Sharp ill-posedness result for the periodic Benjamin-Ono equation (ERRATUM : PAPER WITHDRAWN)

Luc Molinet

ERRATUM : This paper has been withdrawn by the author since there were errors in the calculus of the defect coefficient in Page 11. The corrected calculus gives actually zero which do not lead to a contradiction on the continuity of the flow-map of the Benjamin-Ono equation. The author warmly thank Professor Patrick Gérard for having pointing out this error to him.

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Luc Molinet
L.M.P.T., UMR 6083, Faculté des Sciences et Techniques, Université de Tours,
Parc de Grandmont, 37200 Tours, FRANCE.

E-mail : Luc.Molinet@lmpt.univ-tours.fr

Abstract. We prove the discontinuity for the weak $L^2(\mathbb{T})$-topology of the flow-map associated with the periodic Benjamin-Ono equation. This ensures that this equation is ill-posed in $H^s(\mathbb{T})$ as soon as $s < 0$ and thus completes exactly the well-posedness result obtained in [12].

AMS Subject Classification : 35Q53, 35B20.

Key words : Benjamin-Ono equation; Cauchy problem; Ill-posedness result, Continuity with respect to initial data.

1 Introduction

In this paper we continue our study of the Cauchy problem associated with the Benjamin-Ono equation on the one-dimensional torus (cf. [11], [12]) by proving the ill-posedness character of this Cauchy problem in Sobolev spaces with negative index. Our ill-posedness result is a strong one in the sense that for any $T > 0$ and any non constant function $\varphi \in L^2(\mathbb{T})$, there exist an infinite numbers of times $t \in]0, T[$ such that the map $u_0 \mapsto u(t)$ is discontinuous in $H^s(\mathbb{T})$, $s < 0$, at φ. Recall that in [12] it is proven that this Cauchy problem is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$ with a flow-map that is real analytic on hyperplans of functions with a given mean value.

The Benjamin-Ono equation describes the evolution of the interface between two inviscid fluids under some physical conditions (see [2]). It reads

$$u_t + \mathcal{H}u_{xx} + uu_x = 0 \, . \tag{1}$$

In the periodic setting $u = u(t,x)$ is a function from $\mathbb{R} \times \mathbb{T}$ to $\mathbb{R}$, with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, and $\mathcal{H}$ is the Hilbert transform defined for 2π-periodic functions with mean value zero by

$$\widehat{\mathcal{H}(f)}(0) = 0 \quad \text{and} \quad \widehat{\mathcal{H}(f)}(k) = -i \, \mathrm{sgn}(k) \hat{f}(k), \quad k \in \mathbb{Z}^* \, .$$

This equation enjoys the same dilation symmetry : $u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x)$ as the cubic Schrödinger equation. Recall that the homogeneous Sobolev space $\dot{H}^{-1/2}(\mathbb{R})$ stays invariant by this symmetry. This suggests that the associated Cauchy problem should be ill-posed at least in Sobolev spaces with index less than $s_c^* = -1/2$. On the other hand, as far as the author knows and contrary to the cubic Schrödinger equation, no other symmetry is known for this equation.

The Benjamin-Ono equation is integrable (cf [1]) and so it seems interesting to compare our result to the ones for other classical integrable equations on the one-dimensional torus. For this let us introduce another index, s_c^∞, that is the index of the Sobolev space above which the Cauchy problem is well-posed with a flow-map[1] that is of class C^∞. For the KdV equation , $s_c^* = -3/2$ and $s_c^\infty = -1/2$ (cf. [10]) but using integrability, Kappeler and Topalov [8] recently proved that the flow-map can be continuously extended in $H^{-1}(\mathbb{T})$. For mKdV, $s_c^* = -1/2$ and $s_c^\infty = 1/2$, and the situation is even more intrinquing. Indeed, it was proved by Tsutsumi and Takaoka that mKdV is still well-posed in $H^s(\mathbb{T})$, $s > 1/4$, but with a flow-map that is not uniformly continuous on bounded set for $1/4 < s < 1/2$. Moreover, it was also proved by Kappeler and Topalov (cf. [9]) that, as for KdV, the flow-map can be continuously extended in $L^2(\mathbb{T})$. So for these both integrable equations, the flow-map can be continuously extended below s_c^∞. As proved in [4] (see also [13]), this is not the case for the cubic Schrödinger equation and, as our result shows, this is also not the case for the Benjamin-Ono equation for which $s_c^* = -1/2$ and $s_c^\infty = 0$.

[1] For dispersive periodic equations whose nonlinear term is of the form $u^q u_x$ the smoothness of the flow-map holds not for the original equation but for the equation satisfied by $\tilde{u}(t,x) = u(t, x - \int_0^t \fint u^q)$. Note, however, that for $q = 1$, since the mean-value of u is conserved, the smoothness of the flow-map associated with $\tilde{u}$ ensures the smoothness of the flow-map associated with the original equation on hyperplans of functions with a given mean-value.

Our proof deeply relies on the well-posedness result in $L^2(\mathbb{T})$ established in [11]. Recall that the proof of this result used in a crucial way that some gauge transform of the solution, first introduced by T. Tao (cf. [14]), satisfies the equation (13) (see Section 3.1) which enjoys better smoothing effects that the original one (see [7] for a note on the bad behavior of the original equation with respect to classical bilinear estimates). Here we will also used the special structure of this equation. We proceed by contradiction. Assuming that the flow-map associated with the Benjamin-Ono equation is continuous from $L^2(\mathbb{T})$ equipped with its weak topology into the space of distributions $(C^\infty(\mathbb{T}))^*$ we will first check directly from the expression of the gauge transform that the flow-map associated with it should also be continuous with respect to these topologies. Then, proceeding as in [13] we will pass to the limit on the equation (13) for some subsequence of gauge transforms by separating resonant and non resonant parts of the nonlinear terms. We will prove that its limit does not satisfy exactly (13) but a modified version of this equation. This will lead to the desired contradiction.

1.1 Main results

Our main theorem is a result of discontinuity of the flow-map associated with (1) for the weak $L^2(\mathbb{T})$-topology. Since $L^2(\mathbb{T})$ is compactly embedded in $H^s(\mathbb{T})$ for $s < 0$, it ensures the ill-posedness of the periodic Benjamin-Ono equation in $H^s(\mathbb{T})$ with $s < 0$ (see Remark 1.2 below).

Theorem 1.1 *Let $u_0 \in L^2(\mathbb{T})$ be a non constant function and $\{\tilde{u}_{0,n}\}$ be any sequence of $L^2(\mathbb{T})$ converging strongly in $L^2(\mathbb{T})$ to u_0. We set $u_{0,n} := \tilde{u}_{0,n} + \cos(nx)$ so that $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$ and denote respectively by u_n and u the solution of the Benjamin-Ono equation (1) emanating respectively from $u_{0,n}$ and u_0. Then for any $T > 0$ there exists $t \in]0, T[$ such that $\{u_n(t)\}$ does not converge towards $u(t)$ in the distribution sense.*

Remark 1.2 *Since $L^2(\mathbb{T})$ is compactly embedded in $H^s(\mathbb{T})$ for $s < 0$, Theorem 1.1 ensures that for all non constant function $\varphi \in L^2(\mathbb{T})$ and all $T > 0$, there exists $t \in]0, T[$ such that the map $u_0 \mapsto u(t)$ associated with the Benjamin-Ono equation is discontinuous at φ in any Sobolev space with negative index. This proves the strong ill-posedness of the periodic Benjamin-Ono equation in $H^s(\mathbb{T})$ with $s < 0$.*

Remark 1.3 *Note that taking $\tilde{u}_{0,n} := u_0$, for all $n \in \mathbb{N}$, this ensures that the discontinuity result holds also on hyperplans of functions with a given mean value.*

2 Function spaces and notation

Let us first introduce some notations and function spaces we will work with. For $x, y \in \mathbb{R}$, $x \sim y$ means that there exists C_1, $C_2 > 0$ such that $C_1|x| \leq |y| \leq C_2|x|$ and $x \lesssim y$ means that there exists $C_2 > 0$ such that $|x| \leq C_2|y|$. $[x]$ will denote the entire part of a real number x.
For a 2π-periodic function φ, we define its space Fourier transform by

$$\hat{\varphi}(k) := \mathcal{F}_x(\varphi)(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx}\, \varphi(x)\, dx, \quad \forall k \in \mathbb{Z}$$

and denote by $V(\cdot)$ the free group associated with the linearized Benjamin-Ono equation,

$$\widehat{V(t)\varphi}(k) := e^{-ik|k|t}\, \hat{\varphi}(k), \quad k \in \mathbb{Z}\,.$$

The Sobolev spaces $H^s(\mathbb{T})$ for 2π-periodic functions are defined as usually and endowed with

$$\|\varphi\|_{H^s(\mathbb{T})} := \|\langle k\rangle^s \widehat{\varphi}(k)\|_{L^2(\mathbb{Z})} = \|J_x^s \varphi\|_{L^2(\mathbb{T})}\,,$$

where $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$ and $\widehat{J_x^s\varphi}(k) := \langle k\rangle^s \widehat{\varphi}(k)$. We will denote by $H_0^s(\mathbb{T})$ the closed subspace of $H^s(\mathbb{T})$ that contains the functions of $H^s(\mathbb{T})$ with mean value zero.
For a function $u(t,x)$ on $\mathbb{R}\times\mathbb{T}$, we define its space-time Fourier transform by

$$\hat{u}(\tau,k) := \mathcal{F}_{t,x}(u)(\tau,k) := \frac{1}{(2\pi} \int_{\mathbb{R}\times\mathbb{T}} e^{-i(\tau t + kx)}\, u(t,x), \quad \forall (\tau,k) \in \mathbb{R}\times\mathbb{Z}$$

and define the Bourgain spaces $X^{b,s}$, $Z^{b,s}$, A^b and Y^s of functions on $\mathbb{T}^2$ endowed with the norm

$$\|u\|_{X^{b,s}} := \|\langle \tau + |k|k\rangle^b \langle k\rangle^s \hat{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})} = \|\langle\tau\rangle^b \langle k\rangle^s \mathcal{F}_{t,x}(V(-t)u)\|_{L^2(\mathbb{R}\times\mathbb{Z})} \quad (2)$$

$$\|u\|_{Z^{b,s}} := \|\langle \tau + |k|k\rangle^b \langle k\rangle^s \hat{u}\|_{l^2(\mathbb{Z};L^1(\mathbb{R}))} = \|\langle\tau\rangle^b \langle k\rangle^s \mathcal{F}_{t,x}(V(-t)u)\|_{l^2(\mathbb{Z};L^1(\mathbb{R}))}\,, \quad (3)$$

$$\|u\|_{A^b} \;:=\; \|\langle \tau + |k|k\rangle^b \hat{u}\|_{L^1(\mathbb{R}\times\mathbb{Z})} = \|\langle\tau\rangle^b \mathcal{F}_{t,x}(V(-t)u)\|_{L^1(\mathbb{R}\times\mathbb{Z})} \quad (4)$$

and

$$\|u\|_{Y^{b,s}} := \|u\|_{X^{b,s}} + \|u\|_{Z^{b-1/2,s}} \quad , \quad (5)$$

where we will denote the Wiener algebre A^0 simply by A. Recall that $Y^{1/2,s} \hookrightarrow Z^{0,s} \hookrightarrow C(\mathbb{R}; H^s)$.
$L^p(\mathbb{R}; L^q(\mathbb{T}))$ will denote the Lebesgue spaces endowed with the norm

$$\|u\|_{L^p(\mathbb{R};L^q(\mathbb{T}))} := \Big(\int_{\mathbb{R}} \|u(t,\cdot)\|^p_{L^q(\mathbb{T})}\, dt\Big)^{1/p}$$

with the obvious modification for $p = \infty$.

