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# Large deviations for transient random walks in random environment on a Galton–Watson tree

by

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**Summary.** Consider a random walk in random environment on a super-critical Galton–Watson tree, and let  $\tau_n$  be the hitting time of generation  $n$ . The paper presents a large deviation principle for  $\tau_n/n$ , both in quenched and annealed cases. Then we investigate the subexponential situation, revealing a polynomial regime similar to the one encountered in one dimension. The paper heavily relies on estimates on the tail distribution of the first regeneration time.

**Key words.** Random walk in random environment, law of large numbers, large deviations, Galton–Watson tree.

**AMS subject classifications.** 60K37, 60J80, 60F15, 60F10.

## 1 Introduction

We consider a super-critical Galton–Watson tree  $\mathbb{T}$  of root  $e$  and offspring distribution  $(q_k, k \geq 0)$  with finite mean  $m := \sum_{k \geq 0} kq_k > 1$ . For any vertex  $x$  of  $\mathbb{T}$ , we call  $|x|$  the generation of  $x$ , ( $|e| = 0$ ) and  $\nu(x)$  the number of children of  $x$ ; we denote these children by  $x_i, 1 \leq i \leq \nu(x)$ . We let  $\nu_{min}$  be the minimal integer such that  $q_{\nu_{min}} > 0$  and we suppose that  $\nu_{min} \geq 1$  (thus  $q_0 = 0$ ). In particular, the tree survives almost surely. Following Pemantle and Peres [14], on each vertex  $x$ , we pick independently and with the same distribution a random variable  $A(x)$ , and we define

- $\omega(x, x_i) := \frac{A(x_i)}{1 + \sum_{i=1}^{\nu(x)} A(x_i)}, \forall 1 \leq i \leq \nu(x),$
- $\omega(x, \overleftarrow{x}) := \frac{1}{1 + \sum_{i=1}^b A(x_i)}.$

To deal with the case  $x = e$ , we add a parent  $\overleftarrow{e}$  to the root and we set  $\omega(\overleftarrow{e}, e) = 1$ . Once the environment built, we define the random walk  $(X_n, n \geq 0)$  starting from  $y \in \mathbb{T}$  by

$$\begin{aligned} P_\omega^y(X_0 = y) &= 1, \\ P_\omega^y(X_{n+1} = z | X_n = x) &= \omega(x, z). \end{aligned}$$

The walk  $(X_n, n \geq 0)$  is a  $\mathbb{T}$ -valued Random Walk in Random Environment (RWRE). To determine the transience or recurrence of the random walk, Lyons and Pemantle [11] provides us with the following criterion. Let  $A$  be a generic random variable having the distribution of  $A(e)$ .

**Theorem A (Lyons and Pemantle [11])** *The walk  $(X_n)$  is transient if  $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$ , and is recurrent otherwise.*

In the transient case, let  $v$  denote the speed of the walk, which is the deterministic real  $v \geq 0$  such that

$$\lim_{n \rightarrow \infty} \frac{|X_n|}{n} = v, \quad a.s.$$

Define

$$\begin{aligned} i &:= \text{ess inf } A, \\ s &:= \text{ess sup } A. \end{aligned}$$

We make the hypothesis that  $0 < i \leq s < \infty$ . Under this assumption, we gave a criterion in [1] for the positivity of the speed  $v$ . Let

$$(1.1) \quad \Lambda := \text{Leb} \left\{ t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{q_1} \right\} \quad (\Lambda = \infty \text{ if } q_1 = 0).$$

**Theorem B ([1])** *Assume  $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$ , and let  $\Lambda$  be as in (1.1).*

- If  $\Lambda < 1$ , the walk has zero speed.*
- If  $\Lambda > 1$ , the walk has positive speed.*

When the speed is positive, we would like to have information on how hard it is for the walk to have atypical behaviours, which means to go a little faster or slower than its natural pace. Such questions have been discussed in the setting of biased random walks on Galton–Watson trees, by Dembo et al. in [5]. The authors exhibit a large deviation principle both in quenched and annealed cases. Besides, an uncertainty principle allows them to obtain the equality of the two rate functions. For the RWRE in dimensions one or more, we refer to Zeitouni [17] for a review of the subject. In our case, we consider a random walk which always avoids the parent  $\overleftarrow{e}$  of the root, and we obtain a large deviation principle, which, following [5], has been divided into two parts.

We suppose in the rest of the paper that

$$(1.2) \quad \inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m},$$

$$(1.3) \quad \Lambda > 1,$$

which ensures that the walk is transient with positive speed. Before the statement of the results, let us introduce some notation. Define for any  $n \geq 0$  and  $x \in \mathbb{T}$ ,

$$\begin{aligned} \tau_n &:= \inf \{k \geq 0 : |X_k| = n\}, \\ D(x) &:= \inf \left\{ k \geq 1 : X_{k-1} = x, X_k = \overleftarrow{x} \right\}, \quad (\inf \emptyset := \infty). \end{aligned}$$

Let  $\mathbf{P}$  denote the distribution of the environment  $\omega$  conditionally on  $\mathbb{T}$ , and  $\mathbf{Q} := \int \mathbf{P}(\cdot)GW(d\mathbb{T})$ . Similarly, we denote by  $\mathbb{P}^x$  the distribution defined by  $\mathbb{P}^x(\cdot) := \int P_\omega^x(\cdot)\mathbf{P}(d\omega)$  and by  $\mathbb{Q}^x$  the distribution

$$\mathbb{Q}^x(\cdot) := \int \mathbb{P}^x(\cdot)GW(d\mathbb{T}).$$

**Theorem 1.1 (Speed-up case)** *There exist two continuous, convex and strictly decreasing functions  $I_a \leq I_q$  from  $[1, 1/v]$  to  $\mathbb{R}_+$ , such that  $I_a(1/v) = I_q(1/v) = 0$  and for  $a < b$ ,  $b \in [1, 1/v]$ ,*

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e \left( \frac{\tau_n}{n} \in ]a, b] \right) = -I_a(b),$$

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e \left( \frac{\tau_n}{n} \in ]a, b] \right) = -I_q(b).$$

**Theorem 1.2 (Slowdown case)** *There exist two continuous, convex functions  $I_a \leq I_q$  from  $[1/v, +\infty[$  to  $\mathbb{R}_+$ , such that  $I_a(1/v) = I_q(1/v) = 0$  and for any  $1/v \leq a < b$ ,*

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e \left( \frac{\tau_n}{n} \in [a, b[ \right) = -I_a(a),$$

$$(1.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e \left( \frac{\tau_n}{n} \in [a, b[ \right) = -I_q(a).$$

Besides, if  $i > \nu_{\min}^{-1}$ , then  $I_a$  and  $I_q$  are strictly increasing on  $[1/v, +\infty[$ . When  $i \leq \nu_{\min}^{-1}$ , we have  $I_a = I_q = 0$  on the interval.

As pointed by an anonymous referee, it would be interesting to know when  $I_a$  and  $I_q$  coincide. We do not know the answer in general. However, the computation of the value of the rate functions at  $b = 1$  reveals situations where the rate functions differ. Let

$$\psi(\theta) := \ln \left( E_{\mathbf{Q}} \left[ \sum_{i=1}^{\nu(e)} \omega(e, e_i)^\theta \right] \right).$$

Then  $\psi(0) = \ln(m)$  and  $\psi(1) = \ln \left( E_{\mathbf{Q}} \left[ \sum_{i=1}^{\nu(e)} \omega(e, e_i) \right] \right)$ .

**Proposition 1.3** *We have*

$$(1.8) \quad I_a(1) = -\psi(1),$$

$$(1.9) \quad I_q(1) = -\inf_{\theta \in ]0,1]} \frac{1}{\theta} \psi(\theta).$$

*In particular,  $I_a(1) = I_q(1)$  if and only if  $\psi'(1) \leq \psi(1)$ . Otherwise  $I_a(1) < I_q(1)$ .*

Quite surprisingly, we can exhibit elliptic environments on a regular tree for which the rate functions differ. This could hint that the uncertainty of the location of the first passage in [5] does not hold anymore for a random environment. Here is an explicit example. Consider a binary tree ( $q_2 = 1$ ). Let  $A$  equal 0.01 with probability 0.8 and 500 with probability 0.2. Then we check that the walk is transient, but  $\psi'(1) > \psi(1)$  so that  $I_a(1) \neq I_q(1)$  on such an environment.

Theorem 1.2 exhibits a subexponential regime in the slowdown case when  $i \leq \nu_{\min}^{-1}$ . The following theorem details this regime. Let

$$\mathbb{S}^e(\cdot) := \mathbb{Q}^e(\cdot \mid D(e) = \infty).$$

**Theorem 1.4** *We place ourselves in the case  $i < \nu_{\min}^{-1}$ .*

*(i) Suppose that either “ $i < \nu_{\min}^{-1}$  and  $q_1 = 0$ ” or “ $i < \nu_{\min}^{-1}$  and  $s < 1$ ”. There exist constants  $d_1, d_2 \in (0, 1)$  such that for any  $a > 1/v$  and  $n$  large enough,*

$$(1.10) \quad e^{-n^{d_1}} < \mathbb{S}^e(\tau_n > an) < e^{-n^{d_2}}.$$

(ii) If  $q_1 > 0$  and  $s > 1$  (id est when  $\Lambda < \infty$ ), the regime is polynomial and we have for any  $a > 1/v$ ,

$$(1.11) \quad \lim_{n \rightarrow \infty} \frac{1}{\ln(n)} \ln (\mathbb{S}^e(\tau_n > an)) = 1 - \Lambda.$$

We mention that in one dimension, which can be seen as a critical state of our model where  $q_1 = 1$ , such a polynomial regime is proved by Dembo et al. [6], our parameter  $\Lambda$  taking the place of the well-known  $\kappa$  of Kesten, Kozlov, Spitzer [9]. We did not deal with the critical case  $i = \nu_{\min}^{-1}$ . Furthermore, we do not have any conjecture on the optimal values of  $d_1$  and  $d_2$  and do not know if the two values are equal.

The rest of the paper is organized as follows. Section 2 describes the tail distribution of the first regeneration time, which is a preparatory step for the proof of the different theorems. Then we prove Theorems 1.1 and 1.2 in Section 3, which includes also the computation of the rate functions at speed 1 presented in Proposition 1.3. Section 4 is devoted to the subexponential regime with the proof of Theorem 1.4.

## 2 Moments of the first regeneration time

We define the first regeneration time

$$\Gamma_1 := \inf \{k > 0 : \nu(X_k) \geq 2, D(X_k) = \infty, k = \tau_{|X_k|}\}$$

as the first time when the walk reaches a generation by a vertex having more than two children and never returns to its parent. We propose in this section to give information on the tail distribution of  $\Gamma_1$  under  $\mathbb{S}^e$ . We first introduce some notation used throughout the paper. For any  $x \in \mathbb{T}$ , let

$$(2.1) \quad \begin{aligned} N(x) &:= \sum_{k \geq 0} \mathbb{1}_{\{X_k = x\}}, \\ T_x &:= \inf \{k \geq 0 : X_k = x\}, \\ T_x^* &:= \inf \{k \geq 1 : X_k = x\}. \end{aligned}$$

This permits to define

$$(2.2) \quad \begin{aligned} \beta(x) &:= P_\omega^x(T_x = \infty), \\ \gamma(x) &:= P_\omega^x(T_x = T_x^* = \infty). \end{aligned}$$

The following fact can be found in [5] (Lemma 4.2) in the case of biased random walks, and is directly adaptable in our setting.

**Fact A** The first regeneration height  $|X_{\Gamma_1}|$  admits exponential moments under the measure  $\mathbb{S}^e(\cdot)$ .

## 2.1 The case $i > \nu_{min}^{-1}$

This section is devoted to the case  $i > \nu_{min}^{-1}$ , where  $\Gamma_1$  is proved to have exponential moments.

**Proposition 2.1** *Suppose that  $i > \nu_{min}^{-1}$ . There exists  $\theta > 0$  such that  $E_{\mathbb{S}^e} [e^{\theta\Gamma_1}] < \infty$ .*

*Proof.* The proof follows the strategy of Proposition 1 of Piau [16]. We couple the distance of our RWRE to the root  $(|X_n|)_{n \geq 0}$  with a biased random walk  $(Y_n)_{n \geq 0}$  on  $\mathbb{Z}$  as follows. Let  $p := \frac{i\nu_{min}}{1+i\nu_{min}}$ , and let  $u_n, n \geq 0$ , be a family of i.i.d. uniformly distributed  $[0,1]$  random variables. We set  $X_0 = e$  and  $Y_0 = 0$ . If  $X_k$  and  $Y_k$  are known, we construct

$$\begin{aligned} X_{k+1} &= x_i && \text{if } \sum_{\ell=1}^{i-1} \omega(x, x_\ell) \leq u_k < \sum_{\ell=1}^i \omega(x, x_\ell), \\ X_{k+1} &= \overleftarrow{x} && \text{otherwise,} \\ Y_{k+1} &= y + 2\mathbb{I}_{\{u_k \leq p\}} - 1, \end{aligned}$$

where  $x := X_k \in \mathbb{T}$  and  $y := Y_k \in \mathbb{Z}$ . Then  $(X_n)_{n \geq 0}$  has the distribution of our  $\mathbb{T}$ -RWRE indeed, and  $(Y_n)_{n \geq 0}$  is a random walk on  $\mathbb{Z}$  which increases of one unit with probability  $p > 1/2$  and decreases of the same value with probability  $1 - p$ . Notice also that on the event  $\{D(e) = \infty\}$ , we have

$$|X_{k+1}| - |X_k| \geq Y_{k+1} - Y_k.$$

It implies that the first regeneration time  $\mathcal{R}_1$  of  $(Y_n)_{n \geq 0}$  defined by

$$\mathcal{R}_1 := \inf \{k > 0 : Y_\ell < Y_k \forall \ell < k, Y_m \geq Y_k \forall m > k\}$$

is necessarily a regeneration time for  $(X_n, n \geq 0)$ , which proves in turn that

$$\mathbb{S}^e(\Gamma_1 > n) \leq \mathbb{Q}^e(\mathcal{R}_1 > n).$$

To complete the proof, we must ensure that  $\mathbb{Q}^e(\mathcal{R}_1 > n)$  is exponentially small, which is done in [6] Lemma 5.1.  $\square$

## 2.2 The cases “ $i < \nu_{min}^{-1}$ , $q_1 = 0$ ” and “ $i < \nu_{min}^{-1}$ , $s < 1$ ”

When  $i < \nu_{min}^{-1}$ , if we assume also that  $q_1 = 0$  or  $s < 1$ , we prove that  $\Gamma_1$  has a subexponential tail. This situation covers, in particular, the case of RWRE on a regular tree.

**Proposition 2.2** *Suppose that  $i < \nu_{min}^{-1}$  and  $q_1 = 0$ , then there exist  $1 > \alpha_1 > \alpha_2 > 0$  such that for  $n$  large enough,*

$$(2.3) \quad e^{-n^{\alpha_1}} < \mathbb{S}^e(\Gamma_1 > n) < e^{-n^{\alpha_2}}.$$

*The same relation holds with some  $1 > \alpha_3 > \alpha_4 > 0$  in the case “ $i < \nu_{min}^{-1}$  and  $s < 1$ ”.*

*Proof of Proposition 2.2: lower bound.* We only suppose that  $i < \nu_{min}^{-1}$ , which allows us to deal with both cases of the lemma. Define for some  $p' \in (0, 1/2)$  and  $b \in \mathbb{N}$ ,

$$\begin{aligned} w_+ &:= \mathbf{Q} \left( \sum_{i=1}^{\nu} A(e_i) \geq \frac{1-p'}{p'}, \nu(e) \leq b \right), \\ w_- &:= \mathbf{Q} \left( \sum_{i=1}^{\nu} A(e_i) \leq \frac{p'}{1-p'}, \nu(e) \leq b \right). \end{aligned}$$

By (1.2),  $E_{\mathbf{Q}} \left[ \sum_{i=1}^{\nu(e)} A(e_i) \right] > 1$  and therefore  $\mathbf{Q} \left( \sum_{i=1}^{\nu(e)} A(e_i) > 1 \right) > 0$ . Since  $\text{ess inf } A < \nu_{min}^{-1}$ , it guarantees that  $\mathbf{Q} \left( \sum_{i=1}^{\nu(e)} A(e_i) < 1 \right) > 0$ . Consequently, by choosing  $p'$  close enough of  $1/2$  and  $b$  large, we can take  $w_+$  and  $w_-$  positive. Let  $c := \frac{1}{6 \ln(b)}$ , and define  $h_n := \lfloor c \ln(n) \rfloor$ . A tree  $\mathbb{T}$  is said to be  $n$ -good if

- any vertex  $x$  of the  $h_n$  first generations verifies  $\nu(x) \leq b$  and  $\sum_{i=1}^{\nu(x)} A(x_i) \geq \frac{1-p'}{p'}$ ,
- any vertex  $x$  of the  $h_n$  following generations verifies  $\nu(x) \leq b$  and  $\sum_{i=1}^{\nu(x)} A(x_i) \leq \frac{p'}{1-p'}$ .

