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**Abstract**

We prove that small smooth solutions of semi-linear Klein-Gordon equations with quadratic potential exist over a longer interval than the one given by local existence theory, for almost every value of mass. We use normal form for the Sobolev energy. The difficulty in comparison with some similar results on the sphere comes from the fact that two successive eigenvalues $\lambda, \lambda'$ of $\sqrt{-\Delta + |x|^2}$ may be separated by a distance as small as $\frac{1}{\sqrt{\lambda}}$.

## 0 Introduction

Let $-\Delta + |x|^2$ be the harmonic oscillator on $\mathbb{R}^d$. This paper is devoted to the proof of lower bounds for the existence time of solutions of non-linear Klein-Gordon equations of type

$$\begin{aligned}(\partial_t^2 - \Delta + |x|^2 + m^2)v &= v^{\kappa+1}\\ v|_{t=0} &= \epsilon v_0\\ \partial_t v|_{t=0} &= \epsilon v_1\end{aligned}$$

where $m \in \mathbb{R}_+^*$, $x^\alpha \partial_x^\beta v_j \in L^2$ when $|\alpha| + |\beta| \leq s + 1 - j$ $(j = 0, 1)$ for a large enough integer $s$, and where $\epsilon > 0$ is small enough.

The similar equation without the quadratic potential $|x|^2$, and with data small, smooth and compactly supported, has global solutions when $d \geq 2$ (see Klainerman [18] and Shatah [23] for dimensions $d \geq 3$, Ozawa, Tsutaya and Tsutsumi [22] when $d = 2$). The situation is drastically different when we replace $-\Delta$ by $-\Delta + |x|^2$, since the latter operator has pure point spectrum. This prevents any time decay for solutions of the linear equation. Because of that, the question of long time existence for Klein-Gordon equations associated to the harmonic oscillator is similar to the corresponding problem on compact manifolds.

For the equation $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the circle $\mathbb{S}^1$, it has been proved by Bourgain [6] and Bambusi [1], that for almost every $m > 0$, the above equation has solutions defined on intervals

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of length $c_N\epsilon^{-N}$ for any $N \in \mathbb{N}$, if the data are smooth and small enough (see also the lectures of Grébert [13]). These results have been extended to the sphere $\mathbb{S}^d$ instead of $\mathbb{S}^1$ by Bambusi, Delort, Grébert and Szeftel [2]. A key property in the proofs is the structure of the spectrum of $\sqrt{-\Delta}$ on $\mathbb{S}^d$. It is made of the integers, up to a small perturbation, so that the gap between two successive eigenvalues is bounded from below by a fixed constant.

A natural question is to examine which lower bounds on the time of existence of solutions might be obtained when the eigenvalues of the operator do not satisfy such a gap condition. The problem has been addressed for $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the torus $\mathbb{T}^d$ when $d \geq 2$ by Delort [9]. It has been proved that for almost every $m > 0$, the solution of such an equation exists over an interval of time of length bounded from below by $c\epsilon^{-\kappa(1+2/d)}$ (up to a logarithm) and has Sobolev norms of high index bounded on such an interval. Note that two successive eigenvalues $\lambda, \lambda'$ of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ might be separated by an interval of length as small as $c/\lambda$. A natural question is then to study the same problem for a model for which separation of eigenvalues is intermediate between the cases of the sphere and of the torus. The harmonic oscillator provides such a framework, as the distance between two successive eigenvalues $\lambda, \lambda'$ of $\sqrt{-\Delta + |x|^2}$ is of order $1/\sqrt{\lambda}$. Our goal is to exploit this to get for the corresponding Klein-Gordon equation a lower bound of the time of existence of order $c\epsilon^{-4\kappa/3}$ when $d \geq 2$ (and a slightly better bound if $d = 1$).

Note that the estimate we get for the time of existence is explicit (given by the exponent $-4\kappa/3$) and independent of the dimension $d$. This is in contrast with the case of the torus, where the gain $2/d$ on the exponent brought by the method goes to zero as $d \to +\infty$. The point is that when the dimension increases, the multiplicity of the eigenvalues of $-\Delta + |x|^2$ grows, while the spacing between different eigenvalues remains essentially the same.

The method we use is based, as for similar problems on the sphere and the torus, on normal form methods. Such an idea has been introduced in the study of non-linear Klein-Gordon equations on $\mathbb{R}^d$ by Shatah [23], and is at the root of the results obtained on $\mathbb{S}^1, \mathbb{S}^d, \mathbb{T}^d$ in [6, 1, 3, 2, 9]. In particular, we do not need to use any KAM results, unlike in the study of periodic or quasi-periodic solutions of semi-linear wave or Klein-Gordon equations. For such a line of studies, we refer to the books of Kuksin [20, 21] and Craig [8] in the case of the equation on $\mathbb{S}^1$, to Berti and Bolle [4] for recent results on the sphere, and to Bourgain [7] and Elliasson-Kuksin [12] in the case of the torus.

Finally let us mention that very recently Grébert, Imekraz and Paturel [14] have studied the non-linear Schrödinger equation associated to the harmonic oscillator. They have obtained almost global existence of small solutions for this equation.

# 1 The semi-linear Klein-Gordon equation

## 1.1 Sobolev Spaces

We introduce in this subsection Sobolev spaces we will work with. From now on, we denote by $P = \sqrt{-\Delta + |x|^2}, x \in \mathbb{R}^d, d \geq 1$. The operator $P^2 = -\Delta + |x|^2$ is called the harmonic oscillator on $\mathbb{R}^d$. The eigenvalues of $P^2$ are given by $\lambda_n^2$, where

$$\lambda_n = \sqrt{2n + d}, n \in \mathbb{N}. \tag{1.1.1}$$

Let $\Pi_n$ be the orthogonal projector to the eigenspace associated to $\lambda_n^2$. There are several ways to characterize these spaces. Of course we will show they are equivalent after giving definitions.

**Definition 1.1.1.** *Let $s \in \mathbb{R}$. We define $\mathscr{H}_1^s(\mathbb{R}^d)$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that*
*$(\lambda_n^s ||\Pi_n u||_{L^2})_{n\in\mathbb{N}} \in \ell^2$, equipped with the norm defined by $||u||^2_{\mathscr{H}_1^s} = \sum_{n\in\mathbb{N}} \lambda_n^{2s} ||\Pi_n u||^2_{L^2}$.*

The space $\mathscr{H}_1^s(\mathbb{R}^d)$ is the domain of the operator $g(P)$ on $L^2(\mathbb{R}^d)$, which is defined using functional calculus and where

$$g(r) = (1+r^2)^{\frac{s}{2}}, r \in \mathbb{R}. \tag{1.1.2}$$

Because of (1.1.1), we have

$$||g(P)u||_{L^2} \sim ||u||_{\mathscr{H}_1^s}. \tag{1.1.3}$$

**Definition 1.1.2.** *Let $s \in \mathbb{N}$. We define $\mathscr{H}_2^s(\mathbb{R}^d)$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that $x^\alpha \partial^\beta u \in L^2(\mathbb{R}^d), \forall |\alpha|+|\beta| \leq s$, equipped with the norm defined by $||u||^2_{\mathscr{H}_2^s} = \sum_{|\alpha|+|\beta|\leq s} ||x^\alpha \partial^\beta u||^2_{L^2}$.*

We shall give another definition of the space in the view point of pseudo-differential theory. Let us first list some results from [16].

**Definition 1.1.3.** *We denote by $\Gamma^s(\mathbb{R}^d)$, where $s \in \mathbb{R}$, the set of all functions $u \in C^\infty(\mathbb{R}^d)$ such that: $\forall \alpha \in \mathbb{N}^d$, $\exists\ C_\alpha$, s.t. $\forall z \in \mathbb{R}^d$, we have $|\partial_z^\alpha a(z)| \leq C_\alpha \langle z \rangle^{s-|\alpha|}$, where $\langle z \rangle = (1+|z|^2)^{\frac{1}{2}}$ .*

**Definition 1.1.4.** *Assume $a_j \in \Gamma^{s_j}(\mathbb{R}^d)(j \in \mathbb{N}^*)$ and that $s_j$ is a decreasing sequence tending to $-\infty$. We say a function $a \in C^\infty(\mathbb{R}^d)$ satisfies:*

$$a \sim \sum_{j=1}^{\infty} a_j$$

*if:* $\quad \forall r \geq 2, r \in \mathbb{N}, \quad a - \sum_{j=1}^{r-1} a_j \in \Gamma^{s_r}(\mathbb{R}^d)$.

We now would like to consider operators of the form

$$Au(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi \tag{1.1.4}$$

where $a(x,\xi) \in \Gamma^s(\mathbb{R}^{2d})$. We can also consider a more general formula for the action of the operator

$$Au(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y) dy d\xi \tag{1.1.5}$$

where the function $a(x,y,\xi)$ is called the amplitude. We will describe the class of amplitudes as following:

**Definition 1.1.5.** *Let $s \in \mathbb{R}$ and $\Omega^s(\mathbb{R}^{3d})$ denote the set of functions $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3d})$, which for some $s' \in \mathbb{R}$ satisfy*

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi)| \leq C_{\alpha\beta\gamma} \langle z \rangle^{s-(|\alpha|+|\beta|+|\gamma|)} \langle x-y \rangle^{s'+|\alpha|+|\beta|+|\gamma|},$$

*where $z = (x,y,\xi) \in \mathbb{R}^{3d}$.*

The following proposition is a special case of proposition 1.1.4 in [16].

**Proposition 1.1.6.** *If $b \in \Gamma^s(\mathbb{R}^{2d})$, then $a(x,y,\xi) = b(x,\xi)$ and $a(x,y,\xi) = b(y,\xi)$ belong to $\Omega^s(\mathbb{R}^{3d})$.*

Let $\chi(x,y,\xi) \in C_0^\infty(\mathbb{R}^{3d}), \chi(0,0,0) = 1$. It is shown by lemma 1.2.1 in [16] that (1.1.5) makes sense in the following way:

$$Au(x) = \lim_{\varepsilon \to +0} (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} \chi(\varepsilon x, \varepsilon y, \varepsilon\xi) a(x,y,\xi) u(y) dy d\xi \tag{1.1.6}$$

if $a(x,y,\xi) \in \Omega^s(\mathbb{R}^{3d})$ for some $s$. It is also shown in the same section of it the operator $A$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ and it can be uniquely extended to an operator from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

**Definition 1.1.7.** *The class of pseudo-differential operators $A$ of the form (1.1.5) with amplitudes $a \in \Omega^s(\mathbb{R}^{3d})$ will be denoted by $G^s(\mathbb{R}^d)$.*

We set $G^{-\infty}(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} G^s(\mathbb{R}^d)$.

**Example 1.1.8.** *For $s \in \mathbb{N}$, the constant coefficient differential operator $\sum_{|\alpha|+|\beta|\le s} c_{\alpha\beta} x^\alpha \partial^\beta$ is in the class $G^s(\mathbb{R}^d)$.*

The class $G^s(\mathbb{R}^d)$ has some properties which are just theorems 1.3.1, 1.4.7, 1.4.8 in [16]:

**Theorem 1.1.9.** *Let $s_1, s_2 \in \mathbb{R}$ and $A \in G^{s_1}(\mathbb{R}^d)$, $A' \in G^{s_2}(\mathbb{R}^d)$. Then $A \circ A' \in G^{s_1+s_2}(\mathbb{R}^d)$.*

**Theorem 1.1.10.** *The operator $A \in G^0(\mathbb{R}^d)$ can be extended to a bounded operator on $L^2(\mathbb{R}^d)$.*

**Theorem 1.1.11.** *The operator $A \in G^s(\mathbb{R}^d)$ for $s < 0$ can be extended to a compact operator on $L^2(\mathbb{R}^d)$.*

We shall give a subclass of that of pseudo-differential operators.

**Definition 1.1.12.** *We say $a \in \Gamma^s_{cl}(\mathbb{R}^d)$ if $a \in \Gamma^s(\mathbb{R}^d)$ and $a$ has asymptotic expansion:*

$$a \sim \sum_{j\in\mathbb{N}} a_{s-j}$$

*with $a_{s-j} \in C^\infty(\mathbb{R}^d)$ satisfying for $\theta \ge 1, |x| + |\xi| \ge 1$*

$$a_{s-j}(\theta x, \theta\xi) = \theta^{s-j} a_{s-j}(x,\xi).$$

**Definition 1.1.13.** *Let $A$ be a pseudo-differential operator with amplitude $a \in \Gamma^s_{cl}(\mathbb{R}^d)$. We then call $a_s$ defined above the principle symbol of $A$.*

**Definition 1.1.14.** *We say a pseudo-differential operator $A \in G^s_{cl}(\mathbb{R}^d)$ if its amplitude $a \in \Gamma^s_{cl}(\mathbb{R}^{2d})$.*

By proposition 1.1.6, definition 1.1.14 is meaningful.

**Definition 1.1.15.** *We say that $A \in G^s_{cl}(\mathbb{R}^d)$ is globally elliptic if we have: $\exists R > 0, \exists\, C > 0$ such that $\forall (x,\xi) \in \mathbb{R}^{2d}$ satisfying $|x| + |\xi| \ge R$, we have $|a_s(x,\xi)| \ge C(|x| + |\xi|)^s$, where $a_s$ denotes the principle symbol of $A$.*

We can invert the operator $A \in G^s_{cl}(\mathbb{R}^d)$ up to a regularizing operator, which is just theorem 1.5.7 in [16].

**Theorem 1.1.16.** *Let $A \in G^s_{cl}(\mathbb{R}^d)$ be a globally elliptic operator. Then there is an operator $B \in G^{-s}_{cl}(\mathbb{R}^d)$ such that*

$$B \circ A = I + R_1, \qquad A \circ B = I + R_2, \tag{1.1.7}$$

*where $R_1, R_2$ are regularizing, i.e. $R_1, R_2 \in G^{-\infty}(\mathbb{R}^d)$.*

**Definition 1.1.17.** *Let $A$ be a pseudo-differential operator whose symbol is $\langle \xi, x\rangle^s$ modulo $\Gamma^{s-1}_{cl}$. We define $\mathscr{H}^s_3(\mathbb{R}^d)$ to be the set of all functions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $Au \in L^2(\mathbb{R}^d)$, equipped with the norm defined by $||u||^2_{\mathscr{H}^s_3} = ||Au||^2_{L^2} + ||u||^2_{L^2}$.*

**Remark 1.1.1.** *The pseudo-differential operator $A$ defined above is globally elliptic. Thus by theorem 1.1.16 if $Au \in L^2(\mathbb{R}^d)$, we must have $u \in L^2(\mathbb{R}^d)$.*

**Remark 1.1.2.** *$\mathscr{H}^s_3(\mathbb{R}^d)$ does not depend on the choice of $A$ according to corollary 1.6.5 in [16].*

**Corollary 1.1.18.** *When $s \in \mathbb{N}$, definitions 1.1.1, 1.1.2 and 1.1.17 characterize the same space. Moreover $\mathscr{H}^s_3(\mathbb{R}^d) = \mathscr{H}^s_1(\mathbb{R}^d)$ for any $s \in \mathbb{R}$.*

*Proof.* First let $s \in \mathbb{N}$. Since $A$ in definition 1.1.17 is globally elliptic, by theorem 1.1.16 there is $B \in G^{-s}_{cl}(\mathbb{R}^d)$ such that

$$B \circ A = I + R_1, \qquad A \circ B = I + R_2 \tag{1.1.8}$$

where $R_1, R_2$ are regularizing. Thus for any $\alpha, \beta$ with $|\alpha| + |\beta| \leq s$, by the example after definition 1.1.12 and theorems 1.1.9, 1.1.10 and 1.1.11, we have $||x^\alpha \partial^\beta u||_{L^2} \leq ||x^\alpha \partial^\beta BAu||_{L^2} + ||x^\alpha \partial^\beta R_1 u||_{L^2} \leq C(||Au||_{L^2} + ||u||_{L^2})$, which implies $||u||_{\mathscr{H}^s_2} \leq C||u||_{\mathscr{H}^s_3}$. The inverse inequality follows from the proof of proposition 1.6.6 in [16]. Let us now prove that definition 1.1.1 is equivalent to definition 1.1.17 for any $s \in \mathbb{R}$.

By Theorem 1.11.2 in [16] the operator $g(P)$ defined in (1.1.2) is an essentially self-adjoint globally elliptic operator in the class $G^s(\mathbb{R}^d)$. We have again by theorem 1.1.16 that there is $Q \in G^{-s}_{cl}(\mathbb{R}^d)$ such that

$$g(P) \circ Q = I + R_1', \qquad Q \circ g(P) = I + R_2' \tag{1.1.9}$$

where $R_1', R_2'$ are regularizing. We compute using (1.1.3), (1.1.8), (1.1.9) together with theorem 1.1.9 and theorem 1.1.10

$$\begin{aligned} ||u||_{\mathscr{H}^s_1} \sim ||g(P)u||_{L^2} &\leq ||(g(P) \circ B \circ A)u||_{L^2} + ||(g(P) \circ R_1)u||_{L^2} \\ &\leq C(||Au||_{L^2} + ||u||_{L^2}) \leq C||u||_{\mathscr{H}^s_3} \end{aligned}$$

and

$$\begin{aligned} ||u||_{\mathscr{H}^s_3} &\leq C(||(A \circ Q \circ g(P))u||_{L^2} + ||(A \circ R_2')u||_{L^2} + ||u||_{L^2}) \\ &\leq C(||g(P)u||_{L^2} + ||u||_{L^2}) \leq C||u||_{\mathscr{H}^s_1}, \end{aligned}$$

where the last inequality follows from the fact $\lambda_n \geq 1$. ∎

We denote $\mathscr{H}^s(\mathbb{R}^d) = \mathscr{H}^s_1(\mathbb{R}^d) = \mathscr{H}^s_3(\mathbb{R}^d)$ when $s \in \mathbb{R}$. When $s \in \mathbb{N}$, this space coincides with $\mathscr{H}^s_2(\mathbb{R}^d)$. Let us present some properties of the spaces we shall use.

