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Abstract. — We prove an arithmetic analogue of Fujita's approximation theorem in Arakelov geometry, conjectured by Moriwaki, by using slope method and measures associated to $\mathbb{R}$-filtrations.

Résumé. — On démontre un analogue arithmétique du théorème d'approximation de Fujita en géométrie d'Arakelov — conjecturé par Moriwaki — par la méthode de pentes et les mesures associées aux $\mathbb{R}$-filtrations.

Contents

1. Introduction

Fujita approximation is an approximative version of Zariski decomposition of pseudo-effective divisors [**31**] which holds for smooth projective surfaces but fails in general. Let X be a projective variety defined over a field K and L be a *big* line bundle on X, i.e., the *volume* of L, defined as

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\mathrm{rk}_K H^0(X, L^{\otimes n})}{n^{\dim X}/(\dim X)!},$$

is strictly positive. The Fujita's approximation theorem asserts that, for any $\varepsilon > 0$, there exists a projective birational morphism $\nu : X' \to X$, an integer $p > 0$, together with a decomposition $\nu^*(L^{\otimes p}) \cong A \otimes E$, where A is an ample line bundle, E is effective, such that $p^{-\dim X}\mathrm{vol}(A) \geqslant \mathrm{vol}(L)-\varepsilon$. This theorem had been proved by Fujita himself [**15**] in characteristic 0 case, before its generalization to any characteristic case by Takagi [**28**]. It is the source of many important results concerning big divisors and volume function in algebraic geometry context, such as volume function as a limit, its log-concavity and differentiability, etc. We refer readers to [**19**, 11.4] for a survey, see also [**14, 13, 7, 20**].

The arithmetic analogue of volume function and the arithmetic bigness in Arakelov geometry have been introduced by Moriwaki [**21, 22**]. Let K be a number field and $\mathcal{O}_K$ be its integer ring. Let $\mathscr{X}$ be a projective arithmetic variety of total dimension d over $\operatorname{Spec}\mathcal{O}_K$. For any Hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$, the *arithmetic volume* of $\overline{\mathscr{L}}$ is defined as

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) := \limsup_{n\rightarrow\infty}\frac{\widehat{h}^0(\mathscr{X},\overline{\mathscr{L}}^{\otimes n})}{n^d/d!}, \tag{1}$$

where

$$\widehat{h}^0(\mathscr{X},\overline{\mathscr{L}}^{\otimes n}) := \log\#\{s\in H^0(\mathscr{X},\mathscr{L}^{\otimes n})\,|\,\forall\,\sigma:K\rightarrow\mathbb{C},\ \|s\|_{\sigma,\sup}\leqslant 1\}.$$

Similarly, $\overline{\mathscr{L}}$ is said to be *arithmetically big* if $\widehat{\mathrm{vol}}(\overline{\mathscr{L}})>0$. In [**22, 23**], Moriwaki has proved that the arithmetic volume function is continuous with respect to $\overline{\mathscr{L}}$, and admits a unique continuous extension to $\widehat{\mathrm{Pic}}(\mathscr{X})_{\mathbb{R}}$. In [**22**], he asked the following question (Remark 5.9 *loc. cit.*): *does the Fujita approximation hold in the arithmetic case?*

The validity of arithmetic Fujita approximation has many interesting consequences. For example, assuming that the arithmetic Fujita approximation is true, then by arithmetic Riemann-Roch theorem [**17, 32**], the right side of (1) is actually a limit (see [**22**, Remark 4.1]).

In [**12**], the author proved that the sequence which defines the arithmetic volume function converges. This gives an affirmative answer to a conjecture of Moriwaki [**22**, Remark 4.1]. Rather than applying the arithmetic analogue of Fujita approximation, the proof uses its classical version on the generic fiber and then appeals to an earlier work of the author on the convergence of normalized Harder-Narasimhan polygons, generalizing the arithmetic Hilbert-Samuel formula (in an inexplicit way).

One of the difficulty for establishing arithmetic Fujita approximation is that, if $\overline{\mathscr{A}}$ is a ample Hermitian line subbundle of $\overline{\mathscr{L}}$ which approximates well $\overline{\mathscr{A}}$, then in general the section algebra of A_K does not approximate that of L_K at all. In fact, it approximates only the graded linear series of L generated by small sections.

In this article, we prove the conjecture of Moriwaki on the arithmetic Fujita approximation by using Bost's slope theory [**3, 4, 5**] and the measures associated to $\mathbb{R}$-filtrations [**11, 12**]. The strategy is similar to that in [**12**] except that, instead of using the geometric Fujita approximation in its classical form, we apply a recent result of Lazarsfeld and Mustaţă [**20**] on a very general approximation theorem for graded linear series of a big line bundle on a projective variety, using the theory of Okounkov bodies [**24**]. It permits us to approximate the graded linear series of the generic fiber generated by small sections. Another important ingredient in the proof is the comparison of minimum filtration and slope filtration (Propositions 3.6 and

3.11 *infra*), which relies on the estimations in [**1, 6**] for invariants of Hermitian vector bundles. By the interpretation of the arithmetic volume function by integral with respect to limit of Harder-Narasimhan measures established in [**12**], we prove that the arithmetic volumes of these subalgebras approximate the arithmetic volume of the Hermitian line bundle, and therefore establish the arithmetic Fujita approximation.

Shortly after the first version of this article had been written, X. Yuan told me that he was working on the same subject and has obtained the arithmetic Fujita approximation independently. He also kindly sent me his article [**30**], where he has developed an arithmetic analogue of Okounkov body, inspired by [**20**]. He has also obtained the log-concavity of the arithmetic volume function.

The organization of this article is as follows. In the second section, we introduce the notion of approximable graded algebras and study their asymptotic properties. We then recall the notion of Borel measures associated to filtered vector spaces. At the end of the section, we establish a convergence result for filtered approximable algebras. The third section is devoted to a comparison of filtrations on metrized vector bundles on a number field, which come naturally from the arithmetic properties of these objects. We begin by a reminder on Bost's slope method. Then we introduce the $\mathbb{R}$-indexed minimum filtration and slope filtration for metrized vector bundles and compare them. We also compare the asymptotic behaviour of these two types of filtrations. In the fourth section, we recall the theorem of Lazarsfeld and Mustaţă on the approximability of certain graded linear series. We then describe some approximable graded linear series which come from the arithmetic of a big Hermitian line bundle on an arithmetic variety. The main theorem of the article is established in the fifth section. We prove that the arithmetic volume of a big Hermitian line bundle can be approximated by the arithmetic volume of its graded linear series of finite type, which implies the Moriwaki's conjecture. Finally in the sixth section, we prove that, if a graded linear series generated by small sections approximates well a big Hermitian line bundle $\overline{\mathscr{L}}$, then it also approximates well the asymptotic measure of $\overline{\mathscr{L}}$ truncated at 0.

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2. Approximable algebras and asymptotic measures

In [**11, 12**], the author has used measures associated to filtered vector spaces to study asymptotic invariants of Hermitian line bundles. Several convergence results have been established for graded algebras equipped with $\mathbb{R}$-filtrations, either under the finite generating condition on the underlying graded algebra [**11**, Theorem 3.4.3], or under the geometric condition [**12**, Theorem 4.2] that the underlying graded algebra is the section algebra of a big line bundle. However, as we shall see later in this article, some graded algebras coming naturally from the arithmetic do not satisfy these two conditions. In this section, we generalize the convergence result to a so-called *approximable graded algebra* case.

2.1. Approximable graded algebras. — In the study of projective varieties, graded algebras are natural objects which often appear as graded linear series of a line bundle. In general, such graded algebras are not always finitely generated. However, according to approximation theorems due to Fujita [**15**], Takagi [**28**], Lazarsfeld and Mustaţă [**20**] etc., they can often be approximated arbitrarily closely by its graded subalgebras of finite type. Inspired by [**20**], we formalize this observation as a notion. In this section, K denotes an arbitrary field.

