$. Consequently, we need to measure closeness in some way that is propagated in time: what is needed in the proof of the theorem is that, for $\epsilon > 0$ small enough,

$$\int_0^\infty |g(t, y) - g_\rho(y)| dy \leq \epsilon \quad \text{for all } t \geq 0. \quad (96)$$

We do not know of any simple condition on the initial data that guarantees this to hold except requiring the relative entropy to the equilibrium g_ρ to be small. Then, as the L^1 norm of $g - g_\rho$ is controlled by the relative entropy, we know that (96) also holds. When we consider convergence in norms $\|\cdot\|_{k,\mu}$ with $k \geq 0$ we could control $\int |g - g_\rho|$ by the norm $\|g - g_\rho\|_{k,\mu}$ and obtain a result where closeness to g_ρ is measured in terms of $\|g - g_\rho\|_{k,\mu}$, but this is less convenient for the global result in Section 6. Because of this, we always state our results with an entropy condition.

Before proving our local convergence theorems we will need two previous lemmas. In the first one we show that the function $F[g|g_\rho]$ controls the distance of g to the equilibrium g_ρ in the L^1 norm, and the second one will be used to estimate the nonlinear term $C(g - g_\rho, g - g_\rho)$ which will appear when calculating the time evolution of $\|g - g_\rho\|_{k,\mu}$.

Lemma 5.1. *There is some constant $K \geq 0$ such that*

$$\int_0^\infty |g(y) - g_\rho(y)| dy \leq K \max \left\{ \sqrt{F[g|g_\rho]}, F[g|g_\rho] \right\}.$$

Proof. We can write

$$F[g|g_\rho] = \int_0^\infty g_\rho(y) \Psi \left(\frac{g(y)}{g_\rho(y)} - 1 \right) dy,$$

where $\Psi(y) := (y + 1) \log(y + 1) - y \geq 0$. As it happens that Ψ is a convex function for which $\Psi(y) \geq \Psi(|y|)$, from Jensen's inequality we have

$$\begin{aligned} F[g|g_\rho] &\geq \int_0^\infty g_\rho(y) \Psi \left(\left| \frac{g(y)}{g_\rho(y)} - 1 \right| \right) dy \\ &\geq 2 \Psi \left(\frac{1}{2} \int_0^\infty |g(y) - g_\rho(y)| dy \right), \end{aligned}$$

where the 2 appears from $\int_0^\infty g_\rho = 2$. Hence,

$$\int |g - g_\rho| \leq 2\Psi^{-1} \left(\frac{1}{2} F[g|g_\rho] \right) \leq K \max \left\{ \sqrt{F[g|g_\rho]}, F[g|g_\rho] \right\}$$

for some constant $K \geq 0$, which proves the lemma. \square

Lemma 5.2. *For $\rho > 0$ and $0 < \mu < 4/\rho$ there is a constant $K = K(\mu) > 0$ such that*

$$\|C(h, h)\|_{-1, \mu} \leq K \|h\|_{-1, \mu} \int |h| e^{\frac{\mu}{2}y} dy \quad (h \in X_{-1, \mu}). \quad (97)$$

Also, for integer $k \geq 0$ and $\mu < \nu < 4/\rho$, there is a constant $K = K(k, \mu, \nu) > 0$ such that

$$\|C(h, h)\|_{k, \mu} \leq K \|h\|_{k, \mu} \int |h| e^{\frac{\mu}{2}y} dy + K \|h\|_{k-1, \nu}^2 \quad (98)$$

for all $h \in X_{k, \nu}$.

Proof. Eq. (97) is a consequence of the calculation in eq. (60) with h instead of g_ρ . Now, for $k \geq 1$ and $h \in X_{k, \mu}$, from the expression $C(h, h) = \frac{1}{2}h * h - h \int h$ we have

$$\|C(h, h)\|_{k, \mu} \leq \frac{1}{2} \|h * h\|_{k, \mu} + \|h\|_{k, \mu} \left| \int h \right|, \quad (99)$$

