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Abstract

We prove continuity properties for Riesz and Oseen potentials. As a consequence, we show some new properties on solutions of Poisson's and Oseen equations. The study relies on weighted Sobolev spaces in order to control the behavior of functions at infinity.

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1 Introduction and Notations

The purpose of this paper is to establish some new continuity properties in weighted Sobolev spaces for the Riesz potentials of order one or two and for the convolution with the fundamental solution of Oseen. We focus on those operators when they act on L^p-functions and on distributions that belong to a weighted Sobolev space that will be specified latter. We then use the continuity properties we obtained to find a representation of solutions, respectively, to the Poisson's equation and to the Oseen equations in $\mathbb{R}^3$. It is well known that in unbounded domains, it is important to control the behavior of functions at infinity.

This is the mathematical reason for dealing with weighted spaces. In previous papers (see for instance [15], [16], [18], [19] and [20]), some weighted inequalities for the Riesz potentials are proved under suitable assumptions on the weights. In [5] and [1], it is proved, among other results, that those inequalities still hold for a larger class of weighted Sobolev spaces. The aim of this work was to improve those results by eventually a modification of the Riesz potentials. We then follow the ideas developed for the Riesz potential to prove some continuity properties for the Oseen potential. In this paper, we only consider the three-dimensional case. Note that the two-dimensional case is studied in [3]. The paper is organized as follow. In the next section, we introduce the weighted Sobolev spaces and their main properties for this work. In Section 3, we prove the continuity properties for the Riesz potentials of order one or two and we show the consequences for the Laplace's equation. Finally in Section 4, we deal with the fundamental solution of Oseen.

We end this section with the Notation that we will use all long this work. We denote by $\mathbb{N}$ the set of all positive integer and $\mathbb{Z}$ the set of all integers. In what follows, p is a real number in the interval $]1,+\infty[$. The dual exponent of p denoted by p' is defined by the relation $\frac{1}{p}+\frac{1}{p'}=1$. We will use bold characters for vector or matrix fields. A point in $\mathbb{R}^3$ is denoted by $\boldsymbol{x}=(x_1,x_2,x_3)$ and its distance to the origin by

$$r=|\boldsymbol{x}|=(x_1+x_2+x_3)^{1/2}.$$

We denote by $[k]$ the integer part of k. For any $\ell\in\mathbb{Z}$, $\mathbb{P}_\ell$ stands for the space of polynomials of degree less than or equal to ℓ and $\mathbb{P}_\ell^\Delta$ the harmonic polynomials of $\mathbb{P}_\ell$. If ℓ is a negative integer, we set by convention $\mathbb{P}_\ell=\{0\}$. We denote by $\mathcal{D}(\mathbb{R}^3)$ the space of $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}^3$. We recall that $\mathcal{D}'(\mathbb{R}^3)$ is the well known space of distributions and $L^p(\mathbb{R}^3)$ is the usual Lebesgue space on $\mathbb{R}^3$. For $m\geq 1$, we recall that $W^{m,p}(\mathbb{R}^3)$ is the well-known classical Sobolev spaces. For any $0<\alpha<1$ and for $m\geq 0$, we introduce the space

$$\mathcal{C}^{m,\alpha}(\mathbb{R}^3)=\left\{f\in\mathcal{C}^m(\mathbb{R}^3),\quad \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3,\,\boldsymbol{x}\neq\boldsymbol{y}}\frac{|\partial^m f(\boldsymbol{x})-\partial^m f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha}<+\infty\right\}.$$

Given a Banach space B with its dual space B' and a closed subspace X of B, we denote by $B'\perp X$ the subspace of B' orthogonal to X, *i.e.*,

$$B'\perp X=\{f\in B',\ \ \forall v\in X,\ \ \langle f,v\rangle=0\}=(B/X)'.$$

Finally, we will use the symbol C for generic positive constant whose value may change at each occurrence even at the same line.

2 Weighted Sobolev spaces

We introduce the weight function $\rho(\boldsymbol{x}) = (1+r^2)^{1/2}$. For a nonnegative integer m and $\alpha \in \mathbb{R}$, set

$$k = k(m,p,\alpha) = \begin{cases} -1 & \text{if } \ 3/p+\alpha \notin \{1,...,m\} \\ m-3/p-\alpha & \text{if } \ 3/p+\alpha \in \{1,...,m\} \end{cases}$$

and we define the weighted Sobolev space

$$\begin{aligned} W^{m,p}_{\alpha}(\mathbb{R}^3) =& \{u \in \mathcal{D}'(\mathbb{R}^3); \forall \boldsymbol{\lambda} \in \mathbb{N}^3, 0 \le |\boldsymbol{\lambda}| \le k, \ \rho^{\alpha-m+|\boldsymbol{\lambda}|}(\ln(1+\rho^2))^{-1}\partial^{\boldsymbol{\lambda}} u \in L^p(\mathbb{R}^3), \\ & k+1 \le |\boldsymbol{\lambda}| \le m, \ \rho^{\alpha-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u \in L^p(\mathbb{R}^3)\}, \end{aligned}$$

which is a Banach space equipped with its natural norm given by

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha}(\mathbb{R}^3)} = \Bigg(& \sum_{0\le|\boldsymbol{\lambda}|\le k} \|\rho^{\alpha-m+|\boldsymbol{\lambda}|}(\ln(1+\rho^2))^{-1}\partial^{\boldsymbol{\lambda}} u\|^p_{L^p(\mathbb{R}^3)} \\ & + \sum_{k+1\le|\boldsymbol{\lambda}|\le m} \|\rho^{\alpha-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u\|^p_{L^p(\mathbb{R}^3)} \Bigg)^{1/p}. \end{aligned}$$

We define the semi-norm

$$|u|_{W^{m,p}_{\alpha}(\mathbb{R}^3)} = \left(\sum_{|\boldsymbol{\lambda}|=m} \|\rho^{\alpha}\partial^{\boldsymbol{\lambda}} u\|^p_{L^p(\mathbb{R}^3)} \right)^{1/p}.$$

We shall now point out some properties of those spaces that will be used all long this paper. For a detailed study we refer to [4] and references therein. All the local properties of the space $W^{m,p}_{\alpha}(\mathbb{R}^3)$ coincide with those of the Sobolev space $W^{m,p}(\mathbb{R}^3)$. The space $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^{m,p}_{\alpha}(\mathbb{R}^3)$. As a consequence, its dual space, denoted by $W^{-m,p'}_{-\alpha}(\mathbb{R}^3)$ is a space of distributions. Let q be defined as follow:

$$\begin{aligned} q = & \quad [m-3/p-\alpha]\,, & & \text{if } \ n/p+\alpha \notin \mathbb{Z}^- \\ q = & \quad m-1-3/p-\alpha, & & \text{otherwise.} \end{aligned} \tag{2.1}$$

Then $\mathbb{P}_q$ is the space of all polynomials included in $W^{m,p}_{\alpha}(\mathbb{R}^3)$. Moreover, the following Poincaré-type inequality holds:

$$\forall u \in W^{m,p}_{\alpha}(\mathbb{R}^3), \quad \inf_{\lambda\in\mathbb{P}_{q'}} \|u+\lambda\|_{W^{m,p}_{\alpha}(\mathbb{R}^3)} \le C|u|_{W^{m,p}_{\alpha}(\mathbb{R}^3)}, \tag{2.2}$$

where $q' = \min(q,0)$. From (2.2) and the Sobolev's embedding theorems, we have the algebraic and topological identities

$$W^{1,p}_0(\mathbb{R}^3) = \{v \in L^{\frac{3p}{3-p}}(\mathbb{R}^3), \ \nabla v \in \boldsymbol{L}^p(\mathbb{R}^3)\}, \quad \text{if } 1<p<3 \tag{2.3}$$

and

$$W_0^{2,p}(\mathbb{R}^3) = \left\{ v \in L^{\frac{3p}{3-2p}}(\mathbb{R}^3),\ \ \nabla v \in L^{\frac{3p}{3-p}}(\mathbb{R}^3),\ \ \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^3) \right\}, \quad \text{if } 1 < p < 3/2. \tag{2.4}$$

We introduce the space $\boldsymbol{H}_p = \{\boldsymbol{v} \in \boldsymbol{L}^p(\mathbb{R}^3),\ \text{div}\ \boldsymbol{v} = 0\}$. It follows from (2.2), that the operator

$$\text{div} \ \ : \boldsymbol{L}^p(\mathbb{R}^3)/\boldsymbol{H}_p \mapsto W_0^{-1,p}(\mathbb{R}^3) \perp \mathbb{P}_{[1-3/p']} \tag{2.5}$$

is an isomorphism.