Definition 2.1. — Let $B = \bigoplus_{n\geqslant 0} B_n$ be an integral graded K-algebra. We say that B is *approximable* if the following conditions are verified:

(a) all vector spaces B_n are finite dimensional and $B_n \neq 0$ for sufficiently large n;

(b) for any $\varepsilon \in (0,1)$, there exists an integer $p_0 \geqslant 1$ such that, for any integer $p \geqslant p_0$, one has
$$\liminf_{n\to\infty} \frac{\mathrm{rk}(\mathrm{Im}(S^n B_p \to B_{np}))}{\mathrm{rk}(B_{np})} > 1-\varepsilon,$$
where $S^n B_p \to B_{np}$ is the canonical homomorphism defined by the algebra structure on B.

Remark 2.2. — The condition (a) serves to exclude the degenerate case so that the presentation becomes simpler. In fact, if an integral graded algebra B is not concentrated on B_0, then by choosing an integer $q \geqslant 1$ such that $B_q \neq 0$, we obtain a new graded algebra $\bigoplus_{n\geqslant 0} B_{nq}$ which verifies (a). This new algebra often contains the information about B which interests us.

Example 2.3. — The following are some examples of approximable graded algebras.

1) If B is an integral graded algebra of finite type such that $B_n \neq 0$ for sufficiently large n, then it is clearly approximable.
2) Let X be a projective variety over $\operatorname{Spec} K$ and L be a big line bundle on X. Then by Fujita's approximation theorem, the total graded linear series $\bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ of L is approximable.
3) More generally, Lazarsfeld and Mustaţă have shown that, with the notation of 2), any graded subalgebra of $\bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ containing an ample divisor and verifying the condition (a) above is approximable.

We shall revisit the examples 2) and 3) in §4.1.

The following properties of approximable graded algebras are quite similar to classical results on big line bundles.

Proposition 2.4. — *Let $B = \bigoplus_{n\geqslant 0} B_n$ be an integral graded algebra which is approximable. Then there exists a constant $a \in \mathbb{N}\setminus\{0\}$ such that, for sufficiently large integer p, the algebra $\bigoplus_{n\geqslant 0} \mathrm{Im}(S^n B_p \to B_{np})$ has Krull dimension a. Furthermore, denote by $d(B) := a-1$. The sequence*

$$\Big(\frac{\mathrm{rk}\, B_n}{n^{d(B)}/d(B)!}\Big)_{n\geqslant 1} \tag{2}$$

converges in $\mathbb{R}_+$.

Proof. — Assume that $B_m \neq 0$ for all $m \geqslant m_0$, where $m_0 \in \mathbb{N}$. Since B is integral, for any integer $n \geqslant 1$ and any integer $m \geqslant m_0$, one has

$$\operatorname{rk}(B_{n+m}) \geqslant \operatorname{rk}(B_n). \tag{3}$$

For any integer $p \geqslant m_0$, denote by $a(p)$ the Krull dimension of $\bigoplus_{n\geqslant 0} \operatorname{Im}(S^{np}B_p \to B_{np})$, and define

$$f(p) := \liminf_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{\operatorname{rk} B_{np}}. \tag{4}$$

The approximable condition shows that $\lim\limits_{p\to\infty} f(p) = 1$. Recall that the classical result on Hilbert polynomials implies

$$\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np})) \asymp n^{a(p)-1} \qquad (n \to \infty).$$

Thus, if $f(p) > 0$, then $\operatorname{rk} B_{np} \asymp n^{a(p)-1}$, and hence by (3), one has $\operatorname{rk} B_n \asymp n^{a(p)-1}$ $(n \to \infty)$. Thus $a(p)$ is constant if $f(p) > 0$. In particular, $a(p)$ is constant when p is sufficiently large. We denote by a this constant, and by $d(B) = a - 1$.

In the following, we shall establish the convergence of the sequence (2). It suffices to establish

$$\liminf_{n\to\infty} \frac{\operatorname{rk} B_n}{n^{d(B)}} \geqslant \limsup_{n\to\infty} \frac{\operatorname{rk} B_n}{n^{d(B)}}. \tag{5}$$

By (3), for any integer $p \geqslant 1$, one has

$$\limsup_{n\to\infty} \frac{\operatorname{rk}(B_n)}{n^{d(B)}} = \limsup_{n\to\infty} \frac{\operatorname{rk}(B_{np})}{(np)^{d(B)}} \quad \text{and} \quad \liminf_{n\to\infty} \frac{\operatorname{rk}(B_n)}{n^{d(B)}} = \liminf_{n\to\infty} \frac{\operatorname{rk}(B_{np})}{(np)^{d(B)}} \tag{6}$$

Suppose that $f(p) > 0$. Then one has

$$\begin{aligned}\limsup_{n\to\infty} \frac{\operatorname{rk}(B_{np})}{(np)^{d(B)}} &= \Big(\limsup_{n\to\infty} \frac{\operatorname{rk}(B_{np})}{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}\Big) \cdot \Big(\lim_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{(np)^{d(B)}}\Big) \\ &= f(p)^{-1} \lim_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{(np)^{d(B)}} \leqslant f(p)^{-1} \liminf_{n\to\infty} \frac{\operatorname{rk}(B_{np})}{(np)^{d(B)}}.\end{aligned}$$

Combining with (6) and the approximable hypothesis, we obtain (5). □

Corollary 2.5. — *For any $r \in \mathbb{N}$, one has*

$$\lim_{n\to\infty} \frac{\operatorname{rk}(B_{n+r})}{\operatorname{rk}(B_n)} = 1.$$

Definition 2.6. — Let B be an integral graded K-algebra which is approximable. We denote by $\operatorname{vol}(B)$ the limit

$$\operatorname{vol}(B) := \lim_{n\to\infty} \frac{\operatorname{rk}(B_n)}{n^{d(B)}/d(B)!}.$$

Note that, if B is the total graded linear series of a big line bundle L, then $\operatorname{vol}(B)$ is just the volume of the line bundle L.

Remark 2.7. — It might be interesting to know whether any approximable graded algebra can always be realized as a graded linear series of a big line bundle.

2.2. Reminder on $\mathbb{R}$-filtrations. — Let K be a field and W be a vector space of finite rank over K. For *filtration* on W we mean a sequence $\mathcal{F} = (\mathcal{F}_t W)_{t\in\mathbb{R}}$ of vector subspaces of W, satisfying the following conditions:

1) if $t \leqslant s$, then $\mathcal{F}_s W \subset \mathcal{F}_t W$;
2) $\mathcal{F}_t W = 0$ for sufficiently positive t, $\mathcal{F}_t W = W$ for sufficiently negative t;
3) the function $t \mapsto \operatorname{rk}(\mathcal{F}_t W)$ is left continuous.

The couple $(W, \mathcal{F})$ is called a *filtered vector space*.

If $W \neq 0$, we denote by $\nu_{(W,\mathcal{F})}$ (or simply ν_W if this does not lead to any ambiguity) the Borel probability measure obtained by taking the derivative (in the sense of distribution) of the function $t \mapsto -\operatorname{rk}\mathcal{F}_t W/\operatorname{rk} W$. If $W = 0$, then there is a unique filtration on W and we define ν_0 to be the zero measure by convention. Note that the measure ν_W is actually a linear combination of Dirac measures.

All filtered vector spaces and linear maps preserving filtrations form an exact category. The following proposition shows that mapping $(W, \mathcal{F}) \mapsto \nu_{(W,\mathcal{F})}$ behaves well with respect to exact sequences.