We observe that  $\mathbf{Q}(\mathbb{T} \text{ is } n\text{-good}) \geq w_+^{h_n b^{h_n}} w_-^{h_n b^{2h_n}} \geq e^{-n^{1/3+o(1)}}$  which is stretched exponential, i.e. behaving like  $e^{-n^{r+o(1)}}$  for some  $r \in (0, 1)$ . Define the events

$$\begin{aligned} E_1 &:= \{ \text{at time } \tau_{h_n} \text{ we can't find an edge of level smaller than } h_n \text{ crossed only once} \\ &\quad \cap \{ D(e) > \tau_{h_n} \}, \\ E_2 &:= \{ \text{the walk visits the level } h_n \text{ } n \text{ times before reaching the root or the level } 2h_n \}, \\ E_3 &:= \{ \text{after the } n\text{-th visit of level } h_n, \text{ the walk reaches level } 2h_n \text{ before level } h_n \}, \\ E_4 &:= \{ \text{after time } \tau_{2h_n} \text{ the walk never comes back to level } 2h_n - 1 \}. \end{aligned}$$

Suppose that the tree is  $n$ -good. Since  $A$  is supposed bounded, there exists a constant  $c_1 > 0$  such that for any  $x$  neighbour of  $y$ , we have

$$(2.4) \quad \omega(x, y) \geq \frac{c_1}{\nu(x)}.$$

It yields that  $P_\omega^e(E_1)^{-1} = O(n^K)$  for some  $K > 0$  (where  $O(n^K)$  means that the function is bounded by a factor of  $n \rightarrow n^K$ ). Combine (2.4) with the strong Markov property at time  $\tau_{h_n}$  to see that

$$P_\omega^e(E_3 | E_1 \cap E_2)^{-1} = O(n^K),$$

where  $K$  is taken large enough. We emphasize that the functions  $O(n^K)$  are deterministic. Still by Markov property,

$$(2.5) \quad P_\omega^e(E_1 \cap E_2 \cap E_3 \cap E_4) = E_\omega^e[\mathbb{1}_{E_1 \cap E_2 \cap E_3} \beta(X_{\tau_{2h_n}})].$$

Let  $(Y'_n)_{n \geq 0}$  be the random walk on  $\mathbb{Z}$  starting from zero with

$$P_\omega(Y'_{n+1} = k + 1 | Y'_n = k) = 1 - P_\omega(Y'_{n+1} = k - 1 | Y'_n = k) = p'.$$

We introduce  $T'_i := \inf\{k \geq 0 : Y'_k = i\}$ , and  $p'_n$  the probability that  $(Y'_n)_{n \geq 0}$  visits  $h_n$  before  $-1$ :

$$p'_n := P_\omega(T'_{-1} < T'_{h_n}).$$

By a coupling argument similar to that encountered in the proof of Proposition 2.1, we show that in an  $n$ -good tree,

$$(2.6) \quad P_\omega^e(E_1 \cap E_2) \geq P_\omega^e(E_1)(p'_n)^n = O(n^K)^{-1}(p'_n)^n,$$

which gives

$$(2.7) \quad P_\omega^e(E_1 \cap E_2 \cap E_3) \geq O(n^K)^{-1}(p'_n)^n.$$

Observing that  $\mathbb{Q}^e(\Gamma_1 > n, D(e) = \infty) \geq E_{\mathbf{Q}}[\mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} \mathbb{1}_{E_1 \cap E_2 \cap E_3 \cap E_4}]$ , we obtain by (2.5)

$$\begin{aligned} \mathbb{Q}^e(\Gamma_1 > n, D(e) = \infty) &\geq E_{\mathbb{Q}^e} \left[ \mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} \mathbb{1}_{E_1 \cap E_2 \cap E_3} \beta(X_{\tau_{2h_n}}) \right] \\ &= E_{\mathbb{Q}^e} \left[ \mathbb{1}_{\{\mathbb{T} \text{ is } n\text{-good}\}} P_\omega^e(E_1 \cap E_2 \cap E_3) \right] E_{\mathbf{Q}}[\beta], \end{aligned}$$

by independence. By (2.7),

$$\mathbb{Q}^e(\Gamma_1 > n, D(e) = \infty) \geq O(n^K)^{-1} \mathbf{Q}(\mathbb{T} \text{ is } n\text{-good}) (p'_n)^n.$$

We already know that  $\mathbf{Q}(\mathbb{T} \text{ is } n\text{-good})$  has a stretched exponential lower bound, and it remains to observe that the same holds for  $(p'_n)^n$ . But the method of gambler's ruin shows that  $p'_n \geq 1 - \left(\frac{p'}{1-p'}\right)^{h_n}$ , which gives the required lower bound by our choice of  $h_n$ .  $\square$

Let us turn to the upper bound. We divide the proof in two, depending on which case we deal with.

*Proof of Proposition 2.2: upper bound in the case  $q_1 = 0$ .* Assume that  $q_1 = 0$  (the condition  $i < \nu_{min}^{-1}$  is not required in the proof). The proof of the following lemma is deferred. Recall the notation introduced in (2.2),  $\gamma(e) := P_\omega^e(T_e^- = T_e^* = \infty) \leq \beta(e)$ .

**Lemma 2.3** *When  $q_1 = 0$ , there exists a constant  $c_2 \in (0, 1)$  such that for large  $n$ ,*

$$E_{\mathbf{Q}}[(1 - \gamma(e))^n] \leq e^{-n^{c_2}}.$$

Denote by  $\pi_k$  the  $k$ -th distinct site visited by the walk  $(X_n, n \geq 0)$ . We observe that

$$(2.8) \quad \begin{aligned} \mathbb{Q}^e(\Gamma_1 > n^3) &\leq \mathbb{Q}^e(\Gamma_1 > \tau_n) + \mathbb{Q}^e(\text{more than } n^2 \text{ distinct sites are visited before } \tau_n) \\ &+ \mathbb{Q}^e(\exists k \leq n^2 : N(\pi_k) > n). \end{aligned}$$

Since  $\mathbb{Q}^e(\Gamma_1 > \tau_n) = \mathbb{Q}^e(|X_{\Gamma_1}| > n)$ , it follows from Fact A that  $\mathbb{Q}^e(\Gamma_1 > \tau_n)$  decays exponentially. For the second term of the right-hand side, beware that

$$\begin{aligned} &\mathbb{Q}^e(\text{more than } n^2 \text{ distinct sites are visited before } \tau_n) \\ &\leq \sum_{k=1}^n \mathbb{Q}^e(\text{more than } n \text{ distinct sites are visited at level } k). \end{aligned}$$

If we denote by  $t_i^k$  the first time when the  $i$ -th distinct site of level  $k$  is visited, we have, by the strong Markov property,

$$\begin{aligned} P_\omega^e(\text{more than } n \text{ sites are visited at level } k) &= P_\omega^e(t_n^k < \infty) \\ &\leq P_\omega^e\left(t_{n-1}^k < \infty, D\left(X_{t_{n-1}^k}\right) < \infty\right) \\ &= E_\omega^e\left[\mathbb{1}_{\{t_{n-1}^k < \infty\}} \left(1 - \beta(X_{t_{n-1}^k})\right)\right]. \end{aligned}$$

The independence of the environments entails that

$$E_{\mathbb{Q}^e} \left[ \mathbb{1}_{\{t_{n-1}^k < \infty\}} \left( 1 - \beta(X_{t_{n-1}^k}) \right) \right] = \mathbb{Q}^e (t_{n-1}^k < \infty) E_{\mathbf{Q}}[1 - \beta].$$

Consequently,

$$(2.9) \quad \begin{aligned} \mathbb{Q}^e (t_n^k < \infty) &\leq \mathbb{Q}^e (t_{n-1}^k < \infty) E_{\mathbf{Q}}[1 - \beta] \\ &\leq (E_{\mathbf{Q}}[1 - \beta])^{n-1}, \end{aligned}$$

which leads to

$$(2.10) \quad \mathbb{Q}^e (\text{more than } n^2 \text{ sites are visited before } \tau_n) \leq n (E_{\mathbf{Q}}[1 - \beta])^{n-1},$$

which is exponentially small. We remark, for later use, that equation (2.9) holds without the assumption  $q_1 = 0$ . For the last term of equation (2.8), we write

$$\mathbb{Q}^e (\exists k \leq n^2 : N(\pi_k) > n) \leq \sum_{k=1}^{n^2} \mathbb{Q}^e (N(\pi_k) > n).$$

Let  $U := \bigcup_{n \geq 0} (\mathbb{N}^*)^n$  be the set of words, where  $(\mathbb{N})^0 := \{\emptyset\}$ . Each vertex  $x$  of  $\mathbb{T}$  is naturally associated with a word of  $U$ , and  $\mathbb{T}$  is then a subset of  $U$  (see [13] for a more complete description). For any  $k \geq 1$ ,

$$\begin{aligned} \mathbb{Q}^e (N(\pi_k) > n) &= \sum_{x \in U} \mathbb{Q}^e (x \in \mathbb{T}, N(x) > n, x = \pi_k) \\ &\leq \sum_{x \in U} E_{\mathbf{Q}} [\mathbb{1}_{\{x \in \mathbb{T}\}} P_{\omega}^e (x = \pi_k) (1 - \gamma(x))^n], \end{aligned}$$

with the notation of (2.2). By independence,

$$\begin{aligned} \mathbb{Q}^e (N(\pi_k) > n) &\leq \sum_{x \in U} E_{\mathbf{Q}} [\mathbb{1}_{\{x \in \mathbb{T}\}} P_{\omega}^e (x = \pi_k)] E_{\mathbf{Q}} [(1 - \gamma(e))^n] \\ &= E_{\mathbf{Q}} [(1 - \gamma(e))^n]. \end{aligned}$$

Apply Lemma 2.3 to complete the proof.  $\square$

*Proof of Lemma 2.3.* Let  $\mu > 0$  be such that  $q := \mathbf{Q}(\beta(e) > \mu) > 0$ , and write

$$R := \inf \{k \geq 1 : \exists |x| = k, \beta(x) \geq \mu\}.$$

Let  $x_R$  be such that  $|x_R| = R$  and  $\beta(x_R) \geq \mu$  and we suppose for simplicity that  $x_R$  is a descendant of  $e_1$ . We see that  $\gamma(e) \geq \omega(e, e_1)\beta(e_1) \geq \frac{c_1}{\nu(e)}\beta(e_1)$  by equation (2.4). In turn, equation (2.1) of [1] implies that for any vertex  $x$ , we have

$$\frac{1}{\beta(x)} = 1 + \frac{1}{\sum_{i=1}^{\nu(x)} A(x_i)\beta(x_i)} \leq 1 + \frac{1}{\text{ess inf } A} \frac{1}{\beta(x_i)},$$

for any  $1 \leq i \leq \nu(x)$ . By recurrence on the path from  $e_1$  to  $x_R$ , this leads to

$$\frac{1}{\beta(e_1)} \leq 1 + \frac{1}{\text{ess inf } A} + \dots + \left( \frac{1}{\text{ess inf } A} \right)^{R-1} \frac{1}{\mu}.$$

We deduce the existence of constants  $c_4, c_5 > 0$  such that

$$(2.11) \quad \gamma(e) \geq \frac{c_4}{\nu(e)} e^{-c_5 R}.$$

It yields that

$$E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{I}_{\{\nu(e) < \sqrt{n}\}} \right] \leq \mathbf{Q} \left( R > \frac{1}{4c_5} \ln(n) \right) + e^{-n^{1/4+\alpha(1)}}.$$

We observe that

$$\mathbf{Q} \left( R > \frac{1}{4c_5} \ln(n) \right) \leq \mathbf{Q} \left( \forall |x| = \frac{1}{4c_5} \ln(n), \beta(x) > \mu \right).$$

By assumption,  $q_1 = 0$ ; thus  $\#\{x \in \mathbb{T} : |x| = \frac{1}{4c_5} \ln(n)\} \geq 2^{1/4c_5 \ln(n)} =: n^{c_6}$ . As a consequence,  $\mathbf{Q} \left( \forall |x| = \frac{1}{4c_5} \ln(n), \beta(x) > \mu \right) \leq q^{n^{c_6}}$ . Hence, the proof of our lemma is reduced to find a stretched exponential bound for  $E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{I}_{\{\nu(e) \geq \sqrt{n}\}} \right]$ . For any  $x \in \mathbb{T}$ , denote by  $V_x^\mu$  the number of children  $x_i$  of  $x$  such that  $\beta(x_i) > \mu$ . For  $\varepsilon \in (0, \mathbf{Q}(\beta(e) > \mu))$ ,

$$\begin{aligned} & E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{I}_{\{\nu(e) \geq \sqrt{n}\}} \right] \\ & \leq \mathbf{Q}^e \left( \nu(e) \geq \sqrt{n}, V_e^\mu < \varepsilon \nu(e) \right) + E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{I}_{\{V_e^\mu \geq \varepsilon \nu(e)\}} \right]. \end{aligned}$$

We apply Cramér's Theorem to handle with the first term on the right-hand side. Turning to the second one, the bound is clear once we observe the general inequality,

$$(2.12) \quad \gamma(e) = \sum_{k=1}^{\nu(e)} \omega(e, e_k) \beta(e_k) \geq \frac{c_1}{\nu(e)} \sum_{k=1}^{\nu(e)} \beta(e_k) \geq \frac{c_1 \mu}{\nu(e)} V_e^\mu,$$

which is greater than  $c_1 \mu \varepsilon$  on  $\{V_e^\mu \geq \varepsilon \nu(e)\}$ .  $\square$

**Remark 2.3.** As a by-product, we obtain that  $E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{1}_{\{\nu(e) \geq \sqrt{n}\}} \right] \leq e^{-n^{c_3}}$  without the assumption  $q_1 = 0$ .