We recall that the space $BMO(\mathbb{R}^3)$ stands for the space of functions locally integrable satisfying

$$\|f\|_{BMO(\mathbb{R}^3)} = \sup_Q \frac{1}{|Q|} \int_Q |f(\boldsymbol{x}) - f_Q| d\boldsymbol{x} < \infty,$$

where the supremum is taken all over the cubes Q and $f_Q = \frac{1}{|Q|} \int_Q f(\boldsymbol{x}) d\boldsymbol{x}$.

3 Riesz potentials and Poisson's equation

For any real $\alpha \in]0, 3[$, the Riesz potentials of order α are defined by (*cf.* [21])

$$I_\alpha f = F_\alpha * f = (-\Delta)^{\alpha/2} f, \ \text{ where } \ F_\alpha(\boldsymbol{x}) = \frac{1}{\gamma(\alpha)} |\boldsymbol{x}|^{\alpha-3},$$

with $\gamma(\alpha) = \pi^{3/2} 2^\alpha \Gamma(\alpha/2)/\Gamma(3/2 - \alpha/2)$ and Γ is the Gamma function. Let us recall the following embedding results.

Theorem 3.1. *We have:*

i) $W_0^{1,p}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{3-p}}(\mathbb{R}^3)$, *if* $1 < p < 3$,

ii) $W_0^{1,3}(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3)$,

iii) $W_0^{1,p}(\mathbb{R}^3) \hookrightarrow \mathcal{C}^{0,1-3/p}(\mathbb{R}^3)$, *if* $p > 3$.

We also recall that the Riesz operator $I_1 : L^p(\mathbb{R}^3) \mapsto W_0^{1,p}(\mathbb{R}^3)$ is continuous if $1 < p < 3$ (see for instance [1]) and we have

$$\frac{\partial}{\partial x_j} I_1 f = -R_j f, \tag{3.6}$$

where R_j is the Riesz transform. Observe now that if $f \in L^p(\mathbb{R}^3)$ with $p \geq 3$, then $I_1 f$ does not belong to $\mathcal{D}'(\mathbb{R}^3)$. Moreover, let f be a function defined by

$$f(\boldsymbol{x}) = 0 \ \text{ if } |\boldsymbol{x}| < 1 \ \text{ and } \ f(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|} \ \text{ if } |\boldsymbol{x}| > 1.$$

Then clearly $f \in L^p(\mathbb{R}^3)$ if $p > 3$ and, for any $|\boldsymbol{x}| < \frac{1}{2}$, we have

$$I_1 f(\boldsymbol{x}) = \frac{1}{\gamma(1)} \int_{|\boldsymbol{y}|>1} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2} \frac{1}{|\boldsymbol{y}|}\, d\boldsymbol{y} \geq \frac{C}{\gamma(1)} \int_{|\boldsymbol{y}|>1} \frac{1}{|\boldsymbol{y}|^3}\, d\boldsymbol{y} = +\infty.$$

This is the reason of introducing the operator J_1 defined by

$$J_1 f(\boldsymbol{x}) = \frac{1}{\gamma(1)} \int_{\mathbb{R}^3} \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2} - \frac{1}{|\boldsymbol{y}|^2} \right) f(\boldsymbol{y}) d\boldsymbol{y}. \tag{3.7}$$

Theorem 3.2. *The following operators are continuous:*

i) $$J_1 \,:\, L^p(\mathbb{R}^3) \mapsto \mathcal{C}^{0,1-3/p}(\mathbb{R}^3), \quad \textit{if } p > 3.$$

ii) $$J_1 \,:\, L^p(\mathbb{R}^3) \mapsto W_0^{1,p}(\mathbb{R}^3), \quad \textit{if } p \geq 3.$$

Moreover, in both cases we have

$$\frac{\partial}{\partial x_j} J_1 f = -R_j f. \tag{3.8}$$

Proof. We set

$$K_1(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\gamma(1)} \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2} - \frac{1}{|\boldsymbol{y}|^2} \right).$$

i) Let us first show that

$$\left(\int_{\mathbb{R}^3} |K_1(\boldsymbol{x}, \boldsymbol{y})|^{p'} d\boldsymbol{y} \right)^{1/p'} \leq C |\boldsymbol{x}|^{1-3/p}. \tag{3.9}$$

We can easily write

$$\begin{aligned} \left(\int_{\mathbb{R}^3} |K_1(\boldsymbol{x}, \boldsymbol{y})|^{p'} d\boldsymbol{y} \right)^{1/p'} &\leq \left(\int_{|\boldsymbol{y}|>2|\boldsymbol{x}|} |K_1(\boldsymbol{x}, \boldsymbol{y})|^{p'} d\boldsymbol{y} \right)^{1/p'} + \left(\int_{|\boldsymbol{y}|<2|\boldsymbol{x}|} |K_1(\boldsymbol{x}, \boldsymbol{y})|^{p'} d\boldsymbol{y} \right)^{1/p'} \\ &= K_{11} + K_{12}. \end{aligned}$$

For any $\boldsymbol{y} \in \mathbb{R}^3$ such that $|\boldsymbol{y}| > 2|\boldsymbol{x}|$, we have the inequalities

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2} - \frac{1}{|\boldsymbol{y}|^2} \leq C \frac{|\boldsymbol{x}|}{|\boldsymbol{y}|^3} \leq C \frac{1}{|\boldsymbol{y}|^2}. \tag{3.10}$$

It follows that

$$K_{11} \leq C \left(\int_{|\boldsymbol{y}|>2|\boldsymbol{x}|} \frac{d\boldsymbol{y}}{|\boldsymbol{y}|^{2p'}} \right)^{1/p'} \leq C |\boldsymbol{x}|^{1-3/p}. \tag{3.11}$$

For K_{12}, we can write

$$K_{12} \leq \left(\int_{|\boldsymbol{y}|<2|\boldsymbol{x}|} \frac{d\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{2p'}} \right)^{1/p'} + \left(\int_{|\boldsymbol{y}|<2|\boldsymbol{x}|} \frac{d\boldsymbol{y}}{|\boldsymbol{y}|^{2p'}} \right)^{1/p'}.$$

Now using the fact $|\boldsymbol{y}| < 2|\boldsymbol{x}|$ implies that $|\boldsymbol{y} - \boldsymbol{x}| < 3|\boldsymbol{x}|$, we have

$$K_{12} \leq 2 \left(\int_{|\boldsymbol{y}|<3|\boldsymbol{x}|} \frac{d\boldsymbol{y}}{|y|^{2p'}} \right)^{1/p'} \leq C|\boldsymbol{x}|^{1-3/p}. \tag{3.12}$$

From (3.11) and (3.12), we obtain (3.9). Next, from (3.9) and the Hölder's inequality, we get

$$|J_1\, f(\boldsymbol{x})| = \int_{\mathbb{R}^3} K_1(\boldsymbol{x}, \boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} \leq C|\boldsymbol{x}|^{1-3/p} \|f\|_{L^p(\mathbb{R}^3)}. \tag{3.13}$$