Let $u = \sum_{j\geq 0} \Delta_j u$ be a classical smooth non homogeneous Littlewood-Paley decomposition in space of u, Supp $\mathcal{F}_x(\Delta_0 u) \subset \mathbb{R} \times [-2, 2]$ and

$$\text{Supp } \mathcal{F}_x(\Delta_j u) \subset \mathbb{R} \times [-2^{j+1}, -2^{j-1}] \cup \mathbb{R} \times [2^{j-1}, 2^{j+1}]), \quad j \geq 1 \quad .$$

We defined the Besov type space $\tilde{L}^4_{t,\lambda}$ by

$$\|u\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} := \Big(\sum_{k\geq 0} \|\Delta_k u\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\Big)^{1/2} . \tag{6}$$

Note that by the Littlewood-Paley square function theorem and Minkowski inequality,

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T})} \sim \Big\|\Big(\sum_{k=0}^{\infty}(\Delta_k u)^2\Big)^{1/2}\Big\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \Big(\sum_{k=0}^{\infty} \|\Delta_k u\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\Big)^{1/2} = \|u\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})}$$

and thus $\tilde{L}^4(\mathbb{R} \times \mathbb{T}) \hookrightarrow L^4(\mathbb{R} \times \mathbb{T})$.
We will denote by P_+ and P_- the projection on respectiveley the positive and the negative spatial Fourier modes. Moreover, for $a \geq 0$, we will denote by P_a, Q_a, $P_{>a}$ and $P_{<a}$ the projection on respectively the spatial Fourier modes of absolute value equal or less than a, the spatial Fourier modes of absolute value greater than a, the spatial Fourier modes larger than a and the spatial Fourier modes smaller than a.
We will need the function spaces N and R^θ respectively defined by

$$\|u\|_N := \|u\|_{Z^{0,0}} + \|Q_3 u\|_{X^{7/8,-1}} + \|u\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})}$$

and

$$\|F\|_{R^\theta} := \|F\|_{X^{\theta,0}} + \|F\|_{L^\infty(\mathbb{R};H^1(\mathbb{T}))} + \|F_x\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} + \|F\|_A \quad ,$$

Finally, for any function space B and any $0 < T \leq 1$, we denote by B_T the corresponding restriction in time space endowed with the norm

$$\|u\|_{B_T} := \inf_{v\in B}\{\|v\|_B,\, v(\cdot) \equiv u(\cdot) \text{ on }]-T, T[\,\} .$$

It is worth noticing that the map $u \mapsto \overline{u}$ is an isometry in all our function spaces.

3 Well-posedness result, gauge transform and linear estimates

The proof of Theorem 1.1 uses in a crucial way the well-posedness theorem proved in [12].

Theorem 3.1 *For all $u_0 \in H^s(\mathbb{T})$ with $0 \leq s \leq 1/2$ and all $T > 0$, there exists a solution u of the Benjamin-Ono equation (BO) satisfying*

$$u \in N_T \quad \text{and} \quad w := \partial_x P_+(e^{-i\partial_x^{-1}\tilde{u}/2}) \in X_T^{1/2,s} \tag{7}$$

where

$$\tilde{u} := u(t, x - t \fint u_0) - \fint u_0 \quad \text{and} \quad \widehat{\partial_x^{-1}} := \frac{1}{i\xi},\ \xi \in \mathbb{Z}^* \quad .$$

This solution is unique in the class (7).
Moreover $u \in C(\mathbb{R}; H^s(\mathbb{T})) \cap C_b(\mathbb{R}; L^2(\mathbb{T}))$ and the map $u_0 \mapsto (u, w)$ is continuous from $H^s(\mathbb{T})$ into $(C([0,T]; H^s(\mathbb{T})) \cap N_T) \times X_T^{1/2,s}$ and Lipschitz on every bounded set from $H_0^s(\mathbb{T})$ into $H^s(\mathbb{T})$ into $(C([0,T]; H^s(\mathbb{T})) \cap N_T) \times X_T^{1/2,s}$ follows

Note that the fact that $u_0 \mapsto w$ is continuous from $H^s(\mathbb{T})$ into $X_T^{1/2,s}$ is not explicitly stated in Theorem 1.1 of [12] but follows directly from the estimate (106) page 674 in [12].

3.1 The gauge transform

As indicated in the introduction, we plan to study the behavior of the flow-map constructed in the above theorem with respect to the weak topology of $L^2(\mathbb{T})$. To do so we will use in a crucial way the equation satisfied by the gauge transform

$$w := P_+(e^{-i\partial_x^{-1}\tilde{u}/2}\tilde{u})$$

of the solution u. Let us thus first recall how to get this equation.

Let u be a smooth 2π-periodic solution of (BO) with initial data u_0. In the sequel, we assume that $u(t)$ has mean value zero for all time. Otherwise we do the change of unknown :

$$\tilde{u}(t,x) := u(t, x - t \fint u_0) - \fint u_0 \quad , \tag{8}$$

where $\fint u_0 := P_0(u_0) = \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value of u_0. It is easy to see that $\tilde{u}$ satisfies (BO) with $u_0 - \fint u_0$ as initial data and since $\fint \tilde{u}$ is

preserved by the flow of (BO), $\tilde{u}(t)$ has mean value zero for all time. We define $F = \partial_x^{-1} u$ which is the periodic, zero mean value, primitive of u,

$$\hat{F}(0) = 0 \quad \text{and } \widehat{F}(\xi) = \frac{1}{i\xi}\hat{u}(\xi), \quad \xi \in \mathbb{Z}^* \quad .$$

Following T. Tao [14], we introduce the gauge transform

$$W := P_+(e^{-iF/2}) \quad . \tag{9}$$

Since F satisfies

$$F_t + \mathcal{H}F_{xx} = \frac{F_x^2}{2} - \frac{1}{2}\fint F_x^2 = \frac{F_x^2}{2} - \frac{1}{2}P_0(F_x^2) \quad ,$$

we can check that $w := W_x = -\frac{i}{2}P_+(e^{-iF/2}F_x) = -\frac{i}{2}P_+(e^{-iF/2}u)$ satisfies

$$\begin{aligned} w_t - iw_{xx} &= -\partial_x P_+\Big[e^{-iF/2}\Big(P_-(F_{xx}) - \frac{i}{4}P_0(F_x^2)\Big)\Big] \\ &= -\partial_x P_+\Big(WP_-(u_x)\Big) + \frac{i}{4}P_0(F_x^2)w \; . \end{aligned} \tag{10}$$

On the other hand, one can write u as

$$u = e^{iF/2}e^{-iF/2}F_x = 2i\, e^{iF/2}\partial_x(e^{-iF/2}) = 2ie^{iF/2}w + 2ie^{iF/2}\partial_x P_-(e^{-iF/2}) \quad . \tag{11}$$

Recalling that u is real-valued, we get

$$u = \overline{u} = -2ie^{-iF/2}\overline{w} - 2ie^{-iF/2}\partial_x\overline{P_-(e^{-iF/2})}$$

and thus

$$P_-(u) = -2iP_-\Big(e^{-iF/2}\overline{w}\Big) - 2iP_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big) \tag{12}$$

since $\overline{P_-(v)} = P_+(\overline{v})$ for any complex-valued function v. Substituing (12) in (10), we obtain the following equation satisfied by w :

$$\begin{aligned} w_t - iw_{xx} &= 2i\partial_x P_+\Big(W\partial_x P_-(e^{-iF/2}\overline{w})\Big) \\ &\quad + 2i\partial_x P_+\Big[W\partial_x P_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big)\Big] + \frac{i}{4}P_0(F_x^2)W_x \\ &= A(G,W) + B(G,W) + \frac{i}{4}P_0(F_x^2)W_x \; . \end{aligned} \tag{13}$$

where $G := e^{-iF/2}$. Note already that the last term in (13) can be rewritten as

$$\frac{i}{4}P_0(F_x^2)W_x = \frac{i}{8\pi}\Big(\int_{\mathbb{T}} u_0^2\Big)w \, , \tag{14}$$

since $\int_{\mathbb{T}} u^2$ is a constant of the motion for (1).

3.2 Linear estimates

Let us state some estimates for the free group and the Duhamel operator. Let $\psi \in C_0^\infty([-2,2])$ be a time function such that $0 \le \psi \le 1$ and $\psi \equiv 1$ on $[-1,1]$. The following linear estimates are well-known (cf. [3], [5]).

Lemma 3.2 *For all $\varphi \in H^s(\mathbb{T})$, it holds :*

$$\|\psi(t)V(t)\varphi\|_{X^{1/2,s}} \lesssim \|\varphi\|_{H^s} \quad , \tag{15}$$

$$\|\psi(t)V(t)\varphi\|_{Y^{1/2,s}} \lesssim \|\varphi\|_{H^s} \quad , \tag{16}$$

$$\|\psi(t)V(t)\varphi\|_A \lesssim \|\hat{\varphi}\|_{l^1(\mathbb{Z})} \quad . \tag{17}$$

Proof. (15) and (16) are classical. (17) can be obtained in the same way. Since $V(t)$ commutes with any time function and

$$\mathcal{F}_{t,x}(V(t)w(t,\cdot))(\tau,k) = \hat{w}(\tau - k|k|,k) \quad ,$$

we infer that

$$\begin{aligned} \|\psi(t)V(t)\varphi\|_A &= \|V(t)\psi(t)\varphi\|_A = \|\mathcal{F}_{t,x}(\psi\varphi)\|_{L^1(\mathbb{R}\times\mathbb{Z})} \\ &= \|\hat{\psi}\|_{L^1(\mathbb{R})}\|\hat{\varphi}\|_{l^1(\mathbb{Z})} \lesssim \|\hat{\varphi}\|_{l^1(\mathbb{Z})} \quad . \end{aligned}$$

Lemma 3.3 *For all $G \in X^{-1/2,s} \cap Z^{-1,s}$, it holds*

$$\|\psi(t)\int_0^t V(t-t')G(t')\,dt'\|_{Y^{1/2,s}} \lesssim \|G\|_{X^{-1/2,s}} + \|G\|_{Z^{-1,s}} \, , \tag{18}$$

$$\|\psi(t)\int_0^t V(t-t')G(t')\,dt'\|_A \lesssim \|G\|_{A^{-1}} \tag{19}$$

and for any $0 < \delta \le 1$ and any $0 \le b < 1/2$,

$$\|\psi(t/\delta)\int_0^t V(t-t')G(t')\,dt'\|_{Y^{1/2,s}} \lesssim \delta^{(1/2-b)-}\|G\|_{X^{-b,s}} \, . \tag{20}$$

Let us recall that (18)-(19) are direct consequences of the following one dimensional (in time) inequalities (cf. [5] and [6]): for any function $f \in \mathcal{S}^\infty(\mathbb{R})$, it holds

$$\|\psi(t)\int_0^t f(t')\,dt'\|_{H_t^{1/2}} \lesssim \|f\|_{H_t^{-1/2}} + \Big\|\frac{\mathcal{F}_t(f)}{\langle\tau\rangle}\Big\|_{L^1(\mathbb{R})}$$

and

$$\Big\|\mathcal{F}_t\Big(\psi(t)\int_0^t f(t')\,dt'\Big)\Big\|_{L^1(\mathbb{R})} \lesssim \Big\|\frac{\mathcal{F}_t(f)}{\langle\tau\rangle}\Big\|_{L^1(\mathbb{R})} \quad .$$

Finally to handle with the nonlinear terms we will make use of the following linear estimate due to Bourgain [3] (Actually, the result in [3] is proven for Bourgain's spaces of functions on $\mathbb{T}^2$ associated with the Schrödinger group but the result for functions on $\mathbb{R}\times\mathbb{T}$ can be obtained in exactly the same way and the result for Bourgain's spaces associated with the Benjamin-Ono group follows directly by projecting on the positive and negative modes) .

$$\|v\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \|v\|_{X^{3/8,0}} \quad . \tag{21}$$

Note that according to [6] this ensures that for $0<T<1$ and $3/8\le b<1/2$,

$$\|v\|_{L^4(]-T,T[\times\mathbb{T})} \lesssim T^{b-3/8}\|v\|_{X_T^{b,0}} \quad . \tag{22}$$

4 Proof of Theorem 1.1

Let $\{u_{0_n}\} := \{\tilde{u}_{0,n}+\cos(nx)\}$, where $\{\tilde{u}_{0,n}\}$ is any sequence converging strongly in $L^2(\mathbb{T})$ to some non constant function $u_0\in L^2(\mathbb{T})$, and let u_n and u be the associated emanating solutions constructed in Theorem 3.1. It is clear that $\{u_{0,n}\}$ converges to u_0 weakly but not strongly in $L^2(\mathbb{T})$. We want to prove that there exists no $T>0$ such that the sequences $\{u_n(t)\}$ do converge weakly in the sense of distributions towards $u(t)$ for all $t\in]0,T[$. In the sequel we will restrict ourselves to the case where the functions $\tilde{u}_{0,n}$ and u_0 have mean value zero. Indeed it is obvious that $u_{0,n}-\fint u_{0,n}$ converges also weakly but not strongly in $L^2(\mathbb{T})$ to $u_0-\fint u_0$ and since the solution emanating from $u_0-\fint u_{0,}$ is given by $u(t,x-t\fint u_0)-\fint u_0$, it is clear that the result for the projections on $H_0^0(\mathbb{T})$ ensures the desired result for $\{u_{0,n}\}$. Theorem 1.1 will be a consequence of the following key proposition.