Proposition 2.8. — *Assume that*

$$0 \longrightarrow (W', \mathcal{F}') \longrightarrow (W, \mathcal{F}) \longrightarrow (W'', \mathcal{F}'') \longrightarrow 0$$

is an exact sequence of filtered vector spaces. Then

$$\nu_W = \frac{\operatorname{rk} W'}{\operatorname{rk} W}\nu_{W'} + \frac{\operatorname{rk} W''}{\operatorname{rk} W}\nu_{W''}.$$

Proof. — For any $t \in \mathbb{R}$, one has

$$\operatorname{rk}(\mathcal{F}_t W) = \operatorname{rk}(\mathcal{F}'_t W') + \operatorname{rk}(\mathcal{F}''_t W''),$$

which implies the proposition by taking the derivative in the sense of distribution. □

Corollary 2.9. — *Let $(W, \mathcal{F})$ be a non-zero filtered vector space, $V \subset W$ be a non-zero subspace, equipped with the induced filtration, $\varepsilon = 1 - \operatorname{rk}(V)/\operatorname{rk}(W)$. Then for any bounded Borel function h on $\mathbb{R}$, one has*

$$\left| \int h \,\mathrm{d}\nu_W - \int h \,\mathrm{d}\nu_V \right| \leqslant 2\varepsilon \|h\|_{\sup}.$$

Proof. — The case where $W = V$ is trivial. In the following, we assume that $U := W/V$ is non-zero, and is equipped with the quotient filtration. By Proposition 2.8, one has

$$\nu_W = (1-\varepsilon)\nu_V + \varepsilon\nu_U = \nu_V + \varepsilon(\nu_U - \nu_V).$$

Therefore

$$\left| \int h \,\mathrm{d}\nu_W - \int h \,\mathrm{d}\nu_V \right| = \varepsilon \left| \int h \,\mathrm{d}\nu_U - \int h \,\mathrm{d}\nu_V \right| \leqslant 2\varepsilon \|h\|_{\sup}.$$

□

Let $(W,\mathcal{F})$ be a filtered vector space. We denote by $\lambda : W \to \mathbb{R}\cup\{+\infty\}$ the mapping which sends $x\in W$ to

$$\lambda(x) := \sup\{a\in\mathbb{R}\,|\,x\in\mathcal{F}_aW\}.$$

The function λ takes values in $\operatorname{supp}(\nu_W)\cup\{+\infty\}$, and is finite on $W\setminus\{0\}$. We define

$$\lambda_{\max}(W) = \max_{x\in W\setminus\{0\}}\lambda(x) \quad\text{and}\quad \lambda_{\min}(W) = \min_{x\in W}\lambda(x). \tag{7}$$

Note that when $W\neq 0$, one always has $\lambda_{\min}(W)\leqslant\int x\,\nu_W(\mathrm{d}x)\leqslant\lambda_{\max}(W)$. However, $\lambda_{\min}(0)=+\infty$ and $\lambda_{\max}(0)=-\infty$.

We introduce an order "$\prec$" on the space $\mathscr{M}$ of all Borel probability measures on $\mathbb{R}$. Denote by $\nu_1\prec\nu_2$, or $\nu_2\succ\nu_1$ the relation:

for any bounded increasing function h on $\mathbb{R}$, $\int h\,\mathrm{d}\nu_1\leqslant\int h\,\mathrm{d}\nu_2$.

For any $x\in\mathbb{R}$, denote by δ_x the Dirac measure concentrated at x. For any $a\in\mathbb{R}$, let τ_a be the operator acting on the space $\mathscr{M}$ which sends a measure ν to its direct image by the map $x\mapsto x+a$.

Proposition 2.10. — *Let $(V,\mathcal{F})$ and $(W,\mathcal{G})$ be non-zero filtered vector spaces. Assume that $\phi : V\to W$ is an isomorphism of vector spaces and a is a real number such that $\phi(\mathcal{F}_tV)\subset\mathcal{G}_{t+a}W$ holds for all $t\in\mathbb{R}$, or equivalently, $\forall x\in V$, $\lambda(\phi(x))\geqslant\lambda(x)+a$, then $\nu_W\succ\tau_a\nu_V$.*

See [**11**, Lemma 1.2.6] for proof.

2.3. Convergence of measures of an approximable algebra. — Let B be an integral graded algebra, assumed to be approximable. Let $f:\mathbb{N}\to\mathbb{R}_+$ be a mapping. Assume that, for each integer $n\geqslant 0$, the vector space B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ such that B is f-quasi-filtered, that is, there exists $n_0\in\mathbb{N}$ such that, for any integer $l\geqslant 2$, and all homogeneous elements $x_1,\cdots,x_l$ in B of degrees $n_1,\cdots,n_l$ in $\mathbb{Z}_{\geqslant n_0}$, respectively, one has

$$\lambda(x_1\cdots x_l)\geqslant\sum_{i=1}^{l}\big(\lambda(x_i)-f(n_i)\big),$$

where λ is the function defined in (7).

For any $\varepsilon>0$, let T_ε be the operator acting on the space $\mathscr{M}$ of all Borel probability measures which sends $\nu\in\mathscr{M}$ to its direct image by the mapping $x\mapsto\varepsilon x$.

The purpose of this subsection is to establish the following convergence result, which is a generalization of [**12**, Theorem 4.2].

Theorem 2.11. — *Let B be an approximable graded algebra equipped with filtrations as above such that B is f-quasi-filtered. Assume in addition that*

$$\sup_{n\geqslant 1}\lambda_{\max}(B_n)/n<+\infty \quad\textit{and}\quad \lim_{n\to\infty}f(n)/n=0.$$

Then the sequence $(\lambda_{\max}(B_n)/n)_{n\geqslant 1}$ converges in $\mathbb{R}$, and the measure sequence $(T_{\frac{1}{n}}\nu_{B_n})_{n\geqslant 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$.

Remark 2.12. — We say that a sequence $(\nu_n)_{n\geqslant 1}$ of Borel measures on $\mathbb{R}$ converges vaguely to a Borel measure ν if, for any continuous function h on $\mathbb{R}$ whose support is compact, one has

$$\lim_{n\rightarrow+\infty}\int h\,\mathrm{d}\nu_n=\int h\,\mathrm{d}\nu.$$

Proof. — The first assertion has been established in [**11**, Proposition 3.2.4] in a more general setting without the approximable condition on B. Here we only prove the second assertion.

Assume that $B_n\neq 0$ holds for any $n\geqslant m_0$, where $m_0\geqslant n_0$ is an integer, and denote by $\nu_n=T_{\frac{1}{n}}\nu_{B_n}$. The supports of ν_n are uniformly bounded from above since $\sup_{n\geqslant 1}\lambda_{\max}(B_n)/n<+\infty$. Let p be an integer such that $p\geqslant m_0$. Denote by $A^{(p)}$ be the graded subalgebra of B generated by B_p. For any integer $n\geqslant 1$, we equipped each vector space $A^{(p)}_{np}$ with the induced filtration, and denote by $\nu^{(p)}_n:=T_{\frac{1}{np}}\nu_{A^{(p)}_{np}}$. Furthermore, we choose, for any $r\in\{p+1,\cdots,2p-1\}$, a non-zero element $e_r\in B_r$, and define

$$M^{(p)}_{n,r}=e_rB_{np}\subset B_{np+r},\quad N^{(p)}_{n,r}=e_{3p-r}M^{(p)}_{n,r}\subset B_{(n+3)p},$$
$$a^{(p)}_{n,r}=\frac{\lambda(e_{3p-r})-f(np+r)-f(3p-r)}{np},\quad b^{(p)}_{n,r}=a^{(p)}_{n,r}+\frac{\lambda(e_r)-f(np)-f(r)}{np},$$
$$\nu^{(p)}_{n,r}=T_{\frac{1}{np}}\nu_{M^{(p)}_{n,r}},\quad \eta^{(p)}_{n,r}=T_{\frac{1}{np}}\nu_{N^{(p)}_{n,r}}.$$

Note that, for all $x\in B_{np}$, $y\in M^{(p)}_{n,r}$, one has

$$\lambda(e_rx)\geqslant\lambda(x)+\lambda(e_r)-f(np)-f(r),$$
$$\lambda(e_{3p-r}y)\geqslant\lambda(y)+\lambda(e_{3p-r})-f(3p-r)-f(np+r).$$

By Proposition 2.10, one has

$$\eta^{(p)}_{n,r}\succ\tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r}\succ\tau_{b^{(p)}_{n,r}}\nu_{np}.$$

Let $h(x)$ be a bounded increasing and continuous function on $\mathbb{R}$ whose support is bounded from below, and which is constant when x is sufficiently positive. One has

$$\int h\,\mathrm{d}\eta^{(p)}_{n,r}\geqslant\int h\,\mathrm{d}\tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r}\geqslant\int h\,\mathrm{d}\tau_{b^{(p)}_{n,r}}\nu_{np}.\tag{8}$$

