



HAL
open science

A local optimal diastolic inequality on the two-sphere

Florent Balacheff

► **To cite this version:**

Florent Balacheff. A local optimal diastolic inequality on the two-sphere. *Journal of Topology and Analysis*, 2010, 2 (1), pp.109-121. hal-00336367

HAL Id: hal-00336367

<https://hal.science/hal-00336367>

Submitted on 3 Nov 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A LOCAL OPTIMAL DIASTOLIC INEQUALITY ON THE TWO-SPHERE

FLORENT BALACHEFF

ABSTRACT. Using a ramified cover of the two-sphere by the torus, we prove a local optimal inequality between the diastole and the area on the two-sphere near a singular metric. This singular metric, made of two equilateral triangles glued along their boundary, has been conjectured by E. Calabi to achieve the best ratio area over the square of the length of a shortest closed geodesic. Our diastolic inequality asserts that this conjecture is to some extent locally true.

1. INTRODUCTION

Let g be a smooth Riemannian metric on the two-sphere S^2 . Following I. Babenko [Bab97] we call *general systole*, or simply *systole* in this context, the least length of a non trivial closed geodesic and denote it by sys_0 . For every Riemannian metric g on S^2 ,

$$\text{area}(S^2, g) \geq \frac{1}{32} \text{sys}_0(S^2, g)^2. \quad (1.1)$$

Such an inequality was first proved by C. Croke (see [Cr88]), and later improved in [NR02], [Sa04] and [Ro05]. A natural question is to ask for the best constant in inequality (1.1). It amounts to find the global infimum of the *systolic area* defined as the ratio $\text{area}/\text{sys}_0^2$ among smooth metrics. A flat metric with three conical singularities was conjectured by E. Calabi (see [Cr88]) to be the global minimum. This metric, denoted by g_c , is made of two equilateral triangles of size 1 glued along their boundary. Its systolic area is equal to $\frac{1}{2\sqrt{3}}$.

We may wonder why such a metric has any chance to achieve the minimal systolic area. In order to answer and then give (if needed) strength to Calabi's systolic conjecture we ask the following question : *Is such a conjecture locally true ?*

Observe that the systolic area is not a continuous function of the metric, even when the space of smooth metrics is topologized by the strong topology (for a definition of strong topology, see [Hi76, p. 34]). In order to prove local results we introduce a quantity that coincides with the systole at g_c and that varies continuously with the metric g . This quantity is the *diastole* (see [BS07]). The diastole of a Riemannian closed surface is defined as the value obtained by a minimax process over the space of one-cycles, see section 2.1.

2000 *Mathematics Subject Classification.* 53C23.

Key words and phrases. Conical singularity, diastole, sphere, systole.

Funded by grant 20-118014/1 of the Swiss SNF.

It is denoted by dias . In the case of a smooth metric the diastole is realized as the length of an union of disjoint closed geodesics (see [Pi74, p. 468]). Observe that the notion of diastole holds for any Riemannian closed surface with a finite number of conical singularities. It is proved in [BS07] that for every Riemannian closed surface (M, g) of genus k the diastole bounds from below the area:

$$\text{area}(M, g) \geq \frac{C}{k+1} \text{dias}(M, g)^2.$$

Here C is a positive constant. In the case of a two-sphere and since $\text{sys}_0 \leq \text{dias}$ for smooth metrics we thus (re)obtain inequality (1.1) albeit with a worse constant. We call the ratio $\text{area}/\text{dias}^2$ the *diastolic area* and observe that it is continuous on the space of smooth metrics with a finite number of conical singularities for the uniform topology on metric spaces (see [AZ67, chapter 7] for a definition of uniform topology). The infimum of the diastolic area is not known for any surface, and we do not know if in the case of the two-sphere it is realized by g_c and so equals $1/(2\sqrt{3})$. It is proved in [Bal06] that the round metric g_0 is a critical point of the diastolic area over the space of smooth metrics conformal to g_0 topologized by the strong topology. As

$$\frac{\text{area}(S^2, g_0)}{\text{dias}(S^2, g_0)^2} = \frac{1}{\pi},$$

the round metric is not the global infimum. So its criticality is very interesting and suggests that it may be a local minimum.

