



**HAL**  
open science

## MAXIMAL BROWNIAN MOTIONS

Jean Brossard, Michel Emery, Christophe Leuridan

► **To cite this version:**

Jean Brossard, Michel Emery, Christophe Leuridan. MAXIMAL BROWNIAN MOTIONS. Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques, 2009, 45 (3), pp.876-886. 10.1214/08-AIHP194 . hal-00337008

**HAL Id: hal-00337008**

**<https://hal.science/hal-00337008>**

Submitted on 5 Nov 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# Maximal Brownian motions

Jean Brossard<sup>1</sup>, Michel Émery<sup>2</sup>, and Christophe Leuridan<sup>3</sup>

<sup>1</sup> Institut Fourier, Université Joseph Fourier et CNRS  
BP 74, 38 402 Saint-Martin-d'Hères Cedex, France  
*jean.brossard@ujf-grenoble.fr*

<sup>2</sup> IRMA, Université Louis Pasteur et CNRS  
7 rue René Descartes, 67 084 Strasbourg Cedex, France  
*emery@math.u-strasbg.fr*

<sup>3</sup> Institut Fourier, Université Joseph Fourier et CNRS  
BP 74, 38 402 Saint-Martin-d'Hères Cedex, France  
*christophe.leuridan@ujf-grenoble.fr*

**Summary.** Let  $Z = (X, Y)$  be a planar Brownian motion,  $\mathcal{Z}$  the filtration it generates, and  $B$  a linear Brownian motion in the filtration  $\mathcal{Z}$ . One says that  $B$  (or its filtration) is maximal if no other linear  $\mathcal{Z}$ -Brownian motion has a filtration strictly bigger than that of  $B$ . For instance, it is shown in [2] that  $B$  is maximal if there exists a linear Brownian motion  $C$  independent of  $B$  and such that the planar Brownian motion  $(B, C)$  generates the same filtration  $\mathcal{Z}$  as  $Z$ .

We give a necessary condition for  $B$  to be maximal, and a sufficient condition which may be weaker than the existence of such a  $C$ . This sufficient condition is used to prove that the linear Brownian motion  $\int (X dY - Y dX)/|Z|$ , which governs the angular part of  $Z$ , is maximal.

**MSC 2000:** 60J65, 60H20

**Key words:** Brownian filtration, Maximal Brownian motion, Exchange property

## 1 Introduction

In the theory of filtered probability spaces, it is natural to study pairs of filtrations, one immersed in the other, and to attempt to understand possible different ways the smaller filtration can be immersed in the larger. A simple case is to consider various ways the filtration of a one-dimensional Brownian motion can be immersed in that of a two-dimensional Brownian motion. This study was started by two of us, who introduced in Section 2 of [2] the notions of complementability and maximality (Definition 1 below), and gave examples and counter-examples. The present paper, which concentrates on criteria for maximality, can be considered a sequel to [2].

Let  $Z = (Z_t)_{t \geq 0}$  denote a planar Brownian motion with  $Z_0 = 0$  (all Brownian motions will be started from the origin); call  $\mathcal{Z} = (\mathcal{Z}_t)_{t \geq 0}$  the filtration generated by  $Z$ . In the sequel,  $Z$  and  $\mathcal{Z}$  are fixed, and interest will focus on one-dimensional  $\mathcal{Z}$ -Brownian motions (shortly:  $\mathcal{Z}$ -BM1) and their filtrations. If  $B$  is a  $\mathcal{Z}$ -BM1, its natural filtration  $\mathcal{F}^B$  is immersed in  $\mathcal{Z}$ , that is, every  $\mathcal{F}^B$ -martingale is also a  $\mathcal{Z}$ -martingale. Consequently,  $\mathcal{F}_t^B = \mathcal{Z}_t \cap \mathcal{F}_\infty^B$  for every  $t \geq 0$ , so  $\mathcal{F}^B$  is characterized by its end  $\sigma$ -field  $\mathcal{F}_\infty^B = \sigma(B)$ . Many properties that  $\mathcal{F}^B$  may enjoy can equivalently be stated either in terms of  $\mathcal{F}_t^B$  for all  $t$ , or in terms of  $\sigma(B)$  only.

An example of such equivalent statements is the following double definition, borrowed from [2].

**Definition 1.** *Let  $B$  be a  $\mathcal{Z}$ -BM1.*

*One says that  $B$  is complementable if there exists a  $\mathcal{Z}$ -BM1  $C$  independent of  $B$ , and such that the  $\mathcal{Z}$ -BM2  $(B, C)$  generates the filtration  $\mathcal{Z}$  (equivalently, the  $\mathcal{Z}$ -BM2  $(B, C)$  generates the  $\sigma$ -field  $\sigma(Z)$ ).*

*One says that  $B$  is maximal if for every  $\mathcal{Z}$ -BM1  $D$ , one has*

$$(\forall t \geq 0 \quad \mathcal{F}_t^B \subset \mathcal{F}_t^D) \quad \implies \quad (\forall t \geq 0 \quad \mathcal{F}_t^B = \mathcal{F}_t^D)$$

*(equivalently,  $\sigma(B) \subset \sigma(D) \implies \sigma(B) = \sigma(D)$ ).*

These two properties (to be complementable and to be maximal) can also be considered as properties of the filtration  $\mathcal{F}^B$  generated by  $B$ ; in the sequel, we shall sometimes say that a filtration is complementable or maximal when it is generated by a complementable or maximal  $\mathcal{Z}$ -BM1.

Considering  $Z = (X, Y)$  as the complex BM  $Z = X + iY$  and calling  $R$  the modulus of  $Z$ , we get an interesting example by setting

$$U_t = \int_0^t \frac{X_s dX_s + Y_s dY_s}{R_s} \quad \text{and} \quad V_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{R_s}.$$

Indeed,  $U$  and  $V$  are independent  $\mathcal{Z}$ -BM1 which govern the radial part and angular part of  $Z$ , but  $(U, V)$  does not generate the whole filtration. Actually, as recalled in Proposition 5, it generates only the quotient of  $\mathcal{F}^Z$  by  $SO_2$ , that is, the filtration generated by  $Z$  up to an arbitrary rotation. Yet,  $U$  is complementable: an independent complement is exhibited in [2].