*Proof of Proposition 2.2 : upper bound in the case  $s < 1$ .* We follow the strategy of the case “ $q_1 = 0$ ”. The proof boils down to the estimate of

$$\begin{aligned} & \mathbb{Q}^e(N(\pi_k) > n, D(e) = \infty) \\ = & \mathbb{Q}^e(N(\pi_k) > n, \nu(\pi_k) < \sqrt{n}, D(e) = \infty) + \mathbb{Q}^e(N(\pi_k) > n, \nu(\pi_k) \geq \sqrt{n}, D(e) = \infty). \end{aligned}$$

Let  $x \in \mathbb{T}$  and consider the RWRE  $(X_n, n \geq 0)$  when starting from  $\overleftarrow{x}$ . Inspired by Lyons et al. [12], we propose to couple it with a random walk  $(Y''_n, n \geq 0)$  on  $\mathbb{Z}$ . We first define  $X''_n$  as the restriction of  $X_n$  on the path  $\llbracket \overleftarrow{e}, x \rrbracket$ . Beware that  $X''_n$  exists only up to a time  $T$ , which corresponds to the time when the walk  $(X_n, n \geq 0)$  escapes the path  $\llbracket \overleftarrow{e}, x \rrbracket$ , id est leaves the path and never comes back to it. After this time, we set  $X''_n = \Delta$  for some  $\Delta$  in some space  $\mathcal{E}$ . Then  $(X''_n)_{n \geq 0}$  is a random walk on  $\llbracket \overleftarrow{e}, x \rrbracket \cup \{\Delta\}$ , whose transition probabilities are, if  $y \notin \{\overleftarrow{e}, x, \Delta\}$ ,

$$\begin{aligned} P_{\omega}^{\overleftarrow{x}}(X''_{n+1} = y_+ | X''_n = y) &= \frac{\omega(y, y_+)}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k) \beta(y_k)}, \\ P_{\omega}^{\overleftarrow{x}}(X''_{n+1} = \overleftarrow{y} | X''_n = y) &= \frac{\omega(y, \overleftarrow{y})}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k) \beta(y_k)}, \\ P_{\omega}^{\overleftarrow{x}}(X''_{n+1} = \Delta | X''_n = y) &= \frac{\sum_{k=1}^{\nu(y)} \omega(y, y_k) \beta(y_k)}{\omega(y, y_+) + \omega(y, \overleftarrow{y}) + \sum_{y_k \neq y_+} \omega(y, y_k) \beta(y_k)}, \end{aligned}$$

where  $y_+$  is the child of  $y$  which lies on the path  $\llbracket \overleftarrow{e}, x \rrbracket$ . Besides, the walk is absorbed on  $\Delta$  and reflected on  $\overleftarrow{e}$  and  $x$ . We recall that  $s := \text{ess sup } A$ . We construct the adequate coupling with a biased random walk  $(Y''_n)_{n \geq 0}$  on  $\mathbb{Z}$ , starting from  $|x| - 1$ , increasing with probability  $s/(1+s)$ , decreasing otherwise and such that  $Y''_n \geq |X''_n|$  as long as  $X''_n \neq \Delta$  (which is always possible since  $P_{\omega}(X''_{n+1} = y_+ | X''_n = y) \leq \frac{s}{1+s}$ ). After time  $T$ , we let  $Y''_n$  move independently. By coupling and then by gambler’s ruin method, it leads to

$$P_{\omega}^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}}) \leq P_{\omega}^{|x|-1}(\exists n \geq 0 : Y''_n = |x|) = s.$$

It follows that

$$1 - P_{\omega}^x(T_x^* < T_{\overleftarrow{e}}) \geq \omega(x, \overleftarrow{x}) \left( 1 - P_{\omega}^{\overleftarrow{x}}(T_x < T_{\overleftarrow{e}}) \right) \geq \frac{c_1(1-s)}{\nu(x)},$$

by equation (2.4). Hence,

$$\begin{aligned}
& \mathbb{Q}^e(N(\pi_k) > n, \nu(\pi_k) \leq \sqrt{n}, D(e) = \infty) \\
&= \sum_{x \in U} E_{\mathbf{Q}} \left[ \mathbb{1}_{\{\nu(x) \leq \sqrt{n}\}} P_{\omega}^e(x = \pi_k, D(e) > T_x) P_{\omega}^x(N(x) > n, D(e) = \infty) \right] \\
&\leq \sum_{x \in U} E_{\mathbf{Q}} \left[ P_{\omega}^e(x = \pi_k) \left( 1 - \frac{c_1(1-s)}{\sqrt{n}} \right)^n \right] = \left( 1 - \frac{c_1(1-s)}{\sqrt{n}} \right)^n,
\end{aligned}$$

which decays stretched exponentially. On the other hand,

$$\begin{aligned}
& \mathbb{Q}^e(N(\pi_k) > n, \nu(\pi_k) \geq \sqrt{n}, D(e) = \infty) \\
&\leq \mathbb{Q}^e(\nu(\pi_k) \geq \sqrt{n}, V_{\pi_k}^{\mu} < \varepsilon \nu(\pi_k)) + \mathbb{Q}^e(N(\pi_k) > n, V_{\pi_k}^{\mu} \geq \varepsilon \nu(\pi_k)).
\end{aligned}$$

with the notation introduced in the proof of Lemma 2.3. We have

$$\mathbb{Q}^e(\nu(\pi_k) \geq \sqrt{n}, V_{\pi_k}^{\mu} < \varepsilon \nu(\pi_k)) = \mathbf{Q}(\nu(e) \geq \sqrt{n}, V_e^{\mu} < \varepsilon \nu(e)),$$

which is stretched exponential by Cramér's Theorem. We also observe that

$$\begin{aligned}
\mathbb{Q}^e(N(\pi_k) > n, V_{\pi_k}^{\mu} \geq \varepsilon \nu(\pi_k)) &\leq E_{\mathbb{Q}^e} \left[ \mathbb{1}_{\{V_{\pi_k}^{\mu} \geq \varepsilon \nu(x)\}} (1 - \gamma(\pi_k))^n \right] \\
&= E_{\mathbf{Q}} \left[ \mathbb{1}_{\{V_e^{\mu} \geq \varepsilon \nu(x)\}} (1 - \gamma(e))^n \right] \leq (1 - c\mu\varepsilon)^n,
\end{aligned}$$

by equation (2.12). This completes the proof.  $\square$

### 2.3 The case $\Lambda < \infty$

In this part, we suppose that  $\Lambda < \infty$ , where  $\Lambda$  is defined by

$$\Lambda := \text{Leb} \left\{ t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{q_1} \right\}.$$

We prove that the tail distribution of  $\Gamma_1$  is polynomial.

**Proposition 2.4** *If  $\Lambda < \infty$ , then*

$$(2.13) \quad \lim_{n \rightarrow \infty} \frac{1}{\ln(n)} \ln(\mathbb{S}^e(\Gamma_1 > n)) = -\Lambda.$$

*Proof of Proposition 2.4.* Lemma 3.3 of [1] already gives

$$\liminf_{n \rightarrow \infty} \frac{1}{\ln(n)} \ln(\mathbb{S}^e(\Gamma_1 > n)) \geq -\Lambda.$$

Hence, the lower bound of (2.13) is known. The rest of the section is dedicated to the proof of the upper bound.

We start with three preliminary lemmas. We first prove an estimate for one-dimensional RWRE, that will be useful later on. Denote by  $(R_n, n \geq 0)$  a generic RWRE on  $\mathbb{Z}$  such that the random variables  $A(i), i \geq 0$  are independent and have the distribution of  $A$ , when we set for  $i \geq 0$ ,

$$A(i) := \frac{\omega_R(i, i+1)}{\omega_R(i, i-1)}$$

with  $\omega_R(y, z)$  the quenched probability to jump from  $y$  to  $z$ . We denote by  $P_{\omega, R}^k$  the quenched distribution associated with  $(R_n, n \geq 0)$  when starting from  $k$ , and by  $\mathbf{P}_R$  the distribution of the environment  $\omega_R$ . Let  $c_7 \in (0, 1)$  be a constant whose value will be given later on. For any  $k \geq \ell \geq 0$  and  $n \geq 0$ , we introduce the notation

$$(2.14) \quad p(\ell, k, n) := E_{\mathbf{P}_R} \left[ (1 - c_7 P_{\omega, R}^\ell (T_\ell^* > T_0 \wedge T_k))^n \right].$$

**Lemma 2.5** *Let  $0 < r < 1$ , and  $\Lambda_r := \text{Leb} \{t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{r}\}$ . Then, for any  $\varepsilon > 0$ , we have for  $n$  large enough,*

$$\sum_{k \geq \ell \geq 0} r^k p(\ell, k, n) \leq n^{-\Lambda_r + \varepsilon}.$$

*Proof.* The method used is very similar to that of Lemma 5.1 in [1]. We feel free to present a sketch of the proof. We consider the one-dimensional RWRE  $(R_n)_{n \geq 0}$ . We introduce for  $k \geq \ell \geq 0$ , the potential  $V(0) = 0$  and

$$\begin{aligned} V(\ell) &= - \sum_{i=0}^{\ell-1} \ln(A(i)), \\ H_1(\ell) &= \max_{0 \leq i \leq \ell} V(i) - V(\ell), \\ H_2(\ell, k) &= \max_{\ell \leq i \leq k} V(i) - V(\ell). \end{aligned}$$

We know (e.g. [17]) that

$$(2.15) \quad \frac{e^{-H_2(\ell+1, k)}}{k+1} \leq P_{\omega, R}^{\ell+1} (T_k < T_\ell) \leq e^{-H_2(\ell+1, k)},$$

$$(2.16) \quad \frac{e^{-H_1(\ell)}}{k+1} \leq P_{\omega, R}^{\ell-1} (T_{-1} < T_\ell) \leq e^{-H_1(\ell)}.$$

It yields that

$$P_{\omega,R}^{\ell}(T_{\ell}^* > T_0 \wedge T_k) \geq e^{-H_1(\ell) \wedge H_2(\ell,k) + O(\ln k)},$$

where  $O(\ln k)$  is a deterministic function. Let  $\eta \in (0, 1)$ .

$$\begin{aligned} p(\ell, k, n) &\leq (1 - c_7 n^{-1+\eta})^n + \mathbf{P}_R(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1 - \eta) \ln(n)) \\ &\leq e^{-c_8 n^{\eta}} + \mathbf{P}_R(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1 - \eta) \ln(n)). \end{aligned}$$

In Section 8.1 of [1], we proved that for any  $s \in (0, 1)$ ,  $E_{\mathbf{P}_R} [e^{\Lambda_s(H_1(\ell) \wedge H_2(\ell,k))}] \leq e^{k \ln(1/s) + o_s(k)}$ , where  $o_s(k)$  is such that  $o_s(k)/k$  tends to 0 at infinity. This implies that, defining  $\tilde{o}_s(k) := o_s(k) - \Lambda_s O(\ln k)$ ,

$$\begin{aligned} &s^k \mathbf{P}_R(H_1(\ell) \wedge H_2(\ell, k) - O(\ln k) \geq (1 - \eta) \ln(n)) \\ &\leq s^k \left(1 \wedge e^{k \ln(1/s) - \Lambda_s(1-\eta) \ln(n) + \tilde{o}_s(k)}\right) \\ &\leq n^{-\Lambda_s(1-\eta)} \exp((k \ln(s) + \Lambda_s(1 - \eta) \ln(n)) \wedge \tilde{o}_s(k)). \end{aligned}$$

Observe that there exists  $M_s$  such that for any  $k$  and any  $n$ , we have  $(k \ln(s) + \Lambda_s(1 - \eta) \ln(n)) \wedge \tilde{o}_s(k) \leq \sup_{i \leq M_s \ln(n)} \tilde{o}(i) + \eta \ln n$ , and notice that  $\sup_{i \leq M_s \ln(n)} \tilde{o}_s(i)$  is negligible towards  $\ln(n)$ . This leads to, for  $n$  large enough,

$$s^k p(\ell, k, n) \leq s^k e^{-c_8 n^{\eta}} + n^{-\Lambda_s(1-\eta) + 2\eta}.$$

Let  $r \in (0, 1)$  and  $s > r$ . We have

$$r^k p(\ell, k, n) \leq r^k e^{-c_8 n^{\eta}} + \left(\frac{r}{s}\right)^k n^{-\Lambda_s(1-\eta) + 2\eta}.$$

Lemma 2.5 follows by choosing  $\eta$  small enough and  $s$  close enough to  $r$ .  $\square$

Let  $Z_n$  represent the size of the  $n$ -th generation of the tree  $\mathbb{T}$ . We have the following result.

**Lemma 2.6** *There exists a constant  $c_9 > 0$  such that for any  $H > 0$ ,  $B > 0$  and  $n$  large enough,*

$$E_{\mathbf{Q}} [(1 - \gamma(e))^n \mathbb{1}_{\{Z_H > B\}}] \leq n^{-c_9 B}.$$

*Proof.* We have

$$\begin{aligned} E_{\mathbf{Q}} [(1 - \gamma(e))^n \mathbb{1}_{\{Z_H > B\}}] &\leq E_{\mathbf{Q}} [(1 - \gamma(e))^n \mathbb{1}_{\{\nu(e) \geq \sqrt{n}\}}] + E_{\mathbf{Q}} [(1 - \gamma(e))^n \mathbb{1}_{\{Z_H > B, \nu(e) \leq \sqrt{n}\}}] \\ &\leq e^{-n^{c_3}} + E_{\mathbf{Q}} [(1 - \gamma(e))^n \mathbb{1}_{\{Z_H > B, \nu(e) \leq \sqrt{n}\}}] \end{aligned}$$

by Remark 2.3. When  $\nu(e) \leq \sqrt{n}$ , we have, by (2.11),

$$\gamma(e) \geq \frac{c_4}{\sqrt{n}} e^{-c_5 R},$$

with  $R := \inf\{k \geq 1 : \exists |x| = k, \beta(x) \geq \mu\}$  as before ( $\mu > 0$  is such that  $q := \mathbf{Q}(\beta(e) > \mu) > 0$ ). Thus,

$$E_{\mathbf{Q}} \left[ (1 - \gamma(e))^n \mathbb{1}_{\{Z_H > B, \nu(e) \leq \sqrt{n}\}} \right] \leq \mathbf{Q} \left( R > \frac{1}{4c_5} \ln(n) + H, Z_H > B \right) + e^{-n^{1/4+o(1)}}.$$

By considering the  $Z_H$  subtrees rooted at each of the individuals in generation  $H$ , we see that

$$\begin{aligned} \mathbf{Q}(R > c_{10} \ln(n) + H, Z_H > B) &= E_{GW} \left[ \mathbf{Q}(R > c_{10} \ln(n))^{Z_H} \mathbb{1}_{\{Z_H > B\}} \right] \\ &\leq \mathbf{Q}(R > c_{10} \ln(n))^B. \end{aligned}$$

If  $R > c_{10} \ln(n)$ , we have in particular  $\beta(x) < \mu$  for each  $|x| = c_{10} \ln(n)$  which implies that

$$\mathbf{Q}(R > c_{10} \ln(n) + H, Z_H > B) \leq E_{GW} \left[ q^{Z_{c_{10} \ln(n)}} \right]^B.$$

Let  $t \in (q_1, 1)$ . For  $n$  large enough,  $E_{GW} \left[ q^{Z_{c_{10} \ln(n)}} \right] \leq t^{c_{10} \ln(n)} = n^{c_{10} \ln(t)}$ , ( $E_{GW}[q^{Z_n}]/q_1^n$  has a positive limit by Corollary 1 page 40 of [2]). The lemma follows.  $\square$

Let  $r \in (q_1, 1)$ ,  $\varepsilon > 0$ ,  $B$  be such that

$$(2.17) \quad c_9 B \varepsilon > 2\Lambda$$

and  $H$  large enough so that

$$(2.18) \quad GW(Z_H \leq B) < r^H \frac{1}{B} < 1.$$

In particular,  $c_{11} := GW(Z_H > B) > 0$ .

Let  $\nu(x, k)$  denote for any  $x \in \mathbb{T}$  the number of descendants of  $x$  at generation  $|x| + k$  ( $\nu(x, 1) = \nu(x)$ ), and let

$$(2.19) \quad \mathcal{S}_H := \{x \in \mathbb{T} : \nu(x, H) > B\}.$$

For any  $x \in \mathbb{T}$ , we call  $F(x)$  the youngest ancestor of  $x$  which lies in  $\mathcal{S}_H$ , and  $G(x)$  an oldest descendant of  $x$  in  $\mathcal{S}_H$ . For any  $x, y \in \mathbb{T}$ , we write  $x \leq y$  if  $y$  is a descendant of  $x$

and  $x < y$  if besides  $x \neq y$ . We define for any  $x \in \mathbb{T}$ ,  $W(x)$  as the set of descendants  $y$  of  $x$  such that there exists no vertex  $z$  with  $x < z \leq y$  and  $\nu(z, H) > B$ . In other words,  $W(x) = \{y : y \geq x, F(y) \leq x\}$ . We define also

$$\begin{aligned}\overset{\circ}{W}(x) &:= W(x) \setminus \{x\}, \\ \partial W(x) &:= \{y : \overleftarrow{y} \in W(x), \nu(y, H) > B\}.\end{aligned}$$

Finally, let  $W_j(e) := \{x : |x| = j, x \in W(e)\}$ .