By similar arguments, we can prove that

$$|J_1\, f(\boldsymbol{x}) - J_1\, f(\boldsymbol{y})| \leq C|\boldsymbol{x} - \boldsymbol{y}|^{1-3/p} \|f\|_{L^p(\mathbb{R}^3)},$$

which shows that $J_1 \in \mathcal{C}^{0,1-3/p}(\mathbb{R}^3)$ if $p > 3$.
ii) We first assume $p > 3$. Let $f \in L^p(\mathbb{R}^3)$ and $(f_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3)$ be a sequence that tends to f in $L^p(\mathbb{R}^3)$. For any $k \in \mathbb{N}$, we have

$$J_1\, f_k(\boldsymbol{x}) = I_1\, f_k(\boldsymbol{x}) - \frac{1}{\gamma(1)} \int_{\text{supp } f_k} \frac{1}{|\boldsymbol{y}|^2} f_k(\boldsymbol{y})\, d\boldsymbol{y}.$$

This implies that $\frac{\partial}{\partial x_j} J_1 f_k = \frac{\partial}{\partial x_j} I_1 f_k = -R_j f_k$. Thanks to the continuity of the Riesz transform $R_j : L^p(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3)$ (see [21]), $-R_j f_k$ tends to $-R_j f$ as k tends to infinity. On the other hand, from (3.13), as k tends to infinity, $J_1 f_k$ tends $J_1 f$ in $W^{0,p}_{-1-\varepsilon}(\mathbb{R}^3)$, for any $\varepsilon > 0$. Hence, we obtain $\frac{\partial}{\partial x_j} J_1 f = -R_j f$ and

$$\|\nabla J_1 f\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}.$$

Besides, since $J_1 f(\mathbf{0}) = 0$, we have (see [4] Lemma 3.2)

$$\|J_1 f\|_{W^{0,p}_{-1}(\mathbb{R}^3)} \leq C\|\nabla J_1 f\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)},$$

which implies that the operator $J_1 : L^p(\mathbb{R}^3) \mapsto W^{1,p}_0(\mathbb{R}^3)$ is continuous if $p > 3$.
We assume now $p = 3$. Then the following inequality holds

$$\int_{\mathbb{R}^3} |K_1(\boldsymbol{x}, \boldsymbol{y})|^{3/2}\, d\boldsymbol{y} \leq C(1 + \ln(2 + |\boldsymbol{x}|)). \tag{3.14}$$

It follows that

$$|J_1 f(\boldsymbol{x}) - J_1 f(\boldsymbol{y})| \leq C(1 + \ln(2 + |\boldsymbol{x} - \boldsymbol{y}|)),$$

which shows that the operator $J_1 : L^3(\mathbb{R}^3) \mapsto W^{1,3}_0(\mathbb{R}^3)$ is continuous. $\square$

The definition below extends the Riesz potential I_1 for a distribution $f \in W^{-1,p}_0(\mathbb{R}^3)$.

Definition 3.3. *Let $f \in W^{-1,p}_0(\mathbb{R}^3)$. For any $\varphi \in L^{p'}(\mathbb{R}^3)$, we set*

$$\langle I_1 f, \varphi\rangle =: \langle f, I_1\varphi\rangle_{W^{-1,p}_0(\mathbb{R}^3)\times W^{1,p'}_0(\mathbb{R}^3)}, \quad \textit{when } p > 3/2.$$

Note that the above relation is well defined since the operator $I_1 : L^{p'}(\mathbb{R}^3) \mapsto W_0^{1,p'}(\mathbb{R}^3)$ is continuous if $1 < p' < 3$ and, by duality,

$$I_1 : W_0^{-1,p}(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3) \text{ is continuous if } 3/2 < p < \infty. \tag{3.15}$$

By the same way, we have the following definition for the operator J_1 acting on a distribution $f \in W_0^{-1,p}(\mathbb{R}^3)$.

Definition 3.4. *Let $f \in W_0^{-1,p}(\mathbb{R}^3)$. For any $\varphi \in L^{p'}(\mathbb{R}^3)$, we set*

$$\langle J_1 f, \varphi \rangle =: \langle f, J_1 \varphi \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)}.$$

Thanks to Theorem 3.2 (ii), we have the following result.

Proposition 3.5. *The operator $J_1 : W_0^{-1,p}(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3)$, is continuous if $1 < p \leq 3/2$.*

Remark 3.6. Let $1 < p \leq 3/2$ and $f \in W_0^{-1,p}(\mathbb{R}^3)$ satisfies

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0.$$

Then we have $J_1 f = I_1 f$. In other words, the operator $I_1 : W_0^{-1,p}(\mathbb{R}^3) \perp \mathbb{R} \mapsto L^p(\mathbb{R}^3)$ is continuous if $1 < p \leq 3/2$.

For any interval $I \subset \mathbb{R}$, let 1_I, be the function defined by

$$1_I(t) = 1 \quad \text{if } t \in I \quad \text{and} \quad 1_I(t) = 0 \quad \text{if } t \notin I.$$

We now introduce

$$\mathcal{E}(\boldsymbol{x}) = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{x}|}$$

the fundamental solution of the Poisson's equation in $\mathbb{R}^3$. Next, if $f \in L^p(\mathbb{R}^3)$, we set

$$\mathcal{P}_i f(\boldsymbol{x}) = \begin{cases} \displaystyle\int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} \mathcal{E}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y}, & \text{if } 1 < p < 3, \\ \displaystyle\int_{\mathbb{R}^3} \left(\frac{\partial}{\partial x_i} \mathcal{E}(\boldsymbol{x} - \boldsymbol{y}) - \frac{\partial}{\partial x_i} \mathcal{E}(-\boldsymbol{y}) \right) f(\boldsymbol{y}) d\boldsymbol{y} & \text{if } p \geq 3. \end{cases}$$

The previous definition will be summarised by the following one:

$$\mathcal{P}_i f\,(\boldsymbol{x}) = \int_{\mathbb{R}^3} \left(\frac{\partial}{\partial x_i}\, \mathcal{E}(\boldsymbol{x} - \boldsymbol{y}) - 1_{[3,\infty[}(p)\, \frac{\partial}{\partial x_i}\, \mathcal{E}(-\boldsymbol{y}) \right) f(\boldsymbol{y})\, d\boldsymbol{y}. \tag{3.16}$$

We will also introduce the operator:

$$\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3), \quad \boldsymbol{\mathcal{P}} \boldsymbol{f}(\boldsymbol{x}) = \mathcal{P}_i f_i(\boldsymbol{x}) \tag{3.17}$$

where the pair of identical indices implies implicit summation.

Proceeding as for Theorem 3.2, we prove the following

Theorem 3.7. *The operators*

$$\mathcal{P}_i : L^p(\mathbb{R}^3) \mapsto W_0^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \boldsymbol{\mathcal{P}} : \boldsymbol{L}^p(\mathbb{R}^3) \mapsto W_0^{1,p}(\mathbb{R}^3)$$

are contiuous for any $1 < p < \infty$.

Definition 3.8. *Let* $1 < p < \infty$ *and* $f \in W_0^{-1,p}(\mathbb{R}^3)$. *For any* $\varphi \in L^{p'}(\mathbb{R}^3)$, *we set*

$$\langle \mathcal{P}_i f, \varphi\rangle =: \langle f, \mathcal{P}_i\varphi\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}.$$

Consequently, from the previous, we have the following result.

Corollary 3.9. *The operator* $\mathcal{P}_i : W_0^{-1,p}(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3)$ *is continuous for any* $1 < p < \infty$.

Remark 3.10. Due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$, if $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$, $1 < p < \infty$, then

$$\Delta \boldsymbol{\mathcal{P}} \boldsymbol{f} = \operatorname{div} \boldsymbol{f} \text{ in } \mathbb{R}^3.$$

Moreover,

(i) If $1 < p < 3$, then we have

$$\boldsymbol{\mathcal{P}}\boldsymbol{f} = -\mathcal{E} * \operatorname{div} \boldsymbol{f}.$$

(ii) If $p \geq 3$, then, for any $i = 1, 2, 3$ we get

$$\frac{\partial}{\partial x_i}\boldsymbol{\mathcal{P}}\boldsymbol{f} = -\mathcal{E} * \frac{\partial}{\partial x_i}\operatorname{div} \boldsymbol{f}.$$

As a consequence of the previous remark we have the following lemma which gives an explicit form for the solution of the Poisson's equation.