In this paper we prove that the metric g_c is a local minimum of the diastolic area over the space of singular metrics C^1 -conformal to g_c with respect to the C^1 -topology. More precisely, let

$$\mathcal{M}_{g_c} := \{e^{2u} \cdot g_c \mid u \in C^1(S^2, \mathbb{R})\}$$

be the space of metrics of class C^1 with three conical singularities of angle $2\pi/3$ on S^2 conformal to g_c . The space \mathcal{M}_{g_c} is naturally in bijection with $C^1(S^2, \mathbb{R}_+^*)$ and we will call C^1 -topology on \mathcal{M}_{g_c} the topology induced by the C^1 compact-open topology on $C^1(S^2, \mathbb{R}_+^*)$ (see [Hi76, p. 34]).

The space \mathcal{M}_{g_c} is adapted to the local study of the diastolic area near the metric g_c . In fact we will observe in subsection (2.2) that, thanks to the Riemann uniformization theorem, any smooth metric can be written (up to an isometry) as a metric which differs from g_c by a conformal factor which lies in $C^1(S^2, \mathbb{R}_+^*)$. Especially finding the infimum of the diastolic area over \mathcal{M}_{g_c} amounts to find the infimum of the diastolic area over the space of smooth metrics, see proposition (2.2).

Theorem 1.1. *There exists an open neighborhood \mathcal{O} of g_c in \mathcal{M}_{g_c} with respect to the C^1 -topology such that for all $g \in \mathcal{O}$,*

$$\text{area}(S^2, g) \geq \frac{1}{2\sqrt{3}} \text{dias}(S^2, g)^2$$

with equality if and only if $g = g_c$.

The proof is based on the study of a degree 3 ramified cover of S^2 by the two-torus \mathbb{T}^2 .

2. PRELIMINARIES AND DEFINITIONS

2.1. Isodiatolic inequality for closed surfaces. Recall from [BS07] that the *diastole* of a closed Riemannian surface (M, g) of class C^1 with a finite number of conical singularities is defined by a minimax process over the space of one-cycles and is denoted by $\text{dias}(M, g)$.

More precisely, we denote by $\mathcal{Z}_1(M; \mathbb{Z}_*)$ the space of one-cycles of M , see [BS07] for a precise definition. Here \mathbb{Z}_* denotes \mathbb{Z} in the orientable case, and \mathbb{Z}_2 in the non orientable one. An isomorphism due to F. Almgren [Al60] between $\pi_1(\mathcal{Z}_1(M; \mathbb{Z}_*), \{0\})$ and $H_2(M; \mathbb{Z}_*) \simeq \mathbb{Z}_*$ permits us to define the diastole over the one-cycle space as

$$\text{dias}(M, g) := \inf_{(z_t)} \sup_{0 \leq t \leq 1} \text{Mass}(z_t)$$

where (z_t) runs over the families of one-cycles inducing a generator of $\pi_1(\mathcal{Z}_1(M; \mathbb{Z}_*), \{0\})$ and $\text{Mass}(z_t)$ represents the mass (or length) of z_t . From a result of J. Pitts [Pi74, p. 468] (see also [CC92]), this minimax principle gives rise to a union of closed geodesics (counted with multiplicity) of total length $\text{dias}(M)$ when the metric g is smooth without conical singularities.

Recall the following result (see [BS07]).

Theorem 2.1. *There exists a constant C such that every closed surface M of genus k endowed with a Riemannian smooth metric g satisfies*

$$\text{area}(M, g) \geq \frac{C}{k+1} \text{dias}(M, g)^2. \tag{2.1}$$

The constant C can be taken equal to 10^{-16} .

The best constant involved in inequality (2.1) is not known for any closed surface. Nevertheless the asymptotic behaviour of the above constant for large genus can not be improved, see [BS07]. The inequality (2.1) remains valid for Riemannian metric of class C^1 with a finite number of conical singularities, as these metrics can be obtained as uniform limit (as metric spaces) of Riemannian smooth metrics and both the area and the diastole are continuous for this topology (see [AZ67, p.224 & 269]).