**Proposition 1.1.19.** *If $s_1 \leq s_2$, then $\mathscr{H}^{s_2}(\mathbb{R}^d) \hookrightarrow \mathscr{H}^{s_1}(\mathbb{R}^d)$.*

**Proposition 1.1.20.** *If $s > d/2$, then $\mathscr{H}^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.*

**Proposition 1.1.21.** *Let $f \in C^\infty(\mathbb{R}), f(0) = 0, u \in \mathscr{H}^s(\mathbb{R}^d), s \in \mathbb{N}, s > d$. Then we have $f(u) \in \mathscr{H}^s(\mathbb{R}^d)$. Moreover if $f$ vanishes at order $p+1$ at 0, where $p \in \mathbb{N}$, then $||f(u)||_{\mathscr{H}^s} \leq C||u||^{p+1}_{\mathscr{H}^s}$.*

*Proof.* Proposition 1.1.19 and 1.1.20 follow respectively from the definition and Sobolev embedding. By the chain rule, for $|\alpha|+|\beta| \leq s$, $x^\alpha \partial^\beta f(u)$ may be written as the sum of terms of following form:

$$x^\alpha f^{(k)}(u)(\partial^{\beta_1} u)\dots(\partial^{\beta_k} u),$$

where $k \leq s, |\alpha| + \sum_{i=1}^k |\beta_i| \leq s, |\beta_i| > 0, i = 1, \dots, k$. Let $j_0$ be the index such that $|\beta_{j_0}|$ is the largest among $|\beta_1|, \dots, |\beta_k|$. Thus we must have $|\beta_i| \leq \frac{s}{2}$, $i \neq j_0$. By the assumption on $s$ and proposition 1.1.20, $\partial^\gamma u \in L^\infty(\mathbb{R}^d)$ if $|\gamma| \leq \frac{d}{2}$. We then estimate the factor $x^\alpha \partial^{\beta_{j_0}} u$ of the above quantities in $L^2$-norm and others in $L^\infty$-norm. Thus we have $f(u) \in \mathscr{H}^s(\mathbb{R}^d)$ by proposition 1.1.20. When $f$ vanishes at 0 at order $p+1$, by Taylor formula there is a smooth function $h$ such that $f(u) = u^{p+1}h(u)$. Then we argue as above to get an upper bound of $||f(u)||_{\mathscr{H}^s}$ by $C||u||^p_{\mathscr{H}^s}||u||_{\mathscr{H}^s}$. This concludes the proof. □

**Remark 1.1.3.** *Proposition 1.1.21 actually holds true for $s > d/2$ if we argue as the proof of corollary 6.4.4 in [17]. Since we will consider only in $\mathscr{H}^s(\mathbb{R}^d)$ for large $s$, the lower bound of $s$ is not important.*

## 1.2 Statement of main theorem

Let $d$ be an integer, $d \geq 1$ and $F : \mathbb{R} \to \mathbb{R}$ a real valued smooth function vanishing at order $\kappa + 1$ at 0, $\kappa \in \mathbb{N}^*$. Let $m \in \mathbb{R}^*_+$. we consider the solution $v$ of the following Cauchy problem:

$$\begin{cases} (\partial_t^2 - \Delta + |x|^2 + m^2)v = F(v) \quad \text{on } [-T, T] \times \mathbb{R}^d \\ v(0, x) = \epsilon v_0 \\ \partial_t v(0, x) = \epsilon v_1, \end{cases} \tag{1.2.1}$$

where $v_0 \in \mathscr{H}^{s+1}(\mathbb{R}^d), v_1 \in \mathscr{H}^s(\mathbb{R}^d)$, and $\epsilon > 0$ is a small parameter. By local existence theory one knows that if $s$ is large enough and $\epsilon \in (0, 1)$, equation (1.2.1) admits for any $(v_0, v_1)$ in the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}^d) \times \mathscr{H}^s(\mathbb{R}^d)$ a unique smooth solution defined on the interval $|t| \leq c\epsilon^{-\kappa}$, for some uniform positive constant c. Moreover, $||v(t, \cdot)||_{\mathscr{H}^{s+1}} + ||\partial_t v(t, \cdot)||_{\mathscr{H}^s}$ may be controlled by $C\epsilon$, for another uniform constant $C > 0$, on the interval of existence. The goal would be to obtain existence over an interval of longer length under convenient condition by controlling the Sobolev energy. Our main result is the following:

**Theorem 1.2.1.** *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}^*_+$ and for every $m \in \mathbb{R}^*_+ - \mathcal{N}$, there are $\epsilon_0 > 0, c > 0, s_0 \in \mathbb{N}$ such that for any $s \geq s_0, s \in \mathbb{N}, \epsilon \in (0, \epsilon_0)$, any pair $(v_0, v_1)$ of real valued functions belonging to the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}^d) \times \mathscr{H}^s(\mathbb{R}^d)$, problem (1.2.1) has a unique solution*

$$u \in C^0((-T_\epsilon, T_\epsilon), \mathscr{H}^{s+1}(\mathbb{R}^d)) \cap C^1((-T_\epsilon, T_\epsilon), \mathscr{H}^s(\mathbb{R}^d)), \tag{1.2.2}$$

*where $T_\epsilon$ has a lower bound $T_\epsilon \geq c\epsilon^{-\frac{4}{3}(1-\rho)\kappa}$ for any $\rho > 0$ if $d \geq 2$ and $T_\epsilon \geq c\epsilon^{-\frac{25}{18}(1-\rho)\kappa}$ for any $\rho > 0$ if $d = 1$. Moreover, the solution is uniformly bounded in $\mathscr{H}^{s+1}(\mathbb{R}^d)$ on $(-T_\epsilon, T_\epsilon)$ and $\partial_t u$ is uniformly bounded in $\mathscr{H}^s(\mathbb{R}^d)$ on the same interval.*

### 1.3 A property of spectral projectors on $\mathbb{R}^d$

As we have pointed out $P$ has eigenvalues given by $\lambda_n = \sqrt{2n+d}, n \in \mathbb{N}$. Remark that $\Pi_n$ is the orthogonal projector of $L^2(\mathbb{R}^d)$ onto the eigenspace associated to $\lambda_n^2$. Let us first introduce some notations. For $\xi_0, \xi_1, \dots, \xi_{p+1}$ $p+2$ nonnegative real numbers, let $\xi_{i_0}, \xi_{i_1}, \xi_{i_2}$ be respectively the largest, the second largest and the third largest elements among them and $\xi'$ the largest element among $\xi_1, \dots, \xi_p$, that is,

$$\begin{aligned} &\xi_{i_0} = \max\{\xi_0, \dots, \xi_{p+1}\}, \qquad \xi_{i_1} = \max(\{\xi_0, \dots, \xi_{p+1}\} - \{\xi_{i_0}\}), \\ &\xi_{i_2} = \max(\{\xi_1, \dots, \xi_{p+1}\} - \{\xi_{i_0}, \xi_{i_1}\}), \qquad \xi' = \max\{\xi_1, \dots, \xi_p\}. \end{aligned} \tag{1.3.1}$$

Denote

$$\mu(\xi_0, \dots, \xi_{p+1}) = (1 + \sqrt{\xi_{i_1}})(1 + \sqrt{\xi_{i_2}}). \tag{1.3.2}$$

Set also

$$S(\xi_0, \dots, \xi_{p+1}) = |\xi_{i_0} - \xi_{i_1}| + \mu(\xi_0, \dots, \xi_{p+1}). \tag{1.3.3}$$

The main result of this subsection is the following one:

**Theorem 1.3.1.** *There is a $\nu \in \mathbb{R}_+^*$, depending only on $p$ ($p \in \mathbb{N}^*$) and dimension $d$, and for any $N \in \mathbb{N}$, there is a $C_N > 0$ such that for any $n_0, \dots, n_{p+1} \in \mathbb{N}$, any $u_0, \dots, u_{p+1} \in L^2(\mathbb{R}^d)$,*

$$|\int \Pi_{n_0} u_0 \dots \Pi_{n_{p+1}} u_{p+1} dx| \leq C_N (1 + \sqrt{n_{i_2}})^\nu \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_{j=0}^{p+1} ||u_j||_{L^2}. \tag{1.3.4}$$

*Furthermore if $d = 1$, we may find for any $\varsigma \in (0,1)$*

$$|\int \Pi_{n_0} u_0 \dots \Pi_{n_{p+1}} u_{p+1} dx| \leq C_N \frac{(1 + \sqrt{n_{i_2}})^\nu}{(1 + \sqrt{n_{i_0}})^{\frac{1}{6}(1-\varsigma)}} \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_{j=0}^{p+1} ||u_j||_{L^2}. \tag{1.3.5}$$

*Proof.* By the symmetries we may assume $n_0 \geq n_1 \geq \dots \geq n_{p+1}$. Then recalling the definition of $\lambda_n$ in (1.1.1), we only need to show under the condition of theorem 1.3.1

$$|\int \Pi_{n_0} u_0 \dots \Pi_{n_{p+1}} u_{p+1} dx| \leq C_N \lambda_{n_2}^\nu \frac{(\lambda_{n_1} \lambda_{n_2})^N}{(|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1} \lambda_{n_2})^N} \prod_{j=0}^{p+1} ||u_j||_{L^2} \tag{1.3.6}$$

and when $d = 1$

$$|\int \Pi_{n_0} u_0 \dots \Pi_{n_{p+1}} u_{p+1} dx| \leq C_N \frac{\lambda_{n_2}^\nu}{\lambda_{n_0}^{\frac{1}{6}(1-\varsigma)}} \frac{(\lambda_{n_1} \lambda_{n_2})^N}{(|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1} \lambda_{n_2})^N} \prod_{j=0}^{p+1} ||u_j||_{L^2} \tag{1.3.7}$$

for any $\varsigma \in (0,1)$. We follow the proof of proposition 3.6 in [14]. Let $A$ be a linear operator which maps $D(P^{2k})$ into itself. We define a sequence of operators

$$A_N = [P^2, A_{N-1}]; \qquad A_0 = A. \tag{1.3.8}$$

Then using integration by parts we have

$$(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \langle A\ \Pi_{n_1} u_1, \Pi_{n_0} u_0 \rangle = \langle A_N \Pi_{n_1} u_1, \Pi_{n_0} u_0 \rangle. \tag{1.3.9}$$

Now we set $A$ to be the multiplication operator generated by the function

$$a(x) = (\Pi_{n_2} u_2) \dots (\Pi_{n_{p+1}} u_{p+1}).$$

Then an induction argument shows

$$A_N = \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} C_{\alpha\beta\gamma} (\partial^\alpha a) x^\beta \partial^\gamma \tag{1.3.10}$$

for constants $C_{\alpha\beta\gamma}$. Therefore we compute for some $\nu' > \frac{d}{2}$

$$\begin{aligned} &|(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \int (\Pi_{n_0} u_0) \dots (\Pi_{n_{p+1}} u_{p+1}) dx| \\ &\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} ||(\partial^\alpha a) x^\beta \partial^\gamma \Pi_{n_1} u_1||_{L^2} ||\Pi_{n_0} u_0||_{L^2} \\ &\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} ||a||_{\mathscr{H}^{\nu'+|\alpha|}} ||\Pi_{n_1} u_1||_{\mathscr{H}^{|\beta|+|\gamma|}} ||\Pi_{n_0} u_0||_{L^2}, \end{aligned} \tag{1.3.11}$$

where in the last estimate we used definition 1.1.2 and proposition 1.1.20. Remark that by definition 1.1.1, one has for any $s \geq 0$

$$||\Pi_n u||_{\mathscr{H}^s} \leq C \lambda_n^s ||\Pi_n u||_{L^2}. \tag{1.3.12}$$

This estimate together with the proof of proposition 1.1.21 gives for $n_2 \geq n_3 \cdots \geq n_{p+1}$

$$||a||_{\mathscr{H}^{\nu'+|\alpha|}} \leq C \lambda_{n_2}^{\nu+|\alpha|} \prod_{j=2}^{p+1} ||\Pi_{n_j} u_j||_{L^2} \tag{1.3.13}$$

for some $\nu > 0$ depending only on $p$ and dimension $d$. Thus we have

$$\begin{aligned} &|(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \int (\Pi_{n_0} u_0) \dots (\Pi_{n_{p+1}} u_{p+1}) dx| \\ &\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} \lambda_{n_2}^{\nu+|\alpha|} \lambda_{n_1}^{|\beta|+|\gamma|} \prod_{j=0}^{p+1} ||\Pi_{n_j} u_j||_{L^2} \\ &\leq C \sum_{|\alpha|\leq N} \lambda_{n_2}^{\nu+2N-|\alpha|} \lambda_{n_1}^{|\alpha|} \prod_{j=0}^{p+1} ||\Pi_{n_j} u_j||_{L^2} \\ &\leq C \lambda_{n_2}^{\nu+2N} (\frac{\lambda_{n_1}}{\lambda_{n_2}})^N \prod_{j=0}^{p+1} ||\Pi_{n_j} u_j||_{L^2} \\ &\leq C \lambda_{n_2}^{\nu} (\lambda_{n_1} \lambda_{n_2})^N \prod_{j=0}^{p+1} ||\Pi_{n_j} u_j||_{L^2}. \end{aligned} \tag{1.3.14}$$

Now if $\lambda_{n_1}\lambda_{n_2} \leq |\lambda_{n_0}^2 - \lambda_{n_1}^2|$, then the last estimate implies (1.3.6), while if $\lambda_{n_1}\lambda_{n_2} > |\lambda_{n_0}^2 - \lambda_{n_1}^2|$, then $\frac{\lambda_{n_1}\lambda_{n_2}}{|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1}\lambda_{n_2}} \geq \frac{1}{2}$ and thus (1.3.6) is trivially true.

On the other hand, we use the property of the eigenfunctions (see[19]), which in dimension $d = 1$ says that if $\phi_n$ is the eigenfunction associated to $\lambda_n^2$, then one has $||\phi_n||_{L^\infty} \leq C \lambda_n^{-\frac{1}{6}}$. Therefore we have

$$||\Pi_n u||_{L^\infty} \leq C \lambda_n^{-\frac{1}{6}} ||\Pi_n u||_{L^2} \tag{1.3.15}$$

since in this case the eigenvalues are simple. This estimate gives us

$$|\int \Pi_{n_0} u_0 \dots \Pi_{n_{p+1}} u_{p+1} dx| \leq C\lambda_{n_0}^{-\frac{1}{6}} \prod_{j=0}^{p+1} ||\Pi_{n_j} u_j||_{L^2}. \tag{1.3.16}$$

Combining (1.3.16) with (1.3.6) one gets (1.3.7) for all $N \geq 1$ and some $\nu > 0$ in the case $d = 1$. This concludes the proof. □

# 2 Long time existence

## 2.1 Definition and properties of multilinear operators

Denote by $\mathcal{E}$ the algebraic direct sum of the ranges of the $\Pi_n{}'s, n \in \mathbb{N}$. With notations (1.3.1), (1.3.2) and (1.3.3) we give the following definition.

**Definition 2.1.1.** *Let $\nu \in \mathbb{R}_+,\ \tau \in \mathbb{R},\ p \in \mathbb{N}^*$. We denote by $\mathcal{M}_{p+1}^{\nu,\tau}$ the space of all $p+1$-linear operators $(u_1, \dots, u_{p+1}) \to M(u_1, \dots, u_{p+1})$, defined on $\mathcal{E} \times \dots \times \mathcal{E}$ with values in $L^2(\mathbb{R}^d)$ such that*

- *For every $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}, u_1, \dots, u_{p+1} \in \mathcal{E}$*

$$\Pi_{n_0}[M(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1})] = 0, \tag{2.1.1}$$

  *if $|n_0 - n_{p+1}| > \frac{1}{2}(n_0 + n_{p+1})$ or $n' \stackrel{def}{=} \max\{n_1, \dots, n_p\} > n_{p+1}$.*

- *For any $N \in \mathbb{N}$, there is a $C > 0$ such that for every $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$, $u_1, \dots, u_{p+1} \in \mathcal{E}$, one has*

$$\begin{aligned} &||\Pi_{n_0}[M(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1})]||_{L^2} \\ &\qquad \leq C(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^\tau (1 + \sqrt{n'})^\nu \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_{j=1}^{p+1} ||u_j||_{L^2}. \end{aligned} \tag{2.1.2}$$

*The best constant in the preceding inequality will be denoted by $||M||_{\mathcal{M}_{p+1,N}^{\nu,\tau}}$.*

We may extend the operators in $\mathcal{M}_{p+1}^{\nu,\tau}$ to Sobolev spaces.