and from Lemmas 3.5 and 3.9,

$$\begin{aligned} \|h * h\|_{k, \mu} &\leq K_1 \left\| D^k(y^{k+1}(h * h)) \right\|_{L^2(e^{\mu y})} \\ &\leq K_1 \sum_{i=0}^{k+1} \binom{k+1}{i} \left\| (D^{i-1}(y^i h)) * (D^{k+1-i}(y^{k+1-i} h)) \right\|_{L^2(e^{\mu y})} \\ &\leq K_2 \|h\|_{k, \mu} \int |h| e^{\frac{\mu}{2}y} + K_3 \|h\|_{k, \mu} \int |h| y e^{\frac{\mu}{2}y} \\ &\quad + K_4 \sum_{i=2}^k \binom{k+1}{i} \|h\|_{i, \mu} \int \left| D^{k+1-i}(y^{k+1-i} h) \right| e^{\frac{\mu}{2}y} \\ &\leq K_5 \|h\|_{k, \mu} \left(\int |h| e^{\frac{\mu}{2}y} + \int |h| y e^{\frac{\mu}{2}y} \right) + K_6 \|h\|_{k-1, \nu}^2. \end{aligned} \quad (100)$$

Here, we have bounded the terms in the sum using Young's inequality; the terms for $i = 0$, $i = 1$ and $i = k + 1$ have been bounded putting the L^2 norm on the part with k derivatives (and an integration by parts for $i = 0$). The rest of the terms have been also bounded using Young's inequality, and we have applied Cauchy-Schwarz for the last inequality, together with the fact that $\|h\|_{k-1, \nu}$ controls all norms $\|\cdot\|_{i, \nu}$ with $0 \leq i < k$. Notice that K_1 through K_5 depend only on k

and μ , and K_6 only on k, μ and ν . Finally, from (99),

$$\begin{aligned} \|C(h, h)\|_{k, \mu} &\leq \|h\|_{k, \mu} \left| \int h \right| \\ &+ K_5 \|h\|_{k, \mu} \left(\int |h| e^{\frac{\mu}{2}y} + \int |h| y e^{\frac{\mu}{2}y} \right) + K_6 \|h\|_{k-1, \nu}^2 \\ &\leq K_7 \|h\|_{k, \mu} \int |h| e^{\frac{\mu}{2}y} + K_6 \|h\|_{k-1, \nu}^2. \end{aligned}$$

This proves our result for $k \geq 1$. For the case $k = 0$, the same calculation proves the inequality, with the difference that in this case the term multiplying K_4 in (100) does not appear and the argument is much simpler. \square

We can finally prove Theorem 1.4:

Proof of Theorem 1.4. Step 1: Proof for $k = -1$. Denote $\|\cdot\|_{-1, \mu}$ by $\|\cdot\|$. We assume that $\|g^0 - g_\rho\| < \infty$, and want to show the exponential convergence in the same norm. We prove an *a priori* estimate below, which can be made rigorous by considering approximate regular solutions instead of solutions to the full equation. The numbers K, K_1, K_2, \dots which appear in the proof are understood to depend only on ρ, E, δ, k, μ and ν .

Consider a solution g of the self-similar equation eq. (8) in the conditions of the theorem, and set $h := g - g_\rho$, $h^0 := g^0 - g_\rho$. Then h is a solution of the equation $\partial_t h = L(h) + C(h, h)$, and by Duhamel's formula we have

$$h(t) = e^{tL} h^0 + \int_0^t e^{(t-s)L} [C(h(s), h(s))] ds \quad (t \geq 0). \quad (101)$$

Using our bound on the exponential decay of the semigroup e^{tL} from Corollary 3.15 we obtain

$$\|h(t)\| \leq K_1 \|h^0\| e^{-t} + K_1 \int_0^t \|C(h(s), h(s))\| e^{-(t-s)} ds,$$

for some $K_1 > 0$ which depends on ρ, E and μ . Now, use the bound in Lemma 5.2 to get

$$\|h(t)\| \leq K_1 \|h^0\| e^{-t} + K_2 \int_0^t E_{\frac{\mu}{2}}[h(s)] \|h(s)\| e^{-(t-s)} ds, \quad (102)$$

where $E_{\frac{\mu}{2}}[h(s)]$ is the exponential moment of order $\mu/2$ of h (defined in eq. (80)). Considering E and ν from the hypotheses of the theorem,