Note that $|h(x+\varepsilon x)-h(x)|$ converges uniformly to zero when $\varepsilon\rightarrow 0$. By Corollaries 2.9 and 2.5, we obtain

$$\lim_{n\rightarrow\infty}\left|\int h\,\mathrm{d}\eta^{(p)}_{n,r}-\int h\,\mathrm{d}\nu_{(n+3)p}\right|=0,\tag{9}$$

$$\lim_{n\rightarrow\infty}\left|\int h\,\mathrm{d}\nu^{(p)}_{n,r}-\int h\,\mathrm{d}\nu_{np+r}\right|=0.\tag{10}$$

Note that $|h(x+u)-h(x)|$ converges uniformly to zero when $u\to 0$. Combining with the fact that $\lim_{n\to\infty} a_{n,r}^{(p)} = \lim_{n\to\infty} b_{n,r}^{(p)} = 0$, we obtain

$$\lim_{n\to\infty}\bigg|\int h\,\mathrm{d}\tau_{a_{n,r}^{(p)}}\nu_{n,r}^{(p)} - \int h\,\mathrm{d}\nu_{n,r}^{(p)}\bigg| = 0, \tag{11}$$

$$\lim_{n\to\infty}\bigg|\int h\,\mathrm{d}\tau_{b_{n,r}^{(p)}}\nu_{np} - \int h\,\mathrm{d}\nu_{np}\bigg| = 0. \tag{12}$$

Thus

$$\begin{split}\limsup_{n\to\infty}\bigg|\int h\,\mathrm{d}\nu_{np+r} - \int h\,\mathrm{d}\nu_{np}| = \limsup_{n\to\infty}\bigg|\int h\,\mathrm{d}\tau_{a_{n,r}^{(p)}}\nu_{n,r}^{(p)} - \int h\,\mathrm{d}\tau_{b_{n,r}^{(p)}}\nu_{np}\bigg| \\ \leqslant \limsup_{n\to\infty}\bigg|\int h\,\mathrm{d}\eta_{n,r}^{(p)} - \int h\,\mathrm{d}\tau_{b_{n,r}^{(p)}}\nu_{np}\bigg| = \limsup_{n\to\infty}\bigg|\int h\,\mathrm{d}\nu_{(n+3)p} - \int h\,\mathrm{d}\nu_{np}\bigg|,\end{split} \tag{13}$$

where the first equality comes from (10), (11) and (12). The inequality comes from (8), and the second equality results from (9) and (12).

Let $\varepsilon\in(0,1)$. By the approximability condition on B, there exists two integers $p\geqslant m_0$ and $n_1\geqslant 1$ such that, for any integer $n\geqslant n_1$, one has

$$\frac{\mathrm{rk}\,A_{np}^{(p)}}{\mathrm{rk}\,B_{np}} > 1-\varepsilon.$$

Therefore, by Corollary 2.9, one has

$$\bigg|\int h\,\mathrm{d}\nu_{np} - \int h\,\mathrm{d}\nu_n^{(p)}\bigg| \leqslant 2\varepsilon\|h\|_{\sup}. \tag{14}$$

As $A^{(p)}$ is an algebra of finite type, by [**11**, Theorem 3.4.3], the sequence of measures $(\nu_n^{(p)})_{n\geqslant 1}$ converges vaguely to a Borel probability measure $\nu^{(p)}$. Note that the supports of measures ν_n^p are uniformly bounded from above. Hence $(\int h\,\mathrm{d}\nu_n^{(p)})_{n\geqslant 1}$ is a Cauchy sequence. After the relations (13) and (14), we obtain that, there exists an integer $n_2\geqslant 1$ such that, for any integers m and n, $m\geqslant n_2$, $n\geqslant n_2$, one has

$$\bigg|\int h\,\mathrm{d}\nu_n - \int h\,\mathrm{d}\nu_m\bigg| \leqslant 8\varepsilon\|h\|_{\sup} + \varepsilon.$$

Since ε is arbitrary, the sequence $(\int h\,\mathrm{d}\nu_n)_{n\geqslant 1}$ converges in $\mathbb{R}$. Denote by $C_0^\infty(\mathbb{R})$ the space of all smooth functions of compact support on $\mathbb{R}$. Since any function in $C_0^\infty(\mathbb{R})$ can be written as the difference of two continuous increasing and bounded functions whose supports are both bounded from below which are constant on a neighbourhood of $+\infty$, we obtain that

$$h \longmapsto \lim_{n\to\infty} h\,\mathrm{d}\nu_n$$

is a well defined positive continuous linear functional on $(C_0^\infty(\mathbb{R}),\|\cdot\|_{\sup})$. As $C_0^\infty(\mathbb{R})$ is dense in the space $C_c(\mathbb{R})$ of all continuous functions of compact support on $\mathbb{R}$ with respect to the topology induced by $\|\cdot\|_{\sup}$, the linear functional extends continuously to a Borel measure ν on $\mathbb{R}$. Finally, by Corollary 2.9 and by passing to the limit, we obtain that for any $p\geqslant m_0$, one has

$$|1-\nu(\mathbb{R})| = |\nu^{(p)}(\mathbb{R}) - \nu(\mathbb{R})| \leqslant 1-f(p),$$

where $f(p)$ was defined in (4). As $\lim_{p\to\infty} f(p) = 1$, ν is a probability measure. $\square$

3. Comparison of filtrations on metrized vector bundles

Let K be a number field and $\mathcal{O}_K$ be its integer ring. Denote by δ_K the degree of K over $\mathbb{Q}$. For *metrized vector bundle* on $\operatorname{Spec}\mathcal{O}_K$ we mean a projective $\mathcal{O}_K$-module E together with a family $(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$, where $\|\cdot\|_\sigma$ is a norm on $E_{\sigma,\mathbb{C}}$, assumed to be invariant by the complex conjugation. We often use the expression $\overline{E}$ to denote the couple $(E,(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}})$. If for each σ, $\|\cdot\|_\sigma$ is a Hermitian norm, we say that $\overline{E}$ is a *Hermitian vector bundle*. Any metrized vector bundle of rank one is necessarily a Hermitian vector bundle, and we call it a *Hermitian line bundle*.

As pointed out by Gaudron [**16**, §3], the category of metrized vector bundles on $\operatorname{Spec}\mathcal{O}_K$ is equivalent to that of adelic vector bundles on $\operatorname{Spec} K$.

On a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$, one has a natural filtration defined by successive minima. On a Hermitian vector bundle, there is another filtration defined by successive slopes. In this section, we compare these two filtrations.

3.1. Reminder on the slope method. — In this section, we recall some notions and results of Bost's slope method. The references are [**3, 4, 9, 5**].

Let $\overline{L}$ be a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$. The *Arakelov degree* of $\overline{L}$ is defined as

$$\widehat{\deg}(\overline{L}) := \log\#(L/\mathcal{O}_K s) - \sum_{\sigma:K\to\mathbb{C}} \log\|s\|_\sigma,$$

where s is an arbitrary non-zero element in L. This definition does not depend on the choice of s, thanks to the product formula. An equivalent definition is

$$\widehat{\deg}(\overline{L}) = -\sum_{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K} \log\|s\|_{\mathfrak{p}} - \sum_{\sigma:K\to\mathbb{C}} \log\|s\|_\sigma, \tag{15}$$

this time s could be an arbitrary element in L_K, $\|\cdot\|_{\mathfrak{p}}$ is induced by the $\mathcal{O}_K$-module structure on L. For a Hermitian vector bundle $\overline{E}$ of arbitrary rank, the *Arakelov degree* of $\overline{E}$ is just $\widehat{\deg}(\overline{E}) := \widehat{\deg}(\Lambda^{\operatorname{rk} E}\overline{E})$, where the metrics of $\Lambda^{\operatorname{rk} E}\overline{E}$ are exterior product metrics. The Arakelov degree of the zero vector bundle is zero. Furthermore, it is additive with respect to short exact sequences.

