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# THE NON-SYMMETRIC OPERAD PRE-LIE IS FREE

NANTEL BERGERON AND MURIEL LIVERNET

ABSTRACT. We prove that the pre-Lie operad is a free non-symmetric operad.

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## INTRODUCTION

Operads are a specific tool for encoding type of algebras. For instance there are operads encoding associative algebras, commutative and associative algebras, Lie algebras, pre-Lie algebras, dendriform algebras, Poisson algebras and so on. A usual way of describing a type of algebras is by giving the generators and relations. For instance a Lie algebra  $L$  is a vector space together with a bilinear product, the bracket (the generator) satisfying the relations  $[x, y] = -[y, x]$  and  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$  for all  $x, y, z \in L$ . The vector space of all operations one can perform on  $n$  distinct variables in a Lie algebra is  $\mathcal{L}ie(n)$ , the building block of the symmetric operad  $\mathcal{L}ie$ . Composition in the operad corresponds to composition of operations. The vector space  $\mathcal{L}ie(n)$  has a natural action of the symmetric group, so it is a symmetric operad. The case of associative algebras can be considered in two different ways. An associative algebra  $A$  is a vector space together with a product satisfying the relation  $(xy)z = x(yz)$ . The vector space of all operations one can perform on  $n$  distinct variables in an associative algebra is  $\mathcal{A}s(n)$ , the building block of the symmetric operad  $\mathcal{A}s$ . The vector space  $\mathcal{A}s(n)$  has for basis the symmetric group  $S_n$ . But, in view of the relation, one can look also at the vector space of all ordered-preserving operations one can perform on  $n$  distinct ordered variables in an associative algebra: this is a vector space of dimension 1 generated by the only operation  $x_1 \cdots x_n$ . So the non-symmetric operad  $\widetilde{\mathcal{A}s}$  describing associative algebras is 1-dimensional for each  $n$ : this is the terminal object in the category of non-symmetric operads.

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Here is the picture between symmetric and non-symmetric operads. A symmetric operad  $\mathcal{P}$  starts with a graded vector space  $(\mathcal{P}(n))_{n \geq 0}$  together with an action of the symmetric group  $S_n$  on  $\mathcal{P}(n)$  for each  $n$ . This data is called a symmetric sequence or an  $\mathbb{S}$ -module or a vector species. There is a forgetful functor from vector species to graded vector spaces, forgetting the action of the symmetric group. This functor has a left adjoint  $\mathcal{S}$  which corresponds to tensoring by the regular representation of the symmetric group. A symmetric operad is an operad in the category of vector species and a non-symmetric operad is an operad in the category of graded vector spaces. The symmetric operad  $\mathcal{A}s$  is the image by  $\mathcal{S}$  of  $\star$ . It is clear that  $\mathcal{L}ie$  is not in the image of  $\mathcal{S}$  since the Jacobi relation does not respect the order of the variables  $x < y < z$  nor the anti-symmetry relation. Still one can regard  $\mathcal{L}ie$  as a non-symmetric operad applying the forgetful functor. Salvatore and Tauraso proved in [6] that the operad  $\mathcal{L}ie$  is a free non-symmetric operad.

A free non-symmetric operad describes type of algebras which have a set of generating operations and no relations between them. For instance, magmatic algebras are vector spaces together with a bilinear product. There is a well known free non-symmetric operad, built on Stasheff polytopes, see e.g. [7]. Algebras over this operad are vector spaces  $V$  together with an  $n$ -linear product:  $V^{\otimes n} \rightarrow V$  for each  $n$ . In a homotopy theory point of view, the category of operads is a Quillen category and free operads play an essential role in the homotopy category. One wants to replace an operad  $\mathcal{P}$  by a quasi-free resolution, that is, a free operad together with a differential such that its homology gives  $\mathcal{P}$ . For instance, a quasi-free resolution of  $\widetilde{\mathcal{A}s}$ , the non-symmetric operad for associative algebras is given by the Stasheff polytopes and algebras over it are  $A_\infty$ -algebras (associative algebras up to homotopy). This gives us the motivation for studying whether a given symmetric operad is free as a non-symmetric operad or not.

In this paper we prove that the operad pre-Lie is a free non-symmetric operad. Pre-Lie algebras are vector spaces together with a bilinear product satisfying the relation  $(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y)$ . The operad pre-Lie is based on labelled rooted trees which are of combinatorial interest. In the process of proving the main result, we describe another operad denoted  $\mathcal{T}_{\text{Max}}$  also based on rooted trees and having the advantage to be a set operad. We prove that it is a free non-symmetric operad. The link between the two operads is made via a gradation on labelled rooted trees.

## 1. THE PRE-LIE OPERAD AND ROOTED TREES

We first recall the definition of the pre-Lie operad based on labelled rooted trees as in [3]. For  $n \in \mathbb{N}^*$ , the set  $\{1, \dots, n\}$  is denoted by  $[n]$  and  $[0]$  denotes the empty set. The symmetric group on  $k$  letters is denoted by  $S_k$ . There are many equivalent definitions of operads and we refer to [5] for basics on operads. Here are the definitions needed for the sequel.