**Proposition 2.1.2.** *Let $\nu \in \mathbb{R}_+,\ \tau \in \mathbb{R},\ p \in \mathbb{N}^*,\ s \in \mathbb{N},\ s > \nu + 3$. Then any element $M \in \mathcal{M}_{p+1}^{\nu,\tau}$ extends as a bounded operator from $\mathscr{H}^s(\mathbb{R}^d) \times \dots \times \mathscr{H}^s(\mathbb{R}^d)$ to $\mathscr{H}^{s-\tau-1}(\mathbb{R}^d)$. Moreover, for any $s_0 \in (\nu + 3, s]$, there is $C > 0$ such that for any $M \in \mathcal{M}_{p+1}^{\nu,\tau}$, and any $u_1, \dots, u_{p+1} \in \mathscr{H}^s(\mathbb{R}^d)$,*

$$||M(u_1, \dots, u_{p+1})||_{\mathscr{H}^{s-\tau-1}} \leq C||M||_{\mathcal{M}_{p+1,N}^{\nu,\tau}} \sum_{j=1}^{p+1} \Big[ ||u_j||_{\mathscr{H}^s} \prod_{k \neq j} ||u_k||_{\mathscr{H}^{s_0}} \Big]. \tag{2.1.3}$$

*Proof.* The proof is a modification of proposition 4.4 in [10]. There is one derivative lost compared to that case. We give it for the convenience of the reader. Using definition 1.1.1 we write

$$\begin{aligned} &||M(u_1, \dots, u_{p+1})||^2_{\mathscr{H}^{s-\tau-1}} \\ &\qquad \leq C \sum_{n_0} || \sum_{n_1} \dots \sum_{n_{p+1}} \Pi_{n_0} M(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1})||^2_{L^2} (1 + \sqrt{n_0})^{2s-2\tau-2} \end{aligned} \tag{2.1.4}$$

Because of (2.1.1) and using the symmetries we may assume

$$n_0 \sim n_{p+1} \qquad \text{and} \qquad n_1 \le \cdots \le n_p \le n_{p+1} \le Cn_0 \tag{2.1.5}$$

when estimating the above quantity. Consequently, we have

$$\begin{aligned} \mu(n_0,\ldots,n_{p+1}) &\sim (1+\sqrt{n_p})(1+\sqrt{n_{p+1}}),\\ S(n_0,\ldots,n_{p+1}) &\sim |n_0-n_{p+1}| + \mu(n_0,\ldots,n_{p+1}). \end{aligned} \tag{2.1.6}$$

By (2.1.2) the square root of the general term over $n_0$ sum in (2.1.4) is smaller than

$$C\sum_{n_1\le\cdots\le n_{p+1}} (1+\sqrt{n_0})^{s-1}\frac{(1+\sqrt{n_p})^{\nu}\mu(n_0,\ldots,n_{p+1})^N}{S(n_0,\ldots,n_{p+1})^N}\prod_1^{p+1}||\Pi_{n_j}u_j||_{L^2}. \tag{2.1.7}$$

We have by (2.1.5) and (2.1.6)

$$\frac{\mu(n_0,\ldots,n_{p+1})}{S(n_0,\ldots,n_{p+1})} \sim \frac{1+\sqrt{n_p}}{|\sqrt{n_0}-\sqrt{n_{p+1}}|+1+\sqrt{n_p}}. \tag{2.1.8}$$

The following fact will be useful in this section: For $q\in\mathbb{N}, A\ge 1$ and $N>1$, there is a $C>0$ independent of $q$ and $A$ such that

$$\sum_{n\in\mathbb{N}}\frac{1}{(|\sqrt{n}-\sqrt{q}|+A)^N} \le C\frac{1+\sqrt{q}}{A^{N-2}}. \tag{2.1.9}$$

Let $\iota>2$ be a constant as close to 2 as wanted. Using (2.1.8) and (2.1.9) we deduce

$$\begin{aligned} \sum_{n_0}\frac{\mu(n_0,\ldots,n_{p+1})^{\iota}}{S(n_0,\ldots,n_{p+1})^{\iota}} &\le C(1+\sqrt{n_{p+1}})(1+\sqrt{n_p})^2,\\ \sum_{n_{p+1}}\frac{\mu(n_0,\ldots,n_{p+1})^{\iota}}{S(n_0,\ldots,n_{p+1})^{\iota}} &\le C(1+\sqrt{n_0})(1+\sqrt{n_p})^2. \end{aligned} \tag{2.1.10}$$

We estimate the sum over $n_1\le\cdots\le n_{p+1}$ in (2.1.7) by

$$\begin{aligned} &C\Big(\sum_{n_1\le\cdots\le n_{p+1}}\frac{(1+\sqrt{n_p})^{\nu}\mu^{\iota}}{S^{\iota}}\prod_{j=1}^{p}||\Pi_{n_j}u_j||_{L^2}\Big)^{1/2}\\ &\times\Big(\sum_{n_1\le\cdots\le n_{p+1}}(1+\sqrt{n_0})^{2s-2}(1+\sqrt{n_p})^{\nu}\frac{\mu^{2N-\iota}}{S^{2N-\iota}}\prod_{j=1}^{p}||\Pi_{n_j}u_j||_{L^2}||\Pi_{n_{p+1}}u_{p+1}||_{L^2}^2\Big)^{1/2}. \end{aligned} \tag{2.1.11}$$

Using (2.1.10) to handle $n_{p+1}$ sum, we bound the first factor in (2.1.11) from above by $C(1+\sqrt{n_0})^{\frac12}\Pi_{j=1}^p||u_j||_{\mathscr{H}^{s_0}}^{\frac12}$ if $s_0>\nu+3$ using definition 1.1.1. Incorporating $(1+\sqrt{n_0})^{\frac12}$ into the second factor, we have to bound the quantity

$$\Big(\sum_{n_1\le\cdots\le n_{p+1}}(1+\sqrt{n_0})^{2s-1}(1+\sqrt{n_p})^{\nu}\frac{\mu^{2N-\iota}}{S^{2N-\iota}}\prod_{j=1}^{p}||\Pi_{n_j}u_j||_{L^2}||\Pi_{n_{p+1}}u_{p+1}||_{L^2}^2\Big)^{1/2}. \tag{2.1.12}$$

By (2.1.5) and $\mu\le S$ we have

$$(1+\sqrt{n_0})^{2s-1}(\frac{\mu}{S})^{2N-\iota} \le C(1+\sqrt{n_{p+1}})^{2s-1}(\frac{\mu}{S})^{\iota} \tag{2.1.13}$$

if $N > \iota$. Plugging in (2.1.12), (2.1.11) and then (2.1.4) we bound from above the $n_0$ sum in (2.1.4) by

(2.1.14)
$$C\prod_1^p ||u_j||_{\mathscr{H}^{s_0}} \sum_{n_1\leq\cdots\leq n_{p+1}\leq Cn_0} (1+\sqrt{n_{p+1}})^{2s-1}(1+\sqrt{n_p})^{\nu}(\frac{\mu}{S})^{\iota}\prod_{j=1}^p ||\Pi_{n_j}u_j||_{L^2}||\Pi_{n_{p+1}}u_{p+1}||_{L^2}^2.$$

Changing the order of sums for $n_0$ and $n_{p+1}$, we then use (2.1.10) to handle $n_0$ sum and get a control of (2.1.14) by $C\prod_{j=1}^p ||u_j||_{\mathscr{H}^{s_0}}^2||u_{p+1}||_{\mathscr{H}^s}^2$ according to definition 1.1.1 if $s > \nu + 3$. This concludes the proof. □

Let us define convenient subspaces of the spaces of definition 2.1.1.

**Definition 2.1.3.** *Let $\nu \in \mathbb{R}_+,\ \tau \in \mathbb{R},\ p \in \mathbb{N}^*, \omega : \{0,\dots,p+1\} \to \{-1,1\}$ be given.*

- *If $\sum_{j=0}^{p+1}\omega(j) \neq 0$, we set $\widetilde{\mathcal{M}}_{p+1}^{\nu,\tau}(\omega) = \mathcal{M}_{p+1}^{\nu,\tau}$;*
- *If $\sum_{j=0}^{p+1}\omega(j) = 0$, we denote by $\widetilde{\mathcal{M}}_{p+1}^{\nu,\tau}(\omega)$ the closed subspace of $\mathcal{M}_{p+1}^{\nu,\tau}$ given by those $M \in \mathcal{M}_{p+1}^{\nu,\tau}$ such that*

  (2.1.15) $$\Pi_{n_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) \equiv 0$$

  *for any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection $\sigma$ from $\{j; 0 \leq j \leq p+1, \omega(j) = -1\}$ to $\{j; 0 \leq j \leq p+1, \omega(j) = 1\}$ so that for any $j$ in the first set $n_{\sigma(j)} = n_j$.*

We shall have to use also classes of remainder operators. If $n_1,\dots,n_{p+1} \in \mathbb{N}$ and $j_0 \in \{1,\dots,p+1\}$ is such that $n_{j_0} = \max\{n_1,\dots,n_{p+1}\}$, we denote

(2.1.16) $$\max\nolimits_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}}) = 1 + \max\{\sqrt{n_j};\ 1 \leq j \leq p+1, j \neq j_0\}.$$

**Definition 2.1.4.** *Let $\nu \in \mathbb{R}_+,\ \tau \in \mathbb{R},\ p \in \mathbb{N}^*$. We denote by $\mathcal{R}_{p+1}^{\nu,\tau}$ the space of $\mathbb{C}$ $(p+1)$-linear maps from $\mathcal{E}\times\cdots\times\mathcal{E} \to L^2(\mathbb{R}^d), (u_1,\dots,u_{p+1}) \to R(u_1,\dots,u_{p+1})$ such that for any $N \in \mathbb{N}$, there is a $C > 0$ such that for any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$, any $u_1,\dots,u_{p+1} \in \mathcal{E}$,*

(2.1.17) $$||\Pi_{n_0}R(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})||_{L^2} \leq C(1+\sqrt{n_0})^{\tau}\frac{\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}})^{\nu+N}}{(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^N}\prod_{j=1}^{p+1}||u_j||_{L^2}.$$

The elements in $\mathcal{R}_{p+1}^{\nu,\tau}$ also extend as bounded operators on Sobolev spaces.

**Proposition 2.1.5.** *Let $\nu \in \mathbb{R}_+,\ \tau \in \mathbb{R},\ p \in \mathbb{N}^*$ be given. There is $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, any $R \in \mathcal{R}_{p+1}^{\nu,\tau}$, $(u_1,\dots,u_{p+1}) \to R(u_1,\dots,u_{p+1})$ extends as a bounded map from $\mathscr{H}^s(\mathbb{R}^d)\times\cdots\times\mathscr{H}^s(\mathbb{R}^d) \to \mathscr{H}^{2s-\nu-\tau-7}(\mathbb{R}^d)$. Moreover one has*

(2.1.18) $$||R(u_1,\dots,u_{p+1})||_{\mathscr{H}^{2s-\nu-\tau-7}} \leq C\sum_{1\leq j_1<j_2\leq p+1}\Big[||u_{j_1}||_{\mathscr{H}^s}||u_{j_2}||_{\mathscr{H}^s}\prod_{k\neq j_1,k\neq j_2}||u_k||_{\mathscr{H}^{s_0}}\Big].$$

*Proof.* We may assume $\tau = 0$. By definition 1.1.1 we have to bound $||\Pi_{n_0}R(u_1,\dots,u_{p+1})||_{L^2}$ from above by $(1+\sqrt{n_0})^{-2s+\nu+7}c_{n_0}$ for a sequence $(c_{n_0})_{n_0}$ in $\ell^2$. To do that we decompose $u_j$ as $\sum_{n_j}\Pi_{n_j}u_j$ and use (2.1.17). By symmetry we limit ourselves to summation over

(2.1.19) $$n_1 \leq \cdots \leq n_{p+1},$$

from which we deduce

$$\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}}) = 1+\sqrt{n_p}. \tag{2.1.20}$$

Therefore we are done if we can bound from above

$$C\sum_{n_1\le\cdots\le n_{p+1}} \frac{(1+\sqrt{n_p})^{\nu+N}}{(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^N}\prod_{j=1}^{p-1}(1+\sqrt{n_j})^{-s_0}(1+\sqrt{n_p})^{-s}(1+\sqrt{n_{p+1}})^{-s} \tag{2.1.21}$$

by $(1+\sqrt{n_0})^{-2s+\nu+7}c_{n_0}$ for $s_0, s$ large enough with $s\ge s_0$ since $||\Pi_{n_j}u_j||_{L^2}\le C(1+\sqrt{n_j})^{-s}||u_j||_{\mathscr{H}^s}$. Using (2.1.19) we get an upper bound of (2.1.21) by

$$C\sum_{n_1\le\cdots\le n_{p+1}} \frac{(1+\sqrt{n_p})^{\nu+N-2s}}{(1+\sqrt{n_0}+\sqrt{n_{p+1}})^N}\prod_{j=1}^{p-1}(1+\sqrt{n_j})^{-s_0} \tag{2.1.22}$$

Using the fact $\sum_{n\in\mathbb{N}}\frac{1}{(\sqrt{n}+A)^N}\le\frac{C}{A^{N-2}}$ for $N>2$ and $A\ge 1$ , we take the sum over $n_{p+1}$ to get an upper bound of (2.1.21) by

$$C\sum_{n_1\le\cdots\le n_p} \frac{(1+\sqrt{n_p})^{\nu+N-2s}}{(1+\sqrt{n_0})^{N-2}}\prod_{j=1}^{p-1}(1+\sqrt{n_j})^{-s_0} \tag{2.1.23}$$

if $N>2$. Now take $N=2s-\nu-\frac{5}{2}$ and sum over $n_1,\dots,n_p$. This gives the upper bound we want and thus concludes the proof. □

**Definition 2.1.6.** *Let $\nu\in\mathbb{R}_+,\ \tau\in\mathbb{R},\ p\in\mathbb{N}^*, \omega:\{0,\dots,p+1\}\to\{-1,1\}$ be given.*

- *If $\sum_{j=0}^{p+1}\omega(j)\neq 0$, we set $\widetilde{\mathcal{R}}^{\nu,\tau}_{p+1}(\omega)=\mathcal{R}^{\nu,\tau}_{p+1}$;*
- *If $\sum_{j=0}^{p+1}\omega(j)=0$, we denote by $\widetilde{\mathcal{R}}^{\nu,\tau}_{p+1}(\omega)$ the closed subspace of $\mathcal{R}^{\nu,\tau}_{p+1}$ given by those $R\in\mathcal{M}^{\nu,\tau}_{p+1}$ such that*

  $$\Pi_{n_0}R(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\equiv 0 \tag{2.1.24}$$

  *for any $(n_0,\dots,n_{p+1})\in\mathbb{N}^{p+2}$ such that there is a bijection $\sigma$ from $\{j;0\le j\le p+1,\omega(j)=-1\}$ to $\{j;0\le j\le p+1,\omega(j)=1\}$ so that for any $j$ in the first set $n_{\sigma(j)}=n_j$.*

## 2.2 Rewriting of the equation and the energy

In this subsection we will write the time derivative of the energy in terms of multilinear operators defined in the previous subsection. To do that, we shall need to analyze the nonlinearity. Decompose

$$-F(v) = -\sum_{p=\kappa}^{2\kappa-1}\frac{\partial_v^{p+1}F(0)}{(p+1)!}v^{p+1}+G(v) \tag{2.2.1}$$

where $G(v)$ vanishes at order $2\kappa+1$ at $v=0$. One has

$$cv^{p+1} = c\sum_{n_1}\cdots\sum_{n_{p+1}}(\Pi_{n_1}v)\dots(\Pi_{n_{p+1}}v)$$

for a real constant $c$. One may also write this as $A_p(v)\cdot v$ where $A_p(v)$ is an operator of form

$$A_p(v)\cdot w=\sum_{n_1}\cdots\sum_{n_{p+1}}B(n_1,\dots,n_{p+1})(\Pi_{n_1}v)\dots(\Pi_{n_p}v)(\Pi_{n_{p+1}}w), \tag{2.2.2}$$

where $B(n_1,\dots,n_{p+1})$ is a real valued bounded function supported on $\max\{n_1,\dots,n_p\}\leq n_{p+1}$ and $B$ is constant valued on the domain $\max\{n_1,\dots,n_p\}<n_{p+1}$. For instance, when $p=2$, one may write

$$\{(n_1,n_2,n_3);n_j\in\mathbb{N}\}=\{\max\{n_1,n_2\}\leq n_3\}\cup\{n_1\geq n_2\text{ and }n_1>n_3\}\cup\{n_1<n_2\text{ and }n_2>n_3\}$$

and

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}(\Pi_{n_1}v)(\Pi_{n_2}v)(\Pi_{n_3}v)=\sum\mathbf{1}_{\{\max\{n_1,n_2\}\leq n_3\}}(\Pi_{n_1}v)(\Pi_{n_2}v)(\Pi_{n_3}v)$$
$$+\sum\mathbf{1}_{\{n_3\geq n_2\text{ and }n_3>n_1\}}(\Pi_{n_1}v)(\Pi_{n_2}v)(\Pi_{n_3}v)+\sum\mathbf{1}_{\{n_3>n_2\text{ and }n_3>n_1\}}(\Pi_{n_1}v)(\Pi_{n_2}v)(\Pi_{n_3}v)$$

using the symmetries, so that in this case

$$B(n_1,n_2,n_3)=c(\mathbf{1}_{\{\max\{n_1,n_2\}\leq n_3\}}+\mathbf{1}_{\{n_3\geq n_2\text{ and }n_3>n_1\}}+\mathbf{1}_{\{n_3>n_2\text{ and }n_3>n_1\}}).$$