When $\overline{E}$ is non-zero, the *slope* of $\overline{E}$ is by definition the quotient

$$\widehat{\mu}(\overline{E}) := \frac{\widehat{\deg}(E)}{\delta_K \operatorname{rk}(E)},$$

where $\delta_K = [K:\mathbb{Q}]$. As in the case of vector bundles on curves, the *maximal slope* $\widehat{\mu}_{\max}(\overline{E})$ and the *minimal slope* $\widehat{\mu}_{\min}(\overline{E})$ of $\overline{E}$ are defined as the maximal value of slopes of all non-zero Hermitian subbundles of $\overline{E}$ and the minimal value of all non-zero Hermitian quotient bundles of $\overline{E}$, respectively. The existence of these extremal slopes are due to Stuhler [**27**] and Grayson [**18**]. One has $\widehat{\mu}_{\min}(\overline{E}) = -\widehat{\mu}_{\max}(\overline{E}^\vee)$. A non-zero Hermitian vector bundle is said to be *semistable* if the equality $\widehat{\mu}_{\max}(\overline{E}) = \widehat{\mu}(\overline{E})$ holds. The results of Stuhler and of Grayson mentioned above permit to establish the analogue of Harder-Narasimhan filtration in Arakelov geometry:

Proposition 3.1. — *There exists a unique flag*

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E$$

of E such that each subquotient $\overline{E_i/E_{i-1}}$ ($1 \in \{1, \cdots, n\}$) is semistable, and that, by writing $\mu_i = \widehat{\mu}(\overline{E_i/E_{i-1}})$, the inequalities of successive slopes $\mu_1 > \mu_2 > \cdots > \mu_n$ hold.

Let $\overline{E}$ and $\overline{F}$ be two Hermitian vector bundles. The *height* of a homomorphism $\phi : E_K \to F_K$ is defined as the sum of the logarithms of norms of all local homomorphism induced from ϕ by extending scalars, divided by δ_K, that is,

$$h(\phi) := \frac{1}{\delta_K}\left(\sum_{\mathfrak{p}} \log \|\phi_{\mathfrak{p}}\|_{\mathfrak{p}} + \sum_{\sigma : K \to \mathbb{C}} \log \|\phi\|_{\sigma}\right).$$

It is negative or zero notably when ϕ is *effective*, i.e., ϕ gives rise to an $\mathcal{O}_K$-linear homomorphism, and $\|\phi_\sigma\| \leqslant 1$ for any embedding $\sigma : K \to \mathbb{C}$.

The following slope inequality compares the slopes of two Hermitian vector bundles, given an injective homomorphism between them.

Proposition 3.2. — *Let $\overline{E}$ and $\overline{F}$ be two non-zero Hermitian vector bundles and $\phi : E_K \to F_K$ be an injective K-linear homomorphism. Then then following inequality holds:*

$$\widehat{\mu}_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{F}) + h(\phi),$$

where $h(\phi)$ is the height of ϕ.

By passing to dual Hermitian vector bundles, we obtain the following corollary.

Corollary 3.3. — *Let $\overline{E}$ and $\overline{F}$ be two non-zero Hermitian vector bundles and $\psi : E_K \to F_K$ be a surjective homomorphism. Then*

$$\widehat{\mu}_{\min}(\overline{F}) \geqslant \widehat{\mu}_{\min}(\overline{E}) - h(\psi).$$

3.2. Minimum filtration and slope filtration. — Let $\overline{E}$ be a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Let r be the rank of E and $i \in \{1, \cdots, r\}$. Recall the i^{th} (logarithmic) *minimum* of $\overline{E}$ is defined as

$$e_i(\overline{E}) := -\log\inf\{a > 0 \mid \operatorname{rk}(\operatorname{Vect}_K\{\mathbb{B}(\overline{E}, a)\}) \geqslant i\},$$

where $\mathbb{B}(\overline{E}, a) = \{s \in E \mid \forall \sigma : K \to \mathbb{C}, \|s\|_\sigma \leqslant a\}$. Denote by $e_{\max}(\overline{E}) = e_1(\overline{E})$ and $e_{\min}(\overline{E}) = e_r(\overline{E})$. Define an $\mathbb{R}$-filtration $\mathcal{F}^M$ on E_K as

$$\mathcal{F}^M_t E_K := \operatorname{Vect}_K\{\mathbb{B}(\overline{E}, e^{-t})\},$$

called the *minimum filtration* of $\overline{E}$. Note that $\lambda_{\max}(E_K, \mathcal{F}^M) = e_{\max}(\overline{E})$ and $\lambda_{\min}(E_K, \mathcal{F}^M) = e_{\min}(\overline{E})$.

Assume that $\overline{E}$ is a Hermitian vector bundle, we define another $\mathbb{R}$-filtration $\mathcal{F}^S$ on E_K such that

$$\mathcal{F}^S_t E_K := \sum_{\substack{F \subseteq E \\ \widehat{\mu}_{\min}(\overline{F}) \geqslant t}} F_K, \qquad (16)$$

where $\widehat{\mu}_{\min}(0) = +\infty$ by convention. The filtration $\mathcal{F}^S$ is called the *slope filtration* of $\overline{E}$. The slope filtration is just a reformulation of the Harder-Narasimhan filtration of $\overline{E}$ in considering the successive slopes at the same time. In particular, if $\overline{E}_t$ is the saturated Hermitian vector subbundle of E such that $E_{t,K} = \mathcal{F}^S_t E_K$, then one has

$$\widehat{\mu}_{\min}(\overline{E}_t) \geqslant t. \tag{17}$$

Moreover, one has

$$\lambda_{\max}(E_K, \mathcal{F}^S) = \widehat{\mu}_{\max}(\overline{E}), \qquad \lambda_{\min}(E_K, \mathcal{F}^S) = \widehat{\mu}_{\min}(\overline{E}).$$

See [**11**, §2.2] for details.

Remark 3.4. — É. Gaudron has generalized the notions of maximal slope, minimal slope and Harder-Narasimhan filtration for metrized vector bundles, cf. [**16**, §5.2]. However, it is not clear if the $\mathbb{R}$-indexed version of his definition of Harder-Narasimhan filtration (by using maximal slopes) coincides with (16) when the metrized vector bundle is not Hermitian.

The slope filtration has the following functorial property, which is an application of the slope inequality. For proof, see [**11**, Proposition 2.2.4].

Proposition 3.5. — *Let $\overline{E}$ and $\overline{F}$ be two Hermitian vector bundles on* $\operatorname{Spec}\mathcal{O}_K$*, $\phi : E_K \to F_K$ be a homomorphism. Then for any real number t, one has*

$$\phi(\mathcal{F}^S_t E_K) \subset \mathcal{F}^S_{t-h(\phi)} F_K. \tag{18}$$

Let E be a non-zero projective $\mathcal{O}_K$-module of finite rank. Let $g = (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$ and $g' = (\|\cdot\|'_\sigma)_{\sigma:K\to\mathbb{C}}$ be two families of norms on E. We assume that all metrics $\|\cdot\|'_\sigma$ are Hermitian. Define

$$D(E, g, g') := \max_{\sigma:K\to\mathbb{C}} \sup_{0\neq s\in E_{\sigma,\mathbb{C}}} \big|\log\|s\|_\sigma - \log\|s\|'_\sigma\big|$$

Denote by $\mathcal{F}^M$ the minima filtration of (E, g), and by $\mathcal{F}^S$ the slope filtration of (E, g').

Proposition 3.6. — *One has, for any $t \in \mathbb{R}$,*

$$\mathcal{F}^M_t E_K \subset \mathcal{F}^S_{t-\alpha} E_K,$$

where $\alpha = \log\sqrt{r} + D(E, g, g')$.