Proposition 4.1 *Let $u_0\in H_0^0(\mathbb{T})$ and $\{u_{0,n}\}\subset H_0^0(\mathbb{T})$ be a sequence converging weakly in $L^2(\mathbb{T})$ towards u_0. Then there exist $v\in N_1$ and a subsequence $\{u_{n_k}\}$ of solutions to (1) emanating from $\{u_{0,n}\}$ such that $u_{n_k}(t)\rightharpoonup v(t)$ for all $t\in[-1,1]$.*

Moreover, if we assume that v satisfies (1) on $]0,T[$, with $0<T<1$, then the following assertions hold on the sequence of gauge functions $\{w_{n_k}:=\partial_x P_+(e^{-i\partial_x^{-1}u_{n_k}})\}$:

i) $w_{n_k}\rightharpoonup\partial_x P_+(e^{-iF/2})$ in $X_1^{1/2,0}$ that is continuous from $[0,T]$ into $L^2(\mathbb{T})$ and satisfies (13) on $]0,T[$.

ii) There exists an increasing subsequence $\{n_{k'}\}$ *of* $\{n_k\}$ *such that* $w_{n_{k'}} \rightharpoonup w$ *in* $X_1^{1/2,0}$ *that is solution of*

$$\begin{cases} w_t - iw_{xx} = 2i\partial_x P_+\Big(W\partial_x P_-(e^{-iF/2}\overline{w})\Big) \\ \qquad\qquad +2i\partial_x P_+\Big[W\partial_x P_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big)\Big] + \frac{i}{4}P_0(v^2)w \quad \text{on }]0,T[, \\ +\frac{i}{8\pi}\Big((\alpha^2 - \|u_0\|^2_{L^2(\mathbb{T})}) - 8(a(t) - \|w(t)\|^2_{L^2(\mathbb{T})})\Big)w \\ w(0) = \partial_x P_+(e^{i\partial_x^{-1}u_0/2}) \end{cases} \tag{23}$$

where $F := \partial_x^{-1}v$, $W := \partial_x^{-1}w$, $\alpha^2 := \lim\limits_{n_k\to+\infty}\int_{\mathbb{T}}|u_{0,n_{k'}}|^2$ *and* $t \mapsto a(t) := \lim\limits_{n_k\to+\infty}\int_{\mathbb{T}}|w_{n_{k'}}(t)|^2$ *is a continuous function.*

The proof of this proposition is the aim of the next section. The first part will follow directly from Theorem 3.1. Then assuming that v satisfies (1) on $]0,T[$ we will prove the two assertions in the following way. On one hand, we will observe that due to the expression of the gauge transform, the sequence $\{w_n\}$ has to converge weakly in $X_1^{1/2,0}$ to $\partial_x P_+(e^{-i\partial_x^{-1}v/2})$ which must satisfy equation (13) on $]0,T[$. On the other hand, passing to the limit in the equation (13) for some subsequence of $\{w_n\}$ we prove that its weak limit in $X_1^{1/2,0}$ must satisfy the equation (23) which is a kind of perturbation of (13) by some terms that measure some defect of strong $L^2(\mathbb{T})$-convergence.

From Proposition 4.1 we deduce that there exists $v \in N_1$ and a subsequence of emanating solutions $\{u_{n_k}\}$ such that $u_{n_k}(t) \rightharpoonup v(t)$ in $L^2(\mathbb{T})$ for all $t \in [-1,1]$. If there exists no $T > 0$ such that $v \equiv u$ on $[0,T]$ then Theorem 1.1 is proven and so we are done. We can thus assume that $v \equiv u$ on $[0,T]$ and thus v verifies (1) on $]0,T[$. Let us now prove that the assertions *i)* and *ii)* cannot hold in the same time. For this, let us compute the defect terms at the initial time for our sequence of initial data. First, since $\tilde{u}_{0,n} \to u_0$ in $L^2(\mathbb{T})$, it is obvious that

$$\lim_{n\to\infty}\int_{\mathbb{T}}|u_{0,n}|^2 = \int_{\mathbb{T}}|u_0|^2 + \lim_{n\to\infty}\int_{\mathbb{T}}|\cos(nx)|^2 = \int_{\mathbb{T}}|u_0|^2 + 2\pi\,. \tag{24}$$

The computation of the second term is done in the following lemma.

Lemma 4.2 *Setting* $w_{0,n} := \partial_x P_+(e^{-iF_{0,n}/2})$ *and* $w_0 := \partial_x P_+(e^{-iF_0/2})$ *with* $F_{0,n} := \partial_x^{-1}u_{0,n}$ *and* $F_0 := \partial_x^{-1}u_0$ *, it holds :*

$$\lim_{n\to\infty}\int_{\mathbb{T}}|w_{0,n}|^2 = \int_{\mathbb{T}}|w_0|^2 + \pi/2\,. \tag{25}$$

Proof. We observe that

$$\begin{aligned}\int_{\mathbb{T}}|w_{0,n}|^2 &= \int_{\mathbb{T}}\Big|P_+(u_{0,n}e^{-iF_{0,n}/2})\Big|^2\\ &= \int_{\mathbb{T}}\Big|P_+(\tilde{u}_{0,n}e^{-iF_{0,n}/2})\Big|^2+\int_{\mathbb{T}}\Big|P_+(\cos(nx)e^{-iF_{0,n}/2})\Big|^2\\ &\quad +2\Re\int_{\mathbb{T}}P_+(\cos(nx)e^{-iF_{0,n}/2})\overline{P_+(\tilde{u}_{0,n}e^{-iF_{0,n}/2})}\\ &= A_n+B_n+C_n\ .\end{aligned}$$

where $F_{0,n}=\partial_x^{-1}\tilde{u}_0+\frac{\sin(nx)}{n}$.

Now, since $u_{0,n}\rightharpoonup u_0$ in $L^2(\mathbb{T})$, $F_{0,n}$ and $e^{-iF_{0,n}/2}$ converge respectively to F_0 and $e^{-iF_0/2}$ in any $H^s(\mathbb{T})$ with $s<1$ and thus in $L^\infty(\mathbb{T})$. It is then easy to check that $\tilde{u}_{0,n}e^{-iF_{0,n}/2}\to u_0e^{-iF_0/2}$ in $L^2(\mathbb{T})$ and thus $A_n\to\int_{\mathbb{T}}|w_0|^2$. To compute the limit of C_n we notice that C_n can be rewritten as

$$C_n=2\Re\int_{\mathbb{T}}\cos(nx)e^{-iF_{0,n}/2}\overline{P_+(\tilde{u}_{0,n}e^{-iF_{0,n}/2})}$$

and thus, in the same way, $C_n\to 0$ since $\cos(nx)\rightharpoonup 0$ in $L^2(\mathbb{T})$. Finally, for the same reasons, we get

$$\lim_{n\to\infty}B_n=\lim_{n\to\infty}\int_{\mathbb{T}}\Big|P_+(\cos(nx)e^{-iF_0/2})\Big|^2=\lim_{n\to\infty}\int_{\mathbb{T}}\Big|\frac{e^{inx}}{2}e^{-iF_0/2}\Big|^2=\pi/2\ ,$$

where we used that for any $g\in L^2(\mathbb{T})$, it holds

$$\lim_{n\to\infty}\int_{\mathbb{T}}\Big|P_-\Big(e^{inx}g\Big)\Big|^2=\lim_{n\to\infty}\int_{\mathbb{T}}\Big|P_+\Big(e^{-inx}g\Big)\Big|^2=0\ .$$

□

Gathering (24) and (25) we infer that for our choice of the sequence $\{u_{0,n}\}$ it holds

$$\frac{i}{8\pi}(\alpha^2-\|u_0\|^2_{L^2(\mathbb{T})})-\frac{i}{\pi}(a(0)-\|w(0)\|^2_{L^2(\mathbb{T})})=-\frac{i}{4\pi}\ .$$

Since $t\mapsto a(t)$ and $t\to\partial_xP_+(e^{-iF/2})$ are continuous functions from $[0,T]$ into respectively $\mathbb{R}_+$ and $L^2(\mathbb{T})$, this leads to a contradiction between the assertions *i)* and *ii)* of Proposition 4.1 as soon as $\partial_xP_+(e^{-i\partial_x^{-1}u_0/2})\neq 0$. But this is always the case as shown in the following lemma which completes the proof of the theorem.

Lemma 4.3 *For any non identically vanishing $u_0 \in L^2(\mathbb{T})$ with mean-value zero, it holds*

$$P_+(e^{-i\partial_x^{-1}u_0/2}) \neq Cst\ .$$

Proof. Since $\partial_x^{-1}u_0 \in H^1(\mathbb{T})$ and $P_+(g) = \overline{P_-(\overline{g})}$ for any complex-valued function g, it is equivalent to prove that $P_-(e^{if}) \neq Cst$ for any non identically vanishing function $f \in H^1(\mathbb{T})$ with mean-value zero. We proceed by contradiction by assuming that there exists such f for which $P_-(e^{if}) = Cst$. We could then write

$$e^{if} = \sum_{n=0}^{\infty} a_n e^{in\theta}$$

and thus $e^{if(\theta)} = F(e^{i\theta})$ where

$$F(z) := \sum_{n=0}^{\infty} a_n z^n$$

is an holomorphic function on the unit disk. It is well known that the number of zeros in the unit disk of an holomorphic function H is given by

$$\frac{1}{2\pi i}\int_{C^1} \frac{H'(z)}{H(z)}\, dz$$

where C^1 is the unit circle. Noticing that $\partial_\theta(F(e^{i\theta})) = e^{if(\theta)}if'(\theta)$ on one hand and $\partial_\theta(F(e^{i\theta})) = F'(e^{i\theta})e^{i\theta}$ on the other hand, we infer that

$$\frac{1}{2\pi i}\int_{C^1} \frac{F'(z)}{F(z)}\, dz = \frac{1}{2\pi}\int_0^{2\pi} f'(\theta)\, d\theta = \frac{1}{2\pi}(f(2\pi) - f(0)) = 0\ .$$

Hence, F does not vanish in the unit disk and thus there exists an holomorphic function G on the unit disk such that $F = e^{iG}$. It follows that

$$G(e^{i\theta}) = f(\theta) \bmod 2\pi,\ \forall \theta \in [0, 2\pi],$$

and, since f is continuous, this implies that actually $G(e^{i\theta}) = f(\theta) + Cst$. Therefore G is an holomorphic function on the unit disk that takes only real-values on the unit circle. This is clearly impossible unless $G \equiv Cst$ which forces $f = Cst = 0$ since f has mean-value zero. □

5 Proof of Proposition 4.1

First we observe that, on account of Banach-Steinhaus theorem and Theorem 3.1, the sequence of emanating solutions $\{u_n\}$ is bounded in N_1 and the corresponding sequence of gauge functions $\{w_n\}$ is bounded in $X_1^{1/2,0}$. Therefore, up to the extraction of a subsequence, $\{u_{n_k}\}$ converges weakly in $\tilde{L}^4(]-2,2[\times\mathbb{T}))$ and weakly star in $L^\infty(]-2,2[;L^2(\mathbb{T}))$ to some v in $L^\infty(]-2,2[;L^2(\mathbb{T}))\cap\tilde{L}^4(]-2,2[\times\mathbb{T}))$. Let us check that $v\in N_1$. To do this, we take time extensions $\tilde{\tilde{u}}_n$ of u_n such that $\|\tilde{\tilde{u}}_n\|_N\leq 2\|u_n\|_{N_1}$ and set $\tilde{u}_n:=\psi\tilde{\tilde{u}}_n$ where ψ is the bump time function defined in Section 3.2. Obviously $Q_3\tilde{u}_n\rightharpoonup Q_3\tilde{v}$ in $X^{7/8,-1}$ where $\tilde{v}$ is a time extension of v. It remains to prove that $\tilde{v}\in Z^{0,0}$. Since $\tilde{L}^4(]-2,2[\times\mathbb{T}))\hookrightarrow L^2(]-2,2[\times\mathbb{T})$, it follows from Parseval theorem that $\mathcal{F}_{t,x}(\tilde{u}_n)\rightharpoonup\mathcal{F}_{t,x}(\tilde{v})$ in $l^2(\mathbb{Z};L^2(\mathbb{R}))$. On the other hand, since $\{\tilde{u}_n\}$ is bounded in $Z^{0,0}$, we infer that this convergence holds also weakly in $l^2(\mathbb{Z};\mathcal{M}(\mathbb{R}))$. Therefore, $\mathcal{F}_{t,x}(\tilde{v})\in l^2(\mathbb{Z};\mathcal{M}(\mathbb{R}))\cap l^2(\mathbb{Z};L^2(\mathbb{R}))\hookrightarrow l^2(\mathbb{Z};L^1(\mathbb{R}))$. This ensures that $v\in N_1$. Moreover in view of (1) it is easy to check that for any smooth space function ϕ, the sequence $\{t\mapsto(u_{n_k}(t),\phi)_{L^2}\}$ is bounded in $C([-1,1])$ and uniformly equi-continuous. Hence, from Ascoli's theorem, $(u_{n_k}(t),\phi)$ converges towards $(v(t),\phi)_{L^2}$ in $C([-1,1])$ and thus $u_{n_k}(t)\rightharpoonup v(t)$ in $L^2(\mathbb{T})$ for any $t\in[-1,1]$. Hence, in particular, $v(0)=u_0$.

We will now assume that v satisfies (1) on $]0,T[$ for some $0<T<1$ and prove the assertions $i)$ and $ii)$.