So if we make a change of unknown $u=(D_t+\Lambda_m)v$ with

$$D_t=-i\partial_t,\qquad \Lambda_m=\sqrt{-\Delta+|x|^2+m^2},$$

we may write using (2.2.1)

$$(D_t-\Lambda_m)u=-\sum_{p=\kappa}^{2\kappa-1}A_p\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}(\frac{u+\bar u}{2})+G\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big). \tag{2.2.3}$$

Denote $C(u,\bar u)=-\frac12\sum_{p=\kappa}^{2\kappa-1}A_p\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}$ so that

$$(D_t-\Lambda_m)u=C(u,\bar u)u+C(u,\bar u)\bar u+G\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big). \tag{2.2.4}$$

We have to estimate for the solution $u$ of (2.2.3)

$$\Theta_s(u(t,\cdot))=\frac12\langle\Lambda_m^s u(t,\cdot),\Lambda_m^s u(t,\cdot)\rangle. \tag{2.2.5}$$

Now comes the main result of this subsection:

**Proposition 2.2.1.** *There are $\nu\in\mathbb{R}_+$ and large enough $s_0$ such that for any natural number $s\geq s_0$, there are:*

- *Multilinear operators $M_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$, $\kappa\leq p\leq 2\kappa-1$, $0\leq\ell\leq p$ with $\omega_\ell$ defined by $\omega_\ell(j)=-1$, $j=0,\dots,\ell$, $\omega_\ell(j)=1, j=\ell+1,\dots,p+1$ and $a=2$ if $d\geq 2$ and $a=\frac{13}{6}-\varsigma$ for any $\varsigma\in(0,1)$ if $d=1$;*
- *Multilinear operators $\widetilde{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\widetilde\omega_\ell)$, $\kappa\leq p\leq 2\kappa-1$, $0\leq\ell\leq p$ with $\widetilde\omega_\ell$ defined by $\widetilde\omega_\ell(j)=-1$, $j=0,\dots,\ell,p+1$, $\widetilde\omega_\ell(j)=1, j=\ell+1,\dots,p$;*

- *Multilinear operators* $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell), \widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$, $\kappa \leq p \leq 2\kappa-1$, $0 \leq \ell \leq p$;
- *A map* $u \to T(u)$ *defined on* $\mathscr{H}^s(\mathbb{R}^d)$ *with values in* $\mathbb{R}$, *satisfying when* $||u||_{\mathscr{H}^s} \leq 1$, $|T(u)| \leq C||u||_{\mathscr{H}^s}^{2\kappa+2}$

*such that*

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle M_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle \widetilde{M}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar{u}),u\rangle + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle R_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle \widetilde{R}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar{u}),u\rangle + T(u).\end{aligned} \tag{2.2.6}$$

*Proof.* We compute according to (2.2.4)

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= Re\ i\langle \Lambda_m^s D_t u, \Lambda_m^s u\rangle \\ &= Re\ i\langle \Lambda_m^s C(u,\bar{u})u, \Lambda_m^s u\rangle + Re\ i\langle \Lambda_m^s C(u,\bar{u})\bar{u}, \Lambda_m^s u\rangle + Re\ i\langle \Lambda_m^s G(\Lambda_m^{-1}(\frac{u+\bar{u}}{2})), \Lambda_m^s u\rangle.\end{aligned} \tag{2.2.7}$$

The last term in the right hand side of (2.2.7) contributes to the last term in (2.2.6) by proposition 1.1.21. Let us treat the other two terms in the right hand side of (2.2.7).

**Lemma 2.2.2.** *There are* $M_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$, $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, $\kappa \leq p \leq 2\kappa-1$, $0 \leq \ell \leq p$ *with* $\omega_\ell$ *defined by* $\omega_\ell(j) = -1$, $j = 0,\dots,\ell$, $\omega_\ell(j) = 1, j = \ell+1,\dots,p+1$ *and* $a = 2$ *if* $d \geq 2$ *and* $a = \frac{13}{6} - \varsigma$ *for any* $\varsigma \in (0,1)$ *if* $d = 1$, *such that*

$$\begin{aligned}Re\ i\langle \Lambda_m^s C(u,\bar{u})u, \Lambda_m^s u\rangle &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle M_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle R_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle.\end{aligned} \tag{2.2.8}$$

*Proof of Lemma 2.2.2:* Let $\chi$ be a cut-off function near 0 with small support and $\lambda_n$ defined in (1.1.1). We may decompose the operator $A_p(v)$ defined in (2.2.2) as

$$A_p(v) = A_p^1(v) + A_p^2(v) + A_p^3(v), \tag{2.2.9}$$

where $A_p^j(v)(j = 1,2,3)$ are operators of form

$$\begin{aligned}A_p^1(v)\cdot w &= \sum_{n_0}\cdots\sum_{n_{p+1}} B_1(n_0,\dots,n_{p+1})\Pi_{n_0}[(\Pi_{n_1}v)\dots(\Pi_{n_p}v)(\Pi_{n_{p+1}}w)], \\ A_p^2(v)\cdot w &= \sum_{n_0}\cdots\sum_{n_{p+1}} B_2(n_0,\dots,n_{p+1})\Pi_{n_0}[(\Pi_{n_1}v)\dots(\Pi_{n_p}v)(\Pi_{n_{p+1}}w)], \\ A_p^3(v)\cdot w &= \sum_{n_1}\cdots\sum_{n_{p+1}} B_3(n_1,\dots,n_{p+1})\Pi_{n_0}[(\Pi_{n_1}v)\dots(\Pi_{n_p}v)(\Pi_{n_{p+1}}w)],\end{aligned} \tag{2.2.10}$$

with

$$
\begin{aligned}
B_1(n_0,\dots,n_{p+1}) &= B(n_1,\dots,n_{p+1})\chi\Big(\frac{|\lambda_{n_0}^2-\lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2+\lambda_{n_{p+1}}^2}\Big)\mathbf{1}_{\{\max\{n_1,\dots,n_p\}<\delta n_{p+1}\}},\\
B_2(n_0,\dots,n_{p+1}) &= B(n_1,\dots,n_{p+1})\Big(1-\chi\Big(\frac{|\lambda_{n_0}^2-\lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2+\lambda_{n_{p+1}}^2}\Big)\Big)\mathbf{1}_{\{\max\{n_1,\dots,n_p\}<\delta n_{p+1}\}},\\
B_3(n_1,\dots,n_{p+1}) &= B(n_1,\dots,n_{p+1})\mathbf{1}_{\{\max\{n_1,\dots,n_p\}\geq\delta n_{p+1}\}},
\end{aligned}
\tag{2.2.11}
$$

with some small $\delta>0$. Therefore for the operator $C(u,\bar u)$ defined above (2.2.4), we have

$$
C(u,\bar u)=-\frac{1}{2}\sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1}A_p^j\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}.
\tag{2.2.12}
$$

So the left hand side of (2.2.8) may be written as

$$
-\frac{1}{2}\sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1}Re\ i\langle\Lambda_m^{2s}A_p^j\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}u,u\rangle:=\sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1}I_p^j.
\tag{2.2.13}
$$

Let us treat these quantities term by term.

**(i)** The term $I_p^1$.

Note that $-4I_p^1$ equals to

$$
Re\ i\langle\Big[\Lambda_m^{2s}A_p^1\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}-\Big(A_p^1\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}\Big)^*\Lambda_m^{2s}\Big]u,u\rangle,
\tag{2.2.14}
$$

which may be written as

$$
\begin{aligned}
&Re\ i\langle\Big[\Lambda_m^{2s},A_p^1\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}\Big]u,u\rangle\\
&+Re\ i\langle\Big[A_p^1\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}-\Big(A_p^1\Big(\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Lambda_m^{-1}\Big)^*\Big]\Lambda_m^{2s}u,u\rangle:=I+II
\end{aligned}
\tag{2.2.15}
$$

We expand the first term in (2.2.15) using (2.2.10) to get

$$
\begin{aligned}
I&=Re\ i\langle\sum_{n\in\mathbb{N}^{p+2}}\pi_1\Pi_{n_0}\Big[\Big(\Pi_{n_1}\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\dots\Big(\Pi_{n_p}\Lambda_m^{-1}(\frac{u+\bar u}{2})\Big)\Big(\Pi_{n_{p+1}}\Lambda_m^{-1}u\Big)\Big],u\rangle\\
&=Re\ i\langle\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_2\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar u)\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar u)(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)],u\rangle\\
&=Re\ i\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_2\int(\Pi_{n_0}\bar u)(\Pi_{n_1}\Lambda_m^{-1}\bar u)\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar u)(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)dx,
\end{aligned}
\tag{2.2.16}
$$

where we have used notations

$$
\begin{aligned}
n&=(n_0,\dots,n_{p+1}),\\
\pi_1&=B_1(n_0,\dots,n_{p+1})[(m^2+\lambda_{n_0}^2)^s-(m^2+\lambda_{n_{p+1}}^2)^s],
\end{aligned}
\tag{2.2.17}
$$

$$\pi_2 = \frac{1}{2^p}\binom{p}{\ell} B_1(n_0,\dots,n_{p+1})[(m^2+\lambda_{n_0}^2)^s - (m^2+\lambda_{n_{p+1}}^2)^s].$$

Let $\omega_\ell$ be defined in the statement of the lemma and set

$$\begin{aligned} S_p^\ell = \{(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2};\ \text{there exists a bijection } \sigma \text{ from} \\ \{j; 0\le j\le p+1, \omega_\ell(j) = -1\} \text{ to } \{j; 0\le j\le p+1, \omega_\ell(j)=1\} \\ \text{such that for each } j \text{ in the first set } n_j = n_{\sigma(j)}\}. \end{aligned} \tag{2.2.18}$$

Now we look at the integral in the last line of (2.2.16). If $n \in S_p^\ell$ with $S_p^\ell \neq \emptyset$, there is a bijection $\sigma$ from $\{0,\dots,\ell\}$ to $\{\ell,\dots,p+1\}$ such that $n_j = n_{\sigma(j)}, j = 0,\dots,\ell$. So we may couple $\Pi_{n_j}\bar{u}, j = 0,\dots,\ell$ with $\Pi_{n_{\sigma(j)}}u, j = 0,\dots,\ell$. Since $\pi_2$ is real, we get zero if we take the sum over $n \in S_p^\ell$ when computing the right hand side of (2.2.16). Therefore we may assume $n \notin S_p^\ell$ when computing $I$. Now we define

$$M_\ell^{p,1}(u_1,\dots,u_{p+1}) = -\frac{1}{4}\sum_{n\notin S_p^\ell} \pi_2 \Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})]. \tag{2.2.19}$$

It follows from the second equality in (2.2.16) that

$$I = -4\sum_{\ell=0}^{p} Re\ i\langle M_\ell^{p,1}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}), u\rangle. \tag{2.2.20}$$

Let us turn to the term $II$ in (2.2.15). Note that $A_p^1(v)^*$ is an operator of form

$$A_p^1(v)^*\cdot w = \sum_{n\in\mathbb{N}^{p+2}} B_1(n_{p+1},n_1,\dots,n_p,n_0)\Pi_{n_0}[(\Pi_{n_1}v)\dots(\Pi_{n_p}v)(\Pi_{n_{p+1}}w)]. \tag{2.2.21}$$

Thus we may compute using (2.2.10)

(2.2.22)

$$\begin{aligned} II &= Re\ i\langle \sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p} \pi_3\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{2s}u)], u\rangle \\ &= Re\ i\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_3\int (\Pi_{n_0}\bar{u})(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{2s}u)dx, \end{aligned}$$

where

(2.2.23)

$$\pi_3 = \frac{1}{2^p}\binom{p}{\ell}[B_1(n_0,n_1,\dots,n_p,n_{p+1})(m^2+\lambda_{n_{p+1}}^2)^{-\frac{1}{2}} - B_1(n_{p+1},n_1,\dots,n_p,n_0)(m^2+\lambda_{n_0}^2)^{-\frac{1}{2}}].$$

With the same reasoning as in the paragraph above (2.2.19) we get zero if we take the sum over $n \in S_p^\ell$ when computing the right hand side of (2.2.22). So we may assume $n \notin S_p^\ell$ and define

$$M_\ell^{p,2}(u_1,\dots,u_{p+1}) = -\frac{1}{4}\sum_{n\notin S_p^\ell}\pi_3\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\dots(\Pi_{n_p}\Lambda_m^{-1}u_p)(\Pi_{n_{p+1}}\Lambda_m^{2s}u_{p+1})]. \tag{2.2.24}$$

It follows from (2.2.22) that

$$II = -4\sum_{\ell=0}^{p} Re\ i\langle M_\ell^{p,2}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle. \tag{2.2.25}$$

Let us check that $M_\ell^{p,1}, M_\ell^{p,2} \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ for some $\nu > 0$, where $a = 2$ if $d \geq 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0,1)$ if $d = 1$. Since the function $B_1(n_0,\dots,n_{n_{p+1}})$ is supported on domain $n' = \max\{n_1,\dots,n_p\} < \delta n_{p+1}$ and $n_0 \sim n_{p+1}$ (this is because of the cut-off function and (1.1.1)), we see that (2.1.1) holds true if $supp\chi$ and $\delta$ are small. Let us use theorem 1.3.1 to show that (2.1.2) holds true with $\tau = 2s - a$ for $M_\ell^{p,1}$ and $M_\ell^{p,2}$. Remark that we have

$$|\pi_2| \leq C(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|)(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{2s-1}, \tag{2.2.26}$$

$$|\pi_3| \leq C(1 + \sqrt{n'})^2(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|)(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{-2}. \tag{2.2.27}$$

Indeed, (2.2.26) follows from the fact

$$|(m^2 + \lambda_{n_0}^2)^s - (m^2 + \lambda_{n_{p+1}}^2)^s| \leq C(|\lambda_{n_0} - \lambda_{n_{p+1}}|)(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{2s-1}.$$

If $n' < \delta n_0$ and $n' < \delta n_{p+1}$ for small $\delta > 0$, then

$$B_1(n_0, n_1, \dots, n_p, n_{p+1}) = B_1(n_{p+1}, n_1, \dots, n_p, n_0)$$

since $B(n_1,\dots,n_{p+1})$ is constant valued on the domain $n' < n_{p+1}$. Thus (2.2.27) follows from the fact

$$|(m^2 + \lambda_{n_0}^2)^{-\frac{1}{2}} - (m^2 + \lambda_{n_{p+1}}^2)^{-\frac{1}{2}}| \leq C(|\lambda_{n_0} - \lambda_{n_{p+1}}|)(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-2}.$$

Otherwise, assume $n' \geq \delta n_0$ or $n' \geq \delta n_{p+1}$. Then we must have $n' \geq Cn_0$ and $n' \geq Cn_{p+1}$ if $B_1$ is non zero, since $n_0 \sim n_{p+1}$ which is because of the cut-off function. In this case, (2.2.27) holds true trivially.