**Definition 1.1.** A *non-symmetric* operad is a graded vector space  $(\mathcal{P}(n))_{n \geq 1}$  together with compositions  $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \rightarrow \mathcal{P}(n + m - 1)$  for  $1 \leq i \leq n$  satisfying the

following relations: for  $a \in \mathcal{P}(n)$ ,  $b \in \mathcal{P}(m)$  and  $c \in \mathcal{P}(\ell)$

$$\begin{aligned} (a \circ_i b) \circ_{j+i-1} c &= a \circ_i (b \circ_j c), & \text{for } 1 \leq j \leq m, \\ (a \circ_i b) \circ_j c &= (a \circ_j c) \circ_{i+\ell-1} b, & \text{for } j < i. \end{aligned}$$

If in addition each  $\mathcal{P}(n)$  is acted on the right by the symmetric group  $S_n$  and the compositions are equivariant with respect to this action, then the collection  $(\mathcal{P}(n))_n$  forms a *symmetric operad*. An *algebra* over an operad  $\mathcal{P}$  is a vector space  $X$  endowed with evaluation maps

$$\begin{aligned} ev_n : \quad \mathcal{P}(n) \otimes X^{\otimes n} &\rightarrow X \\ p \otimes x_1 \otimes \dots \otimes x_n &\mapsto p(x_1, \dots, x_n) \end{aligned}$$

compatible with the compositions  $\circ_i$ : for  $p \in \mathcal{P}(n)$ ,  $q \in \mathcal{P}(m)$ ,  $x'_i \in X$  one has

$$(p \circ_i q)(x_1, \dots, x_{n+m-1}) = p(x_1, \dots, x_{i-1}, q(x_i, \dots, x_{i+m-1}), x_{i+m}, \dots, x_{n+m-1}).$$

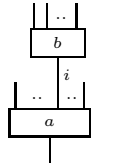
If the operad is symmetric the evaluation maps are required to be equivariant with respect to the action of the symmetric group as follows:

$$(p \cdot \sigma)(x_1, \dots, x_n) = p(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

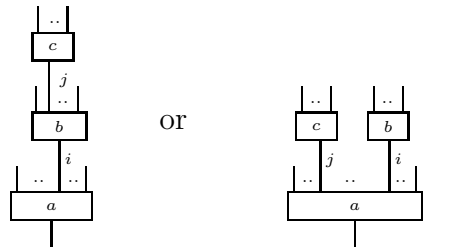
**Remark 1.2.** *Planar tree of operations:* a convenient way to uniquely represent composition of operations in a non-symmetric operad  $\mathcal{P}$  is to use a planar tree. An element  $a \in \mathcal{P}(n)$  is represented by a planar tree with a single trunk (outgoing edge), one vertex labelled by  $a$  and  $n$  leaves (incoming edges):



The  $n$  leaves are counted from left to right as  $1, 2, \dots, n$ . Now if we have  $a \in \mathcal{P}(n)$ ,  $b \in \mathcal{P}(m)$  and  $1 \leq i \leq n$  we represent the composition  $a \circ_i b$  by the planar tree



The resulting tree has  $n + m - 1$  leaves (counted from left to right) and represents an element of  $\mathcal{P}(n + m - 1)$ . The two relations in Definition 1.1 corresponds to the following two trees: for  $a \in \mathcal{P}(n)$ ,  $b \in \mathcal{P}(m)$  and  $c \in \mathcal{P}(\ell)$  we can have



Each relation is obtained by writing down the two ways of interpreting the tree as a composition of operations. In general a planar tree  $\mathbb{T}(a_1, a_2, \dots, a_k)$  with  $k$  vertices

labelled by elements  $a_i \in \mathcal{P}(n_i)$  where  $n_i$  is the number of incoming edges at the  $i$ th vertex, corresponds to a unique composition of operations in  $\mathcal{P}$  independent of any relations. Any full subtree of  $\mathbb{T}(a_1, a_2, \dots, a_k)$  is completely determined by the position of its leaves; they form an interval  $[p, q]$  where  $1 \leq p \leq q \leq n_1 + n_2 + \dots + n_k - k + 1$ .

**Definition 1.3.** Let  $S$  be a set. An  $S$ -labelled rooted tree is a non planar rooted tree whose vertices are in bijection with  $S$ . If  $S = [n]$ , then we talk about  $n$ -labelled rooted trees and denote by  $\mathcal{T}(n)$  the set of those trees. It is acted on by the symmetric group by permuting the labels.

The set  $\mathcal{T}(3)$  has for elements:

$$(1.1) \quad \begin{array}{ccccccc} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad / \\ \circ \\ 1 \end{array} & \begin{array}{c} 1 \quad 3 \\ \diagdown \quad / \\ \circ \\ 2 \end{array} & \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \circ \\ 3 \end{array} & \begin{array}{c} 3 \\ | \\ 2 \\ | \\ 1 \\ \circ \end{array} & \begin{array}{c} 2 \\ | \\ 3 \\ | \\ 1 \\ \circ \end{array} & \begin{array}{c} 1 \\ | \\ 3 \\ | \\ 2 \\ \circ \end{array} & \begin{array}{c} 3 \\ | \\ 1 \\ | \\ 2 \\ \circ \end{array} & \begin{array}{c} 2 \\ | \\ 1 \\ | \\ 3 \\ \circ \end{array} & \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \\ \circ \end{array} \end{array}$$

In general  $\mathcal{T}(n)$  has  $n^{n-1}$  elements (see [1] for more details).