we have $E_{\nu/2}[g(t)] \leq E$ for all $t \geq 0$, so there is some constant $C_1 > 0$ such that

$$E_{\nu/2}[h(t)] \leq C_1 \quad \text{for all } t > 0,$$

(as $h = g - g_\rho$, and $g_\rho(y)e^{\frac{\nu}{2}y}$ is integrable). By Hölder's inequality,

$$E_{\frac{\mu}{2}}[h(s)] \leq M_0[h(s)]^{\frac{\nu-\mu}{\nu}} E_{\nu/2}[h(s)]^{\frac{\mu}{\nu}} \leq \epsilon_1 C_2 =: \epsilon_2 \quad \text{for } s \geq 0, \quad (103)$$

where

$$C_2 := C_1^{\frac{\mu}{\nu}}, \quad \epsilon_1^{\frac{\nu-\mu}{\nu}} := K \max \{ \sqrt{\epsilon}, \epsilon \},$$

and K is the one from Lemma 5.1. We have used here the bound on the relative entropy in the hypotheses, and the fact that it is conserved in time, as the relative entropy is nonincreasing.

Using this bound on eq. (102) we have

$$\|h\| \leq K_1 \|h^0\| e^{-t} + \epsilon_3 \int_0^t \|h(s)\| e^{-(t-s)} ds, \quad (104)$$

where $\epsilon_3 := \epsilon_2 K_2$. Equivalently,

$$\|h\| e^t \leq K_1 \|h^0\| + \epsilon_3 \int_0^t \|h(s)\| e^s ds,$$

and by Gronwall's lemma,

$$\|h\| e^t \leq K_1 \|h^0\| e^{\epsilon_3 t},$$

or

$$\|h\| \leq K_1 \|h^0\| e^{-(1-\epsilon_3)t}.$$

Choosing ϵ so that $1 - \epsilon_3 \geq \delta$ proves the theorem.

Step 2: Proof for $k \geq 0$. For this step, denote by $\|\cdot\|$ the norm $\|\cdot\|_{k,\mu}$. The difference in this case is that the inequality (98) has an additional term, which can be dealt with by using our previous steps. Take $k \geq 0$, and assume that we have proved the result on the convergence in $\|\cdot\|_{k-1,\gamma}$ for $0 < \gamma < 2/\rho$. Repeating the same reasoning as before we arrive at the following instead of (104):

$$\begin{aligned} \|h(t)\| \leq & K_1 \|h^0\| e^{-t} + \epsilon_3 \int_0^t \|h(s)\| e^{-(t-s)} ds \\ & + K_2 \int_0^t \|h(s)\|_{k-1,\nu}^2 e^{-(t-s)} ds, \quad (105) \end{aligned}$$

where the inequality (98) was applied with some ν' such that $\mu < \nu' < \nu$ in the place of ν . Now, as we know that $\|h(s)\|_{k-1,\nu}$ is exponentially decaying in time, for any $\delta < \delta' < 1$ we have

$$\|h(s)\|_{k-1,\nu} \leq K_3 \|h^0\|_{k-1,\nu} e^{-\delta s}.$$

Hence, we get

$$\begin{aligned} \|h(t)\| &\leq K_1 \|h^0\| e^{-t} \\ &\quad + \epsilon_3 \int_0^t \|h(s)\| e^{-(t-s)} ds + K_4 \|h^0\|_{k-1,\nu} e^{-\delta't} \\ &\leq K_5 \|h^0\|_{k,\nu} e^{-\delta't} + \epsilon_3 \int_0^t \|h(s)\| e^{-\delta'(t-s)} ds, \end{aligned}$$

using that $\|h\|_{k,\nu}$ controls both $\|h\|_{k-1,\nu}$ and $\|h\|_{k,\mu}$. Now this inequality has the same form as (104), and the same reasoning followed in the first step finishes the proof. \square

6 Global convergence

To prove global convergence we consider two stages in the evolution of the equation: when the solution is close enough to equilibrium, we can use Theorem 1.4 to get an exponential rate of convergence; when it is far from equilibrium we use the following lemma, which says that the solution converges to equilibrium exponentially fast in the L^2 norm:

Lemma 6.1. *Take a nonnegative initial condition $g_0 \in L^1_2$, with derivative $g'_0 \in L^1$ and mass $\rho > 0$, and consider the solution g to eq. (8). There are explicit constants $C, T > 0$ which depend only on g_0 such that*

$$\|g(t) - g_\rho\|_2 \leq C e^{-\frac{1}{2}t} \quad (t > T). \quad (106)$$

The lemma is independent from the rest of the paper, as it can be proved by using the explicit solution of the Fourier transform of equation (8). The technique to prove it is very similar to the one used in [15] to obtain the uniform convergence to equilibrium; the main difference is that here we are interested in *obtaining a rate*, while in [15] emphasis was placed in giving very general conditions on the initial data under which convergence holds, without any given rate.

Lemma 6.1 does not give the optimal rate of convergence, but we only need it to show that any solution, after some initial time which may be explicitly given in terms of the initial datum, is in the conditions of our local result in Theorem 1.4, and hence prove our global result in Theorem 1.6. However, one could be more careful in the proof below and, imposing slightly stronger decay conditions on the initial datum g_0 , obtain the optimal rate e^{-t} .

We will first prove Theorem 1.6 assuming Lemma 6.1, leaving the proof of the lemma itself for the end.

Proof of Theorem 1.6. Let us show that, for any $\mu < \nu$, we can apply Theorem 1.4 for the exponential decay of the norm $\|\cdot\|_{k,\mu}$ after an certain initial time t_0 .

Lemma 4.1 and an argument as in the proof of Lemma 4.2 show that for some $0 < \mu' < \nu$, the exponential moment $E_{\mu'/2}[g]$ is uniformly bounded for all times $t \geq 0$. So, take $0 < \mu < \mu'$ with $\mu < 2/\rho$. Then, the first condition in Theorem 1.4 holds. Together with this, Lemma 6.1, shows that the relative entropy $F[g|g_\rho]$ tends to zero at an explicit rate, and hence condition 2 in Theorem 1.4 is also satisfied after some initial time t_0 .

Finally, the same calculation as in Theorem 1.4 shows that, for $0 < \mu < \mu'$ with $\mu < 2/\rho$, the norm $\|g - g_\rho\|_{k,\mu}$ is bounded in bounded time intervals, and in particular it is finite at $t = t_0$. This is enough to apply the Theorem 1.4, so we have

$$\|g(t) - g_\rho\|_{k,\mu} \leq K_1 e^{-\delta(t-t_0)} \quad (t \geq t_0),$$

for any $0 < \mu < \nu$ with $\mu < 2/\rho$. As $\|g(t) - g_\rho\|_{k,\mu}$ is bounded for $t \in [0, t_0]$, this proves the result. \square

Proof of Lemma 6.1. By the scaling and time-translation symmetries of eq. (1), it is enough to prove the result when $\int y g = \int g = 2$ (see, e.g., [15]). We consider the Fourier transform of g ,

$$\phi_t(\mu) := \int_0^\infty e^{-i\mu y} g(t, y) dy. \quad (107)$$

A very similar calculation to that in section 4 shows that ϕ has the following explicit expression, identical to (90) (notice that we are assuming $\int g_0 \equiv M_0^0 = 2$):

$$\phi_t(\mu) = 2 + \frac{2\tau}{\frac{2}{\phi_0(\frac{\mu}{\tau})} - \tau + 1} = \frac{2\phi_0(\frac{\mu}{\tau})}{2 + (\tau - 1)(2 - \phi_0(\frac{\mu}{\tau}))}, \quad (108)$$

for $\mu \in \mathbb{R}$ and $t > 0$, denoting $\tau := e^t$. By Plancherel's theorem, proving our result is equivalent to showing that

$$\|\phi_t - \phi_\infty\|_2 \leq C e^{-\frac{1}{2}t} \quad (t > 0), \quad (109)$$

where ϕ_∞ is the Fourier transform of the equilibrium with mass 2, $g_2(y) := 2e^{-y}$,

$$\phi_\infty(\mu) := \frac{2}{1 + i\mu} \quad (\mu \in \mathbb{R}). \quad (110)$$