Moreover, on the support of $\Pi_{n_0} M_\ell^{p,l}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})(l = 1,2)$, i.e., $n_0 \sim n_{p+1}$ and $n_{p+1} \geq \max\{n_1,\dots,n_p\} = n'$, we have

$$\begin{aligned} 1 + \sqrt{n_{i_2}} &\sim 1 + \sqrt{n'}, \\ \mu(n_0,\dots,n_{p+1}) &\sim (1 + \sqrt{n_{p+1}})(1 + \sqrt{n'}), \\ S(n_0,\dots,n_{p+1}) &\sim |n_0 - n_{p+1}| + (1 + \sqrt{n_{p+1}})(1 + \sqrt{n'}), \end{aligned} \tag{2.2.28}$$

from which we deduce

$$\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \sim \frac{1 + \sqrt{n'}}{|\sqrt{n_0} - \sqrt{n_{p+1}}| + 1 + \sqrt{n'}}. \tag{2.2.29}$$

Thus

$$(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|)\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \leq C(1 + \sqrt{n'}).$$

Then we use theorem 1.3.1 (with dimension $d \geq 2$) to get for $l = 1, 2$

(2.2.30)

$$\begin{aligned} &||\Pi_{n_0} M_\ell^{p,l}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})||_{L^2} \\ &\leq C(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{2s-2}(1 + \sqrt{n'})^{\nu+2}(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|)\frac{\mu(n_0,\dots,n_{p+1})^N}{S(n_0,\dots,n_{p+1})^N}\prod_{j=1}^{p+1}||u_j||_{L^2} \\ &\leq C(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{2s-2}(1 + \sqrt{n'})^{\nu+3}\frac{\mu(n_0,\dots,n_{p+1})^{N-1}}{S(n_0,\dots,n_{p+1})^{N-1}}\prod_{j=1}^{p+1}||u_j||_{L^2}. \end{aligned}$$

So $M_\ell^{p,l} \in \mathcal{M}_\ell^{\nu,2s-2}$ for some other $\nu > 0$ in dimension $d \geq 2$. The case of dimension one is similar. (2.1.15) with $\omega = \omega_\ell$ is satisfied by definition. Thus $M_\ell^{p,1}, M_\ell^{p,2} \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ and we have proved

$$I_p^1 = \sum_{\ell=0}^{p} Re\ i\langle M_\ell^{p,1}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle + \sum_{\ell=0}^{p} Re\ i\langle M_\ell^{p,2}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle. \tag{2.2.31}$$

**(ii)** The term $I_p^2$.

Using (2.2.10) we get

(2.2.32)

$$\begin{aligned} -2I_p^2 &= Re\ i\langle \sum_{n\in\mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_4 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)],u\rangle \\ &= Re\ i \sum_{n\in\mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_4 \int (\Pi_{n_0}\Lambda_m^{2s}\bar{u})(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)dx \end{aligned}$$

where

$$\pi_4 = \frac{1}{2^p}\binom{p}{\ell} B_2(n_0,\dots,n_{p+1}).$$

We may rule out the sum over $n \in S_p^\ell$ in the above computation with the same reasoning as in the paragraph above (2.2.19). Thus if we define

$$R_\ell^{p,1}(u_1,\dots,u_{p+1}) = -\frac{1}{2}\sum_{n\notin S_p^\ell} \pi_4 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})], \tag{2.2.33}$$

we have

$$I_p^2 = \sum_{\ell=0}^{p} Re\ i\langle R_\ell^{p,1}(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle. \tag{2.2.34}$$

From the support property of function $B_2(n_0,\dots,n_{p+1})$ we know that $\Pi_{n_0}R_\ell^{p,1}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})$ is supported on $\max\{n_1,\dots,n_p\} < \delta n_{p+1}$ and $|n_0 - n_{p+1}| \geq c(n_0 + n_{p+1})$ for some small $c > 0$. Therefore, on its support, if $n_0 > Cn_{p+1}$ for a large $C$, we have

$$\begin{aligned} \mu(n_0,\dots,n_{p+1}) &= (1+\sqrt{n_{p+1}})(1+\sqrt{n'}) \leq (1+\sqrt{n_0})(1+\sqrt{n'}), \\ S(n_0,\dots,n_{p+1}) &= |n_0 - n_{p+1}| + (1+\sqrt{n_{p+1}})(1+\sqrt{n'}) \sim (1+\sqrt{n_0})^2 \end{aligned}$$

and if $n_0 \leq Cn_{p+1}$, we have

$$\begin{aligned} \mu(n_0,\dots,n_{p+1}) &\leq (1+\sqrt{n'})(1+\sqrt{n_{p+1}}), \\ S(n_0,\dots,n_{p+1}) &\geq c(|n_0 - n_{p+1}|) \geq c(n_0 + n_{p+1}) \sim (1+\sqrt{n_{p+1}})^2. \end{aligned}$$

In both cases we have

$$\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \leq C\frac{1+\sqrt{n'}}{1+\sqrt{n_0}+\dots+\sqrt{n_{p+1}}} = C\frac{\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}})}{1+\sqrt{n_0}+\dots+\sqrt{n_{p+1}}}, \tag{2.2.35}$$

where $\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}})$ is defined above definition 2.1.4. Thus theorem 1.3.1 allows us to get (2.1.17) with $\tau = 2s$ and some $\nu > 0$. (2.1.24) with $\omega = \omega_\ell$ is satisfied by the definition of $R_\ell^{p,1}$. So $R_\ell^{p,1} \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$.

**(iii)** The term $I_p^3$.

The treatment of $I_p^3$ is similar to that of $I_p^2$. The only difference is that we have different support for $B_2$ and $B_3$. So we define

$$R_\ell^{p,2}(u_1,\dots,u_{p+1}) = -\frac{1}{2}\sum_{n\notin S_p^\ell} \pi_5 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\dots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})] \tag{2.2.36}$$

with $\pi_5$ given by

$$\pi_5 = \frac{1}{2^p}\binom{p}{\ell} B_3(n_1,\dots,n_{p+1}) \tag{2.2.37}$$

and we get

$$I_p^3 = \sum_{\ell=0}^{p} Re\ i\langle R_\ell^{p,2}(\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\rangle. \tag{2.2.38}$$

From the support property of $B_3$ we know that $\Pi_{n_0}R_\ell^{p,2}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})$ is supported on domain $\delta n_{p+1}\le \max\{n_1,\dots,n_p\} = n' \le n_{p+1}$. So on this domain we have

$$\begin{aligned}\mu(n_0,\dots,n_{p+1}) &\le (1+\sqrt{n_{p+1}})(1+\sqrt{n'}),\\ S(n_0,\dots,n_{p+1}) &\sim (1+\sqrt{n_0}+\sqrt{n_{p+1}})^2,\end{aligned}$$

from which we deduce

$$\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \le C\frac{1+\sqrt{n'}}{1+\sqrt{n_0}+\dots+\sqrt{n_{p+1}}}. \tag{2.2.39}$$

Thus we have by theorem 1.3.1, for any $N\in\mathbb{N}$, there exists $C_N>0$, such that (2.1.17) holds true with $\tau = 2s$ and some $\nu>0$. On the other hand, (2.1.24) with $\omega=\omega_\ell$ is satisfied by the definition. So $R_\ell^{p,2}\in\widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$.

Taking $M_\ell^p$ to be the sum of $M_\ell^{p,1}$ and $M_\ell^{p,2}$, and $R_\ell^p$ the sum of $R_\ell^{p,1}$ and $R_\ell^{p,2}$, we get (2.2.8) with $M_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ and $R_\ell^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$. This concludes the proof the lemma. □

We have to treat the second term in the right hand side of (2.2.7).

**Lemma 2.2.3.** *There are multilinear operators $\widetilde{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\widetilde\omega_\ell)$, $\widetilde{R}_\ell^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde\omega_\ell)$, $\kappa\le p\le 2\kappa-1$, $0\le\ell\le p$ with $\widetilde\omega_\ell$ defined by $\widetilde\omega_\ell(j)=-1$, $j=0,\dots,\ell,p+1$, $\widetilde\omega_\ell(j)=1, j=\ell+1,\dots,p$, such that*

$$\begin{aligned} Re\ i\langle\Lambda_m^s C(u,\bar u)\bar u,\Lambda_m^s u\rangle &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\widetilde{M}_\ell^p(\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar u),u\rangle\\ &\quad+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle\widetilde{R}_\ell^p(\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar u),u\rangle.\end{aligned} \tag{2.2.40}$$

*Proof of Lemma 2.2.3:* Let $\widetilde{\omega}_\ell$ be defined in the statement of the lemma. We set

$$\begin{aligned}\widetilde{S}_p^\ell = \{(n_0,\dots,n_{p+1})\in\mathbb{N}^{p+2};\ \text{there exists bijection } \sigma \text{ from}\\ \{j; 0\le j\le p+1, \widetilde{\omega}_\ell(j)=-1\} \text{ to } \{j; 0\le j\le p+1, \widetilde{\omega}_\ell(j)=1\}\\ \text{such that for each } j \text{ in the first set } n_j = n_{\sigma(j)}\}.\end{aligned} \tag{2.2.41}$$

Taking the expression of $C(u,\bar{u})$ defined above (2.2.4) into account, we compute using notation (2.2.2)

$$\begin{aligned} &Re\ i\langle\Lambda_m^{2s}C(u,\bar{u})\bar{u},u\rangle\\ &= Re\ i\langle -\frac{1}{2}\sum_{p=\kappa}^{2\kappa-1}\Lambda_m^{2s}A_p\Big(\Lambda_m^{-1}(\frac{u+\bar{u}}{2})\Big)\Lambda_m^{-1}\bar{u},u\rangle\\ &= Re\ i\langle\sum_{p=\kappa}^{2\kappa-1}\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_6\Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})\\ &\qquad\qquad\times(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{-1}\bar{u})],u\rangle\\ &= Re\ i\sum_{p=\kappa}^{2\kappa-1}\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_6\int(\Pi_{n_0}\Lambda_m^{2s}\bar{u})(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\dots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})\\ &\qquad\qquad\times(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\dots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{-1}\bar{u})dx,\end{aligned} \tag{2.2.42}$$

where $\pi_6$ is given by

$$\pi_6 = -\frac{1}{2^{p+1}}\binom{p}{\ell}B(n_1,\dots,n_{p+1}). \tag{2.2.43}$$

With the same reasoning as in the paragraph above (2.2.19) we may assume $n\notin\widetilde{S}_p^\ell$ in the computation of (2.2.42). Let $\chi\in C_0^\infty(\mathbb{R}),\chi\equiv 1$ near zero, and supp$\chi$ small enough. According to (2.2.42), we define

$$\widetilde{M}_\ell^p(u_1,\dots,u_{p+1}) = \sum_{n\notin\widetilde{S}_p^\ell}\chi\Big(\frac{|\lambda_{n_0}^2-\lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2+\lambda_{n_{p+1}}^2}\Big)\pi_6\Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1),\dots,(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})],$$

$$\widetilde{R}_\ell^p(u_1,\dots,u_{p+1}) = \sum_{n\notin\widetilde{S}_p^\ell}\Big(1-\chi\Big(\frac{|\lambda_{n_0}^2-\lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2+\lambda_{n_{p+1}}^2}\Big)\Big)\pi_6\Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1),\dots,(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})].$$

It follows that (2.2.40) holds true.

Now we are left to check that $\widetilde{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\widetilde{\omega}_\ell)$ and $\widetilde{R}_\ell^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$.

Because of cut-off function and the support property of function $B$ in the definition of $\widetilde{M}_\ell^p$ we know that (2.1.1) holds true for $\widetilde{M}_\ell^p$ and we may assume $n_0\sim n_{p+1}$ when estimating $L^2$ norm of $\Pi_{n_0}\widetilde{M}_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})$. Since there is a $\Lambda_m^{-1}$ following each orthogonal projector $\Pi_{n_j},\quad j=1,\dots,p+1$, we see that (1.3.4) implies (2.1.2) with $\tau=2s-1$ and some $\nu>0$. Moreover, (2.1.15) with $\omega=\widetilde{\omega}_\ell$ is satisfied by the definition of $\widetilde{M}_\ell^p$. So $\widetilde{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\widetilde{\omega}_\ell)$.

Assume $\Pi_{n_0}[R(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})]$ does not vanish. Then we have $|n_0-n_{p+1}|\ge c(n_0+n_{p+1})$ for some small $c>0$ because of the cut-off function and $n_{p+1}\ge\max\{n_1,\dots,n_p\}=n'$ because of

the support property of function $B$. Therefore if $n_0 \geq n'$, we have

$$\mu(n_0,\dots,n_{p+1}) = (1+\sqrt{n'})(1+\min\{\sqrt{n_0},\sqrt{n_{p+1}}\}),$$
$$S(n_0,\dots,n_{p+1}) = |n_0-n_{p+1}| + (1+\sqrt{n'})(1+\min\{\sqrt{n_0},\sqrt{n_{p+1}}\}),$$

and thus

$$\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \leq C\frac{1+\sqrt{n'}}{\sqrt{n_0}+\sqrt{n_{p+1}}+1+\sqrt{n'}} \leq C\frac{\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}})}{1+\sqrt{n_0}+\dots+\sqrt{n_{p+1}}};$$

if $n_0 < n'$, we have

$$\mu(n_0,\dots,n_{p+1}) \leq (1+\sqrt{n'})^2, \qquad S(n_0,\dots,n_{p+1}) = |n'-n_{p+1}| + \mu(n_0,\dots,n_{p+1}),$$

and thus

$$\frac{\mu(n_0,\dots,n_{p+1})}{S(n_0,\dots,n_{p+1})} \leq C\frac{1+\sqrt{n'}}{\sqrt{n'}+\sqrt{n_{p+1}}+1+\sqrt{n'}} \leq C\frac{\max_2(\sqrt{n_1},\dots,\sqrt{n_{p+1}})}{1+\sqrt{n_0}+\dots+\sqrt{n_{p+1}}}.$$

Now using theorem 1.3.1 we see that (2.1.17) holds true with $\tau = 2s$ and some $\nu > 0$. But (2.1.24) with $\omega = \widetilde{\omega}_\ell$ is satisfied according to the definition. So $\widetilde{R}^p_\ell \in \widetilde{\mathcal{R}}^{\nu,2s}_{p+1}(\widetilde{\omega}_\ell)$. This concludes the proof of lemma. □

Summarizing the above analysis gives an end to the proof of the proposition 2.2.1. □

In order to control the energy, let us first turn to some useful estimates in the following subsection.

## 2.3 Geometric bounds

This subsection is a modification of section 2.1 in [9]. We give it for the convenience of the reader. Consider the function on $\mathbb{R}^{p+2}$ depending on the parameter $m \in (0,+\infty)$, defined for $\ell = 0,\dots,p+1$ by

$$F^\ell_m(\xi_0,\dots,\xi_{p+1}) = \sum_{j=0}^{\ell}\sqrt{m^2+\xi_j^2} - \sum_{j=\ell+1}^{p+1}\sqrt{m^2+\xi_j^2}. \tag{2.3.1}$$

The main result of this subsection is the following theorem:

**Theorem 2.3.1.** *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}^*_+$ such that for any integers $0 \leq \ell \leq p+1$, any $m \in \mathbb{R}^*_+ - \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that the lower bound*

$$|F^\ell_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{-3-\rho}(1+|\sqrt{n_0}-\sqrt{n_{p+1}}|+\sqrt{n'})^{-2N_0} \tag{2.3.2}$$

*holds true for any $\rho > 0$ and any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2} - S^\ell_p$. Here $\lambda_n$ are given by (1.1.1), $n' = \max\{n_1,\dots,n_p\}$, and $S^\ell_p$ is defined in (2.2.18), in which we have set $\omega_\ell(j) = -1,\ j = 0,\dots,\ell,\ \omega(j) = 1,\ j = \ell+1,\dots,p+1$.*

The proof of the theorem will rely on some geometric estimates that we shall deduce from results of [10]. Let us show that under the condition of theorem 2.3.1 we have

(2.3.3)
$$|F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{-3-\rho}(1+|\sqrt{n_0}-\sqrt{n_{p+1}}|)^{-N_0}(1+\sqrt{n_1}+\cdots+\sqrt{n_p})^{-N_0}.$$

Let $I \subset (0,+\infty)$ be some compact interval and define for $0 \leq \ell \leq p+1$ functions

$$\begin{aligned} f_\ell &: [0,1]\times[0,1]^{p+2}\times I \longrightarrow \mathbb{R} \\ &\quad (z,x_0,\dots,x_{p+1},y) \to f_\ell(z,x_0,\dots,x_{p+1},y) \\ g_\ell &: [0,1]\times[0,1]^{p}\times I \longrightarrow \mathbb{R} \\ &\quad (z,x_1,\dots,x_p\ ,y) \to g_\ell(z,x_1,\dots,x_p\ ,y) \end{aligned} \tag{2.3.4}$$

by

$$\begin{aligned} f_\ell(z,x_0,\dots,x_{p+1},y) &= \sum_{j=0}^{\ell}\sqrt{z^2+y^2x_j^2} - \sum_{j=\ell+1}^{p+1}\sqrt{z^2+y^2x_j^2} \\ g_\ell(z,x_1,\dots,x_p,y) &= z\Big[\sum_{j=1}^{\ell}\frac{z}{\sqrt{z^2+y^2x_j^2}} - \sum_{j=\ell+1}^{p}\frac{z}{\sqrt{z^2+y^2x_j^2}}\Big] \qquad \text{when } z>0, \\ g_\ell(0,x_1,\dots,x_p,y) &\equiv 0. \end{aligned} \tag{2.3.5}$$

Then the graphs of $f_\ell, g_\ell$ are subanalytic subsets of $[0,1]^{p+3}\times I$ and $[0,1]^{p+1}\times I$ respectively, so that $f_\ell, g_\ell$ are continuous subanalytic functions (refer to Bierstone-Milman [5] for an introduction to subanalytic sets and functions). Let us consider the set $\Gamma$ of points $(z,x) \in [0,1]^{p+3}$(resp. $(z,x)\in[0,1]^{p+1}$) such that $y \to f_\ell(z,x,y)$ (resp. $y \to g_\ell(z,x,y)$) vanishes identically. If $(z,x)\in\Gamma$ and $z\neq 0$, we have

$$\ell = \frac{p}{2} \qquad \text{and} \qquad \sum_{j\leq\ell} x_j^{2\kappa} - \sum_{j\geq\ell+1} x_j^{2\kappa} = 0, \forall \kappa \in \mathbb{N}^*$$

where the sum is taken respectively for $0\leq j\leq p+1$ in the case of $f_\ell$ and $1\leq j\leq p$ for $g_\ell$. This implies that there is a bijection $\sigma:\{0,\dots,\ell\}\to\{\ell+1,\dots,p+1\}$ (resp. $\{1,\dots,\ell\}\to\{\ell+1,\dots,p\}$) such that $x_{\sigma(j)}=x_j$ for any $j=0,\dots,\ell$ (resp. $j=1,\dots,\ell$)—see for instance the proof of lemma 5.6 in [10]. When $p$ is even, denote by $S_p$ the set of all bijections respectively from $\{0,\dots,\frac{p}{2}\}$ to $\{\frac{p}{2}+1,\dots,p+1\}$ and from $\{1,\dots,\frac{p}{2}\}$ to $\{\frac{p}{2},\dots,p\}$. Define for $0\leq\ell\leq p+1$

$$\begin{aligned} \rho_\ell(z,x) &\equiv z \qquad \text{if} \qquad \ell \neq \frac{p}{2}, \\ \rho_\ell(z,x) &= z\prod_{\sigma\in S_p}\Big[\sum_{j\leq p/2}(x_{\sigma(j)}^2 - x_j^2)^2\Big] \qquad \text{if} \qquad \ell=\frac{p}{2}, \end{aligned} \tag{2.3.6}$$

where the sum in the above formula is taken for $j\geq 0$ (resp. $j\geq 1$) when we study $f_\ell$ (resp. $g_\ell$). Then the set $\{\rho_\ell=0\}$ contains those points $(z,x)$ such that $y\to f_\ell(z,x,y)$ (resp. $y\to g_\ell(z,x,y)$) vanishes identically. The following proposition is the same as proposition 2.1.2 in [9].