2.2. The singular metric and the Riemann uniformization theorem.

By the Riemann uniformization theorem there exists only one conformal structure on S^2 (up to diffeomorphism). So we set $S^2 = \mathbb{C} \cup \{\infty\}$. Following [Tro86] (see also [Re01]), g_c can be written as

$$(|z - a_1| \cdot |z - a_2| \cdot |z - a_3|)^{-4/3} |dz|^2$$

up to a scale transformation where (a_1, a_2, a_3) is any triple of pairwise distinct points of S^2 . We fix in the sequel the triple to be $(-1, 0, 1)$.

Let

$$\mathcal{M}_{g_c} := \{e^{2u} \cdot g_c \mid u \in C^1(S^2, \mathbb{R})\}$$

be the space of metrics of class C^1 with three conical singularities of angle $2\pi/3$ on S^2 conformal to g_c .

Proposition 2.2. *The infimum of the diastolic area over \mathcal{M}_{g_c} equals the infimum of the diastolic area over the space of smooth metrics.*

Proof. By the Riemann uniformization theorem, every smooth Riemannian metric g is isometric to a metric of the type $e^{2v} \cdot g_0$ where g_0 denotes the round metric and $v \in C^\infty(S^2, \mathbb{R})$. As

$$g_0 = \left(\frac{2}{1 + |z|^2} \right)^2 |dz|^2,$$

we see that g is homothetic (up to diffeomorphism) to the metric

$$e^{2v} \cdot \left(\frac{2}{1 + |z|^2} \right)^2 \cdot (|z + 1| \cdot |z| \cdot |z - 1|)^{4/3} \cdot g_c$$

which differs from g_c by a conformal factor which lies in $C^1(S^2, \mathbb{R}_+)$. So we can approximate the smooth Riemannian metric g by elements of \mathcal{M}_{g_c} with respect to the C^1 -topology. As the diastolic area is continuous with respect to this topology (the diastolic area is in fact continuous for the uniform topology on metric spaces), we get that the infimum of the diastolic area over \mathcal{M}_{g_c} is less than the infimum of the diastolic area over the space of smooth metrics. The reverse inequality is obtained by approximating elements of \mathcal{M}_{g_c} by Riemannian smooth metrics for the uniform topology on metric spaces. \square

2.3. The ramified cover and the isosystolic inequality on the torus.

Let

$$f : \mathbb{T}^2 \rightarrow S^2$$

be the covering map of degree 3 ramified over the three points $\{-1, 0, 1\}$. If g is in \mathcal{M}_{g_c} , we denote by \tilde{g} the pull-back metric of g by the map f . The metric \tilde{g} is C^1 everywhere. In fact, near the ramified points, the map looks like the map $z \mapsto z^3$. So each conical singularity of angle $2\pi/3$ lift to a conical singularity of angle 2π . Observe that

$$\text{area}(\mathbb{T}^2, \tilde{g}) = 3 \cdot \text{area}(S^2, g).$$

The flat metric \tilde{g}_c given by the hexagonal lattice and appearing as the pullback metric f^*g_c is very special. First recall that for a closed Riemannian surface the *homotopy systole* is defined as the least length of a non-contractible closed curve and is denoted by sys_π . Loewner's theorem (unpublished, see [Pu52]) states the first known isosystolic inequality:

Theorem 2.3. *For all metric g of class C^1 on \mathbb{T}^2 ,*

$$\text{area}(\mathbb{T}^2, g) \geq \frac{\sqrt{3}}{2} \text{sys}_\pi(\mathbb{T}^2, g)^2,$$

with equality if and only if (\mathbb{T}^2, g) is homothetic to the flat torus corresponding to the hexagonal lattice $(\mathbb{T}^2, \tilde{g}_c)$.

The study of such isosystolic inequalities is well developed, see [Be00, p.104] for instance.