**Lemma 2.7** *Recall that  $m := E_{GW}[\nu(e)]$  and  $r$  is a real belonging to  $(q_1, 1)$ . We also recall that  $H$  and  $B$  verify  $GW(Z_H \leq B) < r^H \frac{1}{B}$ . We have for any  $j \geq 0$ ,*

$$E_{GW}[W_j(e)] < m r^{j-1}.$$

*Proof.* We construct the subtree  $\mathbb{T}_H$  of the tree  $\mathbb{T}$  by retaining only the generations  $kH$ ,  $k \geq 0$  of the tree  $\mathbb{T}$ . Let

$$(2.20) \quad \mathbb{W} = \mathbb{W}(\mathbb{T}) := \{x \in \mathbb{T}_H : \forall y \in \mathbb{T}_H, (y < x) \Rightarrow \nu(y, H) \leq B\}.$$

The tree  $\mathbb{W}$  is a Galton–Watson tree whose offspring distribution is of mean  $E_{GW}[Z_H \mathbb{1}_{\{Z_H \leq B\}}] \leq B \times GW(Z_H \leq B) \leq r^H$  by (2.18). Then for each child  $e_i$  of  $e$  (in the original tree  $\mathbb{T}$ ), let  $\mathbb{W}^i := \mathbb{W}(\mathbb{T}_{e_i})$  where  $\mathbb{T}_{e_i}$  is the subtree rooted at  $e_i$ . We conclude by observing that  $W_j(e) \leq \sum_{i=1}^{\nu(e)} \#\{x \in \mathbb{W}^i : |x| = 1 + [(j-1)/H] \times H\}$  hence  $E_{GW}[W_j(e)] \leq E_{GW}[\nu(e)] r^{j-1}$ .  $\square$

We still have  $r \in (q_1, 1)$  and  $\varepsilon > 0$ . We prove that for  $n$  large enough, and  $r$  and  $\varepsilon$  close enough to  $q_1$  and 0, we have

$$(2.21) \quad \mathbb{Q}^e(\Gamma_1 > n, D(e) = \infty) \leq c_{12} n^{-(1-2\varepsilon)\Lambda_r + 3\varepsilon},$$

where  $\Lambda_r := \text{Leb}\{t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{r}\}$  as in Lemma 2.5. This suffices to prove Proposition 2.4 since  $\varepsilon$  and  $\Lambda_r$  can be arbitrarily close to 0 and  $\Lambda$ , respectively. We recall that we defined  $B$ ,  $H$  and  $\mathcal{S}_H$  in (2.17),(2.18) and (2.19).

The strategy is to divide the tree in subtrees in which vertices are constrained to have a small number of children (at most  $B$  children at generation  $H$ ). With  $B = H = 1$ , we would have literally pipes. In general, the traps constructed are slightly larger than pipes. We then

evaluate the time spent in such traps by comparison with a one-dimensional random walk.

We define  $\pi_k^s$  as the  $k$ -th distinct site visited in the set  $\mathcal{S}_H$ . We observe that

$$\begin{aligned}
(2.22) \quad & \mathbb{Q}^e (\Gamma_1 > n, D(e) = \infty) \\
& \leq \mathbb{Q}^e (\Gamma_1 > \tau_{\ln^2(n)}) + \mathbb{Q}^e (\text{more than } \ln^4(n) \text{ distinct sites are visited before } \tau_{\ln^2(n)}) \\
& \quad + \mathbb{Q}^e (\exists k \leq \ln^4(n), \exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)) \\
& \quad + \mathbb{Q}^e (\exists x \in W(e), N(x) > n/\ln^4(n), D(e) = \infty, Z_H \leq B) .
\end{aligned}$$

The first term on the right-hand side decays like  $e^{-\ln^2(n)}$  by Fact A, and so does the second term by equation (2.9). We proceed to estimate the third term on the right-hand side of (2.22). Since

$$\mathbb{Q}^e (\exists k \leq \ln^4(n), \exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)) \leq \sum_{k=1}^{\ln^4(n)} \mathbb{Q}^e (\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n))$$

we look at the rate of decay of  $\mathbb{Q}^e (\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n))$  for any  $k \geq 1$ . We first show that the time spent at the frontier of  $W(\pi_k^s)$  will be negligible. Precisely, we show

$$(2.23) \quad \mathbb{Q}^e (N(\pi_k^s) > n^\varepsilon) \leq c_{14} n^{-2\Lambda} ,$$

$$(2.24) \quad \mathbb{Q}^e (\exists z \in \partial W(\pi_k^s), N(z) > n^\varepsilon) \leq c_{15} n^{-2\Lambda} .$$

As  $P_\omega^y(N(y) > n^\varepsilon) \leq (1 - \gamma(y))^{n^\varepsilon}$  for any  $y \in \mathbb{T}$ , we have,

$$\begin{aligned}
(2.25) \quad \mathbb{Q}^e (N(\pi_k^s) > n^\varepsilon) &= E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) P_\omega^y(N(y) > n^\varepsilon) \right] \\
&\leq E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) (1 - \gamma(y))^{n^\varepsilon} \right] .
\end{aligned}$$

We would like to split the expectation  $E_{\mathbf{Q}} [P_\omega^e(\pi_k^s = y)(1 - \gamma(y))^{n^\varepsilon}]$  in two. However the random variable  $P_\omega^e(\pi_k^s = y)$  depends on the structure of the first  $H$  generations of the subtree rooted at  $y$ . Nevertheless, we are going to show that, for some  $c_{14} > 0$ ,

$$E_{\mathbf{Q}} [P_\omega^e(\pi_k^s = y)(1 - \gamma(y))^{n^\varepsilon}] \leq c_{14} E_{\mathbf{Q}} [P_\omega^e(\pi_k^s = y)] E_{\mathbf{Q}} [(1 - \gamma(y))^{n^\varepsilon} | \nu(y, H) > B] .$$

Let  $U := \bigcup_{n \geq 0} (\mathbb{N}^*)^n$  be, as before, the set of words. We have seen that  $U$  allows us to label the vertices of any tree (see [13]). Let  $y \in U$  and let  $\omega_y$  represent the restriction of

the environment  $\omega$  to the outside of the subtree rooted at  $y$  (when  $y$  belongs to the tree). For  $1 \leq L \leq H$ , we denote by  $y_L$  the ancestor of  $y$  such that  $|y_L| = |y| - L$ . We attach to each  $y_L$  the variable  $\zeta(y_L) := \mathbb{1}_{\{\nu(y_L, H) > B\}}$ . We notice that there exists a measurable function  $f$  such that  $P_\omega^e(\pi_k^s = y) = f(\omega_y, \zeta) \mathbb{1}_{\{\nu(y, H) > B\}}$  where  $\zeta := (\zeta(y_L))_{1 \leq L \leq H}$ . Let  $\mathcal{E}(\omega_y) := \{e \in \{0, 1\}^H : \mathbf{Q}(\zeta = e | \omega_y) > 0\}$ . We have

$$E_{\mathbf{Q}}[f(\omega_y, \zeta) | \omega_y] \geq \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e) \mathbf{Q}(\zeta = e | \omega_y).$$

We claim that there exists a constant  $c_{13} > 0$  such that for almost every  $\omega$  and any  $e \in \mathcal{E}(\omega_y)$ ,

$$\mathbf{Q}(\zeta = e | \omega_y) \geq c_{13}.$$

Let us prove the claim. If  $\omega_y$  is such that  $\nu(\bar{y}) > B$ , then  $\mathcal{E}(\omega_y) = \{(1, \dots, 1)\}$  and  $\mathbf{Q}(\zeta = e | \omega_y) = 1$ . Therefore suppose  $\nu(\bar{y}) \leq B$  and let  $h := \max\{1 \leq L \leq H : \nu(y_L, L) \leq B\}$ . We observe that, for any  $e \in \mathcal{E}(\omega_y)$ , we necessarily have  $e_L = 1$  for  $h < L \leq H$ . We are reduced to the study of

$$\mathbf{Q}(\zeta = e | \omega_y) = \mathbf{Q}\left(\bigcap_{1 \leq L \leq h} \{\zeta(y_L) = e_L\} \mid \omega_y\right).$$

For any tree  $\mathcal{T}$ , we denote by  $\mathcal{T}^j$  the restriction to the  $j$  first generations. Let also  $\mathbb{T}_{y_h}$  designate the subtree rooted at  $y_h$  in  $\mathbb{T}$ . Since  $\nu(y_h, h) \leq B$ , we observe that  $\mathbb{T}_{y_h}^h$  belongs almost surely to a finite (deterministic) set in the space of all trees. We construct the set

$$\begin{aligned} \Psi(\mathbb{T}_{y_h}^h, e) := \{ \text{tree } \mathcal{T} : \mathcal{T}^h = \mathbb{T}_{y_h}^h, GW(\mathcal{T}^{h+H}) > 0, \forall |x| \leq 2H, \nu_{\mathcal{T}}(x) \leq B \\ \forall 1 \leq L \leq h, \nu_{\mathcal{T}}(y_L, h) > B \text{ if and only if } e_L = 1 \}. \end{aligned}$$

We observe that  $\Psi(\mathbb{T}_{y_K}^K, e) \neq \emptyset$  as soon as  $e \in \mathcal{E}(\omega_y)$ . Let  $\tilde{\Psi}(\mathbb{T}_{y_K}^K, e) := \{\mathcal{T}^{h+H}, \mathcal{T} \in \Psi(\mathbb{T}_{y_h}^h, e)\}$  be the same set but where the trees are restricted to the first  $h+H$  generations. Since  $\tilde{\Psi}(\mathbb{T}_{y_K}^K, e)$  is again included in a finite deterministic set in the space of trees, we deduce that there exists  $c_{13} > 0$  such that, almost surely,

$$\inf\{GW(\mathcal{T}^{h+H} | \mathcal{T}^h), \mathcal{T} \in \Psi(\mathbb{T}_{y_h}^h, e), e \in \mathcal{E}(\omega_y)\} \geq c_{13}.$$

Consequently,

$$\mathbf{Q}(\zeta = e | \omega_y) \geq \mathbf{Q}(\mathbb{T}_{y_h}^{h+H} \in \tilde{\Psi}(\mathbb{T}_{y_h}^h, e) | \omega_y) \geq c_{13},$$

as required. We get

$$E_{\mathbf{Q}} [f(\omega_y, \zeta) | \omega_y] \geq c_{13} \max_{e \in \mathcal{E}(\omega_y)} f(\omega_y, e) \geq c_{13} f(\omega_y, \zeta).$$

Finally we obtain, with  $c_{14} := \frac{1}{c_{13}}$ ,

$$f(\omega_y, \zeta) \leq c_{14} E_{\mathbf{Q}} [f(\omega_y, \zeta) | \omega_y].$$

By (2.25), it entails that

$$\begin{aligned} \mathbb{Q}^e (N(\pi_k^s) > n^\varepsilon) &\leq c_{14} \sum_{y \in U} E_{\mathbf{Q}} [\mathbb{1}_{\{\nu(y, H) > B\}} E_{\mathbf{Q}} [f(\omega_y, \zeta) | \omega_y] (1 - \gamma(y))^{n^\varepsilon}] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}} [f(\omega_y, \zeta)] E_{\mathbf{Q}} [\mathbb{1}_{\{\nu(e, H) > B\}} (1 - \gamma(e))^{n^\varepsilon}] \\ &= c_{14} \sum_{y \in U} E_{\mathbf{Q}} [P_\omega^e(\pi_k^s = y)] E_{\mathbf{Q}} [(1 - \gamma(e))^{n^\varepsilon} | \nu(e, H) > B]. \end{aligned}$$

It implies that

$$\mathbb{Q}^e (N(\pi_k^s) > n^\varepsilon) \leq c_{14} E_{\mathbf{Q}} [(1 - \gamma(e))^{n^\varepsilon} | Z_H > B] \leq c_{14} n^{-c_9 \varepsilon B},$$

by Lemma 2.6. Since  $c_9 \varepsilon B > 2\Lambda$ , this leads to, for  $n$  large,

$$\mathbb{Q}^e (N(\pi_k^s) > n^\varepsilon) \leq c_{14} n^{-2\Lambda}$$

which is equation (2.23). Similarly, recalling that  $\partial W(y)$  designates the set of vertices  $z$  such that  $\bar{z} \in W(y)$  and  $\nu(z, H) > B$ , we have that

$$\begin{aligned} &\mathbb{Q}^e (\exists y \in \partial W(\pi_k^s), N(y) > n^\varepsilon) \\ &\leq E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{z \in \partial W(y)} (1 - \gamma(z))^{n^\varepsilon} \right] \\ &\leq c_{14} E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) \right] E_{GW} [\partial W(e)] E_{\mathbf{Q}} [(1 - \gamma(e))^{n^\varepsilon} | Z_H > B] \\ &= c_{14} E_{GW} [\partial W(e)] E_{\mathbf{Q}} [(1 - \gamma(e))^{n^\varepsilon} | Z_H > B]. \end{aligned}$$

We notice that  $E_{GW}[\partial W] \leq E_{GW} \left[ \sum_{x \in W(e)} \nu(x) \right] = m E_{GW} [W(e)]$  which is finite by Lemma 2.7. It yields, by Lemma 2.6,

$$\mathbb{Q}^e (\exists x \in W(\pi_k^s), N(G(x)) > n^\varepsilon) \leq c_{15} n^{-2\Lambda}$$

thus proving (2.24). Our next step is then to find an upper bound to the probability to spend most of our time at a vertex  $x$  belonging to some  $\overset{\circ}{W}(y)$ . To this end, recall that  $G(x)$  is an oldest descendant of  $x$  such that  $\nu(x, H) > B$ . We have just proved that the time spent at  $y(= F(x))$  or  $G(x)$  is negligible. Therefore, starting from  $x$ , the probability to spend much time in  $x$  is not far from the probability to spend the same time without reaching  $y$  neither  $G(x)$ . Then, this probability is bound by coupling with a one-dimensional random walk.