**Proposition 2.3.2.** *(i)There are $\widetilde{N}\in\mathbb{N},\alpha_0>0,\delta>0,C>0$, such that for any $0\leq\ell\leq p+1$, any $\alpha\in(0,\alpha_0)$, any $(z,x)\in[0,1]^{p+3}$ (resp. $(z,x)\in[0,1]^{p+1}$) with $\rho_\ell(z,x)\neq 0$, any $N\geq\widetilde{N}$ the sets*

$$\begin{aligned} I_\ell^f(z,x,\alpha) &= \{y\in I; |f_\ell(z,x,y)| < \alpha\rho_\ell(z,x)^N\} \\ I_\ell^g(z,x,\alpha) &= \{y\in I; |g_\ell(z,x,y)| < \alpha\rho_\ell(z,x)^N\} \end{aligned} \tag{2.3.7}$$

*have Lebesgue measure bounded from above by* $C\alpha^{\delta}\rho_{\ell}(z,x)^{N\delta}$.

*(ii) For any* $N \geq \widetilde{N}$, *there is* $K \in \mathbb{N}$ *such that for any* $\alpha \in (0,\alpha_0)$, *any* $(z,x) \in [0,1]^{p+1}$, *the set* $I^g_{\ell}(z,x,\alpha)$ *may be written as the union of at most* $K$ *open disjoint subintervals of* $I$.

We shall deduce (2.3.3) from several lemmas. Let us first introduce some notations. When $p$ is odd or $p$ is even and $\ell \neq \frac{p}{2}$, we set $\mathbb{N}'^p_{\ell} = \emptyset$. When $p$ is even and $\ell = \frac{p}{2}$, we define

$$\begin{aligned}\mathbb{N}'^p_{\ell} = \{\tilde{n} = (n_1,\dots,n_p) \in \mathbb{N}^p;\ &\text{there is a bijection}\\ \sigma : \{1,\dots,\ell\} \to \{\ell+1,\dots,p\}\ &\text{such that } n_{\sigma(j)} = n_j, j = 1,\dots,\ell\}.\end{aligned} \tag{2.3.8}$$

We set also

$$\mathbb{N}^{p+2}_{\ell} = \{(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2};\ \tilde{n} \in \mathbb{N}'^p_{\ell} \text{ and } n_0 = n_{p+1}\}. \tag{2.3.9}$$

Of course, $\mathbb{N}^{p+2}_{\ell} = \emptyset$ if $p$ is odd or $p$ is even and $\ell \neq \frac{p}{2}$.

We remark first that it is enough to prove (2.3.3) for those $(n_1,\dots,n_p)$ which do not belong to $\mathbb{N}'^p_{\ell}$: actually if $p$ is even, $\ell = \frac{p}{2}$ and $(n_1,\dots,n_p) \in \mathbb{N}'^p_{\ell}$, we have $|F^{\ell}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| = |\sqrt{m^2+\lambda^2_{n_0}} - \sqrt{m^2+\lambda^2_{n_{p+1}}}|$ which is bounded from below, when $m$ stays in some compact interval, by

$$\frac{2|n_0 - n_{p+1}|}{\sqrt{m^2+\lambda^2_{n_0}} + \sqrt{m^2+\lambda^2_{n_{p+1}}}} \geq \frac{c}{1+\lambda_{n_0}+\lambda_{n_{p+1}}}$$

since from $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2} - S^{\ell}_p$, we have $n_0 \neq n_{p+1}$. Consequently (2.3.3) holds true trivially. From now on, we shall always consider $p-$tuple $\tilde{n}$ which do not belong to $\mathbb{N}'^p_{\ell}$.

Let us define for $\ell = 1,\dots,p$ another function on $\mathbb{R}^p$ given by

$$G^{\ell}_m(\xi_1,\dots,\xi_p) = \sum_{j=1}^{\ell}\sqrt{m^2+\xi_j^2} - \sum_{j=\ell+1}^{p}\sqrt{m^2+\xi_j^2}. \tag{2.3.10}$$

Let $J \subset (0,+\infty)$ be a given compact interval. For $\alpha > 0, N_0 \in \mathbb{N}, 0 \leq \ell \leq p+1$, $n = (n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ define

$$\begin{aligned}E^{\ell}_J(n,\alpha,N_0) = \{m \in J; |F^{\ell}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| &< \alpha(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{-3-\rho}\\ &\times(1+|\lambda_{n_0}-\lambda_{n_{p+1}}|)^{-N_0}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_0}\}.\end{aligned} \tag{2.3.11}$$

We set also for $\beta > 0, N_1 \in \mathbb{N}^*, \tilde{n} = (n_1,\dots,n_p) \in \mathbb{N}^p - \mathbb{N}'^p_{\ell}$

$$E'^{\ell}_J(\tilde{n},\beta,N_1) = \{m \in J; \left|\frac{\partial G^{\ell}_m}{\partial m}(\lambda_{n_1},\dots,\lambda_{n_p})\right| < \beta(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_1}\}. \tag{2.3.12}$$

We define for $\gamma > \beta$ a subset of $\mathbb{N}^{p+2}$ by

$$\begin{aligned}S(\beta,\gamma,N_1) = \{(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2} - \mathbb{N}^{p+2}_{\ell} : \lambda_{n_0} &< \frac{\gamma}{3\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1}\\ \text{or} \qquad \lambda_{n_{p+1}} &< \frac{\gamma}{3\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1}\}.\end{aligned} \tag{2.3.13}$$

**Lemma 2.3.3.** *Let $\widetilde{N}, \delta, \alpha_0$ be the constants defined in the statement of proposition 2.3.2. There are constants $C_1 > 0, M \in \mathbb{N}^*$ such that for any $\beta \in (0, \alpha_0)$, any $N_1 \in \mathbb{N}$ with $N_1 > M\widetilde{N}$ and $N_1 > \frac{2pM}{\delta}$, one has*

$$meas\Big[\bigcup_{\tilde{n}\in\mathbb{N}^p-\mathbb{N}_\ell'^p} E_J'^\ell(\tilde{n},\beta,N_1)\Big] \leq C_1\beta^\delta. \tag{2.3.14}$$

*Proof.* Set $y = \frac{1}{m}$ and

$$z = (1+\sum_{j=1}^p \lambda_{n_j})^{-1},\ x_j = \lambda_{n_j} z\ ,\ j = 1,\dots,p.$$

Denote by $X$ the set of points $(z,x) \in [0,1]^{p+1}$ of the preceding form for $(n_1,\dots,n_p)$ describing $\mathbb{N}^p$. When $p$ is even and $\ell = p/2$, let $X_\ell'^p$ be the imagine of $\mathbb{N}_\ell'^p$ defined by (2.3.8) under the map $\tilde{n} \to (z,x)$. Using definition (2.3.6), we see that there are constants $M > 0, C > 0$, depending only on $p$, such that for $0 \leq \ell \leq p+1$

$$\forall (z,x) \in X - X_\ell'^p\ ,\ z^M \leq \rho_\ell(z,x) \leq Cz \tag{2.3.15}$$

since , when $\ell = \frac{p}{2}$ and $(n_1,\dots,n_p) \notin \mathbb{N}_\ell'^p$, $\sum_{j=1}^{\frac{p}{2}} (\lambda^2_{n_{\sigma(j)}} - \lambda^2_{n_j})^2 \geq 1$, by the definition of $\lambda_{n_j}$. Remark that with the above notations

$$\frac{\partial G_m^\ell}{\partial m}(\lambda_{n_1},\dots,\lambda_{n_p}) = \sum_{j=1}^{\ell} \frac{m}{\sqrt{m^2+\lambda^2_{n_j}}} - \sum_{j=\ell+1}^{p} \frac{m}{\sqrt{m^2+\lambda^2_{n_j}}} = \frac{1}{z} g_\ell(z,x_1,\dots,x_p,y).$$

Then if $I = \{m^{-1}; m \in J\}$, we see that $m \in E_J'^\ell(\tilde{n},\beta,N_1)$ for $n \notin \mathbb{N}_\ell'^p$ if and only if $y = \frac{1}{m}$ satisfies

$$|g_\ell(z,x_1,\dots,x_p,y)| < \beta z^{N_1+1} \leq \beta \rho_\ell(z,x)^{\frac{1}{M}(N_1+1)} \tag{2.3.16}$$

using (2.3.15). Applying proposition 2.3.2 (i), we see that for any fixed value of $(z,x) \in X - X_\ell'^p$, the measure of those $y$ such that (2.3.16) holds true is bounded from above by

$$C\beta^\delta \rho_\ell(z,x)^{\frac{N_1+1}{M}\delta} \leq C\beta^\delta z^{\frac{N_1+1}{M}\delta}$$

if we assume $N_1 \geq M\widetilde{N}$ and $\beta \in (0,\alpha_0)$. Consequently, we get with a constant $C'$ depending only on $J$,

$$\begin{aligned} meas(E_J'^\ell(n',\beta,N_1)) &\leq C'\beta^\delta (1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-\frac{N_1+1}{M}\delta} \\ &\leq C'\beta^\delta (1+n_1+\cdots+n_p)^{-\frac{N_1+1}{2M}\delta}. \end{aligned}$$

Inequality (2.3.14) follows from this estimate and the assumption on $N_1$. □

**Lemma 2.3.4.** *Let $\widetilde{N}, \delta, \alpha_0$ be the constants defined in the statement of proposition 2.3.2. There are constants $M \in \mathbb{N}^*, \theta > 1, C_2 > 0$ such that for any $N_0, N_1 \in \mathbb{N}^*$ satisfying $N_0 > \widetilde{N}MN_1$ and $N_0\delta > 2(p+2)MN_1$, any $0 < \beta < \gamma$ with $\frac{\gamma}{\beta} > \theta$, any $\alpha > 0$ satisfying $\alpha(\frac{\beta}{2\gamma})^{-\frac{N_0}{N_1}} < \alpha_0$, one has*

$$meas\Big[\bigcup_{n\in S(\beta,\gamma,N_1)} E_J^\ell(n,\alpha,N_0)\Big] \leq C_2\alpha^\delta \left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}\delta}. \tag{2.3.17}$$

*Proof.* We first remark that if $\lambda_{n_0}+\lambda_{n_{p+1}} > \frac{\gamma}{\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1}$ and $n \in S(\beta,\gamma,N_1)$, then either

$$\lambda_{n_0} \geq \frac{2\gamma}{3\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1} \qquad \text{or} \qquad \lambda_{n_{p+1}} \geq \frac{2\gamma}{3\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1},$$

which implies that

$$|F_m^\ell(\lambda_{n_0},\ldots,\lambda_{n_{p+1}})| \geq c\frac{\gamma}{\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1}$$

for some constant $c>0$ depending only on $p$ and $J$, if $\frac{\gamma}{\beta} > \theta$ large enough. Consequently, if $\alpha < \alpha_0$ small enough relatively to $c$, we see that we have in this case $E_J^\ell(n,\alpha,N_0)=\emptyset$ when $n \in S(\beta,\gamma,N_1)$. We may therefore consider only indices $n$ such that

$$n \in S(\beta,\gamma,N_1) \qquad \text{and} \qquad \lambda_{n_0}+\lambda_{n_{p+1}} \leq \frac{\gamma}{\beta}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{N_1}.$$

Consequently, for $m \in E_J^\ell(n,\alpha,N_0)$ and $n \in S(\beta,\gamma,N_1)$, we have

$$\begin{aligned} |F_m^\ell(\lambda_{n_0},\ldots,\lambda_{n_{p+1}})| &\leq \alpha(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_0} \\ &\leq \alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}(1+\lambda_{n_0}+\cdots+\lambda_{n_{p+1}})^{-\frac{N_0}{N_1}}. \end{aligned} \tag{2.3.18}$$

Define for $n \in \mathbb{N}^{p+2}$

$$z=(1+\sum_{j=0}^{p+1}\lambda_{n_j})^{-1}, \qquad x_j=\lambda_{n_j}z,\ j=0,\ldots,p+1. \tag{2.3.19}$$

Denote by $X \subset [0,1]^{p+3}$ the set of points $(z,x)$ of the preceding form, and let $X_\ell^p$ be the imagine of the set $\mathbb{N}_\ell^{p+2}$ defined by (2.3.9) under the map $n \to (z,x)$. By (2.3.6) we have again

$$\forall (z,x) \in X - X_\ell^p\ ,\ z^M \leq \rho_\ell(z,x) \leq Cz$$

for some large enough $M$, depending only on $p$. Moreover

$$F_m^\ell(\lambda_{n_0},\ldots,\lambda_{n_{p+1}}) = \frac{m}{z}f_\ell(z,x_0,\ldots,x_{p+1},y)$$

and (2.3.18) implies that if $n \in S(\beta,\gamma,N_1)$ and $m \in E_J^\ell(n,\alpha,N_0)$, then $y$ satisfies

$$\begin{aligned} |f_\ell(z,x_0,\ldots,x_{p+1},y)| &\leq C\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}} z^{1+\frac{N_0}{N_1}} \\ &\leq C\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}} \rho_\ell(z,x)^{\frac{1}{M}(1+\frac{N_0}{N_1})} \end{aligned} \tag{2.3.20}$$

We assume that $\alpha, N_0, N_1$ satisfy the conditions of the statement of the lemma. Then by (i) of proposition 2.3.2 we get that the measure of those $y \in J$ satisfying (2.3.20) is bounded from above by

$$C\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^\delta z^{\frac{\delta}{M}(1+\frac{N_0}{N_1})}$$

for some constant $C$, independent of $N_0, N_1, \alpha, \beta, \gamma$. Consequently the measure of $E_J^\ell(n, \alpha, N_0)$ is bounded from above when $n \in S(\beta, \gamma, N_1)$ by

$$C\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^\delta \left(1 + \lambda_{n_0} + \cdots + \lambda_{n_{p+1}}\right)^{-\frac{\delta}{M}(1+\frac{N_0}{N_1})}$$
$$\leq C'\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^\delta \left(1 + n_0 + \cdots + n_{p+1}\right)^{-\frac{\delta}{2M}(1+\frac{N_0}{N_1})}$$

for another constant $C'$ depending on $J$. The conclusion of the lemma follows by summation, using that $\frac{\delta}{M}(1 + \frac{N_0}{N_1}) > 2(p+2)$. □

*Proof of theorem 2.3.1.* : We fix $N_0, N_1$ satisfying the conditions stated in lemmas 2.3.3 and 2.3.4, and such that $N_0 > 2p + N_1$. We write when $n \notin S(\beta, \gamma, N_1),\ 0 \leq \ell \leq p+1$,

$$E_J^\ell(n, \alpha, N_0) \subset [E_J^\ell(n, \alpha, N_0) \cap E_J'^\ell(\tilde{n}, \beta, N_1)] \cup [E_J^\ell(n, \alpha, N_0) \cap (E_J'^\ell(\tilde{n}, \beta, N_1)^c]$$

and estimate, using that we reduced ourselves to those $\tilde{n} \notin \mathbb{N}_\ell'^p$

(2.3.21)
$$meas\Big[\bigcup_{n;\ \tilde{n}\notin\mathbb{N}_\ell'^p} E_J^\ell(n, \alpha, N_0)\Big] \leq meas\Big[\bigcup_{n\in S(\beta,\gamma,N_1)} E_J^\ell(n, \alpha, N_0)\Big] + meas\Big[\bigcup_{\tilde{n}\notin\mathbb{N}_\ell'^p} E_J'^\ell(\tilde{n}, \beta, N_1)\Big]$$
$$+ meas\Big[\bigcup_{n\in S(\beta,\gamma,N_0)^c - \mathbb{N}_\ell^{p+2}} E_J^\ell(n, \alpha, N_0) \cap E_J'^\ell(\tilde{n}, \beta, N_1)^c\Big].$$