Lemma 3.11. *Let* $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$. *Then the Poisson's equation*

$$\Delta u = \operatorname{div} \boldsymbol{f} \text{ in } \mathbb{R}^3,$$

has a solution $u \in W_0^{1,p}(\mathbb{R}^3)$, *unique up to an element of* $\mathbb{P}_{[1-3/p]}$, *satisfying*

$$\|\nabla u\|_{L^p(\mathbb{R}^3)} \leq C\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)}.$$

Moreover,

(i) *If* $1 < p < 3$, *then* $u = -\mathcal{E} * \operatorname{div} \boldsymbol{f} = \boldsymbol{\mathcal{P}}\boldsymbol{f}$.

(ii) *If* $p \geq 3$, *then* $u = \boldsymbol{\mathcal{P}}\boldsymbol{f} + \lambda$, $\lambda \in \mathbb{R}$ *and* $\frac{\partial u}{\partial x_i} = -\mathcal{E} * \frac{\partial}{\partial x_i}\operatorname{div} \boldsymbol{f}$.

We now consider the Riesz potential of order 2 and, for $f \in L^p(\mathbb{R}^3)$, we introduce the operator

$$J_2 f(\boldsymbol{x}) = \int_{\mathbb{R}^3} (\mathcal{E}(\boldsymbol{x}-\boldsymbol{y}) - 1_{[\frac{3}{2},\infty[}(p)\,\mathcal{E}(-\boldsymbol{y}) - 1_{[3,\infty[}(p)\,\boldsymbol{x}.\nabla\mathcal{E}(-\boldsymbol{y}))f(\boldsymbol{y})d\boldsymbol{y}, \tag{3.18}$$

where we observe that the operator J_2 coincides with I_2 when $1 < p < 3/2$. Below we recall further embedding results in weighted Sobolev spaces.

Theorem 3.12. *We have*

i) $W_0^{2,p}(\mathbb{R}^3) \hookrightarrow W_0^{\frac{3p}{3-p}}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{3-2p}}(\mathbb{R}^3)$, *if* $1 < p < 3/2$,

ii) $W_0^{2,3/2}(\mathbb{R}^3) \hookrightarrow W_0^{1,3}(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3)$,

iii) $W_0^{2,p}(\mathbb{R}^3) \hookrightarrow W_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3) \hookrightarrow \mathcal{C}^{0,2-3/p}(\mathbb{R}^3)$, *if* $3/2 < p < 3$.

iv) $W_0^{2,p}(\mathbb{R}^3) \hookrightarrow \mathcal{C}^{1,1-3/p}(\mathbb{R}^3)$ *if* $p > 3$.

Proceeding again as for Theorem 3.2, we prove the following result.

Theorem 3.13. *The operator* $J_2 : L^p(\mathbb{R}^3) \mapsto W_0^{2,p}(\mathbb{R}^3)$ *is continuous for any* $1 < p < \infty$. *Moreover, if* $f \in L^p(\mathbb{R}^3)$, *then the solutions of the Poisson's equation*

$$-\Delta u = f \quad \textit{in } \mathbb{R}^3$$

are in the form

$$u = J_2 f + \lambda, \;\; \lambda \in \mathbb{P}_{[2-3/p]}.$$

We are now interested in the operator I_2 when acting on distributions of $W_0^{-1,p}(\mathbb{R}^3)$.

Definition 3.14. *Let* $f \in W_0^{-1,p}(\mathbb{R}^3)$. *For any* $\varphi \in W_0^{1,p'}(\mathbb{R}^3)$, *we set*

$$\langle I_2 f, \varphi \rangle =: \langle f, I_2\varphi \rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}.$$

We easily verify that if $\varphi \to 0$ in $\mathcal{D}(\mathbb{R}^3)$, then $I_2\varphi \to 0$ in $W_0^{1,p'}(\mathbb{R}^3)$ if and only if $p' > 3/2$ (and $I_2\varphi \to 0$ in a space of the type $W_0^{1,3/2}(\mathbb{R}^3)$ if $p' = 3/2$, where $L^{3/2}(\mathbb{R}^3)$ is replaced by $L^{3/2,\infty}(\mathbb{R}^3)$). Hence, we have $I_2 f \in \mathcal{D}'(\mathbb{R}^3)$ if and only if $1 < p \leq 3$ and in this case, $-\Delta(I_2 f) = f$.

Proposition 3.15. *The following operators are continuous*

(i) $\quad I_2 : W_0^{-1,p}(\mathbb{R}^3) \to W_0^{1,p}(\mathbb{R}^3)$ *if* $3/2 < p < 3$,

(ii) $\quad I_2 : W_0^{-1,p}(\mathbb{R}^3)\perp\mathbb{R} \to W_0^{1,p}(\mathbb{R}^3)$ *if* $1 < p \leq 3/2$.

Proof. Assume $1 < p < 3$ and let $f \in W_0^{-1,p}(\mathbb{R}^3)$ satisfying $\langle f, 1\rangle = 0$ if $1 < p \leq 3/2$. Then $f = \operatorname{div} \boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$ and

$$\|\boldsymbol{F}\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}.$$

Hence for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle \frac{\partial}{\partial x_j} I_2 f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= \langle f, I_2 \frac{\partial \varphi}{\partial x_j}\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}\\ &= -\langle F_k, \frac{\partial^2}{\partial x_j \partial x_k} I_2\varphi\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},\end{aligned}$$

i.e.

$$\left\| \frac{\partial}{\partial x_j} I_2 f \right\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}.$$

By the Calderón-Zygmund inequality, we get

$$\begin{aligned} \left| \langle \frac{\partial}{\partial x_j} I_2 f, \varphi \rangle \right| &\leq \|F_k\|_{L^p(\mathbb{R}^3)} \left\| \frac{\partial^2}{\partial x_j \partial x_k} I_2 \varphi \right\|_{L^{p'}(\mathbb{R}^3)} \\ &\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)} \|\varphi\|_{L^{p'}(\mathbb{R}^3)}. \end{aligned} \tag{3.19}$$

Next, we can write

$$\langle I_2 f, \varphi \rangle = -\langle F_k, \frac{\partial}{\partial x_k} I_2 \varphi \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)} = \langle F_k, I_1 R_k \varphi \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)}.$$

But, since the operator $I_1 : L^{(p^*)'}(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is continuous if $\frac{1}{p'} = \frac{1}{(p^*)'} - \frac{1}{3}$, where $p^* = \frac{3p}{3-p}$, we have

$$\begin{aligned} |\langle F_k, I_1 R_k \varphi \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)}| &\leq \|F_k\|_{L^p(\mathbb{R}^3)} \|I_1 R_k \varphi\|_{L^{p'}(\mathbb{R}^3)} \\ &\leq C\|F_k\|_{L^p(\mathbb{R}^3)} \|R_k \varphi\|_{L^{(p^*)'}(\mathbb{R}^3)}. \\ &\leq C\|F_k\|_{L^p(\mathbb{R}^3)} \|\varphi\|_{L^{(p^*)'}}. \end{aligned} \tag{3.20}$$

We thus have the following inequality

$$\|I_2 f\|_{L^{p^*}(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \quad \square$$

Remark 3.16. If $1 < p \leq 3/2$, the continuity of I_2 on the space $W_0^{-1,p}(\mathbb{R}^3)$ does not hold. On the other hand, the mapping $f \mapsto I_2 f - \langle f, 1 \rangle F_2$ is continuous from $W_0^{-1,p}(\mathbb{R}^3)$ onto $W_0^{1,p}(\mathbb{R}^3)$ if $1 < p < 3/2$ and the mapping $f \mapsto I_2 f - \langle f, 1 \rangle$ is continuous from $W_0^{-1,p}(\mathbb{R}^3)$ onto $W_0^{1,p}(\mathbb{R}^3)$. if $1 < p \leq 3/2$, with $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \varphi = 1$.