Define  $\tilde{T}_x^{(\ell)}$  as the  $\ell$ -th time the walk visits  $x$  after visiting either  $F(x)$  or  $G(x)$ , id est  $\tilde{T}_x^{(1)} = T_x$  and,

$$\tilde{T}_x^{(\ell)} := \inf\{k > \tilde{T}_x^{(\ell-1)} : X_k = x, \exists i \in (\tilde{T}_x^{(\ell-1)}, k), X_i = F(x) \text{ or } G(x)\}.$$

Let also  $N^{(\ell)}(x) = \sum_{k=\tilde{T}_x^{(\ell)}(x)}^{\tilde{T}_x^{(\ell+1)}(x)-1} \mathbb{1}_{\{X_k=x\}}$  be the time spent at  $x$  between  $\tilde{T}_x^{(\ell)}$  and  $\tilde{T}_x^{(\ell+1)}$ . We observe that, for any  $k \geq 1$ ,

$$\begin{aligned} & \mathbb{Q}^e(\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)) \\ & \leq \mathbb{Q}^e(N(\pi_k^s) > n^\varepsilon) + \mathbb{Q}^e(\exists x \in W(\pi_k^s), N(G(x)) > n^\varepsilon) \\ & \quad + \mathbb{Q}^e\left(\exists x \in \overset{\circ}{W}(\pi_k^s), \exists \ell \leq 2n^\varepsilon, N^{(\ell)}(x) > n^{1-2\varepsilon}\right) \\ (2.26) \quad & \leq (c_{14} + c_{15})n^{-2\Lambda} + \sum_{\ell \leq 2n^\varepsilon} \mathbb{Q}^e\left(\exists x \in \overset{\circ}{W}(\pi_k^s), N^{(\ell)}(x) > n^{1-2\varepsilon}\right). \end{aligned}$$

Since

$$\mathbb{Q}^e(\exists x \in W(\pi_k^s), N^{(\ell)}(x) > n^{1-2\varepsilon}) \leq E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{x \in \overset{\circ}{W}(y)} P_\omega^x(N^{(\ell)}(x) > n^{1-2\varepsilon}) \right],$$

and by the strong Markov property at  $\tilde{T}_x^{(\ell)}$ ,

$$\begin{aligned} P_\omega^x(N^{(\ell)}(x) > n^{1-2\varepsilon}) &= P_\omega^x(\tilde{T}_x^{(\ell)} < \infty) P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}) \\ &\leq P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}), \end{aligned}$$

this yields

$$\begin{aligned} & \mathbb{Q}^e(\exists x \in W(\pi_k^s), N^{(\ell)}(x) > n^{1-2\varepsilon}) \\ & \leq E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_\omega^e(\pi_k^s = y) \sum_{x \in \overset{\circ}{W}(y)} P_\omega^x(N^{(1)}(x) > n^{1-2\varepsilon}) \right] \end{aligned}$$

$$\begin{aligned}
&\leq c_{14} E_{\mathbf{Q}} \left[ \sum_{y \in \mathcal{S}_H} P_{\omega}^e(\pi_k^s = y) \right] E_{\mathbf{Q}} \left[ \sum_{x \in \overset{\circ}{W}(e)} P_{\omega}^x(N^{(1)}(x) > n^{1-2\varepsilon}) \mid Z_H > B \right] \\
(2.27) \quad &= c_{14} E_{\mathbf{Q}} \left[ \sum_{x \in \overset{\circ}{W}(e)} P_{\omega}^x(N^{(1)}(x) > n^{1-2\varepsilon}) \mid Z_H > B \right].
\end{aligned}$$

For any  $x \in W(e)$ , define, for any  $y \in \llbracket e, G(x) \rrbracket$ ,

$$\begin{aligned}
\tilde{\omega}(y, y_+) &:= \frac{\omega(y, y_+)}{\omega(y, y_+) + \omega(y, \overleftarrow{y})}, \\
\tilde{\omega}(y, \overleftarrow{y}) &:= \frac{\omega(y, \overleftarrow{y})}{\omega(y, y_+) + \omega(y, \overleftarrow{y})},
\end{aligned}$$

where as before  $y_+$  represents the child of  $y$  on the path. We let  $(\tilde{X}_n)_{n \geq 0}$  be the random walk on  $\llbracket e, G(x) \rrbracket$  with the transition probabilities  $\tilde{\omega}$  and we denote by  $\tilde{P}_{\omega, x}(\cdot)$  the probability distribution of  $(\tilde{X}_n, n \geq 0)$ . By Lemma 4.4 of [1], we have the following comparisons:

$$\begin{aligned}
P_{\omega}^{\overleftarrow{x}}(T_x < T_e) &\leq \tilde{P}_{\omega, x}^{\overleftarrow{x}}(T_x < T_e), \\
P_{\omega}^{x_+}(T_{G(x)} < T_x) &\leq \tilde{P}_{\omega, x}^{x_+}(T_{G(x)} < T_x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P_{\omega}^x(T_x^* < T_e \wedge T_{G(x)}) \\
&= \omega(x, \overleftarrow{x}) P_{\omega}^{\overleftarrow{x}}(T_x < T_e) + \omega(x, x_+) P_{\omega}^{x_+}(T_x < T_{G(x)}) + \sum_{i \leq \nu(x): x_i \neq x_+} \omega(x, x_i)(1 - \beta(x_i)) \\
&\leq \omega(x, \overleftarrow{x}) \tilde{P}_{\omega, x}^{\overleftarrow{x}}(T_x < T_e) + \omega(x, x_+) \tilde{P}_{\omega, x}^{x_+}(T_x < T_{G(x)}) + \sum_{i \leq \nu(x): x_i \neq x_+} \omega(x, x_i) \\
&= 1 - \left( \omega(x, \overleftarrow{x}) + \omega(x, x_+) \right) \tilde{P}_{\omega, x}^x(T_x^* > T_e \wedge T_{G(x)}).
\end{aligned}$$

Since  $\nu(x) \leq B$  (for  $x \in \overset{\circ}{W}(e)$ ), we find by (2.4) a constant  $c_{16} \in (0, 1)$  such that  $\omega(x, \overleftarrow{x}) + \omega(x, x_+) \geq c_{16}$ . It yields that

$$P_{\omega}^x(T_x^* < T_e \wedge T_{G(x)}) \leq 1 - c_{16} \tilde{P}_{\omega, x}^x(T_x^* > T_e \wedge T_{G(x)}).$$

We observe that, for any  $x \in W(e)$ , with the notation of (2.14) and taking  $c_7 := c_{16}$ ,

$$E_{\mathbf{P}} \left[ \left( 1 - c_{16} \tilde{P}_{\omega, x}^x(T_x^* > T_e \wedge T_{G(x)}) \right)^n \right] = p(|x|, |G(x)|, n).$$

It follows that

$$E_{GW} \left[ \sum_{x \in \overset{\circ}{W}(e)} \mathbb{P}^x(N^{(1)}(x) > n^{1-2\varepsilon}) \right] \leq E_{GW} \left[ \sum_{x \in \overset{\circ}{W}(e)} p(|x|, |G(x)|, n^{1-2\varepsilon}) \right].$$

On the other hand,  $\sum_{x \in W(e)} p(|x|, |G(x)|, n^{1-2\varepsilon}) \leq \sum_{y \in \partial W(e)} \sum_{x \leq y} p(|x|, |y|, n^{1-2\varepsilon})$ . It implies that

$$\begin{aligned} E_{GW} \left[ \sum_{x \in \overset{\circ}{W}(e)} \mathbb{P}^x(N^{(1)}(x) > n^{1-2\varepsilon}) \right] &\leq \sum_{j \geq 0} E_{GW} [\#\{y \in \partial W(e), |y| = j\}] \left( \sum_{i \leq j} p(i, j, n^{1-2\varepsilon}) \right) \\ &\leq m \sum_{j \geq 0} E_{GW} [W_{j-1}(e)] \left( \sum_{i \leq j} p(i, j, n^{1-2\varepsilon}) \right). \end{aligned}$$

By Lemmas 2.5 and 2.7, for  $n$  large enough,

$$E_{GW} \left[ \sum_{x \in \overset{\circ}{W}(e)} \mathbb{P}^x(N^{(1)}(x) > n^{1-2\varepsilon}) \right] \leq m^2 \sum_{j \geq 0} r^{j-2} \left( \sum_{i \leq j} p(i, j, n^{1-2\varepsilon}) \right) \leq n^{-(1-2\varepsilon)\Lambda_r + \varepsilon}. \quad (2.28)$$

Supposing  $r$  and  $\varepsilon$  close enough to  $q_1$  and 0, equation (2.28) combined with (2.26) and (2.27), shows that, for any  $k \geq 1$ ,

$$\mathbb{Q}^e (\exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)) \leq c_{17} n^{-(1-2\varepsilon)\Lambda_r + 2\varepsilon}.$$

We arrive at

$$(2.29) \quad \mathbb{Q}^e (\exists k \leq \ln^4(n), \exists x \in W(\pi_k^s), N(x) > n/\ln^4(n)) \leq c_{18} n^{-(1-2\varepsilon)\Lambda_r + 3\varepsilon}.$$

Finally, the estimate of  $\mathbb{Q}^e (\exists x \in W(e), N(x) > n/\ln^4(n), D(e) = \infty, Z_H \leq B)$  in (2.22) is similar. Indeed,

$$\begin{aligned} &\mathbb{Q}^e (\exists x \in W(e), N(x) > n/\ln^4(n), D(e) = \infty, Z_H \leq B) \\ &\leq \mathbb{Q}^e (N(e) > n^\varepsilon, D(e) = \infty, \nu(e) \leq B) + \mathbb{Q}^e (\exists x \in W(e), N(G(x)) > n^\varepsilon) \\ &\quad + \mathbb{Q}^e (\exists x \in W(e), \exists \ell \leq 2n^\varepsilon, N^{(\ell)}(x) > n^{1-2\varepsilon}). \end{aligned}$$

We have

$$\begin{aligned} \mathbb{Q}^e (N(e) > n^\varepsilon, D(e) = \infty, \nu(e) \leq B) &\leq E_{\mathbf{Q}} \left[ (1 - \omega(e, \bar{e}))^{n^\varepsilon} \mathbb{1}_{\{\nu(e) \leq B\}} \right] \\ &\leq (1 - c_1/B)^{n^\varepsilon}, \end{aligned}$$

by (2.4). By equation (2.24),

$$\mathbb{Q}^e (\exists x \in W(\pi_k^s), N(G(x)) > n^\varepsilon) \leq c_{15} n^{-2\Lambda}.$$

Finally,

$$\begin{aligned} \mathbb{Q}^e \left( \exists x \in \mathring{W}(e), \exists \ell \leq 2n^\varepsilon, N^{(\ell)}(x) > n^{1-2\varepsilon} \right) &\leq \sum_{\ell \leq 2n^\varepsilon} \mathbb{Q}^e \left( \exists x \in \mathring{W}(e), N^{(\ell)}(x) > n^{1-2\varepsilon} \right) \\ &\leq 2n^\varepsilon \mathbb{Q}^e \left( \exists x \in \mathring{W}(e), N^{(1)}(x) > n^{1-2\varepsilon} \right) \\ &\leq 2n^\varepsilon E_{GW} \left[ \sum_{x \in \mathring{W}(e)} \mathbb{P}^x (N^{(1)}(x) > n^{1-2\varepsilon}) \right] \\ &\leq c_{17} n^{-(1-2\varepsilon)\Lambda_r + 2\varepsilon}, \end{aligned}$$

by (2.28). We deduce that, for  $n$  large enough,

$$(2.30) \quad \mathbb{Q}^e (\exists x \in W(e), N(x) > n/\ln^4(n), D(e) = \infty, Z_H \leq B) \leq n^{-(1-2\varepsilon)\Lambda_r + 3\varepsilon}.$$

In view of (2.22) combined with (2.29) and (2.30), equation (2.21) is proved, and Proposition 2.4 follows.  $\square$

### 3 Large deviations principles

We recall the definition of the first regeneration time

$$\Gamma_1 := \inf \{k > 0 : \nu(X_k) \geq 2, D(X_k) = \infty, k = \tau_{|X_k|}\}.$$

We define by iteration

$$\Gamma_n := \inf \{k > \Gamma_{n-1} : \nu(X_k) \geq 2, D(X_k) = \infty, k = \tau_{|X_k|}\}$$

for any  $n \geq 2$ . We have the following fact (points (i) to (iii) are already discussed in [1]; point (iv) is shown in [8] in the case of regular trees and in [12] in the case of biased random walks, and is easily adaptable to our case).

**Fact B**

(i) For any  $n \geq 1$ ,  $\Gamma_n < \infty$   $\mathbb{Q}^e$ -a.s.

(ii) Under  $\mathbb{Q}^e$ ,  $(\Gamma_{n+1} - \Gamma_n, |X_{\Gamma_{n+1}}| - |X_{\Gamma_n}|)$ ,  $n \geq 1$  are independent and distributed as  $(\Gamma_1, |X_{\Gamma_1}|)$  under the distribution  $\mathbb{S}^e$ .

(iii) We have  $E_{\mathbb{S}^e}[|X_{\Gamma_1}|] < \infty$ .

(iv) The speed  $v$  verifies  $v = \frac{E_{\mathbb{S}^e}[|X_{\Gamma_1}|]}{E_{\mathbb{S}^e}[\Gamma_1]}$ .

The rest of the section is devoted to the proof of Theorems 1.1 and 1.2. It is in fact easier to prove them when conditioning on never returning to the root. Our theorems become

**Theorem 3.1 (Speed-up case)** *There exist two continuous, convex and strictly decreasing functions  $I_a \leq I_q$  from  $[1, 1/v]$  to  $\mathbb{R}_+$ , such that  $I_a(1/v) = I_q(1/v) = 0$  and for  $a < b$ ,  $b \in [1, 1/v]$ ,*

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in ]a, b[ \mid D(e) = \infty \right) \right) = -I_a(b),$$

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in ]a, b[ \mid D(e) = \infty \right) \right) = -I_q(b).$$

**Theorem 3.2 (Slowdown case)** *There exist two continuous, convex functions  $I_a \leq I_q$  from  $[1/v, +\infty[$  to  $\mathbb{R}_+$ , such that  $I_a(1/v) = I_q(1/v) = 0$  and for any  $1/v \leq a < b$ ,*

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in [a, b[ \mid D(e) = \infty \right) \right) = -I_a(a),$$

$$(3.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in [a, b[ \mid D(e) = \infty \right) \right) = -I_q(a).$$

If  $\text{ess inf } A =: i > \nu_{\min}^{-1}$ , then  $I_a$  and  $I_q$  are strictly increasing on  $[1/v, +\infty[$ . If  $i \leq \nu_{\min}^{-1}$ , then  $I_a = I_q = 0$ .

Theorems 1.1 and 1.2 follow from Theorems 3.1 and 3.2 and the following proposition.

**Proposition 3.3** *We have, for  $a < b \leq 1/v$ ,*

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in ]a, b[ \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in ]a, b[ \mid D(e) = \infty \right) \right),$$

$$(3.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in ]a, b[ \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in ]a, b[ \mid D(e) = \infty \right) \right).$$

Similarly, in the slowdown case, we have for  $1/v \leq a < b$ ,

$$(3.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in [a, b] \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( \mathbb{Q}^e \left( \frac{\tau_n}{n} \in [a, b] \mid D(e) = \infty \right) \right),$$

$$(3.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in [a, b] \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in [a, b] \mid D(e) = \infty \right) \right).$$

Theorems 3.1 and 3.2 are proved in two distinct parts for sake of clarity. Proposition 3.3 is proved in subsection 3.3.