Corollary 6 of [2] states that *every complementable  $\mathcal{Z}$ -BM1 is maximal*. Thus,  $U$  is maximal. This corollary also gives many examples of non complementable BM1, for instance  $B' = \int \text{sgn}(B) dB$ , where  $B$  is any  $\mathcal{Z}$ -BM1. Indeed, the filtration generated by  $B'$  is also generated by  $|B|$ , so it is strictly included in that of  $B$ , and not maximal; by the corollary it cannot be complementable.

Conversely, *is every maximal  $\mathcal{Z}$ -BM1 complementable?* We do not know. This question is already open in the simpler case when  $B$  is complementable except for a germ property at time  $0+$  (this will be made precise in Definition 3).

In that particular case, we shall give a necessary and sufficient condition for maximality (Theorem 1). Interestingly, this condition is an exchange property: it says that supremum and intersection of certain  $\sigma$ -fields commute.

The necessary and sufficient condition of Theorem 1 is no longer sufficient for an arbitrary  $\mathcal{Z}$ -BM1 to be maximal, but it is still necessary (Corollary 1).

Using a variation on this condition (Proposition 4), we establish maximality for the  $\mathcal{Z}$ -BM1  $V = \int (X \, dY - Y \, dX)/R$ . When the present paper was first submitted, the question whether  $V$  is complementable, a stronger property, was open. We now know that the answer is positive (September 2008); our construction of an independent complement is rather involved and we hope to simplify it before publication.

## 2 Notation and reminders

In a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , all sub- $\sigma$ -fields are assumed to be  $(\mathcal{A}, \mathbb{P})$ -complete. Similarly, all filtrations are right-continuous and  $(\mathcal{A}, \mathbb{P})$ -complete; a *raw filtration* is a filtration which is not necessarily right-continuous.

If  $S = (S_t)_{t \geq 0}$  is any process, the filtration generated by  $S$  is denoted by  $\mathcal{F}^S$ . For  $0 \leq a < b < \infty$ , we define new processes as follows:

$$\begin{aligned} S^{[a,b]} &= (S_t)_{t \in [a,b]} & S^{]a,b]} &= (S_{a+t} - S_a)_{t \in [0,b-a]} \\ S^{[a,\infty[} &= (S_t)_{t \in [a,\infty)} & S^{]a,\infty[} &= (S_{a+t} - S_a)_{t \geq 0} \end{aligned}$$

If  $\mathcal{G}$  is a filtration and  $\mathcal{S}$  a  $\sigma$ -field, the *exchange property*

$$\bigcap_{h>0} (\mathcal{G}_{t+h} \vee \mathcal{S}) = \mathcal{G}_t \vee \mathcal{S}$$

does not hold in general; but it does when  $\mathcal{G}$  and  $\mathcal{S}$  are independent (see [5]). In plain words, independent enlargements preserve right-continuity. A key ingredient in the proof of this fact is that independent enlargements preserve conditional expectations: if  $\mathcal{T}$  and  $\mathcal{U}$  are two independent  $\sigma$ -fields, if  $T$  is  $\mathcal{T}$ -measurable and if  $\mathcal{S}$  is a sub- $\sigma$ -field of  $\mathcal{T}$ , then  $\mathbb{E}[T | \mathcal{S}] = \mathbb{E}[T | \mathcal{S} \vee \mathcal{U}]$ .

All Brownian motions are started at the origin. A planar Brownian motion  $Z = (X, Y)$  is fixed; its natural filtration is denoted by  $\mathcal{Z} = (\mathcal{Z}_t)_{t \geq 0}$ .

Recall the previsible representation property: every  $\mathcal{Z}$ -martingale has the form  $c + \int H \, dX + \int K \, dY$ , where  $c$  is a constant and  $H$  and  $K$  are  $\mathcal{Z}$ -previsible.

### 3 A necessary condition for maximality

**Definition 2.** Two  $\mathcal{Z}$ -BM1  $B$  and  $C$  are said to be parallel if  $C = \int H dB$  for some  $\mathcal{Z}$ -previsible process  $H$ .

Since  $B$  and  $C$  are linear Brownian motions,  $H = \pm 1$  on a  $(dt \times d\mathbb{P})$ -full set; so one also has  $B = \int H dC$ . Parallelism is an equivalence relation.

Any two  $\mathcal{Z}$ -BM1 generating the same filtration, or more generally any two  $\mathcal{Z}$ -BM1  $B$  and  $C$  such that the filtration  $\mathcal{F}^B$  is included in  $\mathcal{F}^C$ , are parallel. Indeed,  $B$  is a  $\mathcal{Z}$ -martingale adapted to the sub-filtration  $\mathcal{F}^C$ , hence also an  $\mathcal{F}^C$ -martingale; so it is a stochastic integral w.r.t.  $C$ .

**Proposition 1.** 1) Given any  $\mathcal{Z}$ -BM1  $B$ , there always exists some  $\mathcal{Z}$ -BM1 independent of  $B$ .

2) Given two  $\mathcal{Z}$ -BM1  $B$  and  $C$ , the following are equivalent:

- (i)  $B$  and  $C$  are parallel;
- (ii) there exists a  $\mathcal{Z}$ -BM1 independent of  $B$  and independent of  $C$ ;
- (iii) every  $\mathcal{Z}$ -BM1 independent of  $B$  is also independent of  $C$ .

*Proof.* 1)  $B$  has the form  $\int H dX + \int K dY$ , where  $H$  and  $K$  are  $\mathcal{Z}$ -previsible, and  $H^2 + K^2 = 1$ . The martingale  $D = -\int K dX + \int H dY$  is also a  $\mathcal{Z}$ -BM1 since  $\langle D, D \rangle = K^2 + H^2 = 1$ ; and  $D$  is independent of  $B$  since  $\langle D, B \rangle = -KH + HK = 0$ .

(i)  $\Rightarrow$  (iii): if  $B$  and  $C$  are parallel,  $C = \int H dB$  for some previsible  $H$ ; so if  $D$  is any  $\mathcal{Z}$ -BM1 independent of  $B$ , one has  $d\langle C, D \rangle = H d\langle B, D \rangle = 0$ , showing that  $D$  is independent of  $C$  too.