We denote by  $k\mathcal{T}(n)$  the  $k$ -vector space spanned by  $\mathcal{T}(n)$ .

**Theorem 1.4.** [3, theorem 1.9] *The collection  $(k\mathcal{T}(n))_{n \geq 1}$  forms a symmetric operad, the operad pre-Lie denoted by  $\mathcal{PL}$ . Algebras over this operad are pre-Lie algebras, that is, vector spaces  $L$  together with a product  $*$  satisfying the relation*

$$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y), \quad \forall x, y, z \in L.$$

We recall the operad structure of  $\mathcal{PL}$  as explained in [3]. A rooted tree is naturally oriented from the leaves to the root, so it makes sense to define the set of incoming vertices of a vertex  $i$  denoted by  $\text{In}(T, i)$ . There is also at most one outgoing vertex of a vertex  $i$  depending whether  $i$  is the root of  $T$  or not. For  $T \in \mathcal{T}(n)$  and  $S \in \mathcal{T}(m)$ , we define

$$T \circ_i S = \sum_{f: \text{In}(T, i) \rightarrow [m]} T \circ_i^f S,$$

where  $T \circ_i^f S$  is the rooted tree obtained by substituting the tree  $S$  for the vertex  $i$  in  $T$ . The outgoing vertex of  $i$ , if it exists, becomes the outgoing vertex of the root of  $S$ , whereas the incoming vertices of  $i$  are grafted on the vertices of  $S$  according to the map  $f$ . The root of  $T \circ_i^f S$  is the root of  $T$  or the root of  $S$  in case  $i$  is the root of  $T$ . There is also a relabelling of the vertices of  $T$  and  $S$  in  $T \circ_i^f S$ : we add  $i - 1$  to the labels of  $S$  and  $m - 1$  to the ones of  $T$  which are greater than  $i$ . Here is an example:

$$(1.2) \quad \begin{array}{c} \begin{array}{c} 1 \quad 3 \\ \diagdown \quad / \\ \circ \\ 2 \end{array} \circ_2 \begin{array}{c} 1 \\ | \\ 2 \\ \circ \end{array} = \begin{array}{c} 1 \quad 4 \\ \diagdown \quad / \\ \circ \\ 2 \end{array} \circ_2 \begin{array}{c} 2 \\ | \\ 3 \\ \circ \end{array} = \begin{array}{c} 1 \quad 4 \\ \diagdown \quad / \\ \circ \\ 3 \end{array} + \begin{array}{c} 1 \\ | \\ 4 \\ | \\ 3 \\ \circ \end{array} + \begin{array}{c} 4 \\ | \\ 1 \\ | \\ 3 \\ \circ \end{array} + \begin{array}{c} 1 \quad 2 \quad 4 \\ \diagdown \quad / \quad / \\ \circ \\ 3 \end{array} \end{array}$$



2. A GRADATION ON LABELLED ROOTED TREES

We introduce a gradation on labelled rooted trees. We prove that in the expansion of the composition of two rooted trees in the operad pre-Lie there is a unique rooted tree of maximal degree and a unique tree of minimal degree, yielding new non-symmetric operad structures on labelled rooted trees.

**Definition 2.1.** Let  $T$  be an  $n$ -labelled rooted tree. Let  $\{a, b\}$  denote a pair of two adjacent vertices labelled by  $a$  and  $b$ . The degree of  $\{a, b\}$  is  $|a - b|$ . The degree of  $T$  denoted by  $\deg(T)$  is the sum of the degrees of its pairs of adjacent vertices. For instance

$$\deg(\begin{array}{c} 1 \quad 3 \\ \diagdown \quad / \\ \circ \\ / \quad \backslash \\ 2 \end{array}) = 2, \quad \deg(\begin{array}{c} 1 \quad 4 \\ \diagdown \quad / \\ \circ \\ | \\ \circ \\ / \quad \backslash \\ 3 \end{array}) = 4, \quad \deg(\begin{array}{c} 4 \\ | \\ \circ \\ / \quad \backslash \\ 2 \quad 1 \\ | \\ \circ \\ / \quad \backslash \\ 3 \end{array}) = 5, \quad \deg(\begin{array}{c} 1 \\ | \\ \circ \\ / \quad \backslash \\ 2 \quad 4 \\ | \\ \circ \\ / \quad \backslash \\ 3 \end{array}) = 3$$

**Proposition 2.2.** *In the expansion of  $T \circ_i S$  in the operad pre-Lie, there is a unique tree of minimal degree and a unique tree of maximal degree.*

For instance, in the equation (1.2) the rooted tree of minimal degree 3 is  and the one of maximal degree 5 is . The other ones are of degree 4.