5.1 Proof of the first assertion

We set $F_n:=\partial_x^{-1}u_n$. From the hypotheses, $\{\partial_xF_n\}$ is bounded in N_1 and thus $\{F_n\}$ is bounded in $X_1^{0,1}$. Since from the equation,

$$\partial_tF_n+\mathcal{H}\partial_x^2F_n=F_{n,x}^2/2-\frac{1}{2}\fint F_{n,x}^2 \tag{26}$$

and $\{\partial_xF_n\}$ is bounded in $L^4(]-1,1[\times\mathbb{T})$, it follows that $\{F_n\}$ is also bounded in $X_1^{1,0}$. By interpolation with the bound above, it follows that $\{F_n\}$ is bounded in $X_1^{1/2,1/2}$. Since $L^p(]-1,1[\times\mathbb{T})$ is compactly included into $X_1^{1/2,1/2}$, we deduce that $\{F_n\}$ converges to $\partial_x^{-1}v$ in $L^p(]-1,1[\times\mathbb{T})$, $2\leq p<\infty$ and thus also almost everywhere in $[-1,1]\times\mathbb{T}$. Therefore, $\{e^{-iF_n/2}\}$ converges almost everywhere to $e^{-iF/2}$ and since it is obviously bounded by 1 in $L^\infty(]-1,1[\times\mathbb{T})$, the convergence also holds in $\mathcal{D}'$ by the dominated convergence theorem. This ensures that $W_n:=P_+(e^{-iF_n/2})$ converges to $P_+(e^{-i\partial_x^{-1}v/2})$ in $\mathcal{D}'$ and thus w_n, which is bounded in $X_1^{1/2,0}$,

converges to $\partial_x P_+(e^{-i\partial_x^{-1}v/2})$ weakly in $X_1^{1/2,0}$. Moreover, since $v \in N_1 \hookrightarrow C([-1,1];L^2(\mathbb{T}))$ and $v \mapsto \partial_x P_+(e^{-i\partial_x^{-1}v/2})$ is clearly continuous in $L^2(\mathbb{T})$, $t \mapsto \partial_x P_+(e^{-iF/2})$ belongs to $C([0,T];L^2(\mathbb{T}))$.
Let us check that $\partial_x P_+(e^{-i\partial_x^{-1}v/2})$ satisfies the equation (13) on $]0,T[$ with $F := \partial_x^{-1}v$ (note that this is implicitly contains in Theorem 3.1 since v satisfies (1) and belongs to the class of uniqueness). Since $v \in C([0,T];L^2(\mathbb{T}))$ and satisfies (BO) on $]0,T[$, $v_t \in C([0,T];H^{-2}(\mathbb{T}))$. Therefore $F \in C([0,T];H^1(\mathbb{T}))\cap C^1([0,T];H^{-1}(\mathbb{T}))$ and the following calculations are thus justified:

$$\begin{aligned}\partial_t P_+(e^{-iF/2}) &= -\frac{i}{2}P_+(F_t e^{-iF/2}) \\ &= -\frac{i}{2}P_+\Big(e^{-iF/2}(-\mathcal{H}F_{xx} + F_x^2/2 - P_0(F_x^2)/2)\Big)\end{aligned}$$

and

$$\partial_{xx} P_+(e^{-iF/2}) = P_+\Big(e^{-iF/2}(-F_x^2/4 - iF_{xx}/2)\Big) \quad .$$

Since (11) and (12) also make sense for v, we conclude that $\partial_x P_+(e^{-i\partial_x^{-1}v/2})$ satifies (13) in $\mathcal{D}'(]0,T[\times\mathbb{T})$. □

5.2 Two product lemmas

In the sequel we will have to make use of the two following lemmas that are respectively proven in the appendix of [12] and [11].

Lemma 5.1 *Let $z \in L^\infty(\mathbb{R};H^1(\mathbb{T}))$ and let $v \in \tilde{L}^4(\mathbb{R}\times\mathbb{T})$ then*

$$\|zv\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \lesssim (\|z\|_{L^\infty(\mathbb{R}\times\mathbb{T})} + \|z_x\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}))})\|v\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \quad . \tag{27}$$

Lemma 5.2 *Let $\alpha \geq 0$ and $1 < q < \infty$ then*

$$\Big\|D_x^\alpha P_+\Big(fP_-\partial_x g\Big)\Big\|_{L^q(\mathbb{T})} \lesssim \|D_x^{\gamma_1} f\|_{L^{q_1}(\mathbb{T})}\, \|D_x^{\gamma_2} g\|_{L^{q_2}(\mathbb{T})} \quad , \tag{28}$$

with $1 < q_i < \infty$, $1/q_1 + 1/q_2 = 1/q$ and $\left\{ \begin{array}{l} \gamma_1 \geq \alpha,\ \gamma_2 \geq 0 \\ \gamma_1 + \gamma_2 = \alpha + 1 \end{array} \right.$.

5.3 Proof of the second assertion

As announced, we plan now to pass to the limit in (13). For this our first task consists in proving that the sequence $\{G_n\} := \{e^{-iF_n}\}$ is bounded in $R_1^{7/8}$ and converges weakly in $(X_1^{1/2,0}\cap X_1^{0,1})$ to $G = e^{-i\partial_x^{-1}v/2}$ (see in Section 2 the definition of the space $R_1^{7/8}$). Then, in view of the linear estimates of Section 3.2 we will study the behavior of $A(G_n,W_n)$ and $B(G_n,W_n)$ in $Y_1^{-1/2,0}$ and in some spaces continuously embedded in $X_1^{-1/2+,-2}$.

Lemma 5.3 *The sequences $\{F_n\}$ and $\{W_n\}$ associated with $\{u_n\}$ are respectively bounded in R_1^1 and $X_1^{1/2,1} \cap L^\infty(]-1,1[;H^1(\mathbb{T})$.*

Proof. First, note that the result for $\{W_n\}$ follows directly from the boundedness of $\{F_n\}$ and $\{w_n\}$ in respectively $L^\infty(]-1,1[;H^1(\mathbb{T}))$ and $X_1^{1/2,0}$ together with the continuity of the map $F \mapsto P_+(e^{-iF/2})$ in $H^1(\mathbb{T})$. Let us now prove the result for $\{F_n\}$. We set $\tilde{F}_n := \psi^2 F_n$ where $\psi \in C_0^\infty([-2,2])$ is a time function such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $[-1,1]$. From Theorem 3.1 and (26) we already know that $\{\tilde{F}_n\}$ is bounded in $X^{0,1} \cap X^{1,0}$ and that $\{\partial_x \tilde{F}_n\}$ is bounded in $\tilde{L}^4(\mathbb{R}\times\mathbb{T}) \cap Z^{0,0}$. In particular, $\{\mathcal{F}_{t,x}(\partial_x \tilde{F}_n)\}$ is bounded in $l^2(\mathbb{Z};L^1(\mathbb{R}))$ and applying Cauchy-Schwarz in k it follows directly that $\{Q_3 \partial_x \tilde{F}_n\}$ is bounded in A. On the other hand multiplying (26) by ψ^2 and using Lemmas 3.2-3.3 we infer that

$$\|\widehat{P_3\tilde{F}_n}\|_{L^1(\mathbb{R}\times\mathbb{Z})} \lesssim \|\widehat{P_3 F_{0,n}}\|_{l^1(\mathbb{Z})} + \Big\|\chi_{\{|k|\leq 3\}} \frac{\widehat{\psi} u_n * \widehat{\psi} u}{\langle\sigma\rangle}\Big\|_{L^1(\mathbb{R}\times\mathbb{Z})} + \Big\|\frac{\mathcal{F}_t\Big(P_0((\psi u_n)^2)\Big)}{\langle\sigma\rangle}\Big\|_{L^1(\mathbb{R})}$$

Applying Cauchy-Schwarz inequality in τ and k , it follows that

$$\begin{aligned}
\|\widehat{P_3\tilde{F}_n}\|_{L^1(\mathbb{Z}^2)} &\lesssim \|\widehat{P_3 F_{0,n}}\|_{l^1(\mathbb{Z})} + \Big\|\widehat{\psi u_n} * \widehat{\psi} u_n\Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} + \|P_0((\psi u_n)^2)\|_{L^2(\mathbb{R})} \\
&\lesssim \|\widehat{P_3 u_{0,n}}\|_{l^2(\mathbb{Z})} + \|\psi u_n\|^2_{L^4(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|u_{0,n}\|_{L^2(\mathbb{T})} + \|u_n\|_N^2 \ .
\end{aligned}$$

This ensures that $\{P_3\tilde{F}_n\}$ is bounded in A and completes the proof of the lemma.

Lemma 5.4 *Let $\{F_n\}$ be a sequence bounded in R_1^1 that converges in $(C^\infty(]-1,1[\times\mathbb{T}))^*$ to F then the sequences $\{G_n := e^{-iF_n/2}\}$ and $\{\overline{G_n} = e^{iF_n/2}\}$ are bounded in $R_1^{7/8}$ and converge in $(C^\infty(]-1,1[\times\mathbb{T}))^*$ to respectively $e^{-iF/2}$ and $e^{iF/2}$.*

Proof . Since the sign in front of iF we not play any role in the analysis we choose the positive sign and thus we prove the statement for $F \mapsto G$. We start by proving the continuity of the map $F \mapsto e^{iF/2}$ from R_1^1 into $R_1^{7/8}$. Let $\tilde{F}$ be a time extension of F such that $\|\tilde{F}\|_{R^1} \leq 2\|\tilde{F}\|_{R_1^1}$. To simplify the notations we drop the ˜ in the remaining of the proof. Expanding $e^{iF/2}$ as

$$e^{iF/2} = \sum_{k=0}^{\infty} \frac{i^k}{2^k k!} F^k$$

it suffices to check that this serie is absolutely convergent in $R^{7/8}$. First we notice that thanks to Lemma 5.1, for $i \geq 2$,

$$\|\partial_x(F^i)\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \lesssim i\,\|F^{i-1}\partial_x F\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \lesssim i^2\|\partial_x F\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})}\|F\|^{i-2}_{L^\infty(\mathbb{R}\times\mathbb{T})}(1+\|F\|_{L^\infty(\mathbb{R};H^1(\mathbb{T}))})$$

and thus

$$\sum_{i=0}^{\infty}\Big\|\frac{1}{2^i i!}\partial_x(F^i)\Big\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \lesssim \|\partial_x F\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})}(1+\|F\|_{L^\infty(\mathbb{R};H^1(\mathbb{T}))})e^{\|F\|_A}\ .$$

Next, using that A and $L^\infty(\mathbb{T};H^1(\mathbb{T}))$ are algebras, it clearly holds

$$\sum_{i=0}^{\infty}\Big\|\frac{1}{2^i i!}F^i\Big\|_A \lesssim \sum_{i=0}^{\infty}\frac{1}{2^i i!}\|F\|_A^i \lesssim e^{\|F\|_A}\ .$$

and

$$\sum_{i=0}^{\infty}\Big\|\frac{1}{2^i i!}F^i\Big\|_{L^\infty(\mathbb{R};H^1(\mathbb{T}))} \lesssim e^{\|F\|_{L^\infty(\mathbb{R};H^1(\mathbb{T}))}}\ .$$

It thus remains to estimate

$$I_i := \|F^i\|_{X^{7/8,0}} = \Big\|\langle\sigma\rangle^{7/8}\sum_{q_1+..q_i=q,\,k_1+..k_i=k}\widehat{F}(q_1,k_1)...\widehat{F}(q_i,k_i)\Big\|_{L^2(\mathbb{R}\times\mathbb{Z})}$$

where $i\geq 2$ and $\sigma=\sigma(q,k):=q+k^2$. Since we do not have a control on $\|\mathcal{F}_{t,x}(|k\widehat{F}|)\|_{L^4_{t,x}}$ but on $\|\partial_x F\|_{L^4_{t,x}}$ we have to use a Littlewood-Paley decomposition. We can write F^i as

$$\begin{aligned}
F^i &= \sum_{j_1\geq 0}\sum_{j_2=0}^{[j_1-2-\ln i/\ln 2]}\Delta_{j_1}F\,\Delta_{j_2}F\sum_{0\leq j_3,..,j_i\leq j_2}n(j_1,..,j_i)\prod_{q=3}^{i}\Delta_{j_q}F\\
&\quad+\sum_{j_1\geq 0}\sum_{j_2=[j_1-2-\ln i/\ln 2]+1}^{j_1}\Delta_{j_1}F\,\Delta_{j_2}F\sum_{0\leq j_3,..,j_i\leq j_2}n(j_1,..,j_i)\prod_{q=3}^{i}\Delta_{j_q}F\\
&= \sum_{j_1\geq 0}\sum_{j_2=0}^{[j_1-2-\ln i/\ln 2]}T^i_{j_1,j_2}+\sum_{j_1\geq 0}\sum_{j_2=[j_1-2-\ln i/\ln 2]+1}^{j_1}T^i_{j_1,j_2} \qquad (29)
\end{aligned}$$

where $n(j_1,..,j_i)\in\{1,..,i(i-1)\}$.