Proof. — Without loss of generality, we assume that $\mathcal{F}^M_t E_K \neq 0$. Let $\overline{F}$ be the saturation of $\mathcal{F}^M_t E_K$ in E, equipped with metrics induced from g'. Thus $\overline{F}$ becomes a Hermitian vector subbundle of (E, g'). Let a be the rank of F. As F is generated by elements in $\mathbb{B}((E, g), e^{-t})$, there exists non-zero elements $s_1, \cdots, s_a$ in F which form a basis of F_K and such that $\|s_i\|_\sigma \leqslant e^{-t}$ for any $\sigma : K \to \mathbb{C}$. One has $\|s_i\|'_\sigma \leqslant e^{-t+D(E,g,g')}$. Let $\phi : \overline{\mathcal{O}}_K^{\oplus a} \to \overline{F}$ be the homomorphism defined by $(s_1, \cdots, s_a)$, where the Hermitian metrics on $\overline{\mathcal{O}}_K$ are trivial. One has

$$h(\phi) \leqslant \log\sqrt{a} - t + D(E, g, g') \leqslant \log\sqrt{r} - t + D(E, g, g'),$$

where $r = \operatorname{rk}(E)$. By Corollary 3.3, the inequality $\widehat{\mu}_{\min}(\overline{F}) \geqslant t - \log\sqrt{r} - D(E, g, g')$ holds. Therefore, $F_K \subset \mathcal{F}^S_{t-\alpha} E_K$. $\square$

In order to establish the inverse comparison, we need some notation. Let $f : \operatorname{Spec}\mathcal{O}_K \to \operatorname{Spec}\mathbb{Z}$ be the canonical morphism. For any Hermitian vector bundle $\overline{F}$ on $\operatorname{Spec}\mathcal{O}_K$, denote by $f_*\overline{F}$ the Hermitian vector bundle on $\operatorname{Spec}\mathbb{Z}$ whose underlying $\mathbb{Z}$ module is F and such that, for any $s = (s_\sigma)_{\sigma:K\to\mathbb{C}} \in F\otimes_{\mathbb{Z}}\mathbb{C} = \bigoplus_{\sigma:K\to\mathbb{C}} F\otimes_{\mathcal{O}_K,\sigma}\mathbb{C}$, one has

$$\|s\|^2 = \sum_{\sigma:K\to\mathbb{C}} \|s_\sigma\|_\sigma^2.$$

Proposition 3.7. — *The inequality* $e_{\min}(\overline{F}) \geqslant e_{\min}(f_*\overline{F})$ *holds.*

Proof. — Let s be an arbitrary element in F. By definition, one has $\|s\| \geqslant \|s\|_\sigma$ for any $\sigma : K \to \mathbb{C}$. Thus, for any $u > 0$, one has $\mathbb{B}(\overline{F}, u) \supset \mathbb{B}(f_*\overline{F}, u)$. Furthermore, if $\mathbb{B}(f_*\overline{F}, u)$ generates $F_{\mathbb{Q}}$ as a vector space over $\mathbb{Q}$, it also generates F_K as a vector space over K. Therefore, $e_{\min}(\overline{F}) \geqslant e_{\min}(f_*F)$. □

Recall some results in [**1**] and [**6**].

Proposition 3.8. — *1) Let* $\overline{G}$ *be a Hermitian vector bundle on* $\operatorname{Spec}\mathbb{Z}$. *Then*

$$e_{\min}(\overline{G}) + e_{\max}(\overline{G}^\vee) \geqslant -\log(3\operatorname{rk}(G)/2). \tag{19}$$

2) Let $\overline{F}$ *be a Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$. *Then*

$$\widehat{\mu}_{\max}(\overline{F}) - \frac{1}{2}\log(\delta_K\cdot\operatorname{rk}(F)) - \frac{\log|\Delta_K|}{2\delta_K} \leqslant e_{\max}(\overline{F}) \leqslant \widehat{\mu}_{\max}(\overline{F}) - \frac{1}{2}\log(\delta_K), \tag{20}$$

where Δ_K *is the discriminant of* K.

Proof. — See [**1**, Theorem 3.1 (iii)] For 1) and [**6**, (3.23), (3.24)] for 2). □

Denote by $\omega_{\mathcal{O}_K} := \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$ the canonical module of the number field K. Note that the trace map $\operatorname{tr}_{K/\mathbb{Q}}$ is a non-zero element in $\omega_{\mathcal{O}_K}$. We equip $\omega_{\mathcal{O}_K}$ with the norms such that $\|\operatorname{tr}_{K/\mathbb{Q}}\|_\sigma = 1$ for any $\sigma : K \to \mathbb{Q}$. Thus we obtain a Hermitian line bundle $\overline{\omega}_{\mathcal{O}_K}$. The Arakelov degree of $\overline{\omega}_{\mathcal{O}_K}$ is $\log|\Delta_K|$, where Δ_K is the discriminant of K. By [**6**, Proposition 3.2.2], for any Hermitian vector bundle $\overline{F}$ over $\operatorname{Spec}\mathcal{O}_K$, one has a natural isomorphism

$$f_*(\overline{F}^\vee \otimes \overline{\omega}_{\mathcal{O}_K}) \cong (f_*\overline{F})^\vee. \tag{21}$$

The following lemma compares the logarithmic last minimum and the minimal slope.

Lemma 3.9. — *Let* $\overline{F}$ *be a Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$. *One has*

$$e_{\min}(\overline{F}) \geqslant \widehat{\mu}_{\min}(\overline{F}) - \log|\Delta_K| - \frac{1}{2}\log\delta_K - \log(3/2) - \log(\operatorname{rk}(F)).$$

Proof. — In fact,

$$\begin{aligned}e_{\min}(f_*\overline{F}) &\geqslant -e_{\max}((f_*\overline{F})^\vee) - \log(3\delta_K/2) - \log(\operatorname{rk} F)\\ &= -e_{\max}(f_*(\overline{F}^\vee\otimes\overline{\omega}_{\mathcal{O}_K})) - \log(3\delta_K/2) - \log(\operatorname{rk} F)\\ &\geqslant -\widehat{\mu}_{\max}(\overline{F}^\vee\otimes\overline{\omega}_{\mathcal{O}_K}) + \frac{1}{2}\log(\delta_K) - \log(3\delta_K/2) - \log(\operatorname{rk} F)\\ &= \widehat{\mu}_{\min}(\overline{F}) - \log|\Delta_K| - \frac{1}{2}\log\delta_K - \log(3/2) - \log(\operatorname{rk} F),\end{aligned}$$

where the two inequalities comes from (19) and (20), the first equality results from (21). □

Remark 3.10. — The comparison of minima and slopes has been discussed in [**26, 2, 16, 6**]. Let $\overline{F}$ be an arbitrary Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Up to now, the best upper bound for

$$\max_{1\leqslant i\leqslant \operatorname{rk} F}\big|e_i(\overline{F}) - \mu_i(\overline{F})\big|$$

is of order $\operatorname{rk}(F)\log\operatorname{rk}(F)$, where $\mu_i(\overline{F})$ is the i^{th} slope of $\overline{F}$ (see [**16**, Definition 5.10]). It should be interesting to know if this upper bound can be improved to be of order $\log\operatorname{rk}(F)$.

Proposition 3.11. — *With the notation of Proposition 3.6. One has, for any $t\in\mathbb{R}$,*

$$\mathcal{F}_t^S E_K \subset \mathcal{F}_{t-\beta}^M E_K,$$

where $\beta = D(E,g,g') + \log|\Delta_K| + \frac{1}{2}\log\delta_K + \log(3/2) + \log(\operatorname{rk} E)$.

Proof. — Let $\overline{F}$ be the saturated Hermitian vector subbundle of (E,g') such that $F_K = \mathcal{F}_t^S E_K$. By [**11**, Proposition 2.2.1] one has $\widehat{\mu}_{\min}(\overline{F})\geqslant t$. Lemma 3.9 implies

$$e_{\min}(\overline{F}) \geqslant t - \log|\Delta_K| - \frac{1}{2}\log\delta_K - \log(3/2) - \log(\operatorname{rk} F).$$

Denote by (F,g) the metrized vector bundle whose metrics are induced from (E,g). One has

$$e_{\min}(F,g) \geqslant e_{\min}(\overline{F}) - D(E,g,g') \geqslant t-\beta,$$

which implies that $F_K\subset\mathcal{F}^M_{t-\beta}E_K$. □

3.3. Comparison of asymptotic measures. — Let $B=\bigoplus_{n\geqslant 0}B_n$ be an approximable graded algebra. For any integer $r\geqslant 2$ and any element $\mathbf{n}=(n_i)\in\mathbb{N}^r$, denote by

$$\phi_{\mathbf{n}} : B_{n_1}\otimes\cdots\otimes B_{n_r}\longrightarrow B_{n_1+\cdots+n_r}$$

the canonical homomorphism defined by the algebra structure of B.