(iii)  $\Rightarrow$  (ii) is an immediate consequence of 1.

(ii)  $\Rightarrow$  (i): if  $D$  is a  $\mathcal{Z}$ -BM1 independent of  $B$  and of  $C$ , one has

$$\begin{aligned} dB &= H^B dX + K^B dY, & dC &= H^C dX + K^C dY \\ \text{and} & & dD &= H^D dX + K^D dY, \end{aligned}$$

where the vector  $(H^D, K^D)$  is  $(dt \times d\mathbb{P})$ -a.e. orthogonal to  $(H^B, K^B)$  and  $(dt \times d\mathbb{P})$ -a.e. orthogonal to  $(H^C, K^C)$ . Hence,  $(H^B, K^B)$  and  $(H^C, K^C)$  are  $(dt \times d\mathbb{P})$ -a.e. parallel; so  $(H^B, K^B) = L (H^C, K^C)$  for some previsible  $L$  defined  $(dt \times d\mathbb{P})$ -a.e. Consequently  $B = \int L dC$ , showing parallelism.  $\square$

**Open questions.** Given an arbitrary  $\mathcal{Z}$ -BM1  $B$ , does there always exist a  $C$  parallel to  $B$  and maximal? Does there always exist a  $C$  parallel to  $B$  and complementable? We do not know.

**Proposition 2.** Let  $B$  be a  $\mathcal{Z}$ -BM1.

(i) The  $\sigma$ -fields  $\mathcal{B}'_t = \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B)$  form a filtration (that is, they are right-continuous in  $t$ ).

(ii) The filtration  $\mathcal{B}'$  contains  $\mathcal{F}^B$  and it is generated by some  $\mathcal{Z}$ -BM1  $B'$ .

(iii) Any such  $B'$  is parallel to  $B$ .

(iv) Applying the same procedure to  $B'$  instead of  $B$  does not yield anything new: the filtration  $\mathcal{B}''$  defined by  $\mathcal{B}''_t = \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \mathcal{B}'_t)$  is equal to  $\mathcal{B}'$ .

*Proof.* (i) The raw filtration  $\mathcal{B}'$  is right-continuous at 0 because  $\mathcal{B}'_t \subset \mathcal{Z}_t$  and  $\bigcap_{t > 0} \mathcal{Z}_t$  is degenerate. It is also right-continuous at  $t > 0$  for the following reason. For  $\varepsilon > 0$ , the raw filtration  $(\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B)_{t \in [\varepsilon, \infty)}$  is generated by the  $\sigma$ -field  $\mathcal{Z}_\varepsilon$  and the Brownian motion  $B^{] \varepsilon, \infty [}$  which is independent of that  $\sigma$ -field; by the independent exchange property, that raw filtration is right-continuous. Consequently, for  $0 < \varepsilon < t$  one has

$$\bigcap_{h > 0} \mathcal{B}'_{t+h} \subset \bigcap_{h > 0} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_{t+h}^B) = \mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B,$$

and right-continuity of  $\mathcal{B}'$  follows from  $\bigcap_{h > 0} \mathcal{B}'_{t+h} \subset \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B) = \mathcal{B}'_t$ .

(ii) The filtration  $\mathcal{B}'$  clearly contains  $\mathcal{F}^B$ ; to show that  $\mathcal{B}'$  is generated by some  $\mathcal{Z}$ -BM1, it suffices by Corollary 1 of [3] to verify that  $\mathcal{B}'$  is immersed in  $\mathcal{Z}$  and is “1-Brownian after 0”.

The immersion property will be established by checking that if  $R$  is any bounded,  $\mathcal{B}'_\infty$ -measurable r.v.,  $\mathbb{E}[R | \mathcal{Z}_t]$  is  $\mathcal{B}'_t$ -measurable. This is trivial for  $t = 0$  since  $\mathcal{Z}_0$  is degenerate, so we suppose  $t > 0$ . For any  $\varepsilon \in ]0, t]$ , one can write  $R = \Phi(Z^{[0, \varepsilon]}, B^{[0, t]}, B^{]t, \infty [})$  for some measurable functional  $\Phi$ . Now, the first two arguments of  $\Phi$  are  $\mathcal{Z}_t$ -measurable and the third one is independent of  $\mathcal{Z}_t$ ; so  $\mathbb{E}[R | \mathcal{Z}_t] = \int \Phi(Z^{[0, \varepsilon]}, B^{[0, t]}, b) w(db)$ , where  $w$  stands for the law of  $B^{]t, \infty [}$ . Hence  $\mathbb{E}[R | \mathcal{Z}_t]$  is measurable for  $\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B$ ; as this holds for any  $\varepsilon \in ]0, t]$ ,  $\mathbb{E}[R | \mathcal{Z}_t]$  is  $\mathcal{B}'_t$ -measurable and  $\mathcal{B}'$  is immersed in  $\mathcal{Z}$ .

It remains to see that  $\mathcal{B}'$  is 1-Brownian after 0, that is, given any  $s > 0$ , the filtration  $(\mathcal{B}'_t)_{t \geq s}$  is generated by its initial  $\sigma$ -field  $\mathcal{B}'_s$  and some 1-dimensional BM independent of  $\mathcal{B}'_s$ . Now, for all  $t \geq s$ , one can write

$$\begin{aligned} \mathcal{B}'_t &= \bigcap_{\varepsilon \in ]0, s]} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_s^B \vee \sigma(B^{]s, t]}) \\ &= \left( \bigcap_{\varepsilon \in ]0, s]} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_s^B) \right) \vee \sigma(B^{]s, t]}) = \mathcal{B}'_s \vee \sigma(B^{]s, t]}), \end{aligned}$$

where the second equality (exchanging a supremum with an intersection) is due to the process  $B^{]s, t]}$  being independent of the  $\sigma$ -field  $\mathcal{Z}_s$  and a fortiori of the smaller  $\sigma$ -field  $\mathcal{Z}_\varepsilon \vee \mathcal{F}_s^B$ . The claim is established by observing that  $B^{]s, \infty [}$  is a BM1 independent of  $\mathcal{Z}_s$  and a fortiori of  $\mathcal{B}'_s$ . So  $\mathcal{B}'$  is generated by some  $\mathcal{Z}$ -BM1  $B'$ .