The systolic geometries of (S^2, g_c) and $(\mathbb{T}^2, \tilde{g}_c)$ are intimately related : to each systole of $(\mathbb{T}^2, \tilde{g}_c)$ (that is a closed geodesic realizing the homotopy systole) corresponds a geodesic loop of (S^2, g_c) whose length equals the general systole (see Figure 1). More precisely,

- Either the systole of $(\mathbb{T}^2, \tilde{g}_c)$ passes through a point of ramification, and its image by f is a simple geodesic loop with base point a conical singularity,
- Or the systole of $(\mathbb{T}^2, \tilde{g}_c)$ never goes through a point of ramification, and its image is a figure eight geodesic avoiding conical singularities.

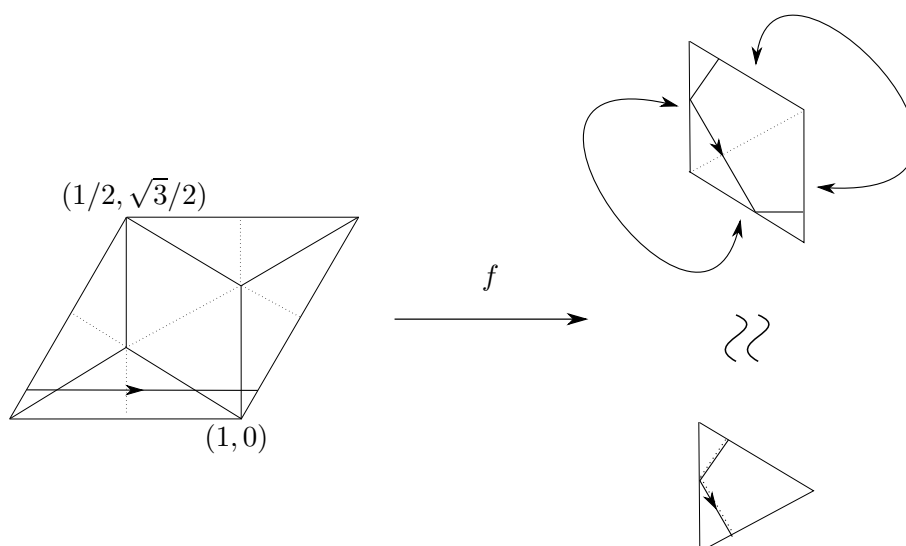


FIGURE 1. Projection of the systoles

The systoles of $(\mathbb{T}^2, \tilde{g}_c)$ can be classified into three one-parameter families. If we think at $(\mathbb{T}^2, \tilde{g}_c)$ as the quotient of \mathbb{R}^2 by the hexagonal lattice spanned by the vectors $(1, 0)$ and $(1/2, \sqrt{3}/2)$, the horizontal lines, the lines parallel to the vector $(1/2, \sqrt{3}/2)$ and the lines parallel to the vector $(-1/2, \sqrt{3}/2)$ correspond to these three families of systoles. Observe now that each of these families projects onto the same one-parameter family of geodesic loops of (S^2, g_c) . We denote by $\gamma_s(t) = f(t, s)$ this family where $(t, s) \in [0, 1] \times [0, \sqrt{3}/2]$.

3. PROOF OF THE LOCAL DIASTOLIC INEQUALITY

Let $g = e^{2u} \cdot g_c$ be a metric in \mathcal{M}_{g_c} where $u \in C^1(S^2, \mathbb{R})$. By application of the Cauchy-Schwarz's inequality and as $dv_{\tilde{g}_c} = dt ds$,

$$\begin{aligned} \int_0^{\sqrt{3}/2} l_g(\gamma_s(t)) ds &= \int_0^{\sqrt{3}/2} \left(\int_0^1 e^{u \circ f(t,s)} dt \right) ds \\ &= \int_{\mathbb{T}^2} e^{u \circ f} dv_{\tilde{g}_c} \\ &= 3 \cdot \int_{S^2} e^u dv_{g_c} \\ &\leq 3 \left(\int_{S^2} e^{2u} dv_{g_c} \right)^{1/2} \left(\int_{S^2} dv_{g_c} \right)^{1/2} \\ &\leq 3 \left(\text{area}(S^2, g) \right)^{1/2} \left(\text{area}(S^2, g_c) \right)^{1/2}. \end{aligned}$$

So

$$\frac{\sqrt{3}}{2} \inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s) \leq 3 \left(\text{area}(S^2, g) \right)^{1/2} \left(\text{area}(S^2, g_c) \right)^{1/2},$$

which gives that

$$\left(\inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s) \right)^2 \leq 2\sqrt{3} \cdot \text{area}(S^2, g)$$

as $\text{area}(S^2, g_c) = \frac{1}{2\sqrt{3}}$.