*Proof*– Any tree in the expansion of  $T \circ_i S$  writes  $U_f := T \circ_i^f S$  for some  $f : \text{In}(T, i) \rightarrow [m]$ . To compute the degree of  $U_f$ , we compute the degree of a pair of two adjacent vertices  $\{a, b\}$  in  $U_f$ . There are 4 cases to consider: a) the pair was previously in  $S$  or b) it was previously in  $T$  and each vertex was different from  $i$ , or c) it was in  $T$  of the form  $\{i, j\}$  for  $j \in \text{In}(T, i)$  or d) if  $i$  is not the root of  $T$  it was of the form  $\{i, k\}$  where  $k$  is the outgoing vertex of  $i$ .

In case a) the degree of the pair in  $U_f$  is the same as it was in  $S$ .

In case b), let  $\{a', b'\}$  be the corresponding pair in  $T$  before relabelling. The degree  $d$  of the pair  $\{a, b\}$  in  $U_f$  is the same as the degree  $d'$  of  $\{a', b'\}$  except if  $a' < i < b'$  or  $b' < i < a'$ , where  $d = d' + m - 1$ . Let  $\text{gap}(T, i)$  be the number of adjacent pairs of vertices in  $T$  satisfying the latter condition.

In case c), let  $\{i, j\}$  be the pair in  $T$  which gives the pair  $\{a, b\}$  in  $U_f$ . Let  $d'$  be the degree of  $\{i, j\}$ . If  $j < i$  then  $\{a, b\} = \{f(j) + i - 1, j\}$ . Its degree  $d$  is minimal and equals  $d'$  if  $f(j) = 1$ . It is maximal and equals  $d' + m - 1$  if  $f(j) = m$ . If  $j > i$  then  $\{a, b\} = \{f(j) + i - 1, j + m - 1\}$ . Its degree  $d$  is minimal and equals  $d'$  if  $f(j) = m$ . It is maximal and equals  $d' + m - 1$  if  $f(j) = 1$ .

In case d), let  $d'$  be the degree of  $\{i, k\}$ . If  $k < i$  then  $\{a, b\} = \{s + i - 1, k\}$  where  $s$  is the label of the root of  $S$ . It has degree  $d' + s - 1$ . If  $k > i$ , then  $\{a, b\} = \{s + i - 1, k + m - 1\}$  and has degree  $(m - s) + d'$ . Let  $\epsilon(T, i, s)$  be  $0, s - 1, m - s$  according to the different situations, 0 corresponding to the one where  $i$  is the root of  $T$ .

As a conclusion

$$\begin{aligned} \deg(T) + \deg(S) + \text{gap}(T, i)(m-1) + \epsilon(T, i, s) &\leq \deg(U_f) \leq \\ \deg(T) + \deg(S) + \text{gap}(T, i)(m-1) + \epsilon(T, i, s) + |\text{In}(T, i)|(m-1). \end{aligned}$$

There is a unique  $f_{\text{Min}}$  such that  $\deg(U_{f_{\text{Min}}})$  is minimal and there is a unique  $f_{\text{Max}}$  such that  $\deg(U_{f_{\text{Max}}})$  is maximal:

$$(2.1) \quad f_{\text{Min}}(k) = \begin{cases} 1 & \text{if } k < i, \\ m & \text{if } k > i, \end{cases}$$

$$(2.2) \quad f_{\text{Max}}(k) = \begin{cases} m & \text{if } k < i, \\ 1 & \text{if } k > i, \end{cases}$$

which ends the proof.  $\square$

**Theorem 2.3.** *There are two different non-symmetric operad structures on the collection  $(k\mathcal{T}(n))_{n \geq 1}$  given by the compositions  $T \circ_i^{f_{\text{Min}}} S$  on the one hand and  $T \circ_i^{f_{\text{Max}}} S$  on the other hand where  $f_{\text{Min}}$  and  $f_{\text{Max}}$  were defined in equations (2.1) and (2.2).*

*Proof*– For a rooted tree  $T$ , let  $E_T$  be the set of the oriented edges of the tree: an edge is denoted  $(a, b)$  where  $a, b$  are the labels of its extremal vertices, and the orientation of the tree is from the leaves to the root. For an integer  $a \neq i$  we denote by  $\tilde{a}_i^m$  the integer  $a$  if  $a < i$  or  $a + m - 1$  if  $a > i$ . Given a map  $f : \text{In}(T, i) \rightarrow [m]$ , the set  $E_{T \circ_i^f S}$  has different type of elements:

- $(a + i - 1, b + i - 1)$  for  $(a, b) \in E_S$ ;
- $(\tilde{a}_i^m, \tilde{b}_i^m)$  for  $(a, b) \in E_T$  and  $a, b \neq i$ ;
- $(\tilde{a}_i^m, f(a) + i - 1)$  for  $(a, i) \in E_T$ ;
- $(i + s - 1, \tilde{b}_i^m)$  for  $(i, b) \in E_T$ .