Let us bound the measure of $E_J^\ell(n, \alpha, N_0) \cap E_J'^\ell(\tilde{n}, \beta, N_1)^c$ for $n \in S(\beta, \gamma, N_0)^c - \mathbb{N}_\ell^{p+2}$. If $m$ belongs to that set, the inequality in (2.3.11) holds true. Remark that we may assume $\ell \leq p$ : if $\ell = p+1$, $|F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \geq c(1 + \lambda_{n_0} + \lambda_{n_{p+1}})$ for some $c > 0$, which is not compatible with (2.3.11) for $\alpha < \alpha_0$ small enough. Let us write (2.3.11) as

$$|\lambda_{n_0} - \lambda_{n_{p+1}} + \widetilde{G}_m(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| < \alpha(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-3-\rho}$$
$$\times(1 + |\lambda_{n_0} - \lambda_{n_{p+1}}|)^{-N_0}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0} \tag{2.3.22}$$

with, using notation (2.3.10)

$$\widetilde{G}_m(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}}) = G_m(\lambda_{n_1}, \ldots, \lambda_{n_p}) + R_m(\lambda_{n_0}, \lambda_{n_{p+1}})$$
$$R_m(\lambda_{n_0}, \lambda_{n_{p+1}}) = (\sqrt{m^2 + \lambda_{n_0}^2} - \lambda_{n_0}) - (\sqrt{m^2 + \lambda_{n_{p+1}}^2} - \lambda_{n_{p+1}}). \tag{2.3.23}$$

Since $n \in S(\beta, \gamma, N_1)^c$, we have by (2.3.13)

$$\lambda_{n_0} \geq \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}\ ,\ \lambda_{n_{p+1}} \geq \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}. \tag{2.3.24}$$

Consequently there is a constant $C > 0$, depending only on $J$, such that

$$|\frac{\partial R_m}{\partial m}(\lambda_{n_0}, \lambda_{n_{p+1}})| \leq C\frac{\beta}{\gamma}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_1}.$$

If $\gamma$ is large enough and $m \in E'^{\ell}_J(\tilde{n},\beta,N_1)^c$, we deduce from (2.3.12) that

$$|\widetilde{\frac{\partial G_m}{\partial m}}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq \frac{\beta}{2}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_1}. \tag{2.3.25}$$

By (ii) of proposition 2.3.2, we know that there is $K \in \mathbb{N}$, independent of $\alpha,\beta,\gamma$ such that the set $J - E'^{\ell}_J(\tilde{n},\beta,N_1)$ is the union of at most $K$ disjoint intervals $J_j(\tilde{n},\beta,N_1)$ , $1 \leq j \leq K$. Consequently, we have

$$E^{\ell}_J(n,\alpha,N_0) \cap (E'^{\ell}_J(\tilde{n},\beta,N_1))^c \subset \bigcup_{j=1}^{K} \{m \in J_j(\tilde{n},\beta,N_1);\ (2.3.22) \text{ holds true}\}, \tag{2.3.26}$$

and on each interval $J_j(n',\beta,N_1)$, (2.3.25) holds true. We may on each such interval perform in the characteristic function of (2.3.22) the change of variable of integration given by $m \to \widetilde{G}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})$. Because of (2.3.25) this allows us to estimate the measure of (2.3.26) by

$$\begin{aligned} &K\frac{2}{\beta}\alpha(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{-3-\rho}(1+|\lambda_{n_0}-\lambda_{n_{p+1}}|)^{-N_0}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_0+N_1} \\ \leq\ & CK\frac{2}{\beta}\alpha(1+n_0+n_{p+1})^{-\frac{1}{2}(3+\rho)}(1+|\sqrt{n_0}-\sqrt{n_{p+1}}|)^{-N_0}(1+n_1+\cdots+n_p)^{-\frac{1}{2}(N_0-N_1)} \end{aligned}$$

Summing in $n_0,\dots,n_{p+1}$, we see that since $N_0 > 2p+N_1$, the last term in (2.3.21) is bounded from above by $C_3\frac{\alpha}{\beta}$ with $C_3$ independent of $\alpha,\beta,\gamma$. By lemmas 2.3.3 and 2.3.4, we may thus bound (2.3.21) by

$$C_2\alpha^{\delta}\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}\delta} + C_1\beta^{\delta} + C_3\frac{\alpha}{\beta}$$

if $\alpha,\beta$ are small enough, $\gamma$ is large enough and $\alpha(\frac{\beta}{\gamma})^{-\frac{N_0}{N_1}}$ is small enough. If we take $\beta = \alpha^{\sigma}$, $\gamma = \alpha^{-\sigma}$ with $\sigma > 0$ small enough, and $\alpha \ll 1$, we finally get for some $\delta' > 0$,

$$meas\left[\bigcup_{n;\ \tilde{n}\notin\mathbb{N}'^{p}_{\ell}} E^{\ell}_J(n,\alpha,N_0)\right] \leq C\alpha^{\delta'} \to 0 \text{ if } \alpha \to 0^+.$$

This implies that in this case the set of those $m \in J$ for which (2.3.3) does not hold true for any $c > 0$ is of zero measure. This concludes the proof. □

We will need a consequence of theorem 2.3.1:.

**Proposition 2.3.5.** *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}^*_+$ such that for any integers $0 \leq \ell \leq p+1$, any $m \in \mathbb{R}^*_+ - \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that the lower bound*

$$|F^{\ell}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{-3-\rho}(1+\sqrt{n'})^{-2N_0}\frac{\mu(n_0,\dots,n_{p+1})^{2N_0}}{S(n_0,\dots,n_{p+1})^{2N_0}} \tag{2.3.27}$$

*holds true for any $\rho > 0$ and any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2} - S^{\ell}_p$ with $n_0 \sim n_{p+1}$ and $n_{p+1} \geq n'$. Here $\lambda_n$, $n'$, $S^{\ell}_p$ are the same as those in theorem 2.3.1.*

*Proof.* By theorem 2.3.1 we know (2.3.2) holds true under the conditions of the proposition. Since we assume $n_0 \sim n_{p+1}$ and $n_{p+1} \geq n'$, we have by (1.3.2) and (1.3.3)

$$\begin{aligned} \mu(n_0,\dots,n_{p+1}) &\sim (1+\sqrt{n_{p+1}})(1+\sqrt{n'}), \\ S(n_0,\dots,n_{p+1}) &\sim |n_0-n_{p+1}| + (1+\sqrt{n_{p+1}})(1+\sqrt{n'}) \\ &\sim (1+\sqrt{n_{p+1}})(1+|\sqrt{n_0}-\sqrt{n_{p+1}}|+\sqrt{n'}). \end{aligned} \tag{2.3.28}$$

Therefore we deduce from (2.3.2)

$$\begin{aligned} |F^\ell_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| &\geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{-3-\rho}\frac{(1+\sqrt{n_{p+1}})^{2N_0}}{S(n_0,\dots,n_{p+1})^{2N_0}} \\ &\geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{-3-\rho}(1+\sqrt{n'})^{-2N_0}\frac{\mu(n_0,\dots,n_{p+1})^{2N_0}}{S(n_0,\dots,n_{p+1})^{2N_0}}. \end{aligned}$$

This concludes the proof of the proposition. □

In the following subsection, we shall also use a simpler version of theorem 2.3.1. Let us introduce some notations. For $m \in \mathbb{R}^*_+$, $\xi_j \in \mathbb{R}$, $j = 0,\dots,p+1$, $e = (e_0,\dots,e_{p+1}) \in \{-1,1\}^{p+2}$, define

$$\widetilde{F}^{(e)}_m(\xi_0,\dots,\xi_{p+1}) = \sum_{j=0}^{p+1} e_j\sqrt{m^2+\xi_j^2}. \tag{2.3.29}$$

When $p$ is even and $\sharp\{j; e_j = 1\} = \frac{p}{2}+1$, denote by $N^{(e)}$ the set of all $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection $\sigma$ from $\{j; 0 \leq j \leq p+1, e_j = 1\}$ to $\{j; 0 \leq j \leq p+1, e_j = -1\}$ so that for any $j$ in the first set $n_j = n_{\sigma(j)}$. In the other cases, set $N^{(e)} = \emptyset$.

**Proposition 2.3.6.** *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}^*_+$ and for any $m \in \mathbb{R}^*_+ - \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that for any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2} - N^{(e)}$ one has*

$$|\widetilde{F}^{(e)}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^{-N_0}. \tag{2.3.30}$$

*Moreover, if $e_0e_{p+1} = 1$, one has the inequality*

$$|\widetilde{F}^{(e)}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\sqrt{n_0}+\sqrt{n_{p+1}})(1+\sqrt{n_1}+\cdots+\sqrt{n_p})^{-N_0}. \tag{2.3.31}$$

*Proof.* With the reasoning as in the proof of proposition 2.1.5 in [9], we get just by replacing $(n_0,\dots,n_{p+1})$ with $(\lambda_0,\dots,\lambda_{p+1})$

$$|\widetilde{F}^{(e)}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\lambda_{n_0}+\cdots+\lambda_{n_{p+1}})^{-N_0}$$

and

$$|\widetilde{F}^{(e)}_m(\lambda_{n_0},\dots,\lambda_{n_{p+1}})| \geq c(1+\lambda_{n_0}+\lambda_{n_{p+1}})(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_0}$$

when $e_0e_{p+1} = 1$. This concludes the proof of the proposition by noting (1.1.1). □

## 2.4 Energy control and proof of main theorem

We shall use the results of subsection 2.3 to control the energy. When $M(u_1,\dots,u_{p+1})$ is a $p+1$-linear form, let us define for $0\leq \ell\leq p+1$,

$$L_\ell^-(M)(u_1,\dots,_{p+1}) = -\Lambda_m M(u_1,\dots,u_{p+1}) \tag{2.4.1}$$
$$-\sum_{j=1}^{\ell} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1}) + \sum_{j=\ell+1}^{p+1} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1})$$

and

$$L_\ell^+(M)(u_1,\dots,_{p+1}) = -\Lambda_m M(u_1,\dots,u_{p+1}) - \sum_{j=1}^{\ell} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1}) \tag{2.4.2}$$
$$+\sum_{j=\ell+1}^{p} M(u_1,\dots,\Lambda_m u_j,\dots,u_{p+1}) - M(u_1,\dots,u_p,\Lambda_m u_{p+1}).$$

We shall need the following lemma:

**Lemma 2.4.1.** *Let $\mathcal{N}$ be the zero measure subset of $\mathbb{R}_+^*$ defined by taking the union of the zero measure subsets defined in proposition 2.3.5 and proposition 2.3.6, and fix $m\in\mathbb{R}_+^*-\mathcal{N}$. Let $\omega_\ell,\widetilde{\omega}_\ell$ be defined in the statement of proposition 2.2.1. There is a $\bar{\nu}\in\mathbb{N}$ such that the following statements hold true for any large enough integer $s$, any integer $p$ with $\kappa\leq p\leq 2\kappa-1$, any integer $\ell$ with $0\leq\ell\leq p$, any $\rho>0$:*

- *Let $\theta\in(0,1)$, $M_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ with $a=2$ if $d\geq 2$ and $a=\frac{13}{6}-\varsigma$ for any $\varsigma\in(0,1)$ if $d=1$ and $\widetilde{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\widetilde{\omega}_\ell)$. Define*

$$M_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}) = \sum_{n_0}\sum_{n_{p+1}} \mathbf{1}_{\{\sqrt{n_0}+\sqrt{n_{p+1}}<\epsilon^{-\theta\kappa}\}}\Pi_{n_0}M_\ell^p(u_1,\dots,u_p,\Pi_{n_{p+1}}u_{p+1}). \tag{2.4.3}$$

*Then there are $\underline{M}_\ell^{p,\epsilon}\in\widetilde{\mathcal{M}}_{p+1}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$ and $\underline{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu,2s-2}(\widetilde{\omega}_\ell)$ satisfying*

$$L_\ell^-(\underline{M}_\ell^{p,\epsilon})(u_1,\dots,u_{p+1}) = M_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}),$$
$$L_\ell^+(\underline{M}_\ell^p)(u_1,\dots,u_{p+1}) = \widetilde{M}_\ell^p(u_1,\dots,u_{p+1}) \tag{2.4.4}$$

*with the estimate for all $N\geq\bar{\nu}$,*

$$||\underline{M}_\ell^{p,\epsilon}||_{\mathcal{M}_{p+1,N}^{\nu+\bar{\nu},2s-1}} \leq C\epsilon^{-(4-a+\rho)\theta\kappa}||M_\ell^p||_{\mathcal{M}_{p+1,N}^{\nu,2s-a}},$$
$$||\underline{M}_\ell^p||_{\mathcal{M}_{p+1,N}^{\nu+\bar{\nu},2s-2}} \leq C||\widetilde{M}_\ell^p||_{\mathcal{M}_{p+1,N}^{\nu,2s-1}}, \tag{2.4.5}$$

*where $||\cdot||_{\mathcal{M}_{p+1,N}^{\nu,\tau}}$ is defined in the statement of definition 2.1.1.*

- *Let $R_\ell^p\in\widetilde{R}_{p+1}^{\nu,2s}(\omega_\ell),\widetilde{R}_\ell^p\in\widetilde{R}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$. Then there are $\underline{R}_\ell^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\omega_\ell)$ and $\underline{R}_\ell'^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\widetilde{\omega}_\ell)$ such that*

$$L_\ell^-(\underline{R}_\ell^p)(u_1,\dots,u_{p+1}) = R_\ell^p(u_1,\dots,u_{p+1}),$$
$$L_\ell^+(\underline{R}_\ell'^p)(u_1,\dots,u_{p+1}) = \widetilde{R}_\ell^p(u_1,\dots,u_{p+1}). \tag{2.4.6}$$

*Proof.* (i) We substitute in (2.4.4) $\Pi_{n_j}u_j$ to $u_j, j=1,\dots,p+1$, and compose on the left with $\Pi_{n_0}$. According to (2.4.1), equalities in (2.4.4) may be written

$$-F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{M}_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})=\Pi_{n_0}M_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.7}$$

$$\widetilde{F}_m^{(e)}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{M}_\ell^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})=\Pi_{n_0}\widetilde{M}_\ell^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.8}$$

where $F_m^\ell$ is defined by (2.3.1) and $\widetilde{F}_m^{(e)}$ is defined by (2.3.29) with $e_0=\cdots=e_\ell=e_{p+1}=-1, e_{\ell+1}=\cdots=e_p=1$.

When considering (2.4.7), we may assume $\mathrm{n}_0\sim n_{p+1}$, $n_{p+1}\geq n'$ and $(n_0,\dots,n_{p+1})\notin S_p^\ell$ if the right hand side of (2.4.7) is non zero since we have (2.1.1) and (2.1.15) for $M_\ell^{p,\epsilon}$. Here $S_p^\ell$ is the same as that in proposition 2.3.5. Thus the assumptions concerning $(n_0,\dots,n_{p+1})$ in proposition 2.3.5 hold true. We deduce from (2.3.27) and the condition $\sqrt{n_0}+\sqrt{n_{p+1}}<\epsilon^{-\theta\kappa}$ that

$$\begin{aligned}|F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})|^{-1}&\leq C(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{3+\rho}(1+\sqrt{n'})^{2N_0}\frac{S(n_0,\dots,n_{p+1})^{2N_0}}{\mu(n_0,\dots,n_{p+1})^{2N_0}}\\&\leq C\epsilon^{-(4-a+\rho)\theta\kappa}(1+\sqrt{n_0}+\sqrt{n_{p+1}})^{a-1}(1+\sqrt{n'})^{2N_0}\frac{S(n_0,\dots,n_{p+1})^{2N_0}}{\mu(n_0,\dots,n_{p+1})^{2N_0}}.\end{aligned} \tag{2.4.9}$$

for any $\rho>0$. Therefore if we define

$$\underline{M}_\ell^{p,\epsilon}(u_1,\dots,u_{p+1})=-\sum_{\substack{n\notin S_p^\ell\\ n_0\sim n_{p+1},n_{p+1}\geq n'}}F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})^{-1}\Pi_{n_0}M_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.10}$$

we obtain according to (2.4.9) and (2.1.2) that $\underline{M}_\ell^{p,\epsilon}\in\widetilde{\mathcal{M}}_{p+1}^{\nu+\bar\nu,2s-1}(\omega_\ell)$ with the first estimate in (2.4.5) with $\bar\nu=2N_0$.

When considering (2.4.8), we may assume $(n_0,\dots,n_{p+1})\notin N^{(e)}$ defined after (2.3.29). Actually, because of (2.1.15), we cannot find a bijection $\sigma$ from $\{0,\dots,\ell,p+1\}$ to $\{\ell+1,\dots,p\}$ such that $n_j=n_{\sigma(j)}$, $j=0,\dots,\ell,p+1$ if the right hand side of (2.4.8) is non zero. Consequently, we may use lower bound (2.3.31). If we define $\underline{M}_\ell^p$ dividing in (2.4.8) by $\widetilde{F}_m^{(e)}$, we thus see that we get an element of $\underline{M}_\ell^p\in\widetilde{\mathcal{M}}_{p+1}^{\nu+\bar\nu,2s-2}(\widetilde{\omega}_\ell)$ for some $\bar\nu$. This completes the proof of (2.4.4) and (2.4.5).