### 3.1 Proof of Theorem 3.1

For any real numbers  $h \geq 0$  and  $b \geq 1$ , any integer  $n \in \mathbb{N}$  and any vertex  $x \in \mathbb{T}$  with  $|x| = n$ , define

$$A(h, b, x) := \left\{ \omega : P_\omega^e \left( \tau_n = T_x, \tau_n \leq bn, T_e^- > \tau_n \right) \geq e^{-hn} \right\},$$

$$e_n(h, b) := E_{\mathbf{Q}} \left[ \sum_{|x|=n} \mathbb{1}_{A(h, b, x)} \right].$$

We define also for any  $b \geq 1$

$$h_c(b) := \inf \{ h \geq 0 : \exists p \in \mathbb{N}, e_p(h, b) > 0 \}.$$

**Lemma 3.4** *There exists for any  $b \geq 1$  and  $h > h_c(b)$ , a real  $e(h, b) > 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln(e_n(h, b)) = \ln(e(h, b)).$$

Moreover, the function  $(h, b) \rightarrow \ln(e(h, b))$  from  $\{(h, b) \in \mathbb{R}_+ \times [1, +\infty[ : h > h_c(b)\}$  to  $\mathbb{R}$  is concave, is nondecreasing in  $h$  and in  $b$ , and

$$\lim_{h \rightarrow \infty} \ln(e(h, b)) = \ln(m).$$

*Proof.* Let  $x \leq y$  be two vertices of  $\mathbb{T}$  with  $|x| = n$  and  $|y| = n + m$ . We observe that

$$\begin{aligned} A(h, b, y) &\supset A(h, b, x) \cap \left\{ \omega : P_\omega^x \left( \tau_{n+m} = T_y, \tau_{n+m} \leq bm, T_x^- > \tau_{n+m} \right) \geq e^{-hm} \right\} \\ &=: A(h, b, x) \cap A_x(h, b, y). \end{aligned}$$

It yields that

$$e_{n+m}(h, b) \geq E_{\mathbf{Q}} \left[ \sum_{|x|=n} \mathbb{1}_{A(h, b, x)} \sum_{|y|=n+m, y \geq x} \mathbb{1}_{A_x(h, b, y)} \right]$$

$$\begin{aligned}
(3.9) \quad &= E_{\mathbf{Q}} \left[ \sum_{|x|=n} \mathbb{I}_{A(h,b,x)} \right] E_{\mathbf{Q}} \left[ \sum_{|x|=m} \mathbb{I}_{A(h,b,x)} \right] \\
&= e_n(h,b) e_m(h,b).
\end{aligned}$$

Let  $h > h_c$  and  $p$  be such that  $e_p(h_c, b) > 0$ , where we write  $h_c$  for  $h_c(b)$ . Then  $e_{np}(h_c, b) > 0$  for any  $n \geq 1$ . We want to show that  $e_k(h, b) > 0$  for  $k$  large enough. By (2.4),  $\omega(e, e_1) \geq c_1$  if  $\nu(e) = 1$  so that  $e_k(-\ln(c_1), b) \geq q_1^k$ . Let  $n_c$  be such that  $e^{-h_c n_c} c_1 \geq e^{-h n_c}$ . We check as before that for any  $n \geq n_c$ , and any  $r \leq p$ , we have indeed

$$\begin{aligned}
e_{np+r}(h, b) &\geq e_{np}(h_c, b) e_r(-\ln(c_1), b) \\
&\geq e_{np}(h_c, b) q_1^r > 0.
\end{aligned}$$

Thus (3.9) implies that

$$(3.10) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(e_n(h, b)) = \sup \left\{ \frac{1}{k} \ln(e_k(h, b)), k \geq 1 \right\} =: \ln(e(h, b)),$$

with  $e(h, b) > 0$ . Similarly, we can check that

$$e_n(th_1 + (1-t)h_2, tb_1 + (1-t)b_2) \geq e_{nt}(h_1, b_1) e_{n(1-t)}(h_2, b_2),$$

which leads to

$$\ln(e(th_1 + (1-t)h_2, tb_1 + (1-t)b_2)) \geq t \ln(e(h_1, b_1)) + (1-t) \ln(e(h_2, b_2)),$$

hence the concavity of  $(h, b) \rightarrow \ln(e(h, b))$ . The fact that  $e(h, b)$  is nondecreasing in  $h$  and in  $b$  is direct. Finally,  $\limsup_{h \rightarrow \infty} \ln(e(h, b)) \leq \ln(m)$  and  $\liminf_{h \rightarrow \infty} \ln(e(h, b)) \geq \liminf_{h \rightarrow \infty} \ln(e_1(h, b)) = \ln(m)$  by dominated convergence.  $\square$

In the rest of the section, we extend  $e(h, b)$  to  $\mathbb{R}_+ \times [1, +\infty[$  by taking  $e(h, b) = 0$  for  $h \leq h_c(b)$ .

**Corollary 3.5** *Let  $S := \{h \geq 0 : e(h, b) > 1\}$  and  $S' := \{h \geq 0 : e(h, b) \geq 1\}$ . We have*

$$\sup\{e^{-h} e(h, b), h \in S\} = \sup\{e^{-h} e(h, b), h \in S'\}.$$

*Proof.* Let  $M := \inf\{h : e(h, b) > 1\}$ . We claim that if  $h < M$ , then  $e(h, b) < 1$ . Indeed, suppose that there exists  $h_0 < M$  such that  $e(h_0, b) \geq 1$ . Then  $e(h_0, b) = 1$  by definition of  $M$ , so that  $e(h, b)$  is constant equal to 1 on  $[h_0, M[$ . By concavity,  $\ln(e(h, b))$  is equal to 0 on  $[h_0, +\infty[$ , which is impossible since it tends to  $\ln(m)$  at infinity. The corollary follows.  $\square$

We have the tools to prove Theorem 1.1.

*Proof of Theorem 1.1.* For  $b \in [1, +\infty[$ , let

$$\begin{aligned} J_a(b) &:= -\sup\{-h + \ln(e(h, b)), h \geq 0\}, \\ J_q(b) &:= -\sup\{-h + \ln(e(h, b)), h \in S\}. \end{aligned}$$

Define then for any  $b \leq 1/v$ ,

$$\begin{aligned} I_a(b) &= J_a(b), \\ I_q(b) &= J_q(b). \end{aligned}$$

We immediately see that  $I_a \leq I_q$ . The convexity of  $J_a$  and  $J_q$  stems from the convexity of the function  $h - \ln(e(h, b))$ . Indeed, let  $J$  represent either  $J_a$  or  $J_q$  and let  $1 \leq b_1 \leq b_2$  and  $t \in [0, 1]$ . Denote by  $h_1, h_2, b$  and  $h$  the reals that verify

$$\begin{aligned} J(b_1) &= h_1 - \ln(e(h_1, b_1)), \\ J(b_2) &= h_2 - \ln(e(h_2, b_2)), \\ h &:= th_1 + (1-t)h_2, \\ b &:= tb_1 + (1-t)b_2. \end{aligned}$$

We observe that

$$\begin{aligned} J(b) &\leq h - \ln(e(h, b)) \\ &\leq t(h_1 - \ln(e(h_1, b_1))) + (1-t)(h_2 - \ln(e(h_2, b_2))) = tJ(b_1) + (1-t)J(b_2) \end{aligned}$$

which proves the convexity. We show now that, for any  $b \geq 1$ ,

$$(3.11) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{Q}^e(\tau_n < T_e^-, \tau_n \leq bn)) = -J_a(b),$$

$$(3.12) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn)) = -J_q(b).$$

We first prove (3.11). Since  $\mathbb{Q}^e(\tau_n < T_e^-, \tau_n \leq bn) \geq e^{-hn} e_n(h, b)$  for any  $h \geq 0$ , we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{Q}^e(\tau_n < T_e^-, \tau_n \leq bn)) \geq -I_a(b).$$

Turning to the upper bound, take a positive integer  $k$ . We observe that

$$\begin{aligned} \mathbb{Q}^e(\tau_n < T_e^-, \tau_n \leq bn) &\leq \sum_{\ell=0}^{k-1} e^{-n\ell/k} e_n((\ell+1)/k, b) \\ &\leq k e^{n/k} \sup\{e^{-hn} e_n(h, b), h \geq 0\}. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{Q}^e(\tau_n < T_e^-, \tau_n \leq bn)) \leq \frac{1}{k} - J_a(b).$$

Letting  $k$  tend to infinity gives the upper bound of (3.11).

To prove equation (3.12), let  $k$  be still a positive integer and  $h \in S$ . Denote by  $V_{pk}(\mathbb{T})$  the set of vertices  $|x| = pk$  such that  $P_\omega^{x_{\ell-1}}(\tau_{\ell k} < T_{x_{\ell-1}}^-, \tau_{\ell k} = T_{x_\ell} \leq bk) \geq e^{-hk}$  for any  $\ell \leq p$ , where  $x_\ell$  represents the ancestor of  $x$  at generation  $\ell k$ . Call  $V(\mathbb{T}) := \cup_{p \geq 0} V_{pk}(\mathbb{T})$  the subtree thus obtained. We observe that  $V$  is a Galton–Watson tree of mean offspring  $e_k(h, b)$ . Let

$$\mathcal{T}_{k,h} := \{\mathbb{T} : V(\mathbb{T}) \text{ is infinite}\}.$$

Take  $\mathbb{T} \in \mathcal{T}_{k,h}$ . For any  $x \in V_{pk}$ , we have

$$\begin{aligned} &P_\omega^e(\tau_{pk} < T_e^-, \tau_{pk} = T_x \leq bpk) \\ &\geq P_\omega^e(\tau_k < T_e^-, \tau_k = T_{x_1} \leq bk) \dots P_\omega^{x_{k-1}}(\tau_{pk} < T_{x_{k-1}}^-, \tau_{pk} = T_x \leq bk) \geq e^{-hpk}. \end{aligned}$$

It implies that

$$P_\omega^e(\tau_{pk} < T_e^-, \tau_{pk} \leq bpk) \geq e^{-hpk} \#V_{pk}(\mathbb{T}).$$

By the Seneta–Heyde Theorem (see [2] page 30 Theorem 3),

$$\lim_{p \rightarrow \infty} \frac{1}{p} \ln(\#V_{pk}(\mathbb{T})) = \ln(e_k(h, b)) \quad \mathbf{Q} - \text{a.s.}$$

It follows that, as long as  $\mathbb{T} \in \mathcal{T}_{k,h}$ ,

$$\liminf_{p \rightarrow \infty} \frac{1}{pk} \ln(P_\omega^e(\tau_{pk} < T_e^-, \tau_{pk} \leq bpk)) \geq -h + \frac{1}{k} \ln(e_k(h, b)).$$

Notice that

$$P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn) \geq P_\omega^e(\tau_{pk} < T_e^-, \tau_{pk} \leq bpk) \min_{|x|=pk} P_\omega^x(\tau_n < T_x^-, \tau_n \leq b(n-pk))$$

where  $p := \lfloor \frac{n}{k} \rfloor$ . Since  $A$  is bounded, there exists  $c_{17} > 0$  such that  $\sum_{i=1}^{\nu(y)} \omega(y, y_i) \geq c_{17}$   $\forall y \in \mathbb{T}$ . It yields that

$$\min_{|x|=pk} P_\omega^x(\tau_n < T_x^-, \tau_n = (n-pk)) \geq c_{17}^k.$$

Hence,

$$(3.13) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn)) \geq -h + \frac{1}{k} \ln(e_k(h, b)).$$

Take now a general tree  $\mathbb{T}$ . Notice that since  $h \in S$ ,  $\mathbf{Q}(\mathcal{T}_{k,h}) > 0$  for  $k$  large enough, and there exists almost surely a vertex  $z \in \mathbb{T}$  such that the subtree rooted at it belongs to  $\mathcal{T}_{k,h}$ . It implies that for large  $k$ , (3.13) holds almost surely. Then letting  $k$  tend to infinity and taking the supremum over all  $h \in S$  leads to

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn)) \geq -J_q(b).$$

For the upper bound in (3.12), we observe that, for any integer  $k$ ,

$$P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn) \leq \sum_{\ell=0}^{k-1} e^{-\ell n/k} \sum_{|x|=n} \mathbb{1}_{A((\ell+1)/k, b, x)}.$$

By Markov's inequality, we have

$$\mathbf{Q} \left( \sum_{|x|=n} \mathbb{1}_{A(h, b, x)} > (e(h, b) + 1/k)^n \right) \leq \frac{e_n(h, b)}{(e(h, b) + 1/k)^n} \leq \left( \frac{e(h, b)}{e(h, b) + 1/k} \right)^n,$$

by (3.10). An application of the Borel–Cantelli lemma proves that  $\sum_{|x|=n} \mathbb{1}_{A(h, b, x)} \leq (e(h, b) + 1/k)^n$  for all but a finite number of  $n$ ,  $\mathbf{Q}$ -a.s. In particular, if  $e(h, b) + 1/k < 1$ , then  $\sum_{|x|=n} \mathbb{1}_{A(h, b, x)} = 0$  for  $n$  large enough. Consequently, for  $n$  large,

$$P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn) \leq e^{n/k} k \sup\{e^{-hn}(e(h, b) + 1/k)^n, h : e(h, b) + 1/k \geq 1\}.$$

We find that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn)) \leq 1/k + \sup\{-h + \ln(e(h, b) + 1/k), h : e(h, b) + 1/k \geq 1\}.$$

Let  $k$  tend to infinity and use Corollary 3.5 to complete the proof of (3.12).

We observe that

$$\begin{aligned} P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn) - P_\omega^e(\tau_n < T_e^- < \infty, \tau_n \leq bn) &\leq P_\omega^e(T_e^- = \infty, \tau_n \leq bn) \\ &\leq P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn). \end{aligned}$$

But  $P_\omega^e(\tau_n < T_e^- < \infty, \tau_n \leq bn) \leq P_\omega^e(\tau_n < T_e^-, \tau_n \leq bn) \max_{i=1, \dots, \nu(e)} (1 - \beta(e_i))$ . Since  $\max_{i=1, \dots, \nu(e)} (1 - \beta(e_i)) < 1$  almost surely, we obtain that

$$(3.14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n \leq bn) \mid D(e) = \infty) = -J_q(b).$$

In the annealed case, notice that  $\mathbb{S}^e(\tau_n < T_e^- < \infty, \tau_n \leq bn) = \mathbb{S}^e(\tau_n < T_e^-, \tau_n \leq bn) E_{\mathbf{P}}[1 - \beta]$  which leads similarly to

$$(3.15) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{S}^e(\tau_n \leq bn)) = -J_a(b).$$

We can now finish the proof of the theorem. The continuity has to be proved only at  $b = 1$  (since  $J_a$  and  $J_q$  are convex on  $[1, +\infty[$ ), which is directly done with the arguments of [5] Section 4. We let  $b < 1/v = E_{\mathbb{S}^e}[\Gamma_1]/E_{\mathbb{S}^e}[|X_{\Gamma_1}|]$  and we observe that for any constant  $c_{18} > 0$ ,

$$\mathbb{S}^e(\tau_n \leq bn) \leq \mathbb{S}^e(\tau_n < \Gamma_{c_{18}n}) + \mathbb{S}^e(\Gamma_{c_{18}n} \leq bn).$$

Choose  $c_{18}$  such that  $b(E_{\mathbb{S}^e}[\Gamma_1])^{-1} < c_{18} < (E_{\mathbb{S}^e}[|X_{\Gamma_1}|])^{-1}$ . Use Cramér's Theorem with Facts A and B to see that  $\mathbb{S}^e(\tau_n < \Gamma_{c_{18}n})$  and  $\mathbb{S}^e(\Gamma_{c_{18}n} \leq bn)$  decrease exponentially. Then,  $\mathbb{S}^e(\tau_n \leq bn)$  has an exponential decay and, by (3.15),  $I_a(b) > 0$  which leads to  $I_q(b) > 0$  since  $I_a \leq I_q$ . We deduce in particular that  $I_a$  and  $I_q$  are strictly decreasing. Furthermore,  $P_\omega^e(\tau_n \leq bn \mid D(e) = \infty)$  tends to 1 almost surely when  $b > 1/v$ , which in virtue of (3.14), implies that  $J_q(b) = 0$ . By continuity,  $I_q(1/v) = 0$  and therefore  $I_a(1/v) = 0$ . Finally, let  $a < b$ ,  $b \in [1, 1/v]$ .

$$P_\omega^e(an < \tau_n \leq bn \mid D(e) = \infty) = P_\omega^e(\tau_n \leq bn \mid D(e) = \infty) - P_\omega^e(\tau_n \leq an \mid D(e) = \infty).$$

Equation (3.2) follows since  $I_q$  is strictly decreasing. The same argument proves (3.1).  $\square$

### 3.2 Proof of Theorem 3.2

The proof is the same as before by taking for  $b \geq 1$ ,

$$\begin{aligned}\tilde{A}(h, b, x) &:= \{\omega : P_\omega^e(\tau_n = T_x, T_e^- > \tau_n \geq bn) \geq e^{-hn}\}, \\ \tilde{e}_n(h, b) &:= E_{\mathbf{Q}} \left[ \sum_{|x|=n} \mathbb{1}_{\tilde{A}(h, b, x)} \right], \\ \tilde{S} &:= \{h : \tilde{e}(h, b) > 1\}.\end{aligned}$$

Define also for any  $b \geq 1$ ,

$$\begin{aligned}\tilde{J}_a(b) &:= -\sup\{-h + \ln(\tilde{e}(h, b)), h \geq 0\}, \\ \tilde{J}_q(b) &:= -\sup\{-h + \ln(\tilde{e}(h, b)), h \in \tilde{S}\},\end{aligned}$$

and for any  $b \geq 1/v$ ,

$$\begin{aligned}I_a(b) &:= \tilde{J}_a(b), \\ I_q(b) &:= \tilde{J}_q(b).\end{aligned}$$