Lemma 3.1. *There exists an open neighborhood \mathcal{O} of g_c in \mathcal{M}_{g_c} such that for all $g \in \mathcal{O}$,*

$$\text{dias}(S^2, g) \leq \inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s).$$

Proof of the Lemma. To each geodesic loop γ_s of (S^2, g_c) we associate a one-parameter family of one-cycles $\{z_s^\alpha\}$ where $\alpha \in [0, 1]$ such that

- $z_s^{1/2} = \gamma_s$,
- $\{z_s^\alpha\}$ starts and ends at one-cycles made of one or two points,
- $\{z_s^\alpha\}$ induces a generator of $\pi_1(\mathcal{Z}_1(S^2; \mathbb{Z}), \{0\}) \simeq \mathbb{Z}$,
- each z_s^α is made of one or two closed curves and have length

$$l_{g_c}(z_s^\alpha) = 1 - 2 \cdot |\alpha - 1/2|$$

where $|\cdot|$ denote the absolute value.

For this we consider two cases.

First case. If $s = 0, \frac{1}{2\sqrt{3}}$ or $\frac{1}{\sqrt{3}}$, then the geodesic loop γ_s goes through a single singularity. Thus γ_s bounds two disks D_1 and D_2 each one containing in its interior a conical singularity. For $i = 1, 2$ we homotope γ_s in D_i to the singularity lying in its interior through a C^0 -family $\{z_{i,s}^\beta\}_{\beta \in [0,1]}$ of geodesic loops (whose base point lies on the edge of the triangle) of length

$$l_{g_c}(z_{i,s}^\beta) = 1 - \beta.$$

Then we set

$$z_s^\alpha = \begin{cases} z_{1,s}^{1-2\alpha} & \text{if } \alpha \in [0, \frac{1}{2}] \\ z_{2,s}^{2\alpha-1} & \text{if } \alpha \in [\frac{1}{2}, 1] \end{cases},$$

see Figure 2, First case.

Second case. If $s = \frac{k}{2\sqrt{3}} + s'$ where $k = 0, 1, 2$ and $s' \in]0, \frac{1}{2\sqrt{3}}[$, then γ_s consists of a figure eight geodesic which avoids singularities. So γ_s decomposes into the concatenation of two simple closed geodesic loops $\gamma_{1,s}$ and $\gamma_{2,s}$. Let D_i be the disk bounding by $\gamma_{i,s}$ and containing a single singularity for $i = 1, 2$. Each $\gamma_{i,s}$ can be homotoped in D_i to the singularity lying in its interior through a family $\{z_{i,s}^\beta\}_{\beta \in [0,1]}$ of geodesic loops (whose base point lies on the edge of the triangle) of length

$$l_{g_c}(z_{i,s}^\beta) = (1 - \beta)l_{g_c}(\gamma_{i,s}).$$

In a similar way, the piecewise geodesic $\tilde{\gamma}_s$ obtained by the concatenation of $\gamma_{1,s}$ and $\gamma_{2,s}^{-1}$ bounds an open disk containing a single singularity in its interior, and we denote by D its closure. Again $\tilde{\gamma}_s$ can be homotoped in D to the singularity lying in the interior of D through a family $\{z_s^\beta\}_{\beta \in [0,1]}$ of piecewise geodesics of length

$$l_{g_c}(z_s^\beta) = (1 - \beta).$$

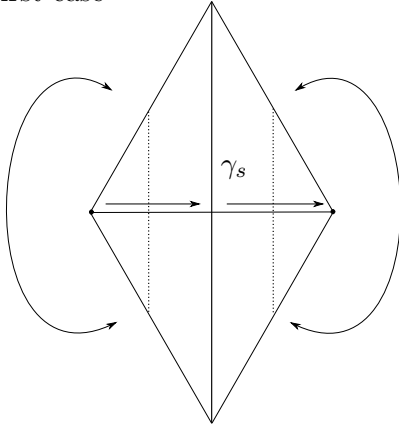
Each piecewise geodesic z_s^β consists of two geodesic arcs $z_{1,s}^\beta$ and $z_{2,s}^\beta$.