(ii) We deduce again from (2.4.6)

$$-F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{R}_\ell^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})=\Pi_{n_0}R_\ell^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.11}$$

$$\widetilde{F}_m^{(e)}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{R}_\ell'^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})=\Pi_{n_0}\widetilde{R}_\ell^{p}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.12}$$

where $F_m^\ell$ and $\widetilde{F}_m^{(e)}$ are the same as in (2.4.7) and (2.4.8). Since $R_\ell^p\in\widetilde{R}_{p+1}^{\nu,2s}(\omega_\ell)$ and thus (2.1.24) implies the right hand side of (2.4.11) vanishes if $(n_0,\dots,n_{p+1})\in S_p^\ell$, where $S_p^\ell$ is defined in (2.2.18), we may assume $(n_0,\dots,n_{p+1})\notin S_p^\ell$. Consequently, the condition of theorem 2.3.1 is satisfied and we have by (2.3.2)

$$|F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})|^{-1}\leq C(1+\sqrt{n_0}+\sqrt{n_1}+\cdots+\sqrt{n_{p+1}})^{2N_0+4}.$$

We then get an element of $\underline{R}_\ell^p\in\widetilde{\mathcal{R}}_{p+1}^{\nu+\bar\nu,2s}(\omega_\ell)$ dividing in (2.4.11) by $-F_m^\ell$ with $\bar\nu=2N_0+4$. Since $\widetilde{R}_\ell^p\in\widetilde{R}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$, we see that the right hand side of (2.4.12) vanishes if $(n_0,\dots,n_{p+1})\in\widetilde{S}_p^\ell$, where $\widetilde{S}_p^\ell$ is defined in (2.2.41). This implies that we may assume $(n_0,\dots,n_{p+1})\notin N^{(e)}$ which is defined after

(2.3.29) with $e_0 = \cdots = e_\ell = e_{p+1} = -1$, $e_{\ell+1} = \cdots = e_p = 1$. Thus the condition of proposition 2.3.6 is satisfied and we have

$$|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})|^{-1} \leq C(1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}})^{N_0}.$$

This allows us to get an element $\underline{R}_\ell'^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\widetilde{\omega}_\ell)$ for some $\bar{\nu}$ by dividing by $\widetilde{F}_m^{(e)}$ in (2.4.12). This concludes the proof. □

**Proposition 2.4.2.** *Let $\mathcal{N}$ be the zero measure subset of $\mathbb{R}_+^*$ defined in lemma 2.4.1, and fix $m \in \mathbb{R}_+^* - \mathcal{N}$. Let $\rho > 0$ be any positive number and $\Theta_s$ defined in (2.2.5). There are for any large enough integer $s$ , a map $\Theta_s^1$, sending $\mathscr{H}^s(\mathbb{R}^d) \times (0, \frac{1}{2})$ to $\mathbb{R}$, and maps $\Theta_s^2$, $\Theta_s^3$, $\Theta_s^4$ sending $\mathscr{H}^s(\mathbb{R}^d)$ to $\mathbb{R}$ such that there is a constant $C_s > 0$ and for any $u \in \mathscr{H}^s(\mathbb{R}^d)$ with $||u||_{\mathscr{H}^s} \leq 1$ and any $\epsilon \in (0, \frac{1}{2})$, one has*

$$\begin{aligned} &|\Theta_s^1(u,\epsilon)| \leq C_s \epsilon^{-(4-a+\rho)\theta\kappa} ||u||_{\mathscr{H}^s}^{\kappa+2}, \qquad (a = 2 \text{ if } d \geq 2 \text{ and} \\ &\qquad\qquad a = \frac{13}{6} - \varsigma \text{ for any } \varsigma \in (0,1) \text{ if } d = 1), \\ &|\Theta_s^2(u)|, |\Theta_s^3(u)|, |\Theta_s^4(u)| \leq C_s ||u||_{\mathscr{H}^s}^{\kappa+2} \end{aligned} \tag{2.4.13}$$

*and such that*

$$R(u) \stackrel{def}{=} \frac{d}{dt}\Big[\Theta_s(u(t,\cdot)) - \Theta_s^1(u(t,\cdot),\epsilon) - \Theta_s^2(u(t,\cdot)) - \Theta_s^3(u(t,\cdot)) - \Theta_s^4(u(t,\cdot))\Big] \tag{2.4.14}$$

*satisfies*

$$|R(u)| \leq C_s \epsilon^{-(4-a+\rho)\theta\kappa} ||u||_{\mathscr{H}^s}^{2\kappa+2} + C_s \epsilon^{(a-1)\theta\kappa} ||u||_{\mathscr{H}^s}^{\kappa+2} + C_s ||u||_{\mathscr{H}^s}^{2\kappa+2}. \tag{2.4.15}$$

*Proof.* Considering the right hand side of (2.2.6), we decompose

$$M_\ell^p(u_1, \ldots, u_{p+1}) = M_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}) + V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}), \tag{2.4.16}$$

where the first term is given by (2.4.3) and the second one by

$$V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}) = \sum_{n_0} \sum_{n_{p+1}} \mathbf{1}_{\{\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}\}} \Pi_{n_0} M_\ell^p(u_1, \ldots, u_p, \Pi_{n_{p+1}} u_{p+1}). \tag{2.4.17}$$

By definition 2.1.1, we get for $a = 2$ if $d \geq 2$ and $a = \frac{13}{6} - \varsigma$ if $d = 1$

(2.4.18)

$$\begin{aligned} ||V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1})||_{\mathscr{H}^{-s}} \leq C_N \sum_{n_0} \cdots \sum_{p+1} (1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{2s-a} \frac{(1+\sqrt{n'})^\nu \mu(n_0, \ldots, n_{p+1})^N}{S(n_0, \ldots, n_{p+1})^N} \\ \times \mathbf{1}_{\{\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}, |n_0 - n_{p+1}| < \frac{1}{2}(n_0 + n_{p+1}), n' \leq n_{p+1}\}} (1 + \sqrt{n_0})^{-s} \prod_{j=1}^{p+1} ||\Pi_{n_j} u_j||_{L^2} \end{aligned}$$

Following the proof of proposition 2.1.2, we know that the gain of $a$ powers of $\sqrt{n_0} + \sqrt{n_{p+1}}$ in the first term in the right hand side, coming from the fact that $M_\ell^p \in \mathcal{M}_{p+1}^{\nu,2s-a}$ , together with the condition $\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}$, allows us to estimate , for $N$ large enough and $s_0$ large enough with respect to $\nu$, (2.4.18) by $C\epsilon^{(a-1)\theta\kappa} \Pi_{j=1}^p ||u_j||_{\mathscr{H}^{s_0}} ||u_{p+1}||_{\mathscr{H}^s}$. Consequently, the quantity

$$\sum_{p=\kappa}^{2\kappa-1} \sum_{\ell=0}^{p} Re\ i\langle V_\ell^{p,\epsilon}(\bar{u}, \ldots, \bar{u}, u, \ldots, u), u\rangle \tag{2.4.19}$$

is bounded form above by the second term of the right hand side of (2.4.15). In the rest of the proof, we may therefore replace in the right hand side of (2.2.6) $M_\ell^p$ by $M_\ell^{p,\epsilon}$.

Apply lemma 2.4.1 to $M_\ell^{p,\epsilon}$, $\widetilde{M}_\ell^p$, $R_\ell^p$, $\widetilde{R}_\ell^p$. This gives $\underline{M}_\ell^{p,\epsilon}, \underline{M}_\ell^p, \underline{R}_\ell^p, \underline{R}_\ell'^p$. We set

$$\begin{aligned}
\Theta_s^1(u(t,\cdot),\epsilon) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle,\\
\Theta_s^2(u(t,\cdot)) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{M}_\ell^{p}(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u}),u\rangle,\\
\Theta_s^3(u(t,\cdot)) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{R}_\ell^{p}(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle,\\
\Theta_s^4(u(t,\cdot)) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{R}_\ell'^{p}(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u}),u\rangle.
\end{aligned} \tag{2.4.20}$$

The general term in $\Theta_s^1(u(t,\cdot),\epsilon)$ has modulus bounded from above by

$$||\underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u)||_{\mathscr{H}^{-s}}||u||_{\mathscr{H}^s} \leq C\epsilon^{-(4-a+\rho)\theta\kappa}||u||_{\mathscr{H}^s}^{\kappa}||u||_{\mathscr{H}^s}^2$$

for $u$ in the unit ball of $\mathscr{H}^s(\mathbb{R}^d)$, using proposition 2.1.2 with $\tau = 2s-1$ and proposition 1.1.19 and (2.4.5). This gives the first inequality of (2.4.13). To obtain the other estimates in (2.4.13), we apply proposition 2.1.2 to $\underline{M}_\ell^p$, remarking that if in (2.1.3) $\tau = 2s-1$ and $s$ is large enough, the left hand side of (2.1.3) controls the $\mathscr{H}^{-s}$ norm of $\underline{M}_\ell^p(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u})$. We also apply proposition 2.1.5 with $\tau = 2s$ in (2.1.18) to $\underline{R}_\ell^p$, $\underline{R}_\ell'^p$. Then if $s_0$ is large enough, the left hand side of (2.1.18) controls $\mathscr{H}^{-s}$ norm of $\underline{R}_\ell^p(\bar{u},\dots,\bar{u},u,\dots,u)$ and $\underline{R}_\ell'^p(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u})$. These give us the other inequalities in (2.4.13). Consequently we are left with proving (2.4.15). Remarking that we may also write the equation as

$$(D_t - \Lambda_m)u = -F\left(\Lambda_m^{-1}(\frac{u+\bar{u}}{2})\right), \tag{2.4.21}$$

we compute using notation (2.4.1)

$$\begin{aligned}
\frac{d}{dt}\Theta_s^1(u,\epsilon) = &\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle L_\ell^-(\underline{M}_\ell^{p,\epsilon})(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle\\
&+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\ i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{F},\dots,\bar{u},u,\dots,u),u\rangle\\
&-\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p+1} Re\ i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,F,\dots,u),u\rangle\\
&+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u),F\rangle.
\end{aligned} \tag{2.4.22}$$

By assumption on $F$, we have by proposition 1.1.19 and 1.1.21 that $||F(v)||_{\mathscr{H}^s} \leq C||u||_{\mathscr{H}^s}^{\kappa}||u||_{\mathscr{H}^s}$ if $s$ is large enough and $||u||_{\mathscr{H}^s} \leq 1$. Since $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathcal{M}}_{p+1}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$, we may apply proposition 2.1.2 with $\tau = 2s-1$ and (2.4.5) to see that the last three terms in (2.4.22) have modulus bounded from above by the first term in the right hand side of (2.4.15). When computing $\frac{d}{dt}\Theta_s(u)$, noting that we have replaced $M_\ell^p$ by $M_\ell^{p,\epsilon}$, the first term in the right hand side of (2.2.6) is the first term in

the right hand side of (2.4.22) because of (2.4.4). Consequently, these contributions will cancel out each other in the expression $\frac{d}{dt}[\Theta_s(u) - \Theta_s^1(u,\epsilon)]$. We compute

$$
\begin{aligned}
\frac{d}{dt}\Theta_s^2(u) = &\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle L_\ell^+(\underline{M}_\ell^p)(\bar u,\dots,\bar u,u,\dots,u,\bar u),u\rangle\\
&+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\ i\langle \underline{M}_\ell^p(\bar u,\dots,\bar F,\dots,\bar u,u,\dots,u,\bar u),u\rangle\\
&-\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p} Re\ i\langle \underline{M}_\ell^p(\bar u,\dots,\bar u,u,\dots,F,\dots,u,\bar u),u\rangle\\
&+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle \underline{M}_\ell^p(\bar u,,\dots,\bar u,u,\dots,u,\bar F),u\rangle\\
&+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\ i\langle \underline{M}_\ell^p(\bar u,,\dots,\bar u,u,\dots,u,\bar u),F\rangle.
\end{aligned}
\tag{2.4.23}
$$

Since $\underline{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu+\bar\nu,2s-2}(\widetilde{\omega}_\ell)$, we have by proposition 2.1.2 with $\tau = 2s-1$, proposition 1.1.19 and (2.4.5) that the last three terms are estimated by the last term in the right hand side of (2.4.15) if $s$ is large enough. The first one, according to lemma 2.4.1, cancels the contribution of $\widetilde{M}_\ell^p$ in (2.2.6) when computing $R(u)$. We may treat $\Theta_s^3(u)$ and $\Theta_s^4(u)$ in the same way using proposition 2.1.5 with $\tau = 2s$, and this will lead to the third term in the right hand side of (2.4.15). Finally, the last term in (2.2.6) contributes to the last term in the right hand side of (2.4.15). This concludes the proof of the proposition. □

*Proof of theorem 2.1.1.* : We deduce from (2.4.13) and (2.4.15)

$$
\begin{aligned}
\Theta_s(u(t,\cdot)) \le\ &\Theta_s(u(0,\cdot)) - \Theta_s^1(u(0,\cdot),\epsilon) - \Theta_s^2(u(0,\cdot)) - \Theta_s^3(u(0,\cdot)) - \Theta_s^4(u(0,\cdot))\\
&+\Theta_s^1(u(t,\cdot),\epsilon) + \Theta_s^2(u(t,\cdot)) + \Theta_s^3(u(t,\cdot)) + \Theta_s^4(u(t,\cdot))\\
&+C_s\epsilon^{-(4-a+\rho)\theta\kappa}\int_0^t ||u(t',\cdot)||_{\mathscr{H}^s}^{2\kappa}||u(t',\cdot)||_{\mathscr{H}^s}^2 dt'\\
&+C_s\epsilon^{(a-1)\theta\kappa}\int_0^t ||u(t',\cdot)||_{\mathscr{H}^s}^{\kappa}||u(t',\cdot)||_{\mathscr{H}^s}^2 dt'\\
&+C_s\int_0^t ||u(t',\cdot)||_{\mathscr{H}^s}^{2\kappa}||u(t',\cdot)||_{\mathscr{H}^s}^2 dt',
\end{aligned}
\tag{2.4.24}
$$

where $a = 2$ if $d \ge 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0,1)$ if $d = 1$. Take $\theta = \frac{1}{3+\rho}$ and $B > 1$ a constant such that for any $(v_0, v_1)$ in the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}^d)\times\mathscr{H}^s(\mathbb{R}^d)$, $u(0,\cdot) = \epsilon(-iv_1 + \Lambda_m v_0)$ satisfies $||u(0,\cdot)||_{\mathscr{H}^s} \le B\epsilon$. Let $K > B$ be another constant to be chosen, and assume that for $\tau'$ in some interval $[0,T]$ we have $||u(\tau',\cdot)||_{\mathscr{H}^s} \le K\epsilon \le 1$. If $d \ge 2$, using (2.4.13) with $a = 2$ we deduce from (2.4.24) and that there is a constant $C > 0$, independent of $B, K, \epsilon$, such that as long as $t \in [0,T]$

$$
||u(t,\cdot)||_{\mathscr{H}^s}^2 \le C[B^2 + \epsilon^{\frac{1}{3+\rho}\kappa}K^{\kappa+2} + t\epsilon^{\frac{4+\rho}{3+\rho}\kappa}(K^{2\kappa+2} + K^{\kappa+2}) + t\epsilon^{2\kappa}K^{2\kappa+2}]\epsilon^2.
$$

If we assume that $T \leq c\epsilon^{-\frac{4+\rho}{3+\rho}}$, where $\rho > 0$ is arbitrary, for a small enough $c > 0$, and that $\epsilon$ is small enough, we get $||u(t,\cdot)||^2_{\mathscr{H}^s} \leq C(2B^2)\epsilon^2$. If $K$ has been chosen initially so that $2CB^2 < K^2$, we get by a standard continuity argument that the priori bound $||u(t,\cdot)||_{\mathscr{H}^s} \leq K\epsilon$ holds true on $[0, c\epsilon^{-\frac{4+\rho}{3+\rho}}]$, in other words, the solution extends to such an interval $|t| \leq c\epsilon^{-\frac{4}{3}(1-\rho)\kappa}$ with another arbitrary $\rho > 0$. If $d = 1$, we may use (2.4.13) with $a = \frac{13}{6} - \varsigma$ to get

$$||u(t,\cdot)||^2_{\mathscr{H}^s} \leq C[B^2 + \epsilon^{\frac{7-6\varsigma}{18+6\rho}\kappa}K^{\kappa+2} + t\epsilon^{\frac{25+6(\rho-\varsigma)}{18+6\rho}\kappa}(K^{2\kappa+2} + K^{\kappa+2}) + t\epsilon^{2\kappa}K^{2\kappa+2}]\epsilon^2.$$

With the same reasoning we may get in this case that the solution extends to an interval of $|t| < c\epsilon^{-\frac{25}{18}(1-\rho)\kappa}$ for some small $c > 0$ and any $\rho > 0$. This concludes the proof of the theorem. □

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