We verify that  $I_a \leq I_q$  and both functions are convex. We have then for any  $b \geq 1$ ,

$$(3.16) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{Q}^e(T_e^- > \tau_n \geq bn)) = -\tilde{J}_a(b),$$

$$(3.17) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(T_e^- > \tau_n \geq bn)) = -\tilde{J}_q(b).$$

As before, we obtain

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n} \ln(\mathbb{S}^e(\tau_n \geq bn)) &= -\tilde{J}_a(b), \\ \lim_{n \rightarrow \infty} \frac{1}{n} \ln(P_\omega^e(\tau_n \geq bn \mid D(e) = \infty)) &= -\tilde{J}_q(b).\end{aligned}$$

We have  $\tilde{J}_a = \tilde{J}_q = 0$  on  $[1, 1/v]$ . In the case  $i > \nu_{min}^{-1}$ , the positivity of  $I_a$  and  $I_q$  on  $]1/v, +\infty[$  comes from Proposition 2.1 and Cramér's Theorem, which implies that they are strictly increasing. Equations (3.3) and (3.4) follow in that case. In the case  $i \leq \nu_{min}^{-1}$ , we follow the strategy of [5]. Let  $\eta > 0$ . As in the proof of Proposition 2.2, we set  $h_n := \lfloor \ln(n)/(6 \ln(b)) \rfloor$ , and for some  $b \in \mathbb{N}$ ,

$$\begin{aligned}w_+ &:= \mathbf{Q} \left( \sum_{i=1}^{\nu} A(e_i) \geq 1 + \eta, \nu(e) \leq b \right), \\ w_- &:= \mathbf{Q} \left( \sum_{i=1}^{\nu} A(e_i) \leq \frac{1}{1 + \eta}, \nu(e) \leq b \right).\end{aligned}$$

Taking  $b$  large enough, we have  $w_+ > 0$  and  $w_- > 0$ . We say that  $\mathbb{T}$  is a  $n$ -good tree if

- any vertex  $x$  of the  $h_n$  first generations verifies  $\nu(x) \leq b$  and  $\sum_{i=1}^{\nu(x)} A(x_i) \geq 1 + \eta$ ,
- any vertex  $x$  of the  $h_n$  following generations verifies  $\nu(x) \leq b$  and  $\sum_{i=1}^{\nu(x)} A(x_i) \leq \frac{1}{1+\eta}$ .

Then we know that  $Q_n := \mathbf{Q}(\mathbb{T} \text{ is } n\text{-good}) \geq \exp(-n^{1/3+o(1)})$ . Let  $Y'$  be a random walk starting from zero which increases (resp. decreases) of 1 with probability  $\frac{1+\eta}{2+\eta}$  (resp.  $\frac{1}{2+\eta}$ ). We define  $p'_n$  as the probability that  $Y'$  reaches  $-1$  before  $h_n$ . We show that (2.6) is still true (by the exactly same arguments), so that there exists a constant  $K > 0$  and a deterministic function  $O(n^K)$  bounded by a factor of  $n \rightarrow n^K$ , such that

$$(3.18) \quad P_\omega^e(T_e^- > \tau_{2h_n} \geq n) \geq O(n^K)^{-1} (p'_n)^n,$$

We have, by gambler's ruin formula,

$$p'_n = 1 - \frac{1}{1 + \left(\frac{1}{1+\eta}\right) + \dots + \left(\frac{1}{1+\eta}\right)^{h_n}} \geq \frac{1}{1 + \eta}.$$

Let  $k_n := \lfloor n^d \rfloor$  with  $d \in (1/3, 1/2)$  and let  $f \in (d, 1-d)$ . We call an  $n$ -slow tree a tree in which we can find a vertex  $|x| = k_n$  such that  $\mathbb{T}_x$  is  $n$ -good (where  $\mathbb{T}_x$  is the subtree rooted at  $x$ ), and for any  $y \leq x$ , we have  $\nu(y) \leq \exp(n^f)$ . We observe that if a tree is not  $n$ -slow, then either there exists a vertex before generation  $k_n$  with more than  $\exp(n^f)$  children, or any subtree rooted at generation  $k_n$  is not  $n$ -good. This leads to

$$\begin{aligned} \mathbf{Q}(\mathbb{T} \text{ is not } n\text{-slow}) &\leq \sum_{\ell=1}^{k_n} E_{GW}[Z_\ell] GW(\nu > e^{n^f}) + E_{GW}[(1 - Q_n)^{Z_{k_n}}] \\ &\leq k_n m^{k_n} m e^{-n^f} + (1 - Q_n)^{(1+\varepsilon)^{k_n}} + GW(Z_{k_n} \leq (1 + \varepsilon)^{k_n}). \end{aligned}$$

We notice that  $(1 - Q_n)^{(1+\varepsilon)^{k_n}} \leq \exp(-(1 + \varepsilon)^{n^{d+o(1)}})$ . Moreover,

$$GW(Z_{k_n} \leq (1 + \varepsilon)^{k_n}) \leq (1 + \varepsilon)^{k_n} E_{GW} \left[ \frac{1}{Z_{k_n}} \right]$$

Observe that for any  $k \geq 0$ ,  $E_{GW} \left[ \frac{1}{Z_{k+1}} \right] \leq q_1 E_{GW} \left[ \frac{1}{Z_k} \right] + (1 - q_1) E_{GW} \left[ \frac{1}{X_1 + X_2} \right]$  where  $X_1$  and  $X_2$  are independent and distributed as  $Z_k$ . We then verify  $E_{GW} \left[ \frac{1}{X_1 + X_2} \right] \leq (u/2) \wedge v$  where

$u := E_{GW} [\min(X_1, X_2)^{-1}]$  and  $v := E_{GW} [\max(X_1, X_2)^{-1}]$ . Since  $u + v = E_{GW} \left[ \frac{2}{Z_k} \right]$ , we deduce that  $E_{GW} \left[ \frac{1}{X_1 + X_2} \right] \leq \frac{2}{3} E_{GW} \left[ \frac{1}{Z_k} \right]$ , leading to  $E_{GW} \left[ \frac{1}{Z_{k+1}} \right] \leq (q_1 + \frac{2}{3}(1 - q_1)) E_{GW} \left[ \frac{1}{Z_k} \right] \leq (q_1 + \frac{2}{3}(1 - q_1))^{k+1}$ . We get

$$GW(Z_{k_n} \leq (1 + \varepsilon)^{k_n}) \leq \left( (1 + \varepsilon)(q_1 + \frac{2}{3}(1 - q_1)) \right)^{k_n},$$

and, taking  $\varepsilon$  small enough,

$$(3.19) \quad \mathbf{Q}(\mathbb{T} \text{ is not } n\text{-slow}) \leq \exp(-n^{d+o(1)}).$$

Let  $1/v \leq a < b$ . We want to show that (under the hypothesis  $i \leq \nu_{min}^{-1}$ ),

$$(3.20) \quad \liminf_{n \rightarrow \infty} \ln P_\omega^e \left( \frac{\tau_n}{n} \in [a, b[, D(e) > \tau_n \right) = 0.$$

If this is proved, the Jensen's inequality gives

$$(3.21) \quad \liminf_{n \rightarrow \infty} \ln \mathbf{Q}^e \left( \frac{\tau_n}{n} \in [a, b[, D(e) > \tau_n \right) = 0.$$

Equations (3.4) and (3.3) follow. Therefore, we focus on the proof of (3.20).

Let  $n_1 := n - k_n - 2h_n$ ,  $\delta > 0$ , and  $G_k := \{|x| = k \text{ s.t. } \mathbb{T}_x \text{ is } n\text{-slow}\}$ . We have

$$\left\{ \frac{\tau_n}{n} \in [a, b[, \tau_e^- > \tau_n \right\} \subset E_5 \cap E_6 \cap E_7,$$

with

$$\begin{aligned} E_5 &:= \left\{ T_e^- > \tau_{n_1}, \frac{\tau_{n_1}}{n_1} \in \left[ \frac{1}{v} - \delta, \frac{1}{v} + \delta \right] \right\}, \\ E_6 &:= \{ X_{\tau_{n_1}} \in G_{n_1} \}, \\ E_7 &:= \left\{ D(X_{\tau_{n_1}}) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta \right) \right\}. \end{aligned}$$

We look at the probability of the event  $E_7$  conditioned on  $E_5$  and  $E_6$ . Therefore, we suppose that  $u := X_{\tau_{n_1}}$  is known, and that the subtree  $\mathbb{T}_u$  rooted at  $u$  is a  $n$ -slow tree. There exists  $x_n$  at generation  $n_1 + k_n$  such that  $\mathbb{T}_{x_n}$  is a  $n$ -good tree and  $\nu(y) \leq e^{n^f}$  for any  $u \leq y < x_n$ . Let also  $n$  be large enough so that  $k_n \leq \delta n$ . It implies that

$$\begin{aligned} & P_\omega^u \left( D(u) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta \right) \right) \\ & \geq P_\omega^u (D(u) > T_{x_n} = k_n) P_\omega^{x_n} \left( D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta \right) \right) \\ & \geq \exp(-c_{21}n^{c_{22}}) P_\omega^{x_n} \left( D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta \right) \right), \end{aligned}$$

for some  $c_{22} \in (0, 1)$ . By definition of a  $n$ -good tree, any vertex  $x$  descendant of  $x_n$  and such that  $|x| \leq n$  verifies  $\nu(x) \leq b$ . Therefore there exists a constant  $c_{23} > 0$  such that  $P_\omega^y(\tau_n \leq 2h_n) \geq c_{23}^{2h_n}$  for any  $y \geq x_n$ ,  $|y| < n$ . By the strong Markov property,

$$\begin{aligned} & P_\omega^{x_n} \left( D(x_n) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - 2\delta \right) \right) \\ & \geq P_\omega^{x_n} \left( D(x_n) > \tau_n, \frac{\tau_n}{n} \geq a - \frac{1}{v} + \delta \right) c_{23}^{2h_n}. \end{aligned}$$

Let  $L := a - \frac{1}{v} + \delta$ . By equation (3.18),

$$P_\omega^{x_n} \left( D(x_n) > \tau_n, \frac{\tau_n}{n} \geq a - \frac{1}{v} + \delta \right) \geq O(n^K)^{-1} \left( \frac{1}{1+\eta} \right)^{Ln}.$$

Hence, by the strong Markov property,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(E_7 | E_5 \cap E_6) &= \liminf_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^u \left( D(u) > \tau_n, \frac{\tau_n}{n} \in \left( a - \frac{1}{v} + \delta, b - \frac{1}{v} - \delta \right) \right) \\ &\geq -L(1+\eta). \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e \left( \frac{\tau_n}{n} \in [a, b], D(e) > \tau_n \right) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(E_5 \cap E_6 \cap E_7) \\ (3.22) \qquad \qquad \qquad &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(E_5 \cap E_6) - L \ln(1+\eta). \end{aligned}$$

Notice that

$$\begin{aligned} E_{\mathbf{Q}} [P_\omega^e(E_5 \cap E_6^c)] &= E_{\mathbf{Q}} [P_\omega^e(E_5) - P_\omega^e(E_5 \cap E_6)] \\ &= \mathbf{Q}(E_5)(1 - \mathbf{Q}(\mathbb{T} \text{ is } n\text{-slow})) \\ &\leq \mathbf{Q}(E_5) \exp(-n^{d+o(1)}), \end{aligned}$$

by equation (3.19). By Markov's inequality,

$$\mathbf{Q}(P_\omega^e(E_5 \cap E_6^c) \geq \frac{1}{n^2}) \leq n^2 \mathbf{Q}(E_5) e^{-n^{d+o(1)}}.$$

The Borel–Cantelli lemma implies that almost surely, for  $n$  large enough,

$$P_\omega^e(E_5 \cap E_6) \geq P_\omega^e(E_5) - \frac{1}{n^2}.$$

We observe that  $P_\omega^e(E_5) \rightarrow P_\omega^e(T_e^- = \infty)$  when  $n$  goes to infinity. Therefore, equation (3.22) becomes

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in [a, b[, D(e) > \tau_n \right) \right) \geq -\left(a - \frac{1}{v} + \delta\right) \ln(1 + \eta).$$

We let  $\eta$  go to 0 to get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( P_\omega^e \left( \frac{\tau_n}{n} \in [a, b[, D(e) > \tau_n \right) \right) = 0$$

which proves (3.20).

### 3.3 Proof of Proposition 3.3

The speed-up case is quite immediate. Indeed, reasoning on the last visit to the root, we have

$$\mathbb{Q}^e(\tau_n \leq bn, D(e) = \infty) \leq \mathbb{Q}^e(\tau_n \leq bn) \leq bn \mathbb{Q}^e(\tau_n \leq bn, D(e) = \infty).$$

Therefore, by Theorem 3.1,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e(\tau_n \leq bn) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e(\tau_n \leq bn \mid D(e) = \infty).$$

It already gives (3.5) since  $I_a$  is strictly decreasing on  $[1, 1/v]$ . We do exactly the same for the quenched inequality. Therefore, let us turn to the slowdown case, beginning with the annealed inequality (3.7). We follow the arguments of [5]. We still write  $i = \text{ess inf } A$ . For technical reasons, we need to distinguish the cases where  $\mathbf{P}(A = i)$  is null or positive. We feel free to deal only with the case  $\mathbf{P}(A = i) = 0$ , the other one following with nearly any change. Moreover, we suppose without loss of generality that  $i > \nu_{\min}^{-1}$ , since the two sides are equal to zero when  $i \leq \nu_{\min}^{-1}$ . Let  $k \geq 1$ . We write  $\ell = k[2]$  to say that  $\ell$  and  $k$  have the same parity. Following [5], we write for  $b > a > 1/v$ ,

$$\begin{aligned} & P_\omega^e(bn > \tau_n \geq an) \\ &= \sum_{\ell=k[2]} \sum_{|x|=k} P_\omega^e(bn > \tau_n \geq an, \tau_n > \ell, X_\ell = x, |X_i| > k, \forall i = \ell + 1 \dots, \tau_n) \\ &= \sum_{\ell=k[2]} \sum_{|x|=k} P_\omega^e(\tau_n > \ell, X_\ell = x) P_\omega^x(bn - \ell > \tau_n > an - \ell, D(x) > \tau_n). \end{aligned}$$

By coupling, we have, for  $p := \nu_{\min} i > 1$ ,

$$\sup_{|x|=k} P_\omega^e(\tau_n > \ell, X_\ell = x) \leq P_\omega^e(|X_\ell| \leq k) \leq P(S_\ell^p \leq k),$$

where  $S_\ell^p$  stands for a reflected biased random walk on the half line, which moves of  $+1$  with probability  $p/1+p$  and of  $-1$  with probability  $1/1+p$ . From (and with the notation of) Lemma 5.2 of [5], we know that for all  $\ell$  of the same parity as  $k$ ,

$$P(S_\ell^p \leq k) \leq c_k(1 + \delta_k)^\ell P(S_\ell^p = k, 1 \leq S_i \leq k-1, i = 1, \dots, \ell-1)$$

where  $c_k < \infty$  and  $\delta = (\delta_k)$  is a sequence independent of all the parameters and tending to zero. In particular, we stress that  $\delta$  do not depend on  $p$ . Hence,  $P_\omega^e(bn > \tau_n \geq an)$  is smaller than

$$c_k(1 + \delta_k)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P(S_\ell^p = k, 1 \leq S_i \leq k-1, i = 1, \dots, \ell-1) W_n(x, \ell)$$

where

$$W_n(x, \ell) := P_\omega^x(bn - \ell > \tau_n \geq an - \ell, D(x) > \tau_n).$$