For each $n\geqslant 0$, let $(\mathscr{B}_n, g_n = (\|\cdot\|_\sigma))$ be a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$ such that $B_n = \mathscr{B}_{n,K}$. Let $(\mathscr{B}_n, g_n' = (\|\cdot\|_\sigma'))$ be a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Define

$$D_n := D(\mathscr{B}_n, g_n, g_n') = \max_{\sigma:K\rightarrow\mathbb{C}}\ \sup_{0\neq s\in B_{n,\sigma,\mathbb{C}}}\big|\log\|s\|_\sigma - \log\|s\|_\sigma'\big|.$$

Denote by $\mathcal{F}^M$ the minima filtration of $(\mathscr{B}_n, g_n)$ and by $\mathcal{F}^S$ the slope filtration of $(\mathscr{B}_n, g_n')$. Let

$$\nu_n^M = T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F}^M)} \qquad \text{and} \qquad \nu_n^S = T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F}^S)}.$$

In this subsection, we study the asymptotic behaviour of measure sequences $(\nu_n^M)_{n\geqslant 1}$ and $(\nu_n^S)_{n\geqslant 1}$.

Proposition 3.12. — *Assume that the following conditions are satisfied:*

(i) there exists an integer $n_0 \geqslant 1$ and a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $f(n) = o(n)$ $(n \to \infty)$ and that, for any integer $l \geqslant 2$ and any element $\mathbf{n} = (n_i)_{i=1}^l \in \mathbb{Z}_{\geqslant n_0}^l$, the height of $\phi_{\mathbf{n}}$ is bounded from above by $f(n_1) + \cdots + f(n_l)$;

(ii) $\sup\limits_{n\geqslant 1} \widehat{\mu}_{\max}(\mathscr{B}_n, g_n')/n < +\infty$.

Then the sequence $(\frac{1}{n}\widehat{\mu}_{\max}(\mathscr{B}_n, g_n'))_{n\geqslant 1}$ converges in $\mathbb{R}$, and the sequence of measures $(\nu_n^S)_{n\geqslant 1}$ converges vaguely to a Borel probability measure ν on $\mathbb{R}$.

Proof. — For any $n \in \mathbb{N}$ and any $t \in \mathbb{R}$, denote by $\overline{\mathscr{B}}_{n,t}$ the Hermitian vector subbundle of $(\mathscr{B}_n, g_n')$ such that $\mathscr{B}_{n,t,K} = \mathcal{F}_t^S B_n$. Let $l \geqslant 2$ be an integer, $\mathbf{n} = (n_i)_{i=1}^l \in \mathbb{Z}_{\geqslant n_0}^l$ and $(t_i)_{i=1}^l \in \mathbb{R}^l$. By using the dual form of [**10**, Theorem 1.1], one obtains

$$\begin{aligned}\widehat{\mu}_{\min}(\overline{\mathscr{B}}_{n_1,t_1} \otimes \cdots \otimes \overline{\mathscr{B}}_{n_l,t_l}) &\geqslant \sum_{i=1}^l \big(\widehat{\mu}_{\min}(\overline{\mathscr{B}}_{n_i,t_i}) - \log \operatorname{rk}(B_{n_i})\big)\\ &\geqslant \sum_{i=1}^l \big(t_i - \log \operatorname{rk}(B_{n_i})\big).\end{aligned}$$

Furthermore, by the assumption (i), the height of $\phi_{\mathbf{n}}$ is no grater than $f(n_1) + \cdots + f(n_l)$. Therefore, the canonical image of $\mathcal{F}_{t_1}^S B_{n_1} \otimes \cdots \otimes \mathcal{F}_{t_l}^S B_{n_l}$ in $B_{n_1+\cdots+n_l}$ lies in $\mathcal{F}_t^S B_{n_1+\cdots+n_l}$ with

$$t = \sum_{i=1}^l \big(t_i - f(n_i) - \log \operatorname{rk}(B_{n_i})\big).$$

Let $\widetilde{f} : \mathbb{N} \to \mathbb{R}_+$ such that $\widetilde{f}(n) = f(n) + \log \operatorname{rk}(B_n)$. The argument above shows that the graded algebra B is $\widetilde{f}$-quasi-filtered. Moreover, by assumption (i) and Proposition 2.4, one has $\lim\limits_{n\to+\infty} \widetilde{f}(n)/n = 0$. By Theorem 2.11, the sequence of measures converges vaguely to a certain Borel probability measure ν. □

Corollary 3.13. — *Under the assumption of Proposition 3.12, if $\lim\limits_{n\to\infty} D_n/n = 0$, then the sequence $(\frac{1}{n}e_{\max}(\mathscr{B}_n, g_n))_{n\geqslant 1}$ converges to $\lim\limits_{n\to\infty} \frac{1}{n}\widehat{\mu}_{\max}(\mathscr{B}_n, g_n')$; and the sequence of measures $(\nu_n^M)_{n\geqslant 1}$ converges vaguely to ν.*

Proof. — By Propositions 2.10, 3.6 and 3.11, for any integer $n \geqslant 1$, one has

$$\tau_{\alpha_n}\nu_n^S \succ \nu_n^M \succ \tau_{-\beta_n}\nu_n^S,$$

where

$$\alpha_n = \frac{1}{2n}\log\mathrm{rk}(B_n) + \frac{D_n}{n},\quad \beta_n = \frac{1}{n}\big(\log|\Delta_K| + \frac{1}{2}\log\delta_K - \log(3/2) - \log\mathrm{rk}(B_n)\big).$$

As $\lim\limits_{n\to\infty}\alpha_n = \lim\limits_{n\to\infty}\beta_n = 0$, the assertions result from Proposition 3.12. □

Remark 3.14. — The assumptions of Proposition 3.12 is fulfilled notably when the following conditions are satisfied:

(a) *the K-algebra structure on B gives rise to an $\mathcal{O}_K$-algebra structure on $\bigoplus_{n\geqslant 0}\mathscr{B}_n$;*

(b) *for any $(m,n)\in\mathbb{N}^2$, any $\sigma: K\to\mathbb{C}$ and for all $s\in B_{n,\sigma,\mathbb{C}}$, $s'\in B_{m,\sigma,\mathbb{C}}$, one has $\|ss'\|_\sigma \leqslant \|s\|_\sigma\|s'\|_\sigma$;*

(c) *$e_{\max}(\mathscr{B}_n, g_n) = O(n)$ $(n\to\infty)$.*

Note that, under the conditions (a) and (b) above, the height of $\phi_{\mathbf{n}}$, $\mathbf{n}=(n_i)_{i=1}^l\in\mathbb{Z}^l_{\geqslant n_0}$, does not exceed $\frac{1}{2}\sum_{i=1}^l\log\mathrm{rk}(B_{n_i})$. The equivalence of the condition (c) and the condition (ii) in Proposition 3.12 results from (20). See [**11**, Remark 4.1.6] for details. One can also compare the conditions above with those in [**25**, page 12].

In this particular case, the graded algebra B, equipped with minimum filtrations, is actually 0-quasi-filtered, where 0 denotes the constant zero function. So we may deduce the convergence of $(\frac{1}{n}e_{\max}(\mathscr{B}_n,g_n))_{n\geqslant 1}$ and $(\nu_n^M)_{n\geqslant 1}$ directly from Theorem 2.11. However, as we shall see in the proof of Proposition 4.6, the comparison of limits established in Corollary 3.13 will play an important role in the study of arithmetic volume function. So we have chosen an indirect approach to emphasis this comparison.

4. Approximable graded linear series in arithmetic

In this section, we recall a result on Fujita approximation for graded linear series due to Lazarsfeld and Mustaţă [**20**]. We then give several examples of approximable graded linear series which come naturally from the arithmetic setting.

4.1. Reminder on geometric Fujita approximation. — Let K be a field and X be a projective variety (i.e. integral projective scheme) defined over K. Let L be a big line bundle on X. Denote by $B:=\bigoplus_{n\geqslant 0}H^0(X,L^{\otimes n})$ the graded K-algebra of global sections of tensor powers of L. For *graded linear series* of L we mean a graded sub-K-algebra of B. The following definition is borrowed from [**20**].