(iii)  $B'$  must be parallel to  $B$  because its filtration contains that of  $B$ .

(iv) The filtration  $\mathcal{B}''$  obtained by iterating this procedure verifies for  $0 < \varepsilon \leq t$  the inclusion  $\mathcal{B}_t'' \subset \mathcal{Z}_\varepsilon \vee \mathcal{B}_t' \subset \mathcal{Z}_\varepsilon \vee (\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B) = \mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B$ . Taking the intersection over  $\varepsilon$  gives  $\mathcal{B}_t'' \subset \mathcal{B}_t'$ , whence equality. This extends to  $t = 0$  by right-continuity, or by observing that  $\mathcal{B}_0''$  is degenerate.  $\square$

With the notation of Proposition 2,  $B$  cannot be maximal unless its filtration equals the (a priori larger) filtration  $\mathcal{B}'$ . Thus we get the following corollary.

**Corollary 1.** *Let  $B$  be a  $\mathcal{Z}$ -BM1. A necessary condition for  $B$  to be maximal is the exchange property*

$$\forall t \geq 0 \quad \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^B) = \mathcal{F}_t^B$$

(or equivalently, the equality

$$\bigcap_{t > 0} (\mathcal{Z}_t \vee \sigma(B)) = \sigma(B)$$

between the final  $\sigma$ -fields of these filtrations).

## 4 Sufficient conditions for maximality

The following lemma was used in the proof of Corollary 6 of [2] (although it was not explicitly stated there).

**Lemma 1.** *Given  $(\Omega, \mathcal{A}, \mathbb{P})$ , let  $\mathcal{S}$ ,  $\mathcal{T}$  and  $\mathcal{U}$  be three  $((\mathcal{A}, \mathbb{P})$ -complete) sub- $\sigma$ -fields of  $\mathcal{A}$  verifying  $\mathcal{S} \subset \mathcal{T}$  and  $\mathcal{S} \vee \mathcal{U} = \mathcal{T} \vee \mathcal{U}$ . If  $\mathcal{U}$  is independent of  $\mathcal{T}$ , then  $\mathcal{S} = \mathcal{T}$ .*

*Proof.* It suffices to use an independent enlargement in conditional expectations so as to write, for any bounded,  $\mathcal{T}$ -measurable r.v.  $T$ ,

$$\mathbb{E}[T | \mathcal{S}] = \mathbb{E}[T | \mathcal{S} \vee \mathcal{U}] = \mathbb{E}[T | \mathcal{T} \vee \mathcal{U}] = T. \quad \square$$

**Definition 3.** *Let  $B$  be a  $\mathcal{Z}$ -BM1. We shall say that  $B$  is complementable after 0 if there exists a  $\mathcal{Z}$ -BM1  $C$  independent of  $B$  such that*

$$\forall t > 0 \quad \forall \varepsilon \in ]0, t] \quad \mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B,C)} = \mathcal{Z}_t \quad (1)$$

(or equivalently,

$$\forall \varepsilon > 0 \quad \mathcal{Z}_\varepsilon \vee \sigma(B, C) = \mathcal{Z}_\infty, \quad (2)$$

since the filtration  $(\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B,C)})_{t \geq \varepsilon}$  is immersed in  $(\mathcal{Z}_t)_{t \geq \varepsilon}$ ).

Clearly, if  $B$  is complementable, it is also complementable after 0. The converse is false; for instance, if  $B$  is complementable, any Brownian motion generating the Goswami-Rao filtration associated to  $B$  (see [1]) is complementable after 0, but not complementable.

Complementability after 0 is easily seen to be a property of the filtration: if  $\bar{B}$  is any other  $\mathcal{Z}$ -BM1 generating  $\mathcal{F}^{\bar{B}}$ , then  $\bar{B}$  is complementable after 0 if and only if  $B$  is (with the same independent complement  $C$ ). Thus we may say that the filtration itself is complementable after 0.

Observe that (2) is equivalent to  $\bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(B, C)) = \mathcal{Z}_\infty$  since  $\mathcal{Z}_\infty$  contains everything. Similarly, (1) is equivalent to

$$\forall t > 0 \quad \bigcap_{\varepsilon \in ]0, t]} (\mathcal{Z}_\varepsilon \vee \mathcal{F}_t^{(B, C)}) = \mathcal{Z}_t .$$

If  $C$  in definition 3 is not a complement to  $B$ , that is, if  $\sigma(B, C)$  is strictly included in  $\mathcal{Z}_\infty$ , one has

$$\left( \bigcap_{\varepsilon>0} \mathcal{Z}_\varepsilon \right) \vee \sigma(B, C) = \mathcal{Z}_0 \vee \sigma(B, C) = \sigma(B, C) \neq \mathcal{Z}_\infty = \bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(B, C)) ,$$

so the exchange property fails for these  $\sigma$ -fields.

Observe also that  $B$  and  $C$  play the same rôle in Definition 3: when (1) or (2) is satisfied, both  $B$  and  $C$  are complementable after 0.

The next two lemmas will be useful to get a sufficient condition for maximality.

**Lemma 2.** *Let  $B$  be a  $\mathcal{Z}$ -BM1. If  $B$  is complementable after 0, then the filtration  $\mathcal{B}'$  introduced in Proposition 2 is complementable after 0.*

*Proof.* Since  $B$  is complementable after 0, let  $C$  be as in Definition 3. For  $\varepsilon > 0$ , one has  $\mathcal{Z}_\infty = \mathcal{Z}_\varepsilon \vee \sigma(B, C) \subset \mathcal{Z}_\varepsilon \vee \sigma(B', C) \subset \mathcal{Z}_\infty$ , wherefrom  $\mathcal{Z}_\varepsilon \vee \sigma(B', C) = \mathcal{Z}_\infty$ . But  $C$  is independent of  $B'$  since  $B'$  is parallel to  $B$  (Propositions 1 and 2 (iii)). Consequently,  $\mathcal{B}'$  is complementable after 0.  $\square$

**Lemma 3.** *Let  $B$  and  $D$  be two  $\mathcal{Z}$ -BM1. If  $B$  is complementable after 0 and  $\sigma(B) \subset \sigma(D)$ , then  $\mathcal{F}^D \subset \mathcal{B}'$ , where  $\mathcal{B}'$  is the filtration defined in Proposition 2.*

*Proof.* As  $B$  is complementable after 0, let  $C$  be as in Definition 3. Consider any  $\mathcal{Z}$ -BM1  $D$  whose filtration contains  $\mathcal{F}^B$ ; to establish  $\mathcal{F}^D \subset \mathcal{B}'$ , it suffices to show inclusion of their end  $\sigma$ -fields only:  $\sigma(D) \subset \mathcal{B}'_\infty = \bigcap_{t>0} (\mathcal{Z}_t \vee \sigma(B))$ .