We deduce that

$$\begin{aligned} P_\omega^e(bn > \tau_n \geq an) &\leq c_k(1 + \delta_k)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell) W_n(x, \ell) \\ (3.23) \quad &= \nu_{\min}^k c_k(1 + \delta_k)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) W_n(x, \ell), \end{aligned}$$

where  $\omega_p$  represents the environment of the biased random walk on the  $\nu_{\min}$ -ary tree such that for any vertex  $x$ ,  $P_{\omega_p}^x(X_1 = x_i) = \frac{p}{\nu_{\min}(1+p)}$  for each child  $x_i$ , and  $P_{\omega_p}^x(X_1 = \bar{x}) = \frac{1}{1+p}$ . Taking the expectations yields that

$$\begin{aligned} \mathbb{Q}^e(bn > \tau_n \geq an) &\leq \nu_{\min}^k c_k(1 + \delta_k)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) E_{\mathbf{Q}}[W_n(x, \ell)]. \\ (3.24) \end{aligned}$$

Moreover, define for any  $|x| = k$ ,

$$\mathcal{S}_{k,\ell}^+(\mathbb{T}, x) = \left\{ \{s_i\}_{i=0}^\ell : |s_{i+1}| - |s_i| = 1, s_0 = 0, k-1 \geq |s_i| > 0, s_\ell = x \right\}$$

the set of paths on  $\mathbb{T}$  which ends at  $x$  in  $\ell$  steps and stays between generation 1 and  $k-1$  before. We notice that, for any environment  $\omega$ ,

$$(3.25) \quad P_\omega^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) = \sum_{\{s\} \in \mathcal{S}_{k,\ell}^+(\mathbb{T}, x)} \sum_{y \in \mathbb{T}} \omega(y, \overleftarrow{y})^{N(y, \overleftarrow{y})} \sum_{i=1}^{\nu(y)} \omega(y, y_i)^{N(y, y_i)}$$

where for each path  $\{s_i\}$ ,  $N(z, y)$  stands for the number of passage from  $z$  to  $y$ . Let  $\varepsilon > 0$ , and  $\mathcal{G}_k$  denote for any  $k$  the set of trees such that any vertex  $x$  of generation less than  $k$  verifies  $\nu(x) = \nu_{\min}$  and  $A(x) \leq \text{ess inf } A + \varepsilon$ . Let  $p' := \nu_{\min}(\text{ess inf } A + \varepsilon)$ . We observe that

$$P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) = \sum_{\{s\} \in \mathcal{S}_{k,\ell}^+(\mathbb{T}, x)} \sum_{y \in \mathbb{T}} \left( \frac{1}{1+p} \right)^{N(y, \bar{y})} \sum_{i=1}^{\nu(y)} \left( \frac{p}{\nu_{\min}(1+p)} \right)^{N(y, y_i)}$$

Therefore, if  $\mathbb{T}$  belongs to  $\mathcal{G}_k$ , we have by equation (3.25),

$$P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) \leq \left( \frac{1+p'}{1+p} \right)^\ell P_{\omega}^e(\tau_k = \ell, D(e) > \ell, X_\ell = k).$$

It entails that

$$\begin{aligned} & \mathbb{I}_{\{\mathbb{T} \in \mathcal{G}_k\}} \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega_p}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) W_n(x, \ell) \\ & \leq \mathbb{I}_{\{\mathbb{T} \in \mathcal{G}_k\}} \left( \frac{1+p'}{1+p} \right)^{bn} \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega}^e(\tau_k = \ell, D(e) > \ell, X_\ell = x) W_n(x, \ell) \\ & = \mathbb{I}_{\{\mathbb{T} \in \mathcal{G}_k\}} \left( \frac{1+p'}{1+p} \right)^{bn} P_{\omega}^e(bn > \tau_n \geq an, D(e) > \tau_n) \\ (3.26) \quad & \leq \left( \frac{1+p'}{1+p} \right)^{bn} P_{\omega}^e(bn > \tau_n \geq an, D(e) > \tau_n). \end{aligned}$$

Taking expectations gives

$$\begin{aligned} & \mathbf{Q}(\mathbb{T} \in \mathcal{G}_k) \sum_{\ell=k[2]} \sum_{|x|=k} P_{\omega_p}^e(\tau_k = \ell, X_\ell = x) E_{\mathbf{Q}}[W_n(x, \ell)] \\ (3.27) \quad & \leq \left( \frac{1+p'}{1+p} \right)^{bn} \mathbb{Q}^e(bn > \tau_n \geq an, D(e) > \tau_n). \end{aligned}$$

As before,

$$\begin{aligned} & \mathbb{Q}^e(bn > \tau_n \geq an, D(e) = \infty) + \mathbb{Q}^e(bn > \tau_n \geq an, \infty > D(e) > \tau_n) \\ & = \mathbb{Q}^e(bn > \tau_n \geq an, D(e) > \tau_n) \\ & \geq \mathbb{Q}^e(bn > \tau_n \geq an, D(e) = \infty). \end{aligned}$$

Since  $\mathbb{Q}^e(bn > \tau_n \geq an, \infty > D(e) > \tau_n) \leq \mathbb{Q}^e(bn > \tau_n \geq an, D(e) > \tau_n) E_{\mathbf{Q}}[1 - \beta]$ , we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an, D(e) > \tau_n) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an \mid D(e) = \infty).$$

Consequently, we have by (3.24) and (3.27)

$$\limsup_{n \rightarrow \infty} \mathbb{Q}^e(bn > \tau_n \geq an) \leq b \ln \left( \frac{1+p'}{1+p} (1 + \delta_k) \right) + \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{Q}^e(bn > \tau_n \geq an \mid D(e) = \infty).$$

Since  $\mathbb{Q}^e(cn > \tau_n > bn) \geq \mathbb{Q}^e(cn > \tau_n > bn, D(e) = \infty)$ , we prove equation (3.7) by taking  $p'$  arbitrarily close to  $p$ , and letting  $k$  tend to infinity.

We prove now the quenched equality (3.8). For any environment  $\omega$ , construct the environment  $f_p(\omega)$  by setting  $A(x) = i$  ( $:= \text{ess inf } A$ ) for any  $|x| \leq k$ . We construct also for  $p' > p$ , an environment  $f_{p'}(\omega)$  by picking independently  $A(x)$  in  $[i, p'/\nu_{\min}]$  for any  $x \leq k$ , such that  $A(x)$  has the distribution of  $A$  conditioned on  $A \in [i, p'/\nu_{\min}]$ . By equation (3.23), we have almost surely

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(bn > \tau_n \geq an) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} P_{f_p(\omega)}^e(bn > \tau_n \geq an, D(e) > \tau_n) + b \ln(1 + \delta_k).$$

Equation (3.26) applied to the environment  $f_{p'}(\omega)$ , together with Theorem 3.2 shows that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln P_{f_{p'}(\omega)}^e(bn > \tau_n \geq an, D(e) > \tau_n) \leq -I_q(b) + b \ln \frac{1+p'}{1+p}.$$

Let  $p'$  tend to  $p$  to get that almost surely,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln P_{f_p(\omega)}^e(bn > \tau_n \geq an, D(e) > \tau_n) \leq -I_q(b).$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(bn > \tau_n \geq an) \leq -I_q(b) + b \ln(1 + \delta_k).$$

When  $k$  goes to infinity, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln P_\omega^e(bn > \tau_n > an) \leq -I_q(b),$$

which gives equation (3.8).

### 3.4 Proof of Proposition 1.3

Recall that, for any  $\theta \in \mathbb{R}$ ,

$$\psi(\theta) := \ln \left( E_{\mathbf{Q}} \left[ \sum_{i=1}^{\nu(e)} \omega(e, e_i)^\theta \right] \right).$$

Obviously, for any  $n \in \mathbb{N}$ ,

$$\frac{1}{n} \ln (\mathbb{Q}^e (\tau_n = n)) = \ln \left( E_{\mathbf{Q}} \left[ \sum_{i=1}^{\nu(e)} \omega(e, e_i) \right] \right) = \psi(1).$$

This proves (1.8). For the quenched case, we have that

$$P_{\omega}^e (\tau_n = n) = \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1}),$$

where  $x_k$  is the ancestor of the vertex  $x$  at generation  $k$ . We observe that we are reduced to the study of a generalized multiplicative cascade, as studied in [10]. The following lemma is well-known in the case of a regular tree (see [7] and [4]). We extend it easily to a Galton–Watson tree.

**Lemma 3.6** *We have  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln (\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})) = \inf_{]0,1[} \frac{1}{\theta} \psi(\theta)$ .*

*Proof.* When  $\psi'(1) < \psi(1)$ , Biggins [3] shows that  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln (\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})) = \psi(1) = \inf_{]0,1[} \frac{1}{\theta} \psi(\theta)$ . Therefore let us assume that  $\psi'(1) \geq \psi(1)$ . By the argument of [7], we obtain,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \left( \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1}) \right) \geq \inf_{]0,1[} \frac{1}{\theta} \psi(\theta).$$

Finally, let  $\theta \in ]0, \theta_c[$  where  $\psi(\theta_c) = \inf_{]0,1[} \frac{1}{\theta} \psi(\theta)$ . Since  $(\sum_i a_i)^\theta \leq \sum_i a_i^\theta$  for any  $(a_i)_i$  with  $a_i \geq 0$ , it yields that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \left( \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1}) \right) \leq \frac{1}{\theta} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \left( \sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})^\theta \right).$$

We see that (still by [3])  $\lim_{n \rightarrow \infty} \frac{1}{n} \ln (\sum_{|x|=n} \prod_{k=0}^{n-1} \omega(x_k, x_{k+1})^\theta) = \psi(\theta)$ . It remains to let  $\theta$  tend to  $\theta_c$ .  $\square$

## 4 The subexponential regime : Theorem 1.4

We prove (1.10) and (1.11) separately. We recall that the speed  $v$  of the walk verifies  $v = \frac{E_{S^e} [ |X_{\Gamma_1}| ]}{E_{S^e} [ \Gamma_1 ]}$ .

*Proof of Theorem 1.4 : equation (1.10).* Suppose that either “ $i < \nu_{\min}^{-1}$  and  $q_1 = 0$ ” or “ $i < \nu_{\min}^{-1}$  and  $s < 1$ ”. Let  $a > 1/v$  and  $c_{24} > 0$  such that  $c_{24} < (E_{\mathbb{S}^e} [X_{\Gamma_1}])^{-1}$ . We have

$$\mathbb{S}^e (\tau_n \geq an) \geq \mathbb{S}^e (\Gamma_{nc_{24}} \geq an) - \mathbb{S}^e (\Gamma_{nc_{24}} > \tau_n) .$$

The second term on the right-hand side decays exponentially by Cramér’s Theorem applied to the random walk  $(|X_{\Gamma_n}|, n \geq 0)$  (recall that  $|X_{\Gamma_1}|$  has exponential moments by Fact A). The simple inequality  $\mathbb{S}^e (\Gamma_{nc_{24}} \geq an) \geq \mathbb{S}^e (\Gamma_1 \geq an)$  thus implies by Proposition 2.2 the lower bound of (1.10). Hence, we turn to the upper bound of (1.10). Part (i) of Lemma 6.3 of [5] states:

**Lemma A (Dembo et al. [5])** *Let  $Y_1, Y_2, \dots$  be an i.i.d. sequence with  $E(Y_1^2) < \infty$ . If  $P(Y_1 \geq x) \leq \exp(-cx^\gamma)$  for some  $0 < \gamma < 1$ ,  $c > 0$  and all  $x$  large enough, then for all  $t > E[Y_1]$ ,*

$$\limsup_{n \rightarrow \infty} n^{-\gamma} \ln P \left( \frac{1}{n} \sum_{j=1}^n Y_j \geq t \right) \leq -c(t - E[Y_1])^\gamma .$$

By Proposition 2.2,  $Y_1 = \Gamma_1$  meets the conditions of the lemma. Therefore, take in lemma A,  $Y_i = \Gamma_i - \Gamma_{i-1}$  and  $t = a/c_{25}$  where  $c_{25}$  is such that

$$(E_{\mathbb{S}^e} [|X_{\Gamma_1}|])^{-1} < c_{25} < a (E_{\mathbb{S}^e} [\Gamma_1])^{-1} .$$

In particular, we have  $t > E_{\mathbb{S}^e} [\Gamma_1]$ . As a result,  $\mathbb{S}^e (\Gamma_n > tn)$  is stretched exponential. We also know that  $\mathbb{S}^e (|X_{\Gamma_{nc_{25}}}| \leq n)$  is exponentially small by Cramér’s Theorem ( $1/c_{25} < E_{\mathbb{S}^e} [|X_{\Gamma_1}|]$ ). The relation  $\mathbb{S}^e (\tau_n \geq an) \leq \mathbb{S}^e (\Gamma_{nc_{25}} \geq an) + \mathbb{S}^e (|X_{\Gamma_{nc_{25}}}| \leq n)$  thus completes the proof.  $\square$

We finish with the case “ $\Lambda < \infty$ ”.

*Proof of Theorem 1.4 : equation (1.11).* Suppose that  $\Lambda < \infty$  and let  $a$ ,  $c_{24}$  and  $c_{25}$  be as before. We write

$$\begin{aligned} \mathbb{S}^e (\Gamma_{nc_{24}} \geq an) &\geq \sum_{k=1}^{nc_{24}} \mathbb{S}^e (\{\Gamma_k - \Gamma_{k-1} \geq an\} \cap \{\Gamma_\ell - \Gamma_{\ell-1} < an, \forall \ell \neq k\}) \\ &= nc_{24} \mathbb{S}^e (\Gamma_1 \geq an) \mathbb{S}^e (\Gamma_1 < an)^{nc_{24}-1} . \end{aligned}$$

By Proposition 2.4,  $\mathbb{S}^e (\Gamma_1 \geq an) = n^{-\Lambda+o(1)}$ . Therefore  $\mathbb{S}^e (\Gamma_1 < an)^{nc_{24}-1}$  tends to 1 (since  $\Lambda > 1$ ). Consequently,

$$\mathbb{S}^e (\Gamma_{nc_{24}} \geq an) \geq n^{1-\Lambda+o(1)} ,$$

which gives the lower bound of (1.11), by the inequality  $\mathbb{S}^e(\tau_n \geq an) \geq \mathbb{S}^e(\Gamma_{nc_{24}} \geq an) - \mathbb{S}^e(\Gamma_{nc_{24}} > \tau_n)$ . Turning, to the upper bound, write as before  $\mathbb{S}^e(\tau_n \geq an) \leq \mathbb{S}^e(\Gamma_{nc_{25}} \geq an) + \mathbb{S}^e(|X_{\Gamma_{nc_{25}}}| \leq n)$ . We already know that  $\mathbb{S}^e(|X_{\Gamma_{nc_{25}}}| \leq n)$  is exponentially small. Let  $H_n := \Gamma_n - E_{\mathbb{S}^e}[\Gamma_1]n$ . When  $E[H_1^p] < \infty$ , example 2.6.5 of [15] says that if  $p \geq 2$ ,

$$P(H_n > x) \leq (1 + 2/p)^p nE[H_1^p]x^{-p} + \exp(-2(p+2)^{-2}e^{-p}x^2/(nE[H_1^2]))$$

and example 2.6.20 of [15], combined with Chebyshev's inequality, shows that if  $1 \leq p \leq 2$ ,

$$P(H_n > x) \leq (2 - 1/n) nE[H_1^p]x^{-p}.$$

By Proposition 2.4,  $E[H_1^p] < \infty$ , for any  $p < \Lambda$ . We take  $x = (\frac{a}{c_{25}}E_{\mathbb{S}^e}[|X_{\Gamma_1}|] - E_{\mathbb{S}^e}[\Gamma_1])n$  to see that  $\mathbb{S}^e(\Gamma_{nc_{25}} \geq an) \leq c(p)n^{1-p}$  for any  $p < \Lambda$ . Let  $p$  tend to  $\Lambda$  in order to complete the proof of equation (1.11).  $\square$

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