Let  $T \in \mathcal{T}(n), S \in \mathcal{T}(m)$  and  $U \in \mathcal{T}(p)$ . In order to avoid confusion, we denote by  $f_{\text{Max}}^{i,p}$  the map sending  $k < i$  to  $p$  and  $l > i$  to 1. We would like to compare the trees

$$V_1 = (T \circ_i^{f_{\text{Max}}^{i,m}} S) \circ_{j+i-1}^{f_{\text{Max}}^{j+i-1,p}} U \quad \text{and} \quad V_2 = T \circ_i^{f_{\text{Max}}^{i,m+p-1}} (S \circ_j^{f_{\text{Max}}^{j,p}} U) :$$

- In  $V_1$  and  $V_2$ , any  $(a, b) \in E_U$  converts to  $(a + j + i - 2, b + j + i - 2)$ .
- In  $V_1$  and  $V_2$ , any  $(a, b) \in E_S$  converts to  $(\tilde{a}_j^p + i - 1, \tilde{b}_j^p + i - 1)$  if  $a, b \neq j$ , or converts to  $(\tilde{a}_j^p + i - 1, f_{\text{Max}}^{j,p}(a) + i + j - 2)$  if  $b = j$  or converts to  $(j + i - 1 + u - 1, \tilde{b}_j^p + i - 1)$  if  $a = j$ .
- In  $V_1$  and  $V_2$ , any  $(a, b) \in E_T$  with  $a, b \neq i$  converts to  $(\tilde{a}_i^{p+m-1}, \tilde{b}_i^{p+m-1})$ .
- In  $V_1$  and  $V_2$ , any  $(a, i) \in E_T$  converts to  $(\tilde{a}_i^{p+m-1}, f_{\text{Max}}^{i,m+p-1}(a) + i - 1)$ .

- In  $V_1$  and  $V_2$ , any  $(i, b) \in E_T$  converts to  $(i - 1 + \text{root}(S \circ_j U), \tilde{b}_i^{m+p-1})$ , where  $\text{root}(S \circ_j U)$  is the root of  $S \circ_j U$ . More precisely

$$\text{root}(S \circ_j U) = \begin{cases} s & \text{if } s < j \\ u + j - 1 & \text{if } s = j \\ s + p - 1 & \text{if } s > j. \end{cases}$$

The proof of

$$(T \circ_i^{f_{\text{Max}}^{i,m}} S) \circ_j^{f_{\text{Max}}^{j,p}} U = (T \circ_j^{f_{\text{Max}}^{j,p}} U) \circ_{i+p-1}^{f_{\text{Max}}^{i+p-1,m}} S, \text{ for } j < i$$

is similar and left to the reader. So is the proof with  $f_{\text{Min}}$  instead of  $f_{\text{Max}}$ .  $\square$

The two operads on labelled rooted trees defined by the theorem are denoted by  $\mathcal{T}_{\text{Max}}$  and  $\mathcal{T}_{\text{Min}}$ . Note that they are set operads, that is, the composition is defined at the level of the sets  $\mathcal{T}(n)$  and not only at the level of the vector spaces  $k\mathcal{T}(n)$ . There is another set operad built on rooted trees: the operad NAP encoding non-associative permutative algebras in [4], in which  $f_{\text{NAP}}$  is the constant map with value the root of  $S$ . This operad has the advantage to be a symmetric operad.

### 3. THE OPERAD PRE-LIE IS FREE AS A NON-SYMMETRIC OPERAD

We show that  $\mathcal{T}_{\text{Max}}$  is a free non-symmetric operad. Using Proposition 2.2, we conclude that the operad pre-Lie is free as a non-symmetric operad. To this end we need to introduce some notation on rooted trees.

**Definition 3.1.** Given two ordered sets  $S$  and  $T$ , an ordered preserving bijection  $\phi : S \rightarrow T$  induces a natural bijection between the set of  $S$ -labelled rooted trees and the set of  $T$ -labelled rooted trees also denoted by  $\phi$ . A  $T$ -labelled rooted tree  $X$  is *isomorphic* to an  $S$ -labelled rooted tree  $Y$  if  $X = \phi(Y)$ .

Given a rooted tree  $T \in \mathcal{T}(n)$  and a subset  $K \subseteq [n]$ , we denote by  $T|_K$  the graph obtained from  $T$  by keeping only the vertices of  $T$  that are labelled by elements of  $K$  and only the edges of  $T$  that have two vertices labelled in  $K$ . Remark that each connected component of  $T|_K$  is a rooted tree itself where the root is given by the unique vertex closest to the root of  $T$  in the component. Also, for  $c \in [n]$  we denote by  $T^{(c)}$  the full subtree of  $T$  derived from the vertex labelled by  $c$ . For example if  $K = \{2, 3, 4, 5, 6\} \subset [7]$  and

$$T = \begin{array}{c} \begin{array}{ccc} 2 & & 7 \\ & \diagdown & / \\ & 6 & \\ & / & \diagdown \\ 5 & & 4 \\ & \diagup & \diagdown \\ & 1 & \\ & / & \diagdown \\ & 3 & \end{array} \end{array}, \quad \text{we have} \quad T|_K = \begin{array}{c} \begin{array}{ccc} 2 & & \\ & \diagdown & / \\ & 6 & \\ & / & \diagdown \\ 5 & & 4 \\ & \diagup & \diagdown \\ & 3 & \end{array} \end{array} \quad \text{and} \quad T^{(1)} = \begin{array}{c} \begin{array}{ccc} 2 & & 7 \\ & \diagdown & / \\ & 6 & \\ & / & \diagdown \\ & 1 & \end{array} \end{array}.$$