Remark 3.17. If $p > 3$ and $f \in W_0^{-1,p}(\mathbb{R}^3)$, it is clear that $I_2 f$ does not belong to $\mathcal{D}'(\mathbb{R}^3)$. But, from (2.5), there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$ such that $f = \text{div } \boldsymbol{F}$ and

$$\|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}.$$

Therefore we introduce the operators

$$\mathcal{I}_1 \boldsymbol{F}(\boldsymbol{x}) = -\frac{1}{\gamma(2)} \int_{\mathbb{R}^3} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^3} F_j(\boldsymbol{y}) d\boldsymbol{y}$$

and

$$\mathcal{J}_1 \boldsymbol{F}(\boldsymbol{x}) = -\frac{1}{\gamma(2)} \int_{\mathbb{R}^3} \left(\frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^3} + \frac{y_j}{|\boldsymbol{y}|^3} \right) F_j(\boldsymbol{y}) d\boldsymbol{y}.$$

Observe first that for any $\boldsymbol{F} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$, we have:

$$\mathcal{I}_1 \boldsymbol{F} = \frac{\partial \mathcal{E}}{\partial x_j} * F_j = \mathcal{E} * \text{div } \boldsymbol{F}$$

and

$$\mathcal{J}_1\boldsymbol{F}(\boldsymbol{x}) = \frac{\partial \mathcal{E}}{\partial x_j} * F_j - \int_{\mathbb{R}^3} \frac{\partial \mathcal{E}(\boldsymbol{x})}{\partial x_j} F_j(\boldsymbol{x}) d\boldsymbol{x} = \mathcal{E} * \operatorname{div} \boldsymbol{F} - \int_{\mathbb{R}^3} \mathcal{E}(\boldsymbol{x}) \operatorname{div} \boldsymbol{F}(\boldsymbol{x}) d\boldsymbol{x}.$$

Now, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$, we have the following properties:

- If $1 < p < 3/2$, then there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$ such that

  $$\boldsymbol{f} - \langle \boldsymbol{f}, 1 \rangle \delta = \operatorname{div} \boldsymbol{F},$$

  in other words $f = \operatorname{div} (\boldsymbol{F} + \langle \boldsymbol{f}, 1 \rangle \nabla \mathcal{E})$ where $\nabla \mathcal{E} \in \boldsymbol{L}^{3/2,\infty}(\mathbb{R}^3)$ *i.e.*

  $$\sup_{\mu > 0} \mu^{3/2}\, mes\{\boldsymbol{x} \in \mathbb{R}^3,\, |\nabla \mathcal{E}(\boldsymbol{x})| > \mu\} < \infty.$$

  In this case we have $\mathcal{I}_1 \boldsymbol{F} \in W_0^{1,p}(\mathbb{R}^3)$ and

  $$-\Delta(\mathcal{I}_1 \boldsymbol{F} + \langle f, 1 \rangle \mathcal{E}) = f \text{ in } \mathbb{R}^3.$$

  Note that $\mathcal{I}_1 \boldsymbol{F} + \langle \boldsymbol{f}, 1 \rangle \mathcal{E}$ does not belong to $W_0^{1,p}(\mathbb{R}^3)$.

- if $p > 3/2$, then $f = \operatorname{div} \boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Moreover, $\mathcal{I}_1 \boldsymbol{F} \in W_0^{1,p}(\mathbb{R}^3)$ if $3/2 < p < 3$, $\mathcal{J}_1 \boldsymbol{F} \in W_0^{1,p}(\mathbb{R}^3)$ if $p \geq 3$ and we have

  $$-\Delta(\mathcal{I}_1 \boldsymbol{F}) = \operatorname{div} \boldsymbol{F} \quad \text{and} \quad -\Delta(\mathcal{J}_1 \boldsymbol{F}) = \operatorname{div} \boldsymbol{F}.$$

Finally, we have the following last result.

Theorem 3.18. *Let $f \in W_0^{-1,p}(\mathbb{R}^3)$ such that $\langle f, 1 \rangle = 0$ if $p \leq 3/2$ and consider the Laplace's equation*

$$-\Delta u = f \quad \text{in } \mathbb{R}^3.$$

i) *if $1 < p < 3$, then the previous equation has a unique solution $u = F_2 * f = \boldsymbol{\mathcal{P}} \boldsymbol{w}$, with* $\operatorname{div} \boldsymbol{w} = f$.
ii) *if $p \geq 3$, then $u = \boldsymbol{\mathcal{P}} \boldsymbol{w}$ is the unique solution, up to a constant of the previous equation and we have*

$$\|\nabla u\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^3)}.$$

4 The three dimensional Oseen potential

In this section, we consider the Oseen problem in $\mathbb{R}^3$:

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + k \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \text{ in } \mathbb{R}^3, \\ \operatorname{div} \boldsymbol{u} = g \text{ in } \mathbb{R}^3. \end{aligned} \tag{4.21}$$

For the investigation of (4.21) in weighted Sobolev spaces, we refer to [10], [12],[11], [7], [9] and [17]. Further works can be found in [8], [13] and [14]. We recall that the fundamental solution of Oseen $(\boldsymbol{\mathcal{O}}, \boldsymbol{e})$ can be written in the form (see [14]):

$$\mathcal{O}_{ij}(\boldsymbol{x}) = \left(\delta_{ij}\Delta - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)\Phi(\boldsymbol{x}), \ \ e_j = \frac{1}{4\pi}\frac{x_j}{|\boldsymbol{x}|^3}, \quad i,j = 1,2,3,$$

where

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi k}\int_0^{ks(\boldsymbol{x})/2\nu}\frac{1-e^{-t}}{t}\,dt, \quad s(\boldsymbol{x}) = |\boldsymbol{x}| - x_1.$$

It is well known that $\mathcal{O}_{ij} \in L^p(\mathbb{R}^3)$ if $2 < p < 3$ and $\nabla\mathcal{O}_{ij} \in \boldsymbol{L}^p(\mathbb{R}^3)$ if $4/3 < p < 3/2$ (see for instance [14], [9] and [2]). Moreover, if the data $(\boldsymbol{f}, g) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)$, then the Oseen problem (4.21) has an explicit solution $(\boldsymbol{u}^*, \pi^*) \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^3) \times \mathcal{C}^\infty(\mathbb{R}^3)$ defined by

$$\begin{aligned} u_i^* &= \mathcal{O}_{ij} * f_j + \frac{\partial\mathcal{E}}{\partial x_i} * g \\ \pi^* &= \frac{\partial\mathcal{E}}{\partial x_j} * f_j + g - \frac{\partial\mathcal{E}}{\partial x_1} * g. \end{aligned} \tag{4.22}$$

It is now natural to inquire about the validity of (4.22) if $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$ and if g belongs to a subspace of $W_0^{1,p}(\mathbb{R}^3)$ that will be specified in the remaining of the paper. For convenience, we introduce the notation $\boldsymbol{\mathcal{O}} * \boldsymbol{f}$ which denotes the vector field defined by $\mathcal{O}_{ij} * f_j$, $i = 1,2,3$. We first have the following properties that extend the results obtained in [2] for the scalar Oseen potential and can be obtained from [9]:

Proposition 4.1. *Let $\boldsymbol{f} \in L^p(\mathbb{R}^3)$. Then, for any $1 < p < \infty$, $\frac{\partial^2\boldsymbol{\mathcal{O}}}{\partial x_i\partial x_j} * \boldsymbol{f} \in L^p(\mathbb{R}^3)$ and $\frac{\partial\boldsymbol{\mathcal{O}}}{\partial x_1} * \boldsymbol{f} \in L^p(\mathbb{R}^3)$, in the sense of principal value, and we have the estimate*

$$\left\|\frac{\partial^2\boldsymbol{\mathcal{O}}}{\partial x_i\partial x_j} * \boldsymbol{f}\right\|_{L^p(\mathbb{R}^3)} + \left\|\frac{\partial\boldsymbol{\mathcal{O}}}{\partial x_1} * \boldsymbol{f}\right\|_{L^p(\mathbb{R}^3)} \leq C\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)}.$$

Morever,

(i) $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in \boldsymbol{L}^{\frac{3p}{3-2p}}(\mathbb{R}^3)$ *if* $1 < p < 3/2$ *and* $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in \boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^3)$ *if* $1 < p < 2$,

(ii) $\nabla\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ *if* $1 < p < 3$ *and* $\nabla\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ *if* $1 < p < 4$.