Step 1: Approximations of $\phi_0(x)$ at $x = 0$ As $\phi(0) = 2$ and

$$\frac{d}{dx}\phi_0(x) = -i \int_0^\infty e^{-ixy} y g_0(y) dy, \quad (111)$$

$$\left| \frac{d}{dx}\phi_0(x) \right| \leq \int_0^\infty y g_0(y) dy = 2, \quad (112)$$

we readily have

$$|\phi_0(x) - 2| \leq 2|x| \quad (x \in \mathbb{R}). \quad (113)$$

Similarly, $\phi'(0) = -2i$ and

$$\frac{d^2}{dx^2}\phi_0(x) = - \int_0^\infty e^{-ixy} y^2 g_0(y) dy, \quad (114)$$

$$\left| \frac{d^2}{dx^2}\phi_0(x) \right| \leq \int_0^\infty y^2 g_0(y) dy =: M_2, \quad (115)$$

and then, using Taylor's series at $x = 0$,

$$\left| \frac{\phi_0(x) - 2}{x} + 2i \right| \leq \frac{1}{2} M_2 |x| \quad (x \in \mathbb{R}). \quad (116)$$

Step 2: Lower bound of the denominator in (108). For $\mu \in \mathbb{R}$ and $\tau > 1$ we have, calling $x := \mu/\tau$,

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq |2 + 2i\mu| - |2i\mu - (\tau - 1)(2 - \phi_0(x))|, \quad (117)$$

and rewriting this second term gives

$$2i\mu - (\tau - 1)(2 - \phi_0(x)) = \mu \left(2i + \frac{\phi_0(x) - 2}{\mu/\tau} \right) + 2 - \phi_0(x),$$

and using eqs. (113) and (116),

$$|2i\mu - (\tau - 1)(2 - \phi_0(x))| \leq \frac{1}{2} M_2 |\mu| |x| + 2|x| \quad (x \equiv \frac{\mu}{\tau} \in \mathbb{R}), \quad (118)$$

and so from (117),

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq 1 + |\mu| - \frac{1}{2} M_2 |\mu| |x| - 2|x|.$$

Finally, for $|x| \leq \epsilon_1 := \min \{1/4, 1/M_2\}$,

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq \frac{1}{2}(1 + |\mu|) \quad (|x| \equiv \left| \frac{\mu}{\tau} \right| \leq \epsilon_1). \quad (119)$$

When $|x| \geq \epsilon_1$, we estimate the denominator the other way round. Call

$$\epsilon_2 := \inf_{|x| \geq \epsilon_1} |2 - \phi_0(x)| > 0.$$

The above defined ϵ_2 is strictly positive because ϕ_0 is continuous, $|\phi_0(x)| < 2$ for $|x| > 0$ and

$$|\phi_0(x)| \leq 2C_1 \frac{1}{|x|} \quad (|x| > 0), \quad (120)$$

with

$$C_1 := \int_0^\infty |g'_0(y)| dy.$$

Hence, for $|x| \geq \epsilon_1$ the denominator can also be estimated as

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq (\tau - 1)|(2 - \phi_0(x))| - 2 \geq (\tau - 1)\epsilon_2 - 2, \quad (121)$$

and then for $\tau - 1 \geq C_2 := 3/\epsilon_2$,

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq 1 \quad (|x| \equiv \left| \frac{\mu}{\tau} \right| \geq \epsilon_1, \quad \tau \geq C_2 + 1). \quad (122)$$

Putting together (119) and (122) we obtain

$$|2 + (\tau - 1)(2 - \phi_0(\mu/\tau))| \geq \frac{1}{2} \quad (\mu \in \mathbb{R}, \quad \tau \geq C_2 + 1). \quad (123)$$

Step 3: Estimate for $|\mu|$ large. Take $K > 0$. We have, for $\mu \in \mathbb{R}$ and $t \geq 0$,

$$|\phi_t(\mu) - \phi_\infty(\mu)| \leq |\phi_t(\mu)| + |\phi_\infty(\mu)|, \quad (124)$$

and

$$\int_{|\mu| > K} |\phi_\infty(\mu)|^2 d\mu \leq \int_{|\mu| > K} \frac{1}{|1 + i\mu|^2} \leq \int_{|\mu| > K} \frac{1}{\mu^2} = \frac{2}{K}. \quad (125)$$