For  $1 \leq a < b \leq n$ ,  $T \in \mathcal{T}_{\text{Max}}(n - b + a)$  and  $S \in \mathcal{T}_{\text{Max}}(b - a + 1)$ , let  $X = T \circ_a S$ . Consider the interval  $[a, b] = \{a, a + 1, \dots, b\}$ , clearly  $X|_{[a,b]}$  is isomorphic to  $S$  under the unique ordered bijection  $[1, b - a + 1] \rightarrow [a, b]$ . Let  $a \leq c \leq b$  be the label of the root of  $X|_{[a,b]}$ . Remark that  $X^{(c)}$  and  $X|_{[a,b]}$  differ only at  $a$  and  $b$  where there could be subtrees of  $X$  that do not belong to  $X|_{[a,b]}$ . We can then characterize trees  $X$  that are obtained from a nontrivial operation  $T \circ_a S$  as follows:



**Definition 3.2.** A tree  $X \in \mathcal{T}_{\text{Max}}(n)$  is called *decomposable* if there exists  $1 \leq a < b \leq n$  with  $(a, b) \neq (1, n)$  such that

- (i)  $X|_{[a,b]}$  is a rooted tree. Let  $c$  be the label of its root. One has  $a \leq c \leq b$ .
- (ii) The subtree  $X^{(c)}$  contains  $X|_{[a,b]}$  and differs from it only at  $a$  and  $b$ .
- (iii) All subtrees in  $X^{(c)} - X|_{[a,b]}$  attached at  $a$  have their root labelled in  $[b+1, n]$ .
- (iv) All subtrees in  $X^{(c)} - X|_{[a,b]}$  attached at  $b$  have their root labelled in  $[1, a-1]$ .

It is clear from the discussion above and the definition of the operad  $\mathcal{T}_{\text{Max}}$  that  $X$  is decomposable if and only if it is the result of a non-trivial composition. Consequently, we say that  $X$  is *indecomposable* if it is not decomposable. That is there is no  $1 \leq a < b \leq n$  such that (i)–(iv) are satisfied. For example let

$$X = \begin{array}{c} \begin{array}{c} 7 \quad 4 \\ \diagdown \quad \diagup \\ 1 \quad 3 \quad 2 \\ \diagdown \quad \diagup \\ 8 \quad 5 \\ \diagdown \quad \diagup \\ \quad 6 \end{array} \end{array}, \quad X|_{[3,5]} = \begin{array}{c} 4 \\ \diagup \\ 3 \\ \diagdown \\ 5 \end{array} \quad \text{and} \quad X^{(5)} = \begin{array}{c} \begin{array}{c} 7 \quad 4 \\ \diagdown \quad \diagup \\ 1 \quad 3 \quad 2 \\ \diagdown \quad \diagup \\ \quad 5 \end{array} \end{array},$$

This tree  $X$  is decomposable since for  $1 \leq 3 < 5 \leq 8$  we have that  $X|_{[3,5]}$  is a single tree and the subtree of  $X^{(5)} - X|_{[3,5]}$  are attached at 3 and 5 only. Moreover, the subtree attached at 3 has root labelled by  $7 \in [6, 8]$  and the subtrees attached at 5 have roots labelled by  $1, 2 \in [1, 2]$ . Indeed, in  $\mathcal{T}_{\text{Max}}$  we have

$$X = \begin{array}{c} \begin{array}{c} 1 \quad 5 \quad 2 \\ \diagdown \quad \diagup \\ 6 \quad 3 \\ \diagdown \quad \diagup \\ \quad 4 \end{array} \end{array} \circ_3 \begin{array}{c} 2 \\ \diagup \\ 1 \\ \diagdown \\ 3 \end{array}.$$

The reader may check that the following are all the indecomposable trees of  $\mathcal{T}_{\text{Max}}$  up to arity 3:

$$\begin{array}{c} 2 \\ \diagup \\ 1 \\ \diagdown \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ \diagup \\ 1 \\ \diagdown \\ 2 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \quad 3 \\ \diagdown \quad \diagup \\ \quad 2 \end{array}.$$

**Theorem 3.3.** *The non-symmetric operad  $\mathcal{T}_{\text{Max}}$  is a free non-symmetric operad.*

*Proof.* If  $\mathcal{T}_{\text{Max}}$  is not free, then for some  $n$  there is a tree  $X \in \mathcal{T}_{\text{Max}}(n)$  with two distinct constructions from indecomposables. In Remark 1.2, a composition of operations is completely determined by a unique planar rooted tree. We then have that  $X = \mathbb{T}(T_1, T_2, \dots, T_r) = \mathbb{Y}(S_1, S_2, \dots, S_k)$  where  $T_1, \dots, T_r, S_1, \dots, S_k$  are indecomposables and  $\mathbb{T}(T_1, T_2, \dots, T_r)$  and  $\mathbb{Y}(S_1, S_2, \dots, S_k)$  are two distinct trees of operations in  $\mathcal{T}_{\text{Max}}$  with  $r, s > 1$ .