In both cases, we have the corresponding estimates.

Let us now introduce the following anisotropic weighted spaces

$$X_p(\mathbb{R}^3) = \left\{v \in L^p(\mathbb{R}^3), \ \ \frac{\partial v}{\partial x_1} \in W_0^{-2,p}(\mathbb{R}^3)\right\},$$

$$Y_0^{1,p}(\mathbb{R}^3) = \left\{v \in W_0^{1,p}(\mathbb{R}^3), \ \ \frac{\partial v}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)\right\}$$

and

$$Z_0^{2,p}(\mathbb{R}^3) = \left\{ v \in W_0^{2,p}(\mathbb{R}^3),\ \ \frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^3) \right\}.$$

These are Banach spaces when endowed respectively with the norms

$$\|v\|_{X_p(\mathbb{R}^3)} = \|v\|_{L^p(\mathbb{R}^3)} + \left\| \frac{\partial v}{\partial x_1} \right\|_{W_0^{-2,p}(\mathbb{R}^3)},$$

$$\|v\|_{Y_0^{1,p}(\mathbb{R}^3)} = \|v\|_{W_0^{1,p}(\mathbb{R}^3)} + \left\| \frac{\partial v}{\partial x_1} \right\|_{W_0^{-1,p}(\mathbb{R}^3)}$$

and

$$\|v\|_{Z_0^{2,p}(\mathbb{R}^3)} = \|v\|_{W_0^{2,p}(\mathbb{R}^3)} + \left\| \frac{\partial v}{\partial x_1} \right\|_{L^p(\mathbb{R}^3)}.$$

Observe that from Proposition 4.1, the operators $\boldsymbol{f} \mapsto \boldsymbol{\mathcal{O}} * \boldsymbol{f}$ from $\boldsymbol{L}^p(\mathbb{R}^3)$ into $\boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and $\boldsymbol{f} \mapsto \nabla \boldsymbol{\mathcal{O}} * \boldsymbol{f}$ from $\boldsymbol{L}^p(\mathbb{R}^3)$ into $\boldsymbol{L}^{\frac{4p}{4-p}}(\mathbb{R}^3)$ are continuous if $1 < p < 2$. This shows that, if $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$, then the explicit forms (4.22) are not necessarily defined for any $1 < p < \infty$. Therefore, following the ideas developed in Section 3, we define the operator $\boldsymbol{\mathcal{O}}$ defined by

$$(\boldsymbol{\mathcal{O}} \boldsymbol{f}(\boldsymbol{x}))_i = \mathcal{O}_{ij} f_j(\boldsymbol{x}) = \int_{\mathbb{R}^3} \big(\mathcal{O}_{ij}(\boldsymbol{x} - \boldsymbol{y}) - 1_{[2,\infty[}\, \mathcal{O}_{ij}(-\boldsymbol{y}) - 1_{[4,\infty[}\, \boldsymbol{x}'.\nabla' \mathcal{O}_{ij}(-\boldsymbol{y}) \big)\, f_j(\boldsymbol{y})\, d\boldsymbol{y},$$

where $\boldsymbol{x}' = (0, x_2, x_3)$ and $\nabla' = (0, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. Proceeding now as in Section 3, we have the following result.

Theorem 4.2. *The operator*

$$\boldsymbol{\mathcal{O}} : \boldsymbol{L}^p(\mathbb{R}^3) \mapsto \boldsymbol{Z}_0^{2,p}(\mathbb{R}^3)$$

is continuous for $1 < p < \infty$.

In order to give an explicit form for the solutions of the Oseen problem (4.21), when the data $(\boldsymbol{f}, g)$ do not belong to $\boldsymbol{\mathcal{D}}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)$ but in the space $\boldsymbol{L}^p(\mathbb{R}^3) \times Y_0^{1,p}(\mathbb{R}^3)$, we need this preliminary lemma.

Lemma 4.3. *Assume* $g \in Y_0^{1,p}(\mathbb{R}^3)$ *and let* $\mathcal{P}_j$ *be defined by* (3.16).
(i) *If* $1 < p < 3$, *then* $\nabla \mathcal{P}_j g \in \mathbf{Y}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3)$, $\mathcal{P}_j g \in Z_0^{2,p}(\mathbb{R}^3)$ *and we have the estimate*

$$\|\nabla \mathcal{P}_j g\|_{\mathbf{Y}_0^{1,p}(\mathbb{R}^3)} + \|\nabla \mathcal{P}_j g\|_{\mathbf{Y}_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3)} + \|\mathcal{P}_j g\|_{Z_0^{2,p}(\mathbb{R}^3)} \leq C \|g\|_{Y_0^{1,p}(\mathbb{R}^3)}. \tag{4.23}$$

Additionally, if $1 < p < 3/2$, *then* $\mathcal{P}_j g \in L^{\frac{3p}{3-2p}}(\mathbb{R}^3)$ *and the following estimate holds*

$$\|\mathcal{P}_j g\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^3)} \leq C \|g\|_{Y_0^{1,p}(\mathbb{R}^3)}. \tag{4.24}$$

(ii) *If $p \geq 3$, then $J_2 \frac{\partial g}{\partial x_j} \in Z_0^{2,p}(\mathbb{R}^3)$, where the operator J_2 is defined by* (3.18)*, and we have*

$$\left\| I_2 \frac{\partial g}{\partial x_j} \right\|_{Z_0^{2,p}(\mathbb{R}^3)} \leq C \|g\|_{Y_0^{1,p}(\mathbb{R}^3)}.$$

Proof. (i) If $1 < p < 3$, then $g \in Y_0^{1,p}(\mathbb{R}^3)$ implies that $g \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ (see (2.3)). Thanks to Proposition 3.7, $\mathcal{P}_j g \in W_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3)$ and we have

$$\|\mathcal{P}_j g\|_{W_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C \|g\|_{Y_0^{1,p}(\mathbb{R}^3)}.$$

Besides, due to the fact that $\nabla g \in L^p(\mathbb{R}^3)$, we also have $\mathcal{P}_j \nabla g \in W_0^{1,p}(\mathbb{R}^3)$. Now, let $(g_k)_{k\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}^3)$ be a sequence that tends to g in $Y_0^{1,p}(\mathbb{R}^3)$. We know that we have

$$\frac{\partial}{\partial x_1} \mathcal{P}_j g_k = \mathcal{P}_j \frac{\partial g_k}{\partial x_1}.$$

Moreover, since $1 < p < 3$, then g_k tends to g in $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ which implies that $\mathcal{P}_j g_k$ tends to $\mathcal{P}_j g$ in $W_0^{1,\frac{3p}{3-p}}(\mathbb{R}^3)$, in particular $\frac{\partial}{\partial x_1}\mathcal{P}_j g_k$ tends to $\frac{\partial}{\partial x_1}\mathcal{P}_j g$ in $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$. Besides, since $\frac{\partial g_k}{\partial x_1}$ tends to $\frac{\partial g}{\partial x_1}$ in $W_0^{-1,p}(\mathbb{R}^3)$, then $\mathcal{P}_j \frac{\partial g_k}{\partial x_1}$ tends to $\mathcal{P}_j \frac{\partial g}{\partial x_1}$ in $L^p(\mathbb{R}^3)$. This implies that $\frac{\partial}{\partial x_1}\mathcal{P}_j g = \mathcal{P}_j \frac{\partial g}{\partial x_1}$. By the same way, we prove that $\nabla \mathcal{P}_j g = \mathcal{P}_j \nabla g$. Thus, we deduce that $\nabla \mathcal{P}_j g \in W_0^{1,p}(\mathbb{R}^3)$ and $\frac{\partial}{\partial x_1}\mathcal{P}_j g \in L^p(\mathbb{R}^3)$ which implies that $\mathcal{P}_j g \in L^{\frac{3p}{3-2p}}(\mathbb{R}^3)$ if $1 < p < 3/2$ (see [8]). Moreover, $\mathcal{P}_j g \in Z_0^{2,p}(\mathbb{R}^3)$ and $\nabla \mathcal{P}_j g \in Y_0^{1,p}(\mathbb{R}^3)$ and the estimates (4.23) and (4.24) hold.