Since  $\mathcal{F}^D$  contains  $\mathcal{F}^B$ ,  $D$  is parallel to  $B$ , and also independent of  $C$  by condition (iii) of Proposition 1; so  $(C, D)$  is a  $\mathcal{Z}$ -BM2. For  $t > 0$ , we shall apply Lemma 1 to the three  $\sigma$ -fields

$$\begin{aligned} \mathcal{S} &= \mathcal{Z}_t \vee \sigma(B) = \mathcal{Z}_t \vee \sigma(B]^{t, \infty[) , \\ \mathcal{T} &= \mathcal{Z}_t \vee \sigma(D) = \mathcal{Z}_t \vee \sigma(D]^{t, \infty[) , \\ \mathcal{U} &= \sigma(C]^{t, \infty[) . \end{aligned}$$

First,  $\mathcal{S}$  is included in  $\mathcal{T}$  because  $\sigma(D)$  contains  $\sigma(B)$ . Second, owing to the definition of  $C$  and to (2),

$$\mathcal{Z}_\infty = \mathcal{Z}_t \vee \sigma(B, C) = \mathcal{Z}_t \vee \sigma(B) \vee \sigma(C^{]t, \infty[}) = \mathcal{S} \vee \mathcal{U} \subset \mathcal{T} \vee \mathcal{U} \subset \mathcal{Z}_\infty,$$

which implies  $\mathcal{S} \vee \mathcal{U} = \mathcal{T} \vee \mathcal{U}$ . Third,  $\mathcal{U}$  is independent of  $\mathcal{T}$  because the three  $\sigma$ -fields  $\mathcal{Z}_t$ ,  $\sigma(C^{]t, \infty[})$  and  $\sigma(D^{]t, \infty[})$  are independent, for  $(C, D)$  is a  $\mathcal{Z}$ -BM2. So Lemma 1 applies, yielding  $\mathcal{S} = \mathcal{T}$ . Therefore  $\sigma(D) \subset \mathcal{T} = \mathcal{S} = \mathcal{Z}_t \vee \sigma(B)$ , and, as  $t > 0$  was arbitrary,  $\sigma(D) \subset \bigcap_{t>0} (\mathcal{Z}_t \vee \sigma(B))$ .  $\square$

From these two lemmas, one deduces immediately the following results.

**Proposition 3.** *Let  $B$  a  $\mathcal{Z}$ -BM1. If  $B$  is complementable after 0, then the filtration  $\mathcal{B}'$  defined in Proposition 2 is maximal.*

*Proof.* If  $\mathcal{F}^D$  contains  $\mathcal{B}'$  for  $D$  some  $\mathcal{Z}$ -BM1, then  $\mathcal{F}^D$  contains a fortiori  $\mathcal{F}^B$  and Lemma 3 gives  $\mathcal{F}^D \subset \mathcal{B}'$ ; so  $\mathcal{F}^D$  cannot strictly contain  $\mathcal{B}'$ .  $\square$

*Remark 1.* Define  $B$  to be *maximal after 0* if for  $D$  any other  $\mathcal{Z}$ -BM1, one has the maximality property

$$\bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(B)) \subset \bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(D)) \implies \bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(B)) = \bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(D))$$

(equivalently, with obvious notation,  $\mathcal{B}' \subset \mathcal{D}' \implies \mathcal{B}' = \mathcal{D}'$ ). This property is clearly necessary for  $\mathcal{B}'$  to be maximal; it is also sufficient, because, if  $\mathcal{B}' \subset \mathcal{F}^D$ , one also has  $\mathcal{B}' \subset \mathcal{F}^D \subset \mathcal{D}'$ , and maximality after 0 forces  $\mathcal{B}' = \mathcal{D}'$ , whence  $\mathcal{B}' = \mathcal{F}^D$ . So Proposition 3 can be thus rephrased: *complementability after 0 implies maximality after 0*.

The next maximality criterion also has two equivalent statements, in terms of the filtrations  $\mathcal{F}^B$  and  $\mathcal{B}'$ , or of their end  $\sigma$ -fields  $\sigma(B)$  and  $\mathcal{B}'_\infty$ .

**Theorem 1.** *Assume that  $B$  is a  $\mathcal{Z}$ -BM1, complementable after 0. Then  $B$  is maximal if and only if the following exchange property holds:*

$$\sigma(B) = \bigcap_{\varepsilon>0} (\mathcal{Z}_\varepsilon \vee \sigma(B))$$

(equivalently, the filtrations  $\mathcal{F}^B$  and  $\mathcal{B}'$  are equal).

*Proof.* Corollary 1 says that the exchange property is necessary for maximality (without supposing  $B$  to be complementable after 0).

Conversely, if the exchange property holds, the filtration  $\mathcal{B}'$  from Proposition 2 has the same terminal  $\sigma$ -field as the filtration  $\mathcal{F}^B$ , so these filtrations are equal; and Proposition 3 says that  $\mathcal{B}'$  is maximal.  $\square$

We shall now give a variant of the sufficient condition in Theorem 1, obtained by modifying both the assumption that  $B$  be complementable after 0 and the exchange property.