• Contribution of the first term of (29). Setting $\alpha^i_{j_1,j_2}:=8+j_1+j_2+\ln i/\ln 2$, we first write

$$\begin{aligned}
\|T^i_{j_1,j_2}\|_{X^{7/8}} &\lesssim \Big\|\chi_{\{|\sigma|\leq 2^{\alpha^i_{j_1,j_2}}\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}(T^i_{j_1,j_2})\Big\|_{L^2}+\Big\|\chi_{\{|\sigma|>2^{\alpha^i_{j_1,j_2}}\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}(T^i_{j_1,j_2})\Big\|_{L^2}\\
&\lesssim 2^{7\alpha^i_{j_1,j_2}/8}\|T^i_{i,j}\|_{L^2}+\Big\|\chi_{\{|\sigma|>2^{\alpha^i_{j_1,j_2}}\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}(T^i_{j_1,j_2})\Big\|_{L^2}\ . \qquad (30)
\end{aligned}$$

Noticing that

$$\begin{aligned}\sum_{0\le j_3,..,j_i\le j_2}\Big\|\prod_{q=3}^{i}\Delta_{j_q}F\Big\|_{L^\infty(\mathbb{R}\times\mathbb{T})} &\lesssim \Big(\sum_{j=0}^{\infty}\|\Delta_j F\|_{L^\infty(\mathbb{R}\times\mathbb{T})}\Big)^{i-2}\\ &\lesssim \|F\|_A^{i-2}\end{aligned}$$

and that by frequency localization,

$$\|\Delta_j F\|_{L^4(\mathbb{R}\times\mathbb{T})}\lesssim 2^{-j}\Big(\|F\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|F_x\|_{L^4(\mathbb{R}\times\mathbb{T})}\Big)$$

we infer that

$$\begin{aligned}\sum_{j_1\ge 0}\sum_{j_2=0}^{[j_1-2-\ln i/\ln 2]} & 2^{7\alpha^i_{j_1,j_2}/8}\|T^i_{i,j}\|_{L^2(\mathbb{R}\times\mathbb{T})}\\ &\lesssim i^2(i-1)\sum_{j_1\ge 0}\sum_{j_2\ge 0}2^{7j_1/8}\|\Delta_{j_1}F\|_{L^4(\mathbb{R}\times\mathbb{T})}2^{7j_2/8}\|\Delta_{j_2}F\|_{L^4(\mathbb{R}\times\mathbb{T})}\|F\|_A^{i-2}\\ &\lesssim i^2(i-1)\sum_{j_1\ge 0}\sum_{j_2\ge 0}2^{-j_1/8}2^{-j_2/8}(\|F\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|F_x\|_{L^4(\mathbb{R}\times\mathbb{T})})^2\|F\|_A^{i-2}\\ &\lesssim i^2(i-1)(\|F\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|F_x\|_{L^4(\mathbb{R}\times\mathbb{T})})^2\|F\|_A^{i-2}\,.\end{aligned}$$

To estimate the second term of the right-hand side of (30) we notice that, since $j_2\le j_1-2-\ln i/\ln 2$,

$$|\mathcal{F}_{t,x}(T^i_{j_1,j_2})(\tau,k)|\lesssim i(i-1)\sum_{B^{q,k}_{i,j_1}}|\widehat{F}(\tau_1,k_1)|\,\widehat{F}(\tau_2,k_2)|\prod_{p=3}^{i}|\widehat{F}(\tau_p,k_p)|$$

where

$$B^{q,k}_{i,j_1}:=\Big\{(\tau_1,..,\tau_i,k_1,..,k_i)\in\mathbb{R}^i\times\mathbb{Z}^i,\sum_{p=1}^{i}\tau_p=\tau,\ \sum_{p=1}^{i}k_p=k,\ |k_1|\in I_{j_1},$$

$$i|k_2|\le|k_1|\text{ and }|k_p|\le\min(2|k_2|+2,|k_1|/i),p=3,..,i\Big\}$$

with $I_0:=[0,2]$ and $I_k:=[2^{k-1},2^{k+1}]$ for $k\ge 1$. Now, setting $\sigma_i:=\sigma(\tau_i,k_i)=\tau_i+|k_i|k_i$, the resonant relation gives

$$\sigma-\sum_{j=1}^{i}\sigma_j=k|k|-\sum_{j=1}^{i}k_j|k_j|=(\sum_{j=1}^{i}k_j)|\sum_{j=1}^{i}k_j|-\sum_{j=1}^{i}k_j|k_j| \qquad (31)$$

and, since k and k_1 have the same sign in $B^{q,k}_{i,j_1,j_2}$, it is not too hard to check that

$$|\sigma - \sum_{j=1}^{i} \sigma_j| \leq 8i|k_1|(|k_2|+1) \ .$$

For $|\sigma| \geq 2^{8+j_1+j_2+\ln i/\ln 2} \geq 10i|k_1|(|k_2|+1)$ it thus results that

$$\langle \sigma \rangle \lesssim i \max_{j=1,,i} \langle \sigma_j \rangle \ .$$

and thus

$$\begin{aligned}
&\sum_{j_1\geq 0} \sum_{j_2=0}^{[j_1-2-\ln i/\ln 2]} \Big\| \chi_{\{|\sigma|>2^{\alpha^i_{j_1,j_2}}\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x}(T^i_{j_1,j_2}) \Big\|_{l^2(\mathbb{Z})} \\
&\qquad \lesssim \sum_{j_1\geq 0} \sum_{j_2=0}^{j_1-2-\ln i/\ln 2} 2^{-j_1/8} 2^{-j_2/8} \Big\| \chi_{\{|\sigma|>2^{\alpha^i_{j_1,j_2}}\}} \langle \sigma \rangle \mathcal{F}_{t,x}(T^i_{j_1,j_2}) \Big\|_{l^2(\mathbb{Z})} \\
&\qquad \lesssim i^2(i-1)\|F\|_{X^{1,0}} \|F\|_A^{i-1} \ . \qquad (32)
\end{aligned}$$

• Contribution of second term of (29) . We proceed in a similar way. We set $\beta^i_{j_1,j_2} := 8 + j_1 + j_2 + 2\ln i/\ln 2$ and notice that

$$\begin{aligned}
&\sum_{j_1\geq 0} \sum_{j_2=[j_1-2-\ln i/\ln 2]+1}^{j_1} \Big\| \chi_{\{|\sigma|\leq \beta^i_{j_1,j_2}\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x}(T^i_{j_1,j_2}) \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} \\
&\quad \lesssim \sum_{j_1\geq 0} \sum_{j_2=0}^{[j_1-2-\ln i/\ln 2]} 2^{7\beta^i_{j_1,j_2}/8} \|T^i_{i,j}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \\
&\quad \lesssim i^3(i-1) \sum_{j_1\geq 0} \sum_{j_2\geq 0} 2^{7j_1/8} \|\Delta_{j_1} F\|_{L^4(\mathbb{R}\times\mathbb{T})} 2^{7j_2/8} \|\Delta_{j_2} F\|_{L^4(\mathbb{R}\times\mathbb{T})} \|F\|_A^{k-1} \\
&\quad \lesssim i^3(i-1)(\|F\|_{L^4(\mathbb{R}\times\mathbb{T})} + \|F_x\|_{L^4(\mathbb{R}\times\mathbb{T})})^2 \|F\|_A^{k-1} \ .
\end{aligned}$$

On the other hand, since this time $j_2 > j_1 - 2 - \ln i/\ln 2$, we have

$$|\mathcal{F}_{t,x}(S^i_{j_1,j_2})(\tau,k)| \lesssim i(i-1) \sum_{C^{\tau,k}_{i,j_1}} |\widehat{F}(\tau_1,k_1)|\, \widehat{F}(\tau_2,k_2)| \prod_{p=3}^{i} |\widehat{F}(\tau_p,k_p)|$$

where

$$C^{q,k}_{i,j_1} := \Big\{ (\tau_1,..,\tau_i,k_1,..,k_i) \in \mathbb{R}^i \times \mathbb{Z}^i, \sum_{p=1}^{i} \tau_p = \tau, \ \sum_{p=1}^{i} k_p = k, \ |k_1| \in I_{j_1},$$

$$10i|k_2| \geq |k_1| \text{ and } |k_p| \leq 2|k_2| + 2, p = 3, .., i \Big\} .$$

(31) then ensures that in $C^{\tau,k}_{i,j_1}$ it holds

$$|\sigma - \sum_{j=1}^{i} \sigma_j| \leq 100 i^2 \max(2, |k_1|) \max(|k_2|, 2) .$$

For $|\sigma| \geq 2^{\beta^i_{j_1,j_2}} \geq 200 i^2 \max(2, |k_1|) \max(|k_2|, 2)$ it results that

$$\langle \sigma \rangle \lesssim i \max_{j=1,,i} \langle \sigma_j \rangle .$$

and we thus obtain an estimate similar to (32). Since $\sum_{i=1}^{\infty} \frac{i^3(i-1)}{2^i i!} < \infty$, this completes the proof of the strong continuity of the map $F \mapsto G$ from R^1_1 into $R^{7/8}_1$.

Let us now prove the convergence result. On account of the continuity result proved above, the sequence $\{e^{iF_n/2}\}$ is bounded in $R^{7/8}_1$. Therefore it is relatively compact in $(C^\infty(]-1,1[\times\mathbb{T}))^*$ and thus it remains to check that the only possible limit is $e^{iF/2}$. Since the serie $\sum_{k=0^\infty} \frac{i^k F^k}{2^k k!}$ converges absolutely in $L^4(]-1,1[\times\mathbb{T})$, by the Lebesgue dominated convergence theorem it suffices to check that for any fixed k, the map $F \mapsto F^k$ is strongly continuous from a function space E, where R^1_1 is compactly embedded, into $(C^\infty(]-1,1[\times\mathbb{T}))^*$. Obviously $E = L^{\max(2,k)}(\mathbb{R}\times\mathbb{T})$ answers the question for $1 \leq k < \infty$. Indeed $X^{1/2,1/2}_1$ is compactly embedded in $L^k(]-1,1[\times\mathbb{T})$ for $2 \leq k < \infty$ and $F \mapsto F^k$ is continuous from $L^{\max(2,k)}(]-1,1[\times\mathbb{T})$ into $L^1(]-1,1[\times\mathbb{T})$. □

Let us now prove the desired continuity result on B.

Lemma 5.5 *Let $\{(G_n, W_n)\}$ be a sequence bounded in $R^{7/8}_1 \times X^{1/2,1}_1$ that converges in the sense of distributions to (G, W) . Then $B(G_n, W_n)$ converges weakly in $X^{-1/2+,0}_1$ to $B(G, W)$.*

Proof . Let (G, W) belonging to $R^{7/8}_1 \times X^{1/2,1}_1$. We take extensions $\tilde{G}$ and $\tilde{W}$ of G and W, supporting in time in $]-2, 2[$, such that $\|\tilde{G}\|_{N^{7/8}} \lesssim \|G\|_{N^{7/8}_1}$ and $\|\tilde{W}\|_{X^{1/2,1}} \lesssim \|W\|_{X^{1/2,1}_1}$, that we still denote by G and W to simplify the notation. From (21) we infer by duality that

$$\|B(G, W)\|_{X^{-1/2+,0}} \lesssim \|B(G, W)\|_{L^{4/3}(\mathbb{R}\times\mathbb{T})} .$$

According to Lemma 5.2 it results that

$$\begin{aligned} \|B(G, W)\|_{X^{-1/2+,0}} &\lesssim \|w\|_{L^4(\mathbb{R}\times\mathbb{T})} \Big\| \partial_x P_- \Big(G \partial_x P_+ \overline{G} \Big) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \\ &\lesssim \|w\|_{L^4(\mathbb{R}\times\mathbb{T})} \|\partial_x G\|_{L^4(\mathbb{R}\times\mathbb{T})} \|\partial_x \overline{G}\|_{L^4(\mathbb{R}\times\mathbb{T})} . \end{aligned} \quad (33)$$

This proves that $B(G_n, W_n)$ remains bounded in $X^{-1/2+,0}$. Now to prove the convergence result, we will argue as in the preceding lemma by proving the strong continuity of B from a function space, where $(X_1^{1/2,0} \cap X_1^{0,1}) \times X_1^{1/2,1}$ is compactly embedded, into $X_1^{-1/2,-1}$. Since $X_1^{1/3,1/3} = [X_1^{1/2,0}, X_1^{0,1}]^{2/3}$ is compactly embedded in $L^6(]-1,1[\times\mathbb{T})$, we infer by interpolating with $u_x \in L^4(]-1,1[\times\mathbb{T})$ that $R_1^{7/8}$ is compactly embedded in the space of functions $u \in L^2(]-1,1[\times\mathbb{T})$ such that $D_x^{2/3}u$ belongs to $L^{9/2}(]-1,1[\times\mathbb{T})$. On the other hand, using again Lemma 5.2 and (21) we observe that

$$\begin{aligned}
\|B(G,W)\|_{X^{-1/2,-1}} &\lesssim \Big\| P_+\Big(W\partial_x P_-(G\partial_x P_+\overline{G})\Big)\Big\|_{L^{4/3}(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|D_x^{2/3}W\|_{L^{\frac{36}{11}}(\mathbb{R}\times\mathbb{T})}\Big\|D_x^{1/3}P_-\Big(G\partial_x P_+\overline{G}\Big)\Big\|_{L^{9/4}(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|W\|_{X^{3/8,2/3}}\|D_x^{2/3}G\|_{L^{9/2}(\mathbb{R}\times\mathbb{T})}\|D_x^{2/3}\overline{G}\|_{L^{9/2}(\mathbb{R}\times\mathbb{T})}\,.
\end{aligned}$$

This concludes the proof since $X_1^{1/2,1}$ is obviously compacty embedded in $X_1^{3/8,2/3}$. □

Let us now study the continuity of A.