(ii) Since $\frac{\partial g}{\partial x_j} \in L^p(\mathbb{R}^3)$, from Theorem 3.13, we have $J_2 \frac{\partial g}{\partial x_j} \in W_0^{2,p}(\mathbb{R}^3)$. Proceeding as in the first part (i), we prove that

$$\frac{\partial}{\partial x_1} J_2 \left(\frac{\partial g}{\partial x_j} \right) = \frac{\partial}{\partial x_j} J_2 \left(\frac{\partial g}{\partial x_1} \right).$$

Since $\frac{\partial g}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ with $p \geq 3$, then we have $J_2(\frac{\partial g}{\partial x_1}) \in W_0^{1,p}(\mathbb{R}^3)$ and $\frac{\partial}{\partial x_j} J_2 \left(\frac{\partial g}{\partial x_1} \right) \in L^p(\mathbb{R}^3)$. $\square$

We now introduce the pair $(\boldsymbol{u}^*, \pi^*)$ defined by

$$\begin{aligned} u_i^* &= \mathcal{O}_{ij} f_j + \mathcal{P}_i g, \quad \pi^* = \mathcal{P}_j f_j + g - \mathcal{E} * \frac{\partial g}{\partial x_1} \quad \text{if } 1 < p < 3, \\ u_i^* &= \mathcal{O}_{ij} f_j - I_2 \frac{\partial g}{\partial x_i}, \quad \pi^* = \mathcal{P}_j (f_j + G_j) + g \quad \text{if } p \geq 3, \end{aligned} \tag{4.25}$$

where $\boldsymbol{G} \in L^p(\mathbb{R}^3)$ is a (non unique) vector field such that div $\boldsymbol{G} = \frac{\partial g}{\partial x_1}$. Next, we introduce the notations used in [7] (see also [6]) for the resolution of the Oseen problem (4.21). let γ, $\delta \in \mathbb{R}$ be such that $\gamma \in [3,4]$, $\gamma > p$, $\delta \in [\frac{3}{2}, 2]$, $\delta > p$. we define two real numbers $r = r(p,\gamma)$, $s = s(p,\delta)$ as follow:

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{\gamma} \quad \text{and} \quad \frac{1}{s} = \frac{1}{p} - \frac{1}{\delta}.$$

Finally, we also introduce the space of polynomials

$$\mathcal{N}_k = \left\{ (\boldsymbol{\lambda}, \mu) \in \mathbb{P}_k \times \mathbb{P}^{\Delta}_{k-1}, \;\; -\Delta \boldsymbol{\lambda} + \frac{\partial \boldsymbol{\lambda}}{\partial x_1} + \nabla \mu = \mathbf{0}, \;\; \operatorname{div} \boldsymbol{\lambda} = 0 \right\}.$$

Combining Theorem 2.6 of [7], Proposition 4.1, Theorem 4.2 and Lemma 4.3, we easily prove the following result which gives an explicit form for the solutions of the Oseen equations for $\boldsymbol{f} \in L^p(\mathbb{R}^3)$.

Theorem 4.4. *Let $(\boldsymbol{f}, g) \in L^p(\mathbb{R}^3) \times Y^{1,p}_0(\mathbb{R}^3)$. Then the Oseen problem (4.21) has at least one solution $(\boldsymbol{u}, p) \in Z^{2,p}_0(\mathbb{R}^3) \times W^{1,p}_0(\mathbb{R}^3)$ defined by*

$$\boldsymbol{u} = \boldsymbol{u}^* + \boldsymbol{\lambda}, \;\; p = p^* + \mu,$$

where $(\boldsymbol{u}^*, p^*)$ is given by (4.25), $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_{[2-3/p]}$ and we have the estimate

$$\left\| \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^3)} + \left\| \frac{\partial \boldsymbol{u}}{\partial x_1} \right\|_{L^p(\mathbb{R}^3)} + \|\nabla p\|_{L^p(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)} + \|g\|_{Y^{1,p}_0(\mathbb{R}^3)} \right).$$

Additionally,

(i) *If $1 < p < \frac{3}{2}$, then $\boldsymbol{u}^* \in L^s(\mathbb{R}^3)$ and*

$$\|\boldsymbol{u}^*\|_{L^s(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)} + \|g\|_{Y^{1,p}_0(\mathbb{R}^3)} \right).$$

(ii) *If $\frac{3}{2} < p < 3$, then $\nabla \boldsymbol{u}^* \in L^r(\mathbb{R}^3)$ and satisfies*

$$\|\nabla \boldsymbol{u}^*\|_{L^r(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)} + \|g\|_{Y^{1,p}_0(\mathbb{R}^3)} \right).$$

We now extend the definition of $\boldsymbol{\mathcal{O}} * \boldsymbol{f}$ in the case where $\boldsymbol{f} \in W^{-1,p}_0(\mathbb{R}^3)$ by setting

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \langle \boldsymbol{\mathcal{O}} * \boldsymbol{f}, \varphi \rangle =: \langle \boldsymbol{f}, \check{\boldsymbol{\mathcal{O}}} * \varphi \rangle_{W^{-1,p}_0(\mathbb{R}^3) \times W^{1,p'}_0(\mathbb{R}^3)}$$

where $\check{\boldsymbol{\mathcal{O}}}(\boldsymbol{x}) = \boldsymbol{\mathcal{O}}(-\boldsymbol{x})$.

Lemma 4.5. *Assume $1 < p < 4$ and $\boldsymbol{f} \in W^{-1,p}_0(\mathbb{R}^3) \perp \mathbb{P}_{[1-3/p']}$. Then $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$, $\nabla \boldsymbol{\mathcal{O}} \in L^p(\mathbb{R}^3)$ and we have*

$$\|\boldsymbol{\mathcal{O}} * \boldsymbol{f}\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} + \|\nabla \boldsymbol{\mathcal{O}} * \boldsymbol{f}\|_{L^p(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{W^{-1,p}_0(\mathbb{R}^3)}.$$

Moreover, the following assertions hold.

(i) *If $1 < p < 3$, then $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ and*

$$\|\boldsymbol{\mathcal{O}} * \boldsymbol{f}\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{W^{-1,p}_0(\mathbb{R}^3)}.$$

(ii) *If $p = 3$, then $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^q(\mathbb{R}^3)$, for any $q \geq 12$.*

(iii) *If $3 < p < 4$, then $\boldsymbol{\mathcal{O}} * \boldsymbol{f} \in L^\infty(\mathbb{R}^3)$.*

Proof. The above properties are proved for the fundamental solution $\mathcal{O}$ (see [2], Theorem 4.9) and the proof is similar for the Oseen fundamental solution $\boldsymbol{\mathcal{O}}$. $\square$

For $p \geq 4$ and $f \in W_0^{-1,p}(\mathbb{R}^3)$, we now define the following operator $\boldsymbol{\mathcal{O}}$ such for any $i, j = 1, 2, 3$,

$$\mathcal{O}_{ij} f(\boldsymbol{x}) = \int_{\mathbb{R}^3} \left(\frac{\partial}{\partial x_k} \mathcal{O}_{ij}(\boldsymbol{x} - \boldsymbol{y}) - \frac{\partial}{\partial x_k} \mathcal{O}_{ij}(-\boldsymbol{y}) \right) F_k(\boldsymbol{y})\, d\boldsymbol{y}, \tag{4.26}$$

where $\boldsymbol{F} \in L^p(\mathbb{R}^3)$ is a vector field such that $f = \operatorname{div} \boldsymbol{F}$. Thanks to Theorem 4.2, we have the following result.