**Proposition 4.** *Let  $W = (B, C)$  be a  $\mathcal{Z}$ -BM2; call  $\mathcal{B}'$  the filtration constructed from  $B$  as in Proposition 2. Assume*

(i) *for each  $t > 0$ ,  $\sigma(Z^{[t, \infty[}) \subset \sigma(Z_t) \vee \sigma(W^{]t, \infty[})$ ;*

(ii) *for each  $t > 0$ , the conditional laws  $\mathcal{L}[Z_t | \mathcal{F}_t^B]$  and  $\mathcal{L}[Z_t | \mathcal{B}'_t]$  are equal.*

*Then  $\mathcal{F}^B = \mathcal{B}'$  and  $B$  is maximal.*

Observe that hypothesis (i) is stronger than supposing  $B$  to be completable after 0, since it features  $\sigma(Z_t)$  instead of the larger  $\sigma$ -field  $\mathcal{Z}_t$  in the right-hand side. On the opposite, (ii) is weaker than the exchange property  $\mathcal{F}^B = \mathcal{B}'$  from Theorem 1.

Remark also that one always has  $\mathcal{L}[Z_t | \mathcal{F}_t^B] = \mathcal{L}[Z_t | \mathcal{F}_\infty^B]$  and  $\mathcal{L}[Z_t | \mathcal{B}'_t] = \mathcal{L}[Z_t | \mathcal{B}'_\infty]$  by independent enlargements; so (ii) in Proposition 4 amounts to

$$\forall t > 0 \quad \mathcal{L}[Z_t | \sigma(B)] = \mathcal{L}\left[Z_t \mid \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \sigma(B))\right]. \quad (3)$$

*Proof of Proposition 4.* It suffices to show that  $\sigma(B) = \mathcal{B}'_\infty$ ; this implies equality of the filtrations  $\mathcal{F}^B$  and  $\mathcal{B}'$ , and maximality of  $B$  follows by Theorem 1. For an arbitrary bounded,  $\mathcal{Z}_\infty$ -measurable r.v.  $R$ , we have to show

$$\mathbb{E}\left[R \mid \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \sigma(B))\right] = \mathbb{E}[R | \sigma(B)]. \quad (4)$$

By  $L^1$ -density, we may restrict ourselves to the case that  $R$  is measurable in  $\sigma(Z^{[t, \infty[})$  for some  $t > 0$ . By hypothesis (i),  $R$  is some measurable function of  $Z_t$  and of the processes  $B^{]t, \infty[}$  and  $C^{]t, \infty[}$ . By linearity and density, we may suppose  $R = f(Z_t)g(B^{]t, \infty[})h(C^{]t, \infty[})$  with  $f$ ,  $g$  and  $h$  bounded and measurable. Calling  $w$  the 1-dimensional Wiener measure, one has

$$\begin{aligned} \mathbb{E}[R | \mathcal{Z}_t \vee \sigma(B)] &= f(Z_t)g(B^{]t, \infty[})\mathbb{E}[h(C^{]t, \infty[}) | \mathcal{Z}_t \vee \sigma(B)] \\ &= f(Z_t)g(B^{]t, \infty[})\int h \, dw \end{aligned}$$

since  $C^{]t, \infty[}$  is a BM1 independent of  $\mathcal{Z}_t \vee \sigma(B^{]t, \infty[}) = \mathcal{Z}_t \vee \sigma(B)$ . Consequently

$$\mathbb{E}\left[R \mid \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \sigma(B))\right] = (\int h \, dw) g(B^{]t, \infty[})\mathbb{E}\left[f(Z_t) \mid \bigcap_{\varepsilon > 0} (\mathcal{Z}_\varepsilon \vee \sigma(B))\right].$$

Using (3), the right-hand side becomes  $(\int h \, dw) g(B^{]t, \infty[})\mathbb{E}[f(Z_t) | \sigma(B)]$ ; as this is  $\sigma(B)$ -measurable, (4) is established.  $\square$

*Remark 2.* The same proof also yields a slightly stronger result: Proposition 4 remains true if (i) and (ii) are no longer supposed to hold for each  $t > 0$ , but only for a sequence of times tending to 0.

## 5 Maximality of the BM1 driving the angular part of $Z$

It will be convenient to consider  $Z = (X, Y)$  as the complex BM  $Z = X + iY$ . Call  $R$  the modulus of  $Z$ . Almost surely  $Z$  does not come back to 0, so another complex BM  $W$  can be defined by

$$W_t = \int_0^t \frac{R_s}{Z_s} dZ_s .$$

The real and imaginary parts of  $W$  are the BM1 given by

$$U_t = \int_0^t \frac{X_s dX_s + Y_s dY_s}{R_s} \quad \text{and} \quad V_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{R_s} .$$

The next proposition recalls some well-known facts concerning  $U$  and  $V$  (see for instance Proposition 3.1 and Théorème 3.4 in [4]).

**Proposition 5 (classical properties of  $U$  and  $V$ ; see [4]).**

- (i) *The linear Brownian motions  $U$  and  $V$  are independent.*
- (ii) *The process  $R$  is a strong solution of the stochastic differential equation*

$$dR_s = dU_s + \frac{ds}{2R_s} ;$$

*in particular,  $R$  and  $U$  generate the same filtration.*

- (iii) *For  $0 < s \leq t$ ,*

$$\frac{Z_t}{R_t} = \frac{Z_s}{R_s} \exp\left(i \int_s^t \frac{dV_r}{R_r}\right) .$$

- (iv) *For each  $t > 0$ ,  $\mathcal{F}_t^Z = \mathcal{F}_t^W \vee \sigma(Z_t/R_t)$ ; moreover the r.v.  $Z_t/R_t$  is independent of  $W$  and uniformly distributed on the unit circle.*

It is shown in [2] that the  $\mathcal{Z}$ -BM1  $U$  is complementable, hence also maximal. Clearly,  $V$  is not an independent complement to  $U$ , because  $(U, V)$  does not generate  $\mathcal{Z}$  (Proposition 5 (iv)); but  $V$  (and also  $U$ ) is complementable after 0, because if  $Z$  has been observed during some time-interval  $[0, \varepsilon]$  (however small), then observing  $U$  and  $V$  suffices to recover  $Z$ .

We shall show that  $V$  is maximal as an application of Proposition 4.

**Theorem 2.** *The  $\mathcal{Z}$ -BM1  $V = \int \frac{X dY - Y dX}{R}$  is maximal.*

*Proof.* We shall check that  $V$  satisfies hypotheses (i) and (ii) of Proposition 4.