Lemma 5.6 *The operator A is continuous from $R_1^{7/8} \times X_1^{1/2,1}$ into $Y_1^{-1/2,0}$.*

Proof. Let (G,W) belonging to $R_1^{7/8} \times X_1^{1/2,1}$. We take extensions $\tilde{G}$ and $\tilde{W}$ of G and W, such that $\|\tilde{G}\|_{R^{7/8}} \leq 2\|G\|_{R_1^{7/8}}$ and $\|\tilde{W}\|_{X^{1/2,1}} \leq 2\|W\|_{X_1^{1/2,1}}$, that we still denote by G and W to simplify the notation. We decompose $A(G,W)$ as

$$\begin{aligned}
A(G,W) &= 2i\sum_{j,p\in\mathbb{N}} \partial_x P_+\Big(W\partial_x P_-(\Delta_j\overline{w}\Delta_p G\Big) \\
&= 2i\sum_{0\leq j<p+5} \partial_x P_+\Big(W\partial_x P_-(\Delta_j\overline{w}\Delta_p G\Big) \\
&\quad +2i\sum_{0\leq p\leq j-5} \partial_x P_+\Big(W\partial_x P_-(\Delta_j\overline{w}\Delta_p G\Big) \\
&= A_1 + A_2\,. \qquad (34)
\end{aligned}$$

To estimate the first term we use again (21) and Lemma 5.2 to get

$$\begin{aligned}
\|A_1(G,W)\|_{X^{-1/2+,0}} &\lesssim \|A_1(G,W)\|_{L^{4/3}(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|w\|_{L^4\mathbb{R}\times\mathbb{T})}\Big\|\sum_p \partial_x P_-(\sum_{j=0}^{p+4}\Delta_j\overline{w}\Delta_p G)\Big\|_{L^2(\mathbb{R}\times\mathbb{T})}
\end{aligned}$$

and the last term of the above right-hand side can be estimated in the following way (we set $\tilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1}$ for $j \geq 1$)

$$
\begin{aligned}
C &:= \Big\| \sum_{p\geq 0} \partial_x P_-(\sum_{j=0}^{p+4} \Delta_j \overline{w} \Delta_p G) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \Big\| \sum_{p=0}^{1}\sum_{i=0}^{2} \Delta_i \Big(\partial_x P_-(\sum_{j=0}^{p+4} \Delta_j \overline{w} \Delta_p G) \Big) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \\
&\quad + \Big\| \sum_{p\geq 2} \tilde{\Delta}_p \Big(\partial_x P_-(\sum_{j=0}^{p-2} \Delta_j \overline{w} \Delta_p G) \Big) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \\
&\quad + \Big\| \sum_{p\geq 2}\sum_{i=-1}^{4} \partial_x \Big(P_-(\Delta_{p+i} \overline{w} \Delta_p G) \Big) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})} \\
&\lesssim \|G\|_{L^4(\mathbb{R}\times\mathbb{T})} \|w\|_{L^4(\mathbb{R}\times\mathbb{T})} + \Big[\sum_{p\geq 2} \Big\| \tilde{\Delta}_p \Big(\partial_x P_-(\sum_{j=0}^{p-2} \Delta_j \overline{w} \Delta_p G) \Big) \Big\|^2_{L^2(\mathbb{R}\times\mathbb{T})} \Big]^{1/2} \\
&\quad + \sum_{p\geq 2}\sum_{i=-1}^{4} \Big\| \partial_x \Big(P_-(\Delta_{p+i} \overline{w} \Delta_p G) \Big) \Big\|_{L^2(\mathbb{R}\times\mathbb{T})}
\end{aligned}
$$

where in the last step we use the quasi-orthogonality of the $\tilde{\Delta}_j$ in $L^2(\mathbb{R}\times\mathbb{T})$. Applying Cauchy-Schwarz in p for the last term of the above right-hand side member, we finally get

$$
\begin{aligned}
C &\lesssim \|G\|_{L^4(\mathbb{R}\times\mathbb{T})} \|w\|_{X^{1/2,0}} + \Big(\sum_p \|\partial_x \Delta_p G\|^2_{L^4(\mathbb{R}\times\mathbb{T})} \Big)^{1/2} \Big[\|w\|_{L^4(\mathbb{R}\times\mathbb{T})} + \Big(\sum_p \|\Delta_p w\|^2_{L^4(\mathbb{R}\times\mathbb{T})} \Big)^{1/2} \Big] \\
&\lesssim \Big(\|G\|_{L^4(\mathbb{R}\times\mathbb{T})} + \|\partial_x G\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})} \Big) \|w\|_{X^{1/2,0}} \ .
\end{aligned}
$$

Now setting

$$
H_p = \sum_{j\geq p+5} \Delta_j \overline{w} \Delta_p G \tag{35}
$$

A_2 can be rewritten as

$$
A_2(G,W) = \sum_{p\geq 0} \partial_x P_+ \Big(W \partial_x P_- H_p \Big) \ .
$$

We thus have to estimate

$$
I := \Big| \Big(A_2(G,W), h \Big)_{L^2} \Big|
$$

$$
\begin{aligned}
&\lesssim \sum_{p\geq 0}\int_{\mathbb{R}^2}\sum_{B}\chi_{\{k\leq 2^{p+5}\}}|\hat{h}(\tau,k)||k||\widehat{W}(\tau_1,k_1)||k-k_1||\widehat{H}_p(\tau_2,k_2)|\,d\tau_1 d\tau_2\\
&\quad+\sum_{p\geq 0}\int_{\mathbb{R}^2}\sum_{B}\chi_{\{k>2^{p+5}\}}|\hat{h}(\tau,k)||k||\widehat{W}(\tau_1,k_1)||k-k_1||\widehat{H}_p(\tau_2,k_2)|\,d\tau_1 d\tau_2\\
&= I_1+I_2
\end{aligned}
$$

where $k=k_1+k_2$, $\tau=\tau_1+\tau_2$ and

$$
B:=\{(k_1,k_2)\in\mathbb{Z}^2,\ k_1>0,\ k_2<0 \text{ and } k_1+k_2>0\}\,.
$$

The idea of this dichotomy is the following : In the domain of integration of I_1, $|k|$ is controlled by 2^p which is the order of the modes of $\Delta_p G$. On the other hand, in the domain of integration of I_2 the modes of h and $\overline{w}$ are very large with respect to the modes of $\Delta_p G$ and then the resonant relation will give a smoothing effect.
We use that $k_1\geq |k_2|$ on B and a Littlewood-Paley decomposition of h to get thanks to (21) and Cauchy-Schwarz inequality in p,

$$
\begin{aligned}
I_1&\lesssim \sum_{p\geq 0}\int_{\mathbb{R}^2}\sum_{B}\sum_{i=-6}^{p}2^{p-i}|\widehat{\Delta_{p-i}h}(\tau,k)||\hat{W}(\tau_1,k_1)||k_2||\hat{H}_p(\tau_2,k_2)|\\
&\lesssim \int_{\mathbb{R}^2}\sum_{B}\sum_{i=-6}^{\infty}2^{-i}\sum_{p\geq\max(0,i)}|\widehat{\Delta_{p-i}h}(\tau,k)||\hat{w}(\tau_1,k_1)|2^p|\hat{H}_p(\tau_2,k_2)|\\
&\lesssim \sup_{i\geq -6}\sum_{p\geq\max(0,i)}\int_{\mathbb{R}^2}\sum_{B}|\widehat{\Delta_{p-i}h}(\tau,k)||\hat{w}(\tau_1,k_1)|2^p|\hat{H}_p(\tau_2,k_2)|\\
&\lesssim \|\mathcal{F}^{-1}(|\hat{w}|)\|_{L^4(\mathbb{R}\times\mathbb{T})}\Big(\sum_{p\geq 0}\|\mathcal{F}^{-1}(|\hat{h}|)\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\Big)^{1/2}\Big(\sum_{p\geq 0}2^{2p}\|\mathcal{F}^{-1}(|\hat{H}_p|)\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\Big)^{1/2}.
\end{aligned}
$$

Note that $X^{3/8,0}\hookrightarrow\tilde{L}^4(\mathbb{R}\times\mathbb{T})$. Moreover, since

$$
\Delta_i H_p=\Delta_i\Big(\sum_{\substack{j\geq p+5\\ i-2\leq j\leq i+2}}\Delta_j\overline{w}\Delta_p G\Big)\,,
$$

we infer that

$$
\begin{aligned}
\|H_p\|^2_{L^2}\sim\sum_{i\geq 0}\|\Delta_i H_p\|^2_{L^2(\mathbb{R}\times\mathbb{T})}&\lesssim\sum_{i\geq 0}\|\Delta_i\overline{w}\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\|\Delta_p G\|^2_{L^4(\mathbb{R}\times\mathbb{T})}\\
&\lesssim\|\overline{w}\|^2_{X^{3/8,0}}\|\Delta_p G\|^2_{L^4(\mathbb{R}\times\mathbb{T})}
\end{aligned}
$$

and thus

$$I_1 \lesssim \|w\|^2_{X^{3/8,0}} \|h\|_{X^{3/8,0}} (\|G\|_{X^{1/2,0}} + \|\partial_x G\|_{\tilde{L}^4(\mathbb{R}\times\mathbb{T})}) \,. \tag{36}$$

On the other hand,

$$I_2 \lesssim \int_{\mathbb{R}^3} \sum_{B_2} |k||\hat{h}(\tau,k)||\hat{W}(\tau_1,k_1)||k-k_1||\widehat{\overline{w}}(\tau_2,k_2)||\hat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3$$

where $\tau := \tau_1+\tau_2+\tau_3$, $k := k_1+k_2+k_3$ and

$$B_2 := \Big\{ (k_1,k_2,k_3) \in \mathbb{Z}^3,\ k_1>0,\ k_2<0,\ k_1+k_2+k_3>0 \\ k_2+k_3<0 \text{ and } \ \min(|k|,|k_2|) \geq 10|k_3|+1 \Big\} \,.$$

Note that on B_2 we have $100k_3^2 \leq |k_2|k$ and $|k-k_1| = |k_2+k_3| \leq 2|k_2|$. Hence, $|k_1| \leq 2\max(|k|,|k-k_1|) \leq 4\max(|k|,|k_2|)$ and thus on B_2, it holds

$$\begin{aligned} |\sigma-\sigma_1-\sigma_2-\sigma_3| &= \Big|(\sum_{i=1}^3 k_i)^2 - k_1^2 + k_2^2 - k_3|k_3|\Big| \\ &= \Big|2k_2k + 2k_1k_3 + k_3^2 - k_3|k_3|\Big| \\ &\geq |k_2k| \,. \end{aligned} \tag{37}$$

Therefore, since clearly $k_1 \geq k$ on B_2,

$$\begin{aligned} I_2 &\lesssim \int_{\mathbb{R}^3} \sum_{B_2} |k|^{1/2}|k_2|^{1/2} |\hat{h}(\tau,k)||\hat{w}(\tau_1,k_1)||\widehat{\overline{w}}(\tau_2,k_2)||\hat{G}(\tau_3,k_3)|, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \int_{\mathbb{R}^3} \sum_{B_2} \max(\langle\sigma\rangle^{1/2}, \langle\sigma_i\rangle^{1/2})\, |\hat{h}(\tau,k)||\hat{w}(\tau_1,k_1)||\widehat{\overline{w}}(\tau_2,k_2)||\hat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3 \,. \end{aligned}$$

This last estimate together with (36) ensure that out of the domain $B_3 := B_2 \cap \{|\sigma| \geq \max(|\sigma_i|)/10\}$ the following estimate holds :

$$\|A_2(F,W)\|_{X^{-1/2+,0}} \lesssim \|W\|^2_{X^{1/2,1}} \Big(\|\partial_x G\|_{\tilde{L}^4(\mathbb{T})} + \|G\|_A + \|G\|_{X^{7/8,0}} \Big) \tag{38}$$

and that on the domain B_3, it holds

$$\|A_2(F,W)\|_{X^{-1/2,0}} \lesssim \|W\|^2_{X^{1/2,1}} (\|\partial_x G\|_{\tilde{L}^4(\mathbb{T})} + \|G\|_A) \,. \tag{39}$$

Since $X^{-1/2+,0}$ is continuously embedded in $Z^{-1,0}$, it thus remains to estimate the $Z^{-1,0}$-norm of $A_2(G,W)$ on B_3. We proceed as in [12]. Note that here we will replace W by $W_\delta := \psi(\cdot/\delta)W$ with $0 < \delta \leq 1$ and make appear a contraction factor in $\delta > 0$ (we will need it in Lemma 5.10). By (37), in this region we have :

$$\langle \sigma \rangle \sim \langle k\, k_2 \rangle \quad . \tag{40}$$

We thus have to estimate

$$I := \Big\| \int_{C(\tau,k)} \chi_{\{k\geq 1\}} \frac{\langle |k|k_1^{-1}|\hat{w_\delta}(\tau_1,k_1)||k-k_1||\widehat{\overline{w_\delta}}(\tau_2,k_2)||\hat{G}(\tau_3,k_3)|}{\langle \sigma \rangle} \Big\|_{l^2_k L^1_q} \tag{41}$$

where $\tau_3 := \tau - \tau_1 - \tau_2$, $k_3 := k - k_1 - k_2$ and

$$C(\tau,k) := \Big\{ (\tau_1,\tau_2,k_1,k_2) \in \mathbb{R}^2 \times \mathbb{Z}^2,\ k_1 \geq 1,\ k_2 \leq -1, \\ \min(k,|k_2|) \geq 10|k-k_1-k_2|+1,\ |\sigma| \geq (k|k_2|) \Big\} .$$

Note that in $C(\tau,k)$ with $k \geq 0$ it holds, as in B_2, $k_1 \geq \max(k,|k-k_1|)$ and $k_1 \leq 4\max(k,|k_2|)$. We divide B_3 into 2 subregions.