Then we set

$$z_s^\alpha = \begin{cases} z_{1,s}^{1-2\alpha} + z_{2,s}^{1-2\alpha} & \text{if } \alpha \in [0, \frac{1}{2}] \\ z_s^{2\alpha-1} & \text{if } \alpha \in [\frac{1}{2}, 1] \end{cases},$$

see Figure 2, Second case.

First case



Second case

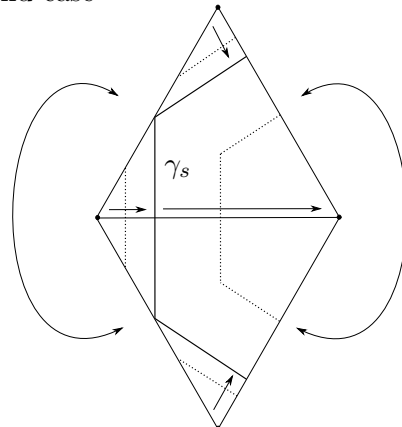


FIGURE 2. Families of one-cycles.

Denote by $\|\cdot\|_\infty$ the uniform norm on $C^1(S^2, \mathbb{R})$. For each $\alpha \in [0, 1/2[\cup]1/2, 1]$, the family of 1-cycles

$$\{z_s^{t/2+(1-t)\alpha} \mid t \in [0, 1]\}$$

covers a domain Ω_s^α whose boundary is the reunion of γ_s and z_s^α . We can lift this domain to a domain $\tilde{\Omega}_s^\alpha$ of \mathbb{T}^2 as in Figure 3 which decomposes into two domains $\tilde{\Omega}_{1,s}^\alpha$ and $\tilde{\Omega}_{2,s}^\alpha$. One of this domain is reduced to a point if $s = 0, 1/(2\sqrt{3})$ or $1/\sqrt{3}$. Each domain $\tilde{\Omega}_{i,s}^\alpha$ is bounded by three curves: $\tilde{\gamma}_{i,s}$, $\tilde{z}_{i,s}^\alpha$ and $\tilde{c}_{i,s}$. Note that $\tilde{c}_{i,s}$ is composed of two connected components. The curve $\tilde{c}_{i,s}$ is oriented such that the concatenation of the first connected component of $\tilde{c}_{i,s}$, $\tilde{z}_{i,s}^\alpha$ and the second component of $\tilde{c}_{i,s}$ makes sense and is an arc homotopic to $\tilde{\gamma}_{i,s}$. We endow $\tilde{\Omega}_{i,s}^\alpha$ with the orientation such that its boundary is the concatenation of $\tilde{\gamma}_{i,s}$, the first connected component of $-\tilde{c}_{i,s}$, $-\tilde{z}_{i,s}^\alpha$ and the second component of $-\tilde{c}_{i,s}$. We can easily compute that

$$\text{area}(\tilde{\Omega}_{i,s}^\alpha, \tilde{g}_c) = \frac{1}{4} \tan \frac{\pi}{6} (l_{g_c}(\gamma_{i,s})^2 - l_{g_c}(z_{i,s}^\alpha)^2).$$

Observe that

$$\text{area}(\tilde{\Omega}_{i,s}^\alpha, \tilde{g}_c) \leq \frac{1}{2} \tan \frac{\pi}{6} (l_{g_c}(\gamma_{i,s}) - l_{g_c}(z_{i,s}^\alpha))$$

as $l_{g_c}(\gamma_{i,s}) + l_{g_c}(z_{i,s}^\alpha) \leq 2$.