Since  $X$  is decomposable, we can find  $1 \leq a < b \leq n$ , such that  $X|_{[a,b]}$  is isomorphic to a single  $T_i$  in position  $[a, b]$  in  $\mathbb{T}(T_1, T_2, \dots, T_r)$ . Moreover  $X|_{[a,b]}$  satisfies (i)–(iv) of Definition 3.2.

If  $X|_{[a,b]}$  is also isomorphic to a tree  $S_j$  in position  $[a, b]$  in  $\mathbb{Y}(S_1, S_2, \dots, S_k)$ , then we replace  $X$  by the smaller tree in  $\mathcal{T}_{\text{Max}}(n - b + a)$  that we obtain by removing  $T_i$  in  $\mathbb{T}(T_1, T_2, \dots, T_r)$  and removing  $S_j$  in  $\mathbb{Y}(S_1, S_2, \dots, S_k)$ . Clearly, this new smaller  $X$  has two distinct constructions from indecomposables. We can thus assume that  $X|_{[a,b]}$  is not isomorphic to a single  $S_j$  in position  $[a, b]$  in  $\mathbb{Y}(S_1, S_2, \dots, S_k)$ .

We now study how  $X|_{[a,b]}$  overlap in the position  $[a, b]$  of  $\mathbb{Y}(S_1, S_2, \dots, S_k)$ . Remark first that since all  $S_j$  are indecomposables, the interval  $[a, b]$  cannot be part of a single  $S_j$  of  $\mathbb{Y}(S_1, S_2, \dots, S_k)$ . Indeed, that would imply that  $S_j$  would contain a subtree satisfying Definition 3.2 which would be a contradiction.

We may assume that  $a > 1$ . To see this, assume that the only sub-interval  $[a, b] \subset [1, n]$  such that  $X|_{[a,b]}$  is isomorphic to a single  $T_i$  in position  $[a, b]$  in  $\mathbb{T}(T_1, T_2, \dots, T_r)$  is such that  $a = 1$ . Assume moreover that the only sub-interval  $[a', b'] \subset [1, n]$  such that  $X|_{[a',b']}$  is isomorphic to a single  $S_j$  in position  $[a', b']$  in  $\mathbb{Y}(S_1, S_2, \dots, S_k)$  is such that  $a' = 1$ . Since  $S_j$  is indecomposable, we must have  $b > b'$ . Similarly, since  $T_i$  is indecomposable, we must have  $b < b'$ . This implies that  $b = b'$  and  $T_i = S_j$ . This possibility was excluded above. So we must have  $a > 1$  or  $a' > 1$ . In the case where  $a = 1$  and  $a' > 1$  we could just interchange the role of  $\mathbb{T}(T_1, T_2, \dots, T_r)$  and  $\mathbb{Y}(S_1, S_2, \dots, S_k)$  and assume that we have  $a > 1$ .

Now, since  $T_j$  is indecomposable, there is no subinterval  $[c, d] \subseteq [a, b]$  such that  $X|_{[c,d]}$  is isomorphic to a full subtree of operations  $\mathbb{Y}'(S_{j_1}, S_{j_2}, \dots, S_{j_\ell})$ . Hence we can find  $c < a \leq d < b$  such that  $X|_{[c,d]} \cong \mathbb{Y}'(S_{j_1}, S_{j_2}, \dots, S_{j_\ell})$  satisfies the Definition 3.2.

The graph  $X|_{[a,d]}$  belongs to both  $X|_{[a,b]}$  and  $X|_{[c,d]}$ . Let  $e$  be the label of the root of  $X|_{[a,b]}$  and  $f$  the root of  $X|_{[c,d]}$ . The two full subtrees  $X^{(e)}$  and  $X^{(f)}$  both contain  $X|_{[a,d]}$ . This implies that either  $X^{(f)}$  is fully contained in  $X^{(e)}$ , or  $X^{(e)}$  is fully contained in  $X^{(f)}$ .

Let us assume that  $X^{(f)}$  is fully contained in  $X^{(e)}$ , that means  $X|_{[a,b]}$  and  $X|_{[c,d]}$  are both subtrees of  $X^{(e)}$ . From Definition 3.2, we know that the difference between  $X^{(e)}$  and  $X|_{[a,b]}$  is only some subtrees at  $a$  and  $b$ . The subtree  $X|_{[c,d]}$  contains  $a$  but does not contain  $b$ . Moreover, the point  $c$  is not in  $X|_{[a,b]}$ , so it must be in a subtree of  $X^{(e)}$  attached to  $a$ . The unique shortest path between  $a$  and  $c$  in  $X|_{[c,d]}$  belongs to that subtree of  $X^{(e)}$ . Since it is attached at  $a$  and is not in  $X|_{[a,b]}$ , by (iii) of Definition 3.2, it must have a root  $r \in [b + 1, n]$ . This is a contradiction, the root  $r$  is part of any path joining  $a$  and  $c$  and  $r \notin [c, d]$ , hence not in  $X|_{[c,d]}$ . The case where  $X^{(e)}$  is fully contained in  $X^{(f)}$  is argued similarly, using condition (iv) of Definition 3.2, and leads to a contradiction as well. We must conclude that  $\mathcal{T}_{\text{Max}}$  is free.  $\square$