Theorem 4.6. *The operator*

$$\boldsymbol{\mathcal{O}} : W_0^{-1,p}(\mathbb{R}^3) \mapsto W_0^{1,p}(\mathbb{R}^3)$$

is continuous if $p \geq 4$.

Before stating our last result, we need two preliminary lemmas that take into account the second equation of (4.21)

Lemma 4.7. *Assume* $g \in W_0^{-2,p}(\mathbb{R}^3) \perp \mathbb{P}_{[2-3/p']}$. *Then* $\mathcal{E} * g \in L^p(\mathbb{R}^3)$ *and we have*

$$\|\mathcal{E} * g\|_{L^p(\mathbb{R}^3)} \leq C \|g\|_{W_0^{-2,p}(\mathbb{R}^3)}.$$

Proof. For any $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\langle \mathcal{E} * g, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} = \langle g, \mathcal{E} * \varphi \rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)}.$$

Next, since $g \in W_0^{-2,p}(\mathbb{R}^3) \perp \mathbb{P}_{[2-3/p']}$, for any $\lambda \in \mathbb{P}_{[2-3/p']}$, we have

$$\langle g, \mathcal{E} * \varphi \rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} = \langle g, \mathcal{E} * \varphi + \lambda \rangle_{W_0^{-2,p}(\mathbb{R}^3)}.$$

It follows that

$$\left| \langle g, \mathcal{E} * \varphi \rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} \right| \leq C \|g\|_{W_0^{-2,p}(\mathbb{R}^3)} \inf_{\lambda \in \mathbb{P}_{[2-3/p']}} \|\mathcal{E} * \varphi + \lambda\|_{W_0^{2,p'}(\mathbb{R}^3)}.$$

Using now (2.2) and the Calderón-Zygmund inequality we can write

$$\begin{aligned}
\left| \langle g, \mathcal{E} * \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} \right| &\leq C \|g\|_{W_0^{-2,p}(\mathbb{R}^3)} \left\| \frac{\partial^2 (\mathcal{E} * \varphi)}{\partial x_i \partial x_j} \right\|_{L^{p'}(\mathbb{R}^3)} \\
&\leq C \|g\|_{W_0^{-2,p}(\mathbb{R}^3)} \|\Delta(\mathcal{E} * \varphi)\|_{L^{p'}(\mathbb{R}^3)} \\
&\leq C \|g\|_{W_0^{-2,p}(\mathbb{R}^3)} \|\|\varphi\|_{L^{p'}(\mathbb{R}^3)},
\end{aligned}$$

which ends the proof. $\square$

Lemma 4.8. *Assume $1 < p < \infty$ and $g \in X_p(\mathbb{R}^3)$ such that*

$$\forall \lambda \in \mathbb{P}_{[2-3/p']}, \quad \langle \frac{\partial g}{\partial x_1}, \lambda \rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} = 0.$$

Then $\mathcal{P}_j g \in Y_0^{1,p}(\mathbb{R}^3)$ and there exists $C > 0$ such that

$$\|\mathcal{P}_j g\|_{Y_0^{1,p}(\mathbb{R}^3)} \leq C \|g\|_{X_p(\mathbb{R}^3)}.$$

Proof. Since $g \in X_p(\mathbb{R}^3)$, then from Theorem 3.7, $\mathcal{P}_j g \in W_0^{1,p}(\mathbb{R}^3)$ and we have

$$\|\mathcal{P}_j g\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C \|g\|_{L^p(\mathbb{R}^3)}.$$

It remains to prove that $\frac{\partial}{\partial x_1} \mathcal{P}_j g \in W_0^{-1,p}(\mathbb{R}^3)$. For any $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\left| \left\langle \frac{\partial}{\partial x_1} \mathcal{P}_j g, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} \right| = \left| \left\langle g, \frac{\partial}{\partial x_1} (\check{\mathcal{P}}_j * \varphi) \right\rangle_{L^p(\mathbb{R}^3), L^{p'}(\mathbb{R}^3)} \right|,$$

where $\check{\mathcal{P}}_j(\boldsymbol{x}) = \mathcal{P}_j(-\boldsymbol{x})$. It follows that

$$\begin{aligned} \left| \left\langle \frac{\partial}{\partial x_1} \mathcal{P}_j g, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} \right| &= \left| \left\langle \frac{\partial g}{\partial x_1}, \mathcal{E} * \frac{\partial \varphi}{\partial x_j} \right\rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} \right| \\ &= \left| \left\langle \frac{\partial g}{\partial x_1} * \mathcal{E}, \frac{\partial \varphi}{\partial x_j} \right\rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)} \right| \end{aligned}$$

From Lemma 4.7, we get

$$\left| \left\langle \frac{\partial}{\partial x_1} \mathcal{P}_j g, \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} \right| \leq C \left\| \frac{\partial g}{\partial x_1} \right\|_{W_0^{-2,p}(\mathbb{R}^3)} \|\varphi\|_{W_0^{1,p'}(\mathbb{R}^3)}. \tag{4.27}$$

Thus we deduce that $\frac{\partial}{\partial x_1} \mathcal{P}_j g \in W_0^{-1,p}(\mathbb{R}^3)$ and

$$\left\| \frac{\partial}{\partial x_1} \mathcal{P}_j g \right\|_{W_0^{-1,p}(\mathbb{R}^3)} \leq C \left\| \frac{\partial g}{\partial x_1} \right\|_{W_0^{-2,p}(\mathbb{R}^3)}. \quad \square$$

For $\boldsymbol{f} \in W_0^{-1,p} \perp \mathbb{P}_{[1-3/p']}$ and $g \in X_p(\mathbb{R}^3)$, we now define the pair $(\boldsymbol{u}^*, p^*)$

$$\begin{aligned} & u_i^* = \mathcal{O}_{ij} * f_j + \mathcal{P}_j g \quad \text{if } 1 < p < 3, \quad u_i^* = \mathcal{O}_{ij} f_j + \mathcal{P}_j g \quad \text{if } p \geq 3, \\ & p^* = -\mathcal{E} * \left(\operatorname{div} \boldsymbol{f} + \frac{\partial g}{\partial x_1} \right) + g. \end{aligned} \tag{4.28}$$

Combining Theorem 2.2 of [7], Lemma 4.5, Theorem 4.6 and the previous lemmas, we easily prove the following result.

Theorem 4.9. *Let* $(\boldsymbol{f}, g) \in W_0^{-1,p}(\mathbb{R}^3) \times X_p(\mathbb{R}^3)$ *satisfy the following compatibility conditions*

$$\forall \boldsymbol{\lambda} \in \mathbb{P}_{[1-3/p']}, \quad \langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0$$

and

$$\forall \lambda \in \mathbb{P}_{[2-3/p']}, \quad \left\langle \frac{\partial g}{\partial x_1}, \lambda \right\rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} = 0.$$

Then the Oseen problem (4.21) *has at least a solution* $(\boldsymbol{u}, p) \in Y_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *defined by*

$$\boldsymbol{u} = \boldsymbol{u}^* + \lambda, \quad p = p^*,$$

where the pair $(\boldsymbol{u}^*, p^*)$ *is given by* (4.28) *and* $\boldsymbol{\lambda} \in \mathbb{P}_{[1-3/p]}$. *The following estimate holds*

$$\inf_{\boldsymbol{\lambda} \in \mathbb{P}_{[1-3/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{Y_0^{1,p}(\mathbb{R}^3)} + \|p^*\|_{L^p(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{W_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{X_p(\mathbb{R}^3)} \right).$$

Moreover, if $1 < p < 3$, *then* $\boldsymbol{u}^* \in L^r(\mathbb{R}^3)$ *and we have*

$$\|\boldsymbol{u}^*\|_{L^r(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{W_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{X_p(\mathbb{R}^3)} \right).$$

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