Hypothesis (i) is readily verified: knowing  $Z_t$  and the increments of  $U$  and  $V$  after  $t$  is sufficient to recover the path of  $Z$  after  $t$ : the modulus  $R = |Z|$  is obtained as the solution to the stochastic differential equation

$$dR_s = dU_s + \frac{ds}{2R_s} ,$$

and the argument  $\Theta$  of  $Z = Re^{i\Theta}$  is then given by  $d\Theta = dV/R$ .

Hypothesis (ii) says that the conditional laws of  $Z_t$  given the  $\sigma$ -fields  $\sigma(V)$  and  $\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \sigma(V))$  are equal. The former,  $\mathcal{L}[Z_t|V]$ , is simply the law of  $Z_t$  because  $Z_t$  is independent of  $V$ . This is a direct consequence of Proposition 5: by (ii),  $R_t$  is independent of  $V$ , and by (iv),  $Z_t/R_t$  is independent of the pair  $(U, V)$ , and a fortiori of  $(V, R_t)$ . Hence the triple  $(V, R_t, Z_t/R_t)$  is independent, and  $Z_t = (Z_t/R_t) \times R_t$  is independent of  $V$ .

So, to verify Hypothesis (ii), we have to check that  $Z_t$  is independent of  $\bigcap_{s>0}(\mathcal{Z}_s \vee \sigma(V))$  too. This property is less straightforward than the independence of  $Z_t$  and  $V$ ; we state it as a lemma. Strictly speaking, the elementary proof, given above, that  $Z_t$  is independent of  $V$ , was not necessary, because it is also a consequence of that lemma, for  $\bigcap_{\varepsilon>0}(\mathcal{Z}_\varepsilon \vee \sigma(V))$  contains  $\sigma(V)$ .

**Lemma 4 (key lemma).** *For any  $t > 0$ , the r.v.  $Z_t$  is independent of the  $\sigma$ -field*

$$\bigcap_{s>0}(\mathcal{Z}_s \vee \sigma(V)) .$$

The end of this paper is devoted to that proof, which consists of four steps; the first three steps are Lemmas 5 to 7. Lemma 5 is certainly well-known.

**Lemma 5.** *If  $(T_p)_{p \geq 1}$  is any sequence of strictly positive (possibly infinite), identically distributed r.v., then  $\limsup_p T_p$  is a.s. strictly positive.*

*Proof.* For  $\delta > 0$ , observe that

$$\left\{ \limsup_p T_p < \delta \right\} = \left\{ \exists p \sup_{q \geq p} T_q < \delta \right\} \subset \liminf_p \{T_p < \delta\} .$$

According to Fatou's lemma, the probability of these events is majorized by  $\liminf_p \mathbb{P}[T_p < \delta]$ , which equals  $\mathbb{P}[T_1 < \delta]$  because the  $T_p$  have the same law. So  $\mathbb{P}[\limsup_p T_p < \delta] \leq \mathbb{P}[T_1 < \delta]$ ; the lemma follows by letting  $\delta$  go to 0.  $\square$

**Lemma 6.** *Let  $(\beta_t)_{t \in [0,1]}$  be a BM1 and  $(\alpha_t)_{t \in [0,1]}$  be a continuous, strictly positive process independent of  $\beta$ , with diffuse law. Then the conditional law of  $\int_0^1 \alpha_t d\beta_t$  given  $\beta$  is almost surely diffuse. A fortiori, for every real  $m \neq 0$ ,*

$$\left| \mathbb{E} \left[ \exp(im \int_0^1 \alpha_t d\beta_t) \mid \beta \right] \right| < 1 \quad a.s.$$

*Proof.* Let  $\tilde{\alpha}$  be a process with the same law as  $\alpha$ , independent of  $(\beta, \alpha)$ . Then

$$\mathcal{L} \left[ \int_0^1 (\alpha_t - \tilde{\alpha}_t) d\beta_t \mid \alpha, \tilde{\alpha} \right] = \mathcal{N} \left( 0, \int_0^1 (\alpha_t - \tilde{\alpha}_t)^2 dt \right)$$

is a.s. non-degenerate. Consequently,

$$\mathbb{P} \left[ \int_0^1 (\alpha_t - \tilde{\alpha}_t) d\beta_t = 0 \right] = 0 ,$$

wherefrom

$$\mathbb{P} \left[ \int_0^1 \alpha_t d\beta_t = \int_0^1 \tilde{\alpha}_t d\beta_t \mid \beta \right] = 0 \quad a.s.$$

But, conditional on  $\beta$ , the r.v.  $\int_0^1 \alpha_t d\beta_t$  and  $\int_0^1 \tilde{\alpha}_t d\beta_t$  are independent and have the same law; so this conditional law must be diffuse.  $\square$

**Lemma 7.** *Call  $\mathcal{R}$  the  $\sigma$ -field  $\sigma((R_{2^n})_{n \in \mathbb{Z}})$ . The r.v.  $Z_1/R_1$  is independent of  $\bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V) \vee \mathcal{R})$  and uniformly distributed on the unit circle.*

*Proof.* Owing to the reverse martingale convergence theorem, it suffices to prove that for every  $m \in \mathbb{Z} \setminus \{0\}$ ,

$$\mathbb{E} \left[ \left( \frac{Z_1}{R_1} \right)^m \mid \mathcal{Z}_{2^n} \vee \sigma(V) \vee \mathcal{R} \right] \longrightarrow 0 \quad \text{a.s. when } n \rightarrow -\infty .$$

To that end, call  $Q_n$  this conditional expectation and, using Proposition 5 (iii), observe that for  $n \leq -1$ ,

$$\frac{Z_1}{R_1} = \frac{Z_{2^n}}{R_{2^n}} \prod_{k=n}^{-1} e^{i\Delta_k} ,$$

where  $\Delta_k$  denotes the angular variation of  $Z$  from time  $2^k$  to  $2^{k+1}$ :

$$\Delta_k = \int_{2^k}^{2^{k+1}} \frac{dV_s}{R_s} .$$

One has

$$Q_n = \left( \frac{Z_{2^n}}{R_{2^n}} \right)^m \mathbb{E} \left[ \prod_{k=n}^{-1} e^{im\Delta_k} \mid \mathcal{F}_{2^n}^R \vee \sigma \left( \frac{Z_{2^n}}{R_{2^n}} \right) \vee \sigma(V) \vee \mathcal{R} \right] ,$$

where  $\mathcal{Z}_{2^n}$  has been replaced with  $\mathcal{F}_{2^n}^R \vee \sigma(Z_{2^n}/R_{2^n})$  in the conditioning  $\sigma$ -field owing to the equality

$$\mathcal{Z}_{2^n} = \mathcal{F}_{2^n}^R \vee \mathcal{F}_{2^n}^V \vee \sigma(Z_{2^n}/R_{2^n}) .$$