**Remark 3.4.** The non-symmetric operads  $\mathcal{T}_{\text{Min}}$  and NAP are not free. Indeed, in the operad  $\mathcal{T}_{\text{Min}}$  one has the following relation:

$$\begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} \circ_1 \begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} = \begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} \circ_2 \begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} = \begin{array}{c} 3 \\ \bullet \\ | \\ \bullet \\ 2 \\ | \\ \bullet \\ 1 \end{array}$$

And in the operad NAP one has the following relation

$$\begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} \circ_1 \begin{array}{c} 1 \\ \bullet \\ | \\ \bullet \\ 2 \end{array} = \begin{array}{c} 1 \\ \bullet \\ | \\ \bullet \\ 2 \end{array} \circ_2 \begin{array}{c} 2 \\ \bullet \\ | \\ \bullet \\ 1 \end{array} = \begin{array}{c} 1 \quad 3 \\ \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \\ 2 \end{array}$$

**Remark 3.5.** Let  $k\mathcal{T}_{\text{Max}}^0(n)$  denote the  $k$ -vector space spanned by the indecomposables of  $\mathcal{T}_{\text{Max}}(n)$  and let  $\beta_n$  be its dimension. Let  $\alpha(x) = \sum_{n \geq 1} \alpha_n x^n$  be the Hilbert serie associated to the free non-symmetric operad generated by the vector spaces  $k\mathcal{T}_{\text{Max}}^0(n)$ . It is well known (see e.g. [6]) that one has the identity

$$\beta(\alpha(x)) + x = \alpha(x),$$

where  $\beta(x) = \sum_{n \geq 2} \beta_n x^n$ . Theorem 3.3 implies that  $\alpha_n = n^{n-1}$ . As a consequence, we get that the Hilbert series for indecomposable of  $\mathcal{T}_{\text{Max}}$  is

$$\begin{aligned} \mathcal{H}_{\mathcal{T}_{\text{Max}}^0}(x) &= \sum_{n \geq 0} \dim(k\mathcal{T}_{\text{Max}}^0(n)) x^n = 2x^2 + x^3 + 14x^4 + 146x^5 + \\ &+ 1994x^6 + 32853x^7 + 630320x^8 + 13759430x^9 + \dots \end{aligned}$$

**Corollary 3.6.** *The non-symmetric operad pre-Lie is a free non-symmetric operad.*

*Proof.* The natural basis of the vector spaces  $\mathcal{PL}(n)$  and  $\mathcal{T}_{\text{Max}}(n)$  are the same and is given by the set of  $n$ -labelled rooted trees  $\mathcal{T}(n)$ . However the operad compositions differ. Following Section 2, we say that a tree in  $\mathcal{PL}$  is indecomposable if it is not the element of maximal degree in any non-trivial composition of trees in  $\mathcal{PL}$ . That is to say it is indecomposable in  $\mathcal{T}_{\text{Max}}$ . From Theorem 3.3, any rooted tree  $X$  is constructed from indecomposables in  $\mathcal{T}_{\text{Max}}$  in a unique way  $\mathbb{T}(T_1, T_2, \dots, T_r)$ . If we consider the sequence of operations  $\mathbb{T}(T_1, T_2, \dots, T_r)$  in  $\mathcal{PL}$ , then  $X$  would be the unique maximal degree element in the result, by Proposition 2.2. By triangularity, we have that the set (in  $\mathcal{PL}$ )

$$\{\mathbb{T}(T_1, T_2, \dots, T_r) : T_i \text{ indecomposable and } \mathbb{T} \text{ any planar tree of operations}\}$$

is linearly independent and span  $\mathcal{PL}$ . This means  $\mathcal{PL}$  is free.  $\square$

**Remark 3.7.** The Hilbert Series for indecomposables and the operad  $\mathcal{PL}$  are the same as in Remark 3.5.

**Remark 3.8.** In the previous work [2] we have shown that as symmetric operads we have that  $\mathcal{PL} = \mathcal{Lie} \circ F$  for some graded vector space  $F$ . We have also shown that  $F$  is naturally equipped with a structure of non-symmetric operad. One can readily adapt the method developed here to show that  $F$  is also free as a non-symmetric operad.

## REFERENCES

- [1] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.
- [2] Nantel Bergeron and Muriel Livernet. A combinatorial basis for the free Lie algebra of the labelled rooted trees. preprint, arXiv:0707.4460, 2007.
- [3] Frédéric Chapoton and Muriel Livernet. Pre-Lie algebras and the rooted trees operad. *Internat. Math. Res. Notices*, 8:395–408, 2001.
- [4] Muriel Livernet. A rigidity theorem for pre-Lie algebras. *J. Pure Appl. Algebra*, 207(1):1–18, 2006.

- [5] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [6] Paolo Salvatore and Roberto Tauraso. The operad Lie is free. arXiv:0802.3010, 2008.
- [7] James Dillon Stasheff. Homotopy associativity of  $H$ -spaces. I, II. *Trans. Amer. Math. Soc.* 108 (1963), 275-292; *ibid.*, 108:293-312, 1963.

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