Definition 4.1. — We say that a graded linear series $W=\bigoplus_{n\geqslant 0}W_n$ of L *contains an ample divisor* if there exists an integer $p\geqslant 1$, an ample line bundle A and an effective line bundle M on X, together with a non-zero section $s\in H^0(X,M)$, such that $L^{\otimes p}\cong A\otimes M$, and that the homomorphism of graded algebras

$$\bigoplus_{n\geqslant 0}H^0(X,A^{\otimes n})\longrightarrow\bigoplus_{n\geqslant 0}H^0(X,L^{\otimes np})$$

induced by s factors through $\bigoplus_{n\geqslant 0}W_{np}$.

Remark 4.2. — In [**20**, Definition 2.9], this condition was called the "condition (C)". As a big divisor is always the sum of an ample divisor and an effective one, the total graded linear series B contains an ample divisor.

Definition 4.3. — Let $W = \bigoplus_{n\geqslant 0} W_n$ be a graded linear series of L. Denote by $\mathrm{vol}(W)$ the number

$$\mathrm{vol}(W) := \limsup_{n\to\infty} \frac{\mathrm{rk}(W_n)}{n^{\dim X}/(\dim X)!}. \tag{22}$$

Note that $\mathrm{vol}(B) = \mathrm{vol}(L)$. For a general linear series W of L, one has $\mathrm{vol}(W) \leqslant \mathrm{vol}(L)$. By using the method of Okounkov bodies introduced in [**24**], Lazarsfeld and Mustaţă have established the following generalization of Fujita's approximation theorem.

Theorem 4.4 **(Lazarsfeld-Mustaţă)**. — *Assume that $W = \bigoplus_{n\geqslant 0} W_n$ is a graded linear series of L which contains an ample divisor and such that $W_n \neq 0$ for sufficiently large n. Then W is approximable.*

In particular, the total graded linear series B is approximable. In [**20**, Remark 3.4], the authors have explained why their theorem implies the Fujita's approximation theorem in its classical form. We include their explanation as the corollary below.

Corollary 4.5 **(Geometric Fujita approximation)**. — *For any $\varepsilon > 0$, there exists an integer $p \geqslant 1$, a birational projective morphism $\varphi : X' \to X$, an ample line bundle A and an effective line bundle M such that*
1) one has $\varphi^*(L^{\otimes p}) \cong A \otimes M$;
2) $\mathrm{vol}(A) \geqslant p^{\dim X}(\mathrm{vol}(L) - \varepsilon)$.

Proof. — For any integer p such that $B_p \neq 0$, let $\varphi_p : X_p \to X$ be the blow-up (twisted by L) of X along the base locus of B_p. That is

$$X_p = \mathrm{Proj}\Big(\mathrm{Im}\Big(\bigoplus_{n\geqslant 0} S^n(\pi^* B_p) \longrightarrow \bigoplus_{n\geqslant 0} L^{\otimes np}\Big)\Big).$$

Denote by E_p the exceptional divisor and by s the global section of $\mathcal{O}(E_p)$ which trivializes $\mathcal{O}(E_p)$ outside the exceptional divisor. By definition, one has $\mathcal{O}_{X_p}(1) \cong \varphi_p^* L^{\otimes p} \otimes \mathcal{O}(-E_p)$. On the other hand, the canonical homomorphism $\varphi_p^*\pi^* B_p \to \mathcal{O}_{X_p}(1)$ is surjective, therefore corresponds to a morphism of schemes $i_p : X_p \to \mathbb{P}(B_p)$ such that $i_p^*\mathcal{O}_{\mathbb{P}(B_p)}(1) = \mathcal{O}_{X_p}(1)$. The restriction of global sections of $\mathcal{O}_{\mathbb{P}(B_p)}(n)$ on X_p gives an injective homomorphism

$$\mathrm{Im}(S^n B_p \to B_{np}) \longrightarrow H^0(X_p, \mathcal{O}_{X_p}(n)),$$

where we have identified $H^0(X_p, \mathcal{O}_{X_p}(n))$ with a subspace of $H^0(X_p, \varphi_p^* L^{\otimes n})$ via s. Since the total grade linear series B is approximable, one has

$$\sup_p \liminf_{n\to\infty} \frac{\mathrm{rk}(\mathrm{Im}(S^n B_p \to B_{np}))}{\mathrm{rk}\, B_{np}} = 1,$$

which implies

$$\sup_p \lim_{n\to\infty} \frac{\operatorname{rk} H^0(X_p, \mathcal{O}_{X_p}(n))}{(np)^d/d!} = \operatorname{vol}(L).$$

The line bundle $\mathcal{O}_{X_p}(1)$ constructed above is actually nef and big. However, a slight perturbation of L permits to conclude. □

4.2. Arithmetic volume of approximable graded linear series. — In the sequel, K denotes a number field and $\mathcal{O}_K$ denotes its integer ring. Let $\delta_K := [K:\mathbb{Q}]$ be the degree of K over $\mathbb{Q}$. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathcal{O}_K$ be a projective arithmetic variety of total dimension d and $X = \mathscr{X}_K$. Let $\overline{\mathscr{L}}$ be a Hermitian line bundle on $\mathscr{X}$, supposed to be big in the sense of Moriwaki [**21**]. Let $L = \mathscr{L}_K$. Note that L is a big line bundle on X.

Let B be a graded linear series of L. For any integer $n \geqslant 0$, denote by $\mathscr{B}_n$ the saturation of B_n in $\pi_*(\mathscr{L}^{\otimes n})$. For any embedding $\sigma : K \to \mathbb{C}$, denote by $\|\cdot\|_{\sigma,\sup}$ the sup-norm on $B_{n,\sigma,\mathbb{C}}$. Thus we obtain a metrized vector bundle $(\mathscr{B}_n, g_n)$ with $g_n = (\|\cdot\|_{\sigma,\sup})_{\sigma:K\to\mathbb{C}}$.

Inspired by [**21**], we define the arithmetic volume function of B as follows:

$$\widehat{\operatorname{vol}}(B) := \limsup_{n\to\infty} \frac{\widehat{h}^0(\mathscr{B}_n, g_n)}{n^d/d!},$$

where for any metrized vector bundle $\overline{E} = (E, (\|\cdot\|_\sigma))$ on $\operatorname{Spec}\mathcal{O}_K$, $\widehat{h}^0(\overline{E})$ is defined as

$$\widehat{h}^0(\overline{E}) := \log \#\{s \in E \,|\, \forall \sigma : K \to \mathbb{C},\ \|s\|_\sigma \leqslant 1\}.$$

Proposition 4.6. — *Assume that the graded linear series B is approximable. Then the sequence $(\frac{1}{n}e_{\max}(\mathscr{B}_n, g_n))_{n\geqslant 1}$ converges in $\mathbb{R}$. Furthermore, for any integer $n \geqslant 1$, let $\nu_n := T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F}^M)}$ be the normalized probability measure associated to the minimum filtration of $(\mathscr{B}_n, g_n)$, then the sequence of measures $(\nu_n)_{n\geqslant 1}$ converges vaguely to a Borel probability measure ν_B. Moreover, one has*

$$\int_{\mathbb{R}} \max\{x, 0\}\,\nu_B(\mathrm{d}x) = \frac{\widehat{\operatorname{vol}}(B)}{\delta_K d \operatorname{vol}(B)}. \tag{23}$$

Proof. — To establish the convergence of $(\frac{1}{n}e_{\max}(\mathscr{B}_n, g_n))_{n\geqslant 1}$ and $(\nu_n)_{n\geqslant 1}$, it suffices to prove that $(\mathscr{B}_n, g_n)$ verify the conditions in Remark 3.14, where (a) and (b) are straightforward. In order to prove the condition (c), we introduce, for any integer $n \geqslant 1$, an auxiliary family $g_n' = (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$ of Hermitian norms on $\pi_*(\mathscr{L}^{\otimes n})$, invariant under complex conjugation, and such that, for any $0 \neq s \in H^0(X_\sigma(\mathbb{C}), L_{\sigma,\mathbb{C}})$.

$$\log\|s\|_\sigma - \frac{3}{2}\log(\operatorname{rk}\pi_*(\mathscr{L}^{\otimes n})) \leqslant \log\|s\|_{\sigma,\sup} \leqslant \log\|s\|_\sigma - \frac{1}{2}\log(\operatorname{rk}\pi_*(\mathscr{L}^{\otimes n})). \tag{24}$$

This is always possible by the argument of the ellipsoids of John or Löwner