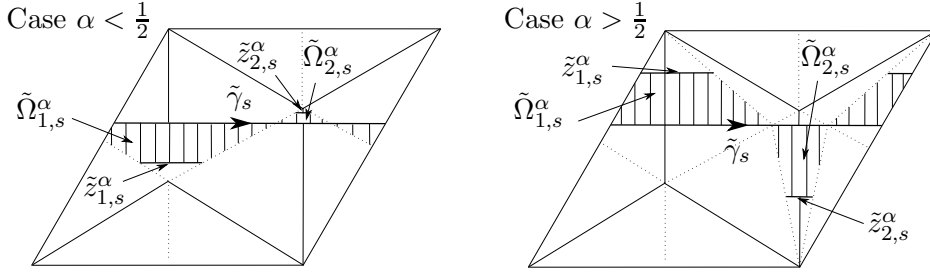


FIGURE 3. Domains.

For all $g = e^{2u} \cdot g_c \in \mathcal{M}_{g_c}$ and by Stokes' Theorem,

$$\begin{aligned} l_g(\gamma_{i,s}) - l_g(z_{i,s}^\alpha) &= \int_{\tilde{\gamma}_{i,s}} e^{u \circ f} dt - \int_{\tilde{z}_{i,s}^\alpha} e^{u \circ f} dt \\ &= \int_{\tilde{c}_{i,s}} e^{u \circ f} dt + \int_{\tilde{\Omega}_{i,s}^\alpha} de^{u \circ f} \wedge dt \\ &= \int_{\tilde{c}_{i,s}} dt + \int_{\tilde{c}_{i,s}} (e^{u \circ f} - 1) dt + \int_{\tilde{\Omega}_{i,s}^\alpha} \frac{\partial}{\partial s} e^{u \circ f} ds \wedge dt \\ &\geq \int_{\tilde{c}_{i,s}} dt - \int_{\tilde{c}_{i,s}} \|e^u - 1\|_\infty dt - \sup_{\tilde{\Omega}_{i,s}^\alpha} \left| \frac{\partial}{\partial s} e^{u \circ f} \right| \cdot \text{area}(\tilde{\Omega}_{i,s}^\alpha, \tilde{g}_c) \\ &\geq (1 - \|e^u - 1\|_\infty - \frac{1}{2} \cdot \tan \frac{\pi}{6} \cdot \sup_{\tilde{\Omega}_{i,s}^\alpha} \left| \frac{\partial}{\partial s} e^{u \circ f} \right|) (l_{g_c}(\gamma_{i,s}) - l_{g_c}(z_{i,s}^\alpha)) \end{aligned}$$

Observe that there exists a positive constant B such that

$$\sup_{\tilde{\Omega}_{i,s}^\alpha} \left| \frac{\partial}{\partial s} e^{u \circ f} \right| \leq B \cdot \|de^u\|_0$$

where $\|de^u\|_0 = \sup\{|de^u_x(v)| \mid (x, v) \in TS^2 \text{ with } g_0(v, v) = 1\}$ is the uniform norm on closed one-forms on S^2 associated to the round metric g_0 . In fact,

$$\left| \frac{\partial}{\partial s} e^{u \circ f} \right| (x) = \left| de^u_{f(x)} \circ df_x \left(\frac{\partial}{\partial s} \right) \right|$$

for all $x \in \mathbb{T}^2$. But $g_0(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s}))$ goes to 0 when x goes to a point of ramification. This is clear from the fact that $g_c(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s})) = 1$ outside singularities and from the following expression of g_0 :

$$g_0 = \left(\frac{2}{1 + |z|^2} \right)^2 \cdot (|z + 1| \cdot |z| \cdot |z - 1|)^{4/3} \cdot g_c$$

where $z \in S^2 = \mathbb{C} \cup \{\infty\}$. So we can set $B = \sup_{x \in \mathbb{T}^2} \sqrt{g_0(df_x(\frac{\partial}{\partial s}), df_x(\frac{\partial}{\partial s}))}$.

Now define the set \mathcal{O} as the reunion of the open sets

$$\{e^{2u} \cdot g_c \in \mathcal{M}_{g_c} \text{ such that } \|e^u - 1\|_\infty < t \text{ and } \frac{1}{2} \tan \frac{\pi}{6} B \|de^u\|_0 < 1 - t\}$$

for $t \in]0, 1[$. The set \mathcal{O} is an open neighborhood of g_c in the C^1 -topology and we get

$$l_g(\gamma_s) \geq l_g(z_s^\alpha)$$

for all $g \in \mathcal{O}$, and for all (s, α) .