• The subregion $\max(|\sigma_1|,|\sigma_2|) \geq (k|k_2|)^{\frac{1}{16}}$. We will assume that $\max(|\sigma_1|,|\sigma_2|) = |\sigma_1|$ since the other case can be treated in exactly the same way. Then, by (40), we get

$$I \lesssim \Big\| \chi_{\{k\geq 1\}} \int_{\tilde{C}(\tau,k)} \frac{|\hat{w_\delta}(\tau_1,k_1)||\widehat{\overline{w_\delta}}(\tau_2,k_2)|\langle k_3\rangle^{-1/128}|\hat{G}(\tau_3,k_3)|}{\langle \sigma \rangle^{1/2+\frac{1}{256}} \langle \sigma_1 \rangle^{-1/8}} \Big\|_{l^2_k L^1_q} \tag{42}$$

where

$$\tilde{C}(\tau,k) = \{ (\tau_1,\tau_2,k_1,k_2) \in C(\tau,k),\ |\sigma_1| \geq (k|k_2|)^{\frac{1}{16}} \} .$$

and by applying Cauchy-Schwarz in τ we obtain thanks to (22),

$$\begin{aligned} I &\lesssim \Big\| \int_{\tilde{C}(\tau,k)} \langle \sigma_1 \rangle^{1/8} |\hat{w_\delta}(\tau_1,k_1)||\widehat{\overline{w_\delta}}(\tau_2,k_2)|\langle k_3\rangle^{-1/128}|\hat{G}(\tau_3,k_3)| \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} \\ &\lesssim \|\mathcal{F}^{-1}(\langle \sigma_1\rangle^{1/8}|\hat{w_\delta}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \|\mathcal{F}^{-1}(|\widehat{\overline{w_\delta}}|)\|_{L^4(\mathbb{R}\times\mathbb{T})} \|\mathcal{F}^{-1}(\langle k_3\rangle^{-1/128}|\hat{G}|)\|_{L^\infty(\mathbb{R}\times\mathbb{T})} \\ &\lesssim \delta^{1/16} \|w_\delta\|^2_{X^{1/2,0}} \left(\|G\|_{X^{0,0}} + \|G\|_{X^{0,1}} \right) , \end{aligned} \tag{43}$$

where in the last step we used that for a function $v \in X^{0,0} \cap X^{0,1}$,

$$\|\mathcal{F}^{-1}(\langle k\rangle^{-1/128}|\hat{v}|)\|_{L^\infty(\mathbb{R}\times\mathbb{T})} \lesssim \|v\|_{X^{1/2+,1/2-}} \lesssim \|v\|_{X^{0,0}} + \|v\|_{X^{0,1}} .$$

• The subregion $\max(|\sigma_1|,|\sigma_2|) \le (k|k_2|)^{\frac{1}{16}}$. Changing the τ, τ_1, τ_2 summation in τ_1, τ_2, τ_3 summation in (41) and using (40), we infer that

$$I \lesssim \Big\| \sum_{C(k)} k_1^{-1} \int_{\tau_1 = -k_1^2 + O(|k\,k_2|^{1/16})} |\hat{w}_\delta(\tau_1, k_1)| \int_{\tau_2 = k_2^2 + O(|k\,k_2|^{1/16})} |\widehat{\overline{w}_\delta}(q_2, k_2)| \int_{\tau_3 \in \mathbb{Z}} |\hat{G}(\tau_3, k_3)| \Big\|_{l_k^2}$$

with $C(k) = \{k_1 \ge 1, k_2 \le -1 \text{ and } k - k_1 \le -1\}$. Applying Cauchy-Schwarz inequality in τ_1 and τ_2 and recalling that $k_1 \ge 1$ we get

$$I \lesssim \Big\| \chi_{\{k \ge 1\}} \sum_{C(k)} \langle k_1 \rangle^{-1} (k|k_2|)^{\frac{1}{16}} K_1(k_1) K_2(k_2) \int_{\mathbb{R}} |\hat{G}(\tau_3, k_3)|\, d\tau_3 \Big\|_{l_k^2}$$

where

$$K_1(k) = \Big(\int_{\mathbb{R}} |\hat{w}_\delta(\tau, k)|^2 \, d\tau \Big)^{1/2} \text{ and } K_2(k) = \Big(\int_{\mathbb{R}} |\widehat{\overline{w}_\delta}(\tau, k)|^2 \, d\tau \Big)^{1/2} .$$

Therefore, by Hölder and then Cauchy-Schwarz inequalities,

$$\begin{aligned}
I &\lesssim \Big\| \langle k \rangle^{-\frac{3}{4}} \int_{\mathbb{R}} \sum_{k_3 \in \mathbb{Z}} |\hat{G}(\tau_3, k_3)| \sum_{k_1 \in \mathbb{Z}} K_1(k_1) K_2(k - k_1 - k_3)\, d\tau_3 \Big\|_{l_k^2} \\
&\lesssim \Big\| \sum_{k_3 \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{G}(\tau_3, k_3)|\, d\tau_3 \sum_{k_1 \in \mathbb{Z}} K_1(k_1) K_2(k - k_1 - k_3) \Big\|_{l_k^\infty} \\
&\lesssim \Big(\sum_{k \in \mathbb{Z}} K_1(k)^2 \Big)^{1/2} \Big(\sum_{k \in \mathbb{Z}} K_2(k)^2 \Big)^{1/2} \sum_{k_3 \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{G}(\tau_3, k_3)|\, d\tau_3 \\
&\lesssim \|w_\delta\|_{L^2(\mathbb{R}\times\mathbb{T})}^2 \sum_{k_3 \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{G}(q_3, k_3)|\, d\tau_3 \\
&\lesssim \delta^{1/16} \|G\|_A \|w_\delta\|_{X^{1/2,0}}^2 . \qquad (44)
\end{aligned}$$

This completes the proof of the lemma. □

We will now decompose $A(G, W)$ in another way to study the behavior of A with respect to weaker topologies. First we note that

$$\begin{aligned}
A(G, W) &= 2i\partial_x^2 P_+ \Big(W P_-(\overline{w} G) \Big) - 2i\partial_x P_+ \Big(w P_-(\overline{w} G) \Big) \\
&= A_1(G, W) + A_2(G, W) . \qquad (45)
\end{aligned}$$

The following lemma ensures that A_1 behaves well for our purpose.

Lemma 5.7 *The operator $A_1(G, W)$ is continuous from $X_1^{3/8,1/8} \times X_1^{3/8,7/8}$ into $X_1^{-3/8,-2}$.*

Proof. Up to the use of classical time extension of W and G, it is equivalent to prove that

$$I := \Big| \int_{\mathbb{R}^3} \sum_{\tilde{D}} k^2 \widehat{h}(\tau,k) \widehat{W}(\tau_1,k_1) i k_2 \widehat{\overline{W}}(\tau_2,k_2) \widehat{G}(\tau_3,k_3)\, d\tau_1 d\tau_2 d\tau_3 \Big|$$

$$\lesssim \|h\|_{X^{1/2,2}} \|W\|^2_{X^{3/8,7/8}} \|G\|_{X^{3/8,1/8}}$$

with $\tau = \tau_1 + \tau_2 + \tau_3$, $k = k_1 + k_2 + k_3$ and

$$\tilde{D} := \{ (k_1,k_2,k_3) \in \mathbb{Z}^3,\ k_1 > 0,\ k_2 < 0,\ k_2 + k_3 \leq 0,\ k_1 + k_2 + k_3 > 0 \}\ .$$

Noticing that on $\tilde{D}$ it holds $|k_2 + k_3| \leq k_1$ and thus $|k_2| \leq k_1 + |k_3|$, it is straightforward using (21) to see that

$$\begin{aligned} I &\lesssim \int_{\mathbb{R}^3} \sum_{\tilde{D}} k^2 |\widehat{h}(\tau,k)| |k_1|^{1/8} |\widehat{W}(\tau_1,k_1)| |k_2|^{7/8} |\widehat{\overline{W}}(\tau_2,k_2)| |k_3|^{1/8} |\widehat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \|h\|_{X^{3/8,2}} \|W\|_{X^{3/8,1/8}} \|W\|_{X^{3/8,7/8}} \|G\|_{X^{3/8,1/8}}\ . \end{aligned}$$

□

We continue the decomposition of A by decomposing $A_2(W,G)$ as

$$\begin{aligned} A_2(W,G) &= -2i\partial_x P_+ \Big(w P_-(\overline{w} G \Big) \\ &= 2 \sum_D (k_1+k_2+k_3) \widehat{w}(k_1) \widehat{\overline{w}}(k_2) \widehat{G}(k_3) e^{i(k_1+k_2+k_3)x} \\ &\quad + 2 \sum_{0<k\leq k_1} k \widehat{w}(k_1) \widehat{\overline{w}}(-k_1) \widehat{G}(k) e^{ikx} \\ &= A_{21}(G,W) + A_{22}(G,W) \end{aligned} \tag{46}$$

where

$$D := \Big\{ (k_1,k_2,k_3) \in \mathbb{Z}^3,\ k_1 > 0,\ k_2 < 0,\ k_2 + k_3 \leq 0,\ k_1 + k_2 + k_3 > 0 \text{ and } k_1 + k_2 \neq 0 \Big\}\ .$$

Lemma 5.8 *A_{21} is continuous from $(X_1^{7/16,0} \cap X_1^{3/8,1/8}) \times X_1^{7/16,31/32}$ into $X_1^{-7/16,-2}$.*

Proof . Up to the choice of suitable time extensions of w and G we have to estimate

$$I := \Big| \int_{\mathbb{R}^3} \sum_D k \widehat{h}(\tau,k) \widehat{w}(\tau_1,k_1)) \widehat{\overline{w}}(\tau_2,k_2) \widehat{G}(\tau_3,k_3)\, d\tau_1 d\tau_2 d\tau_3 \Big|$$

where $k = k_1 + k_2 + k_3$ and $\tau = \tau_1 + \tau_2 + \tau_3$. We divide D into 3 regions.
• $D_1 := \{(k_1, k_2, k_3) \in D,\ k^2 \geq \frac{1}{10}\max(k_1, |k_2|)\}$. In this region we get

$$\begin{aligned} I_{/D_1} &\lesssim \int_{\mathbb{R}^3} \sum_{D_1} |k|^2 |\widehat{h}(\tau,k)||k_1|^{-1/4} |\widehat{w}(\tau_1,k_1))||k_2|^{-1/4} |\widehat{\overline{w}}(\tau_2,k_2)||\widehat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \|h\|_{X^{3/8,2}} \|W\|^2_{X^{3/8,3/4}} \|G\|_{X^{3/8,0}} \ . \end{aligned}$$

• $D_2 := \{(k_1, k_2, k_3) \in D,\ k_3^2 \geq \frac{1}{10}\max(k_1, |k_2|)\}$. In this region we get

$$\begin{aligned} I_{/D_2} &\lesssim \int_{\mathbb{R}^3} \sum_{D_2} |k||\widehat{h}(\tau,k)||k_1|^{-1/32} |\widehat{w}(\tau_1,k_1))||k_2|^{-1/32} |\widehat{\overline{w}}(\tau_2,k_2)||k_3|^{1/8} |\widehat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \|h\|_{X^{3/8,1}} \|W\|^2_{X^{3/8,31/32}} \|G\|_{X^{3/8,1/8}} \ . \end{aligned}$$