But everything in both sides of the conditional expectation is in  $\sigma(R, V)$ , except  $Z_{2^n}/R_{2^n}$  which is independent of  $(R, V)$  (see Proposition 5 (iv)). So, by independent enlargement,  $\sigma(Z_{2^n}/R_{2^n})$  may be removed, yielding

$$Q_n = \left( \frac{Z_{2^n}}{R_{2^n}} \right)^m \mathbb{E} \left[ \prod_{k=n}^{-1} e^{im\Delta_k} \mid \mathcal{F}_{2^n}^R \vee \sigma(V) \vee \mathcal{R} \right] .$$

Now, the random variables  $\Delta_k = \int_{2^k}^{2^{k+1}} dV_s/R_s$  are conditionally independent given  $\mathcal{F}_{2^n}^R \vee \sigma(V) \vee \mathcal{R}$  because  $R$  and  $V$  are independent,  $R$  is Markov and  $V$  has independent increments. This gives

$$Q_n = \left( \frac{Z_{2^n}}{R_{2^n}} \right)^m \prod_{k=n}^{-1} C_k ,$$

where

$$\begin{aligned} C_k &= \mathbb{E}\left[ e^{im\Delta_k} \mid \mathcal{F}_{2^n}^R \vee \sigma(V) \vee \mathcal{R} \right] \\ &= \mathbb{E}\left[ e^{im\Delta_k} \mid \sigma(V]^{2^k, 2^{k+1}}, R_{2^k}, R_{2^{k+1}} \right]. \end{aligned}$$

It remains to prove that the product  $\prod_{k=n}^{-1} C_k$  tends to 0 when  $n \rightarrow -\infty$ . By scaling invariance of  $Z$ , the law of

$$(2^{-k/2}(V_{2^k t} - V_{2^k})_{t \in [1, 2]}, 2^{-k/2}R_{2^k}, 2^{-k/2}R_{2^{k+1}}, \Delta_k)$$

does not depend upon  $k$ ; hence the  $C_k$  are identically distributed. One can apply Lemma 6 to  $\beta = V^{[1, 2]}$  and to  $\alpha$  the inverse of the 2-dimensional Bessel bridge from  $R_1$  to  $R_2$ ; this yields

$$|C_0| = \left| \mathbb{E}\left[ \exp(im \int_1^2 R_s^{-1} dV_s) \mid \sigma(V, R_1, R_2) \right] \right| < 1 \quad \text{a.s.}$$

Applying now Lemma 5 to  $T_k = -\ln |C_k|$  gives  $\liminf_k |C_k| < 1$ ; so the infinite product tends to zero, and Lemma 7 is proved.  $\square$

The fourth step is the final one:

*Proof of Lemma 4.* By scaling, we may suppose  $t = 1$ ; we have to establish that

$$\mathbb{E}\left[ f(Z_1) \mid \bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V)) \right]$$

is constant, where  $f$  is any bounded, Borel function.

By linearity and  $L^1$ -density, we may restrict ourselves to functions of the form  $f(z) = g(|z|)h(z/|z|)$ , with  $g$  and  $h$  bounded. Using Lemma 7, one can write

$$\mathbb{E}\left[ f(Z_1) \mid \bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V) \vee \mathcal{R}) \right] = c g(R_1),$$

with  $c = \int_0^1 h(e^{i2\pi t}) dt$ . Hence

$$\mathbb{E}\left[ f(Z_1) \mid \bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V)) \right] = c \mathbb{E}\left[ g(R_1) \mid \bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V)) \right]. \quad (5)$$

But

$$\mathcal{Z}_s \vee \sigma(V) = \mathcal{F}_s^R \vee \sigma(V) \vee \sigma(Z_s/R_s);$$

therefore, by independence of  $R, V$  and  $Z_s/R_s$ , one has for  $0 < s \leq 1$

$$\mathbb{E}[g(R_1) \mid \mathcal{Z}_s \vee \sigma(V)] = \mathbb{E}[g(R_1) \mid \mathcal{F}_s^R].$$

Let  $s$  go to zero. By reverse martingale convergence, and using the fact that  $\mathcal{F}_{0+}^R = \mathcal{F}_{0+}^U$  is degenerate, one ends up with

$$\mathbb{E}\left[ g(R_1) \mid \bigcap_{s>0} (\mathcal{Z}_s \vee \sigma(V)) \right] = \lim_{\substack{s \rightarrow 0 \\ s > 0}} \mathbb{E}[g(R_1) \mid \mathcal{F}_s^R] = \mathbb{E}[g(R_1)].$$

So the right-hand side of (5) is constant, and we are done.  $\square$

## References

1. Attal S., Burdzy K., Émery M., Hu Y., *Sur quelques filtrations et transformations browniennes*, Séminaire de Probabilités XXIX, Lecture Notes in Mathematics **1613**, Springer (1995), 56–69.
2. Brossard J., Leuridan C., *Filtrations browniennes et compléments indépendants*, Séminaire de Probabilités XLI, Lecture Notes in Mathematics **1934**, Springer (2008), 265–278.
3. Émery M., *On certain almost Brownian filtrations*, Ann. Inst. Henri Poincaré - Probabilités et Statistiques **41** (2005), 285–305.
4. Stroock D.W., Yor M., *Some remarkable martingales*, Séminaire de Probabilités XV, Lecture Notes in Mathematics **850**, Springer (1980), 590–603.
5. von Weizsäcker H. *Exchanging the order of taking suprema and countable intersections of  $\sigma$ -algebras*. Ann. Inst. Henri Poincaré, Section B **19** (1983), 91–100.