Fix $s' \in [0, \frac{\sqrt{3}}{2}]$ such that $l_g(\gamma_{s'}) = \inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s)$. For all $g \in \mathcal{O}$, we obtain

$$\begin{aligned} \text{dias}(S^2, g) &\leq \sup_{\alpha \in [0, 1]} l_g(z_s^\alpha) \\ &\leq l_g(\gamma_{s'}) \\ &\leq \inf_{s \in [0, \sqrt{3}/2]} l_g(\gamma_s). \end{aligned}$$

□

Let us finish the proof of the Theorem 1.1. For all $g \in \mathcal{O}$, we have

$$(\text{dias}(S^2, g))^2 \leq 2\sqrt{3} \cdot \text{area}(S^2, g).$$

Now observe that the equality case imposes to the conformal factor e^{2u} to be constant. So g is homothetic to g_c .

REFERENCES

- [Al60] Almgren, F.: *The homotopy groups of the integral cycle groups*. Topology **1** (1960) 257–299.
- [AZ67] Aleksandrov, A. & Zalgaller, V.: *Intrinsic geometry of surfaces*. Translations of Mathematical Monographs **15**, AMS (1967).
- [Bab97] Babenko, I.: *Topological entropy of geodesic flows on simply connected manifolds, and related problems*. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), 57-74; translation in Izv. Math. **61** (1997), 517–535.

- [Bal06] Balacheff, F.: *Sur la systole de la sphère au voisinage de la métrique standard*. Geometriae Dedicata **121** (2006), 61-71.
- [BS07] Balacheff, F. & Sabourau, S.: *Diastolic inequalities and isoperimetric inequalities on surfaces*. Preprint (2007), available on <http://math.univ-lille1.fr/~balacheff/>.
- [Be00] Berger, M.: *Riemannian Geometry during the Second Half of the Twentieth Century*. University lecture series **17**, AMS (2000).
- [CC92] Calabi, E.; Cao J.: *Simple closed geodesics on convex surfaces*. J. Differential Geom. **36** (1992), 517-549.
- [Cr88] Croke, C.: *Area and the length of the shortest closed geodesic*. J. Differential Geom. **27** (1988), no. 1, 1-21.
- [Hi76] Hirsch, M.: *Differential Topology*. Graduate Texts in Math. **33**, Springer Verlag New-York (1976).
- [NR02] Nabutovsky, A. & Rotman, R.: *The length of the shortest closed geodesic on a 2-dimensional sphere*. Int. Math. Res. Not. **23**, 2002, 1211-1222.
- [Pi74] Pitts, J.: *Regularity and singularity of one dimensional stationary integral varifolds on manifolds arising from variational methods in the large*. Symposia Mathematica, Vol. XIV (1974) 465-472.
- [Pu52] Pu, P.: *Some inequalities in certain nonorientable Riemannian manifolds*. Pacific. J. Math. **2** (1952), 55-72.
- [Re01] Reshetnyak, Yu. G.: *On the conformal representation of Alexandrov surfaces*. Report. Univ. Jyväskylä **83** (2001), 287-304.
- [Ro05] Rotman, R.: *The length of a shortest closed geodesic and the area of a 2-dimensional sphere*. Proc. Amer. Math. Soc. **134** (2006), 3041-3047.
- [Sa04] Sabourau, S.: *Filling radius and short closed geodesic of the sphere*. Bull. SMF **132**, 2004, 105-136.
- [Tro86] Troyanov, M.: *Les surfaces euclidiennes à singularités coniques*. L'Enseignement Mathématique **32** (1986) 79-94.

FLORENT BALACHEFF - LABORATOIRE PAUL PAINLEVÉ, BT. M2, UNIVERSITÉ DES SCIENCES ET TECHNOLOGIES DE LILLE, 59655 VILLENEUVE D'ASCQ, FRANCE
E-mail address: Florent.Balacheff@math.univ-lille1.fr