• $D_3 := \{(k_1, k_2, k_3) \in D,\ \max(k^2, k_3^2) < \frac{1}{10}\max(k_1, |k_2|)\}$. In this region we use the resonant relation. Setting $\sigma = \sigma(\tau, k) := q + |k|k$ and $\sigma_i = \sigma(\tau_i, k_i)$, we have

$$\sigma - \sigma_1 - \sigma_2 - \sigma_3 = k^2 - k_1^2 + k_2^2 - |k_3|k_3 \ .$$

Since on D_3, $|k_1| \neq |k_2|$, it holds

$$|k_1^2 - k_2^2| \geq \max(k_1, |k_2|)$$

and thus

$$\max(|\sigma|, |\sigma_i|) \gtrsim \max(k_1, |k_2|) \ .$$

Therefore in D_3 we get

$$\begin{aligned} I_{/D_3} &\lesssim \int_{\mathbb{R}^3} \sum_{D_2} |k|\langle\sigma\rangle^{1/16} |\widehat{h}(\tau,k)| \langle\sigma_1\rangle^{1/16} |k_1|^{-1/32} |\widehat{w}(\tau_1,k_1))| \\ &\qquad \langle\sigma_2\rangle^{1/16} |k_2|^{-1/32} |\widehat{\overline{w}}(\tau_2,k_2)| \langle\sigma_3\rangle^{1/16} |\widehat{G}(\tau_3,k_3)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \|h\|_{X^{7/16,1}} \|W\|^2_{X^{7/16,31/32}} \|G\|_{X^{7/16,0}} \ . \end{aligned}$$

This completes the proof of the lemma. □

We rewritte now $A_{22}(G, W)$ in the following way :

$$\begin{aligned} A_{22}(G,W) &= -2i \sum_{k_1>0,\, k>0} \widehat{w}(k_1)\widehat{\overline{w}}(-k_1)(ik)\widehat{G}(k) e^{ikx} \\ &\qquad -2 \sum_{0<k_1<k} \widehat{w}(k_1)\widehat{\overline{w}}(-k_1) k \widehat{G}(k) e^{ikx} \\ &= -\frac{i}{\pi}\|w\|^2_{L^2(\mathbb{T})} w + A_{221}(G,W) \end{aligned} \tag{47}$$

since $w = \partial_x P_+ G$. Finally, we notice that A_{221} is a good term on account of the following lemma.

Lemma 5.9 *A_{221} is continuous from $X_1^{3/8,0} \times X_1^{3/8,1/2}$ into $X_1^{-3/8,-2}$.*

Proof. Up to the choice of suitable time extensions of W and G it suffices to estimate

$$\begin{aligned} I &:= \int_{\mathbb{R}^3} \sum_{0<k_1<k} |k||\widehat{h}(\tau,k)||\widehat{w}(\tau_1,k_1)||\widehat{\overline{w}}(\tau_2,-k_1)||\widehat{G}(\tau_3,k)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \int_{\mathbb{R}^3} \sum_{0<k_1<k} |k|^2|\widehat{h}(\tau,k)||k_1|^{-1/2}|\widehat{w}(q_1,k_1)||k_1|^{-1/2}|\widehat{\overline{w}}(\tau_2,-k_1)||\widehat{G}(\tau_3,k)|\, d\tau_1 d\tau_2 d\tau_3 \\ &\lesssim \|h\|_{X^{3/8,2}} \|W\|^2_{X^{3/8,1/2}} \|G\|_{X^{3/8,0}} \ . \end{aligned} \tag{48}$$

□

Let us set now

$$\Lambda(G,W) := A_1(G,W) + A_{21}(G,W) + A_{221}(G,W)$$

so that

$$\Lambda(G,W) = A(G,W) + \frac{i}{\pi}\|w\|^2_{L^2(\mathbb{T})} w \ . \tag{49}$$

Note that the map $W \mapsto \|W_x\|^2_{L^2(\mathbb{T})} W_x$ is clearly continuous from $L^\infty(]-1,1[;H^1(\mathbb{T}))$ into $L^2(]-1,1[\times\mathbb{T})$. We thus deduce from Lemma 5.6 that Λ is continuous from $R_1^{7/8} \times (X_1^{1/2,1} \cap L^\infty(]-1,1[;H^1(\mathbb{T}))$ into $Y_1^{-1/2,0}$. On the other hand, gathering Lemmas 5.7-5.9 we get that Λ is continuous from $(X_1^{7/16,0} \cap X_1^{3/8,1/8}) \times X_1^{7/16,31/32}$ into $X_1^{-7/16,-2}$.

Since $X_1^{1/2,0} \cap X_1^{0,1} \hookrightarrow R_1^{7/8}$ and $(X_1^{1/2,0} \cap X_1^{0,1}) \times X_1^{1/2,1}$ is clearly compactly embedded in $(X_1^{7/16,0} \cap X_1^{3/8,1/8}) \times X_1^{7/16,31/32}$ we thus infer from Lemmas 5.3-5.4 that $\Lambda(G_n,W_n)$ is bounded in $Y_1^{-1/2,0}$ and converges in the sense of distributions towards $\Lambda(G,W)$. Moreover, according to (18), $f \mapsto \int_0^t V(t-t')f(t')\,dt'$ is continuous from $Y_1^{-1/2,s}$ into $X_1^{1/2,s}$ and thus $(G,W) \mapsto \int_0^t V(t-t')\Lambda(G(t'),W(t'))\,dt'$ is continuous from $R_1^{7/8} \times (X_1^{1/2,1} \cap L^\infty(]-1,1[;H^1(\mathbb{T}))$ into $X_1^{1/2,0}$ and from[2] $(X_1^{7/16,1/16} \cap X_1^{3/8,1/8}) \times X_1^{7/16,31/32}$ into $X_1^{1/2,-2}$. It follows that

$$\int_0^t V(t-t')\Lambda(G_n(t'),W_n(t'))\,dt' \rightharpoonup \int_0^t V(t-t')\Lambda(G(t'),W(t'))\,dt' \text{ in } X_1^{1/2,0} \ . \tag{50}$$

[2]Note that $X_1^{-7/16,-2} \hookrightarrow Y_1^{-1/2,-2}$

According to Lemmas 5.4, 5.5 and (18) it is clear that the same convergence results hold for $B(G_n, W_n)$. In particular,

$$\int_0^t V(t-t')B(G_n(t'), W_n(t'))\,dt' \rightharpoonup \int_0^t V(t-t')B(G(t'), W(t'))\,dt' \text{ in } X_1^{1/2,0}\ . \tag{51}$$

Finally, to identify the limit of the terms $\|w_n\|^2_{L^2(\mathbb{T})}w_n$ we will need the following compactness result on sequences of gauge functions $\{w_n\}$.

Lemma 5.10 *Let $\{u_{0,n}\} \subset H^0_0(\mathbb{T})$ be a sequence of initial data that is bounded in $L^2(\mathbb{T})$. Then the associated sequence of norm of gauge functions $\{t \mapsto \|w_n(t)\|_{L^2(T)}\}$ is bounded in $C([-1,1])$ and uniformly equi-continuous on $[-1,1]$.*

Proof. The boundedness follows directly from Theorem 3.1 since $u(t) \mapsto \partial_x P_+(e^{-i\partial_x^{-1}u(t)})$ is clearly continuous on $H^0_0(\mathbb{T})$. Moreover, from (16), (18) and the Duhamel formulation of (13) we infer that for any $t_0 \in [-1,1]$ and $\delta > 0$ small enough,

$$\begin{aligned}&\|w_n(\cdot) - V(\cdot - t_0)w_n(t_0)\|_{L^\infty(t_0-\delta,t_0+\delta;L^2(\mathbb{T}))}\\ &\lesssim \Big\|\psi(\frac{t-t_0}{\delta})\int_{t_0}^t V(t-t')\Big[A(G_n, \psi(\cdot/\delta)W_n) + B(G_n, W_n) + \frac{i}{4}P_0(F^2_{n,x})w_n\Big](t')\,dt'\Big\|_{Y^{1/2,0}}\ .\end{aligned}$$

Therefore, combining (18), (20), (33), (38), (44) and the fact that obviously $\{P_0(F^2_{n,x})w_n\}$ is bounded in $X^{0,0}_{3/2}$, we infer that

$$\|w_n(\cdot) - V(\cdot - t_0)w_n(t_0)\|_{L^\infty(t_0-\delta,t_0+\delta;L^2(\mathbb{T}))} \lesssim \delta^\nu\ ,$$

for some $\nu > 0$. Since $V(\cdot)$ is unitary in $L^2(\mathbb{T})$, it thus results that

$$\sup_{t\in[t_0-\delta,t_0+\delta]}\Big|\|w_n(t)\|_{L^2} - \|w_n(t_0)\|_{L^2(\mathbb{T})}\Big| \lesssim \delta^\nu\ .$$

This ensures that $\{t \mapsto \|w_n(t)\|_{L^2(T)}\}$ is uniformly equi-continuous on $[-1,1]$. □

5.4 End of the proof

Note that, by Banach-Steinhaus theorem, $\{\|u_{0,n}\|_{L^2}\}$ is bounded in $\mathbb{R}_+$ and thus admits at least one adherence value. Let us denote by $\alpha \geq 0$ such an adherence value of $\{\|u_{0,n}\|_{L^2}\}$ and let us denote by $\{\|u_{0,n_k}\|_{L^2}\}$ a subsequence that converges towards α. Setting $w_{0,n} := \partial_x P_+(e^{-i\partial_x^{-1}u_{0,n}/2})$ and

recalling that the $L^2(\mathbb{T})$-norm is a constant of the motion for (1) we infer from (13), (49) and the Duhamel formula that

$$\begin{aligned} w_{n_k}(t) = & V(t)w_{0,n_k} - \int_0^t V(t-t')\Big(\Lambda(G_{n_k},W_{n_k})(t') + B(G_{n_k},W_{n_k})(t')\Big)\,dt' \\ & + \frac{i}{8\pi}\|u_{0,n}\|_{L^2}^2 \int_0^t V(t-t')w_{n_k}(t')\,dt' \\ & - \frac{i}{\pi}\int_0^t V(t-t')(\|w_{n_k}\|_{L^2}^2 w_{n_k})(t')\,dt' \ , \ \forall t \in]-1,1[\, . \end{aligned} \tag{52}$$

Note that Lemma 5.10 ensures that up to another extraction of a subsequence, the sequence of functions $\{t \mapsto \|w_{n_k}(t)\|_{L^2(T)}^2\}$ converges to some positive continuous function $t \mapsto a(t)$ in $C([-1,1])$. Moreover, since obviously $\{\partial_x^{-1}u_{0,n}\}$ converges strongly in $L^\infty(\mathbb{T})$ towards $\partial_x^{-1}u_0$, it is easy to check that $\{w_{0,n}\}$ converges towards $w_0 := \partial_x P_+(e^{-i\partial_x^{-1}u_0/2})$ weakly in $L^2(\mathbb{T})$. From the linear estimates of Lemmas 3.2-3.3 and (50)-(51), it thus follows that

$$\begin{aligned} w(t) = & V(t)w_0 - \int_0^t V(t-t')\Big(\Lambda(G,W)(t') + B(G,W)(t')\Big)\,dt' \\ & + \frac{i}{8\pi}\,\alpha^2 \int_0^t V(t-t')w(t')\,dt' \\ & - \frac{i}{\pi}\int_0^t V(t-t')(a(t')w(t'))\,dt' \ , \ \forall t \in]0,1[\, . \end{aligned} \tag{53}$$

with $G := e^{-i\partial_x^{-1}v/2}$ and $W := \partial_x^{-1}w$. Moreover w is solution of the following Cauchy problem on $]0,1[$:

$$\begin{cases} w_t - iw_{xx} + (\Lambda + B)(G,W) - \dfrac{i}{\pi}\Big(\dfrac{\alpha^2}{8} - a(t)\Big)w = 0 \\ w(0) = \partial_x P_+(e^{-i\partial_x^{-1}u_0/2}) \end{cases} . \tag{54}$$

Finally, since $v \in N_1$ and satisfies (1) on $]0,T[$ with $0 < T < 1$, the $L^2(\mathbb{T})$-norm of v is conserved on $[0,T]$. The equation for w can thus clearly be rewritten on $]0,T[$ as

$$\begin{aligned} w_t - iw_{xx} = & 2i\partial_x P_+\Big(W\partial_x P_-(e^{-iF/2}\overline{w})\Big) \\ & + 2i\partial_x P_+\Big[W\partial_x P_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big)\Big] + \frac{i}{4}P_0(v^2)w \\ & + \frac{i}{8\pi}(\alpha^2 - \|u_0\|_{L^2(\mathbb{T})}^2)w - \frac{i}{\pi}(a(t) - \|w(t)\|_{L^2(\mathbb{T})}^2)w \end{aligned}$$

that concludes the proof.

Acknowledgements

The author was partially supported by the ANR projects "Etude qualitative des EDP dispersives". and "Equa-Disp". The author is grateful to Professors J.-M. Delort, D. Dos Santos and F. Parreau for stimuling discussions.

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