On the other hand, notice that (121) implies that for $\tau \geq C_2 + 1$,

$$|2 + (\tau - 1)(2 - \phi_0(x))| \geq \epsilon_3 \tau \quad (126)$$

for some ϵ_3 depending only on C_2 and ϵ_2 . Then, using the explicit expression (108), eq. (126) and eq. (120),

$$|\phi_t(\mu)| \leq \frac{2}{\epsilon_3} \frac{|\phi_0(\mu/\tau)|}{\tau} \leq 4 \frac{C_1}{\epsilon_3} \frac{1}{|\mu|} \quad (\tau \geq C_2 + 1), \quad (127)$$

so, for $\tau \geq C_2 + 1$,

$$\int_{|\mu| > K} |\phi_t(\mu)|^2 d\mu \leq 16 \frac{C_1^2}{\epsilon_3^2} \int_{|\mu| > K} \frac{1}{|\mu|^2} d\mu \leq 32 \frac{C_1^2}{\epsilon_3^2} \frac{1}{K}. \quad (128)$$

Equations (125) and (128) together now give

$$\int_{|\mu|>K} |\phi_t - \phi_\infty|^2 d\mu \leq \frac{1}{K} \left(2 + 32 \frac{C_1^2}{\epsilon_3^2} \right) \quad (\tau \geq C_2 + 1, \quad K > 0). \quad (129)$$

Step 4: Estimate for $|\mu|$ small. Denoting $x \equiv \mu/\tau$, the difference $\phi_t(\mu) - \phi_\infty(\mu)$ can be written as

$$\phi_t(\mu) - \phi_\infty(\mu) = \frac{2i\mu(\phi_0(x) - 2) + 2\mu \left(2i + \frac{\phi_0(x) - 2}{x} \right)}{(1 + i\mu)(2 + (1 - \tau)(\phi_0(x) - 2))}. \quad (130)$$

For the numerator we use (113) and (116) to get

$$\begin{aligned} \left| 2i\mu(\phi_0(x) - 2) + 2\mu \left(2i + \frac{\phi_0(x) - 2}{x} \right) \right| \\ \leq (4 + M_2) |\mu| |x| = (4 + M_2) |\mu|^2 \frac{1}{\tau}. \end{aligned} \quad (131)$$

For the denominator, we use eq. (119):

$$|2 + (1 - \tau)(\phi_0(x) - 2)| \geq \frac{1}{2}(1 + |\mu|) \quad \left(\left| \frac{\mu}{\tau} \right| \leq \epsilon_1 \right).$$

Then, for $\left| \frac{\mu}{\tau} \right| \leq \epsilon_1$,

$$|\phi_t(\mu) - \phi_\infty(\mu)| \leq C_3 \frac{1}{\tau} \frac{|\mu|^2}{(1 + |\mu|)^2} \leq C_3 \frac{1}{\tau}, \quad (132)$$

with $C_3 := 4(4 + M_2)$. Hence for any $K := \epsilon_1 \tau$ we have

$$\begin{aligned} \int_{|\mu|<K} |\phi_t(\mu) - \phi_\infty(\mu)|^2 d\mu &\leq C_3^2 \frac{1}{\tau^2} \int_{|\mu|<K} d\mu \\ &= 2C_3^2 \frac{K}{\tau^2} = 2C_3^2 \epsilon_1^2 \frac{1}{K} =: C_4 \frac{1}{K}. \end{aligned} \quad (133)$$

Step 5: Final estimate. From (129) and (133), taking $K := \epsilon_1 \tau$ we obtain for $\tau \geq C_2 + 1$:

$$\int_{-\infty}^{+\infty} |\phi_t(\mu) - \phi_\infty(\mu)|^2 d\mu \leq \frac{1}{K} \left(C_4 + 2 + 32 \frac{C_1^2}{\epsilon_3^2} \right) =: C_5 e^{-t}. \quad (134)$$

This shows (109), and finishes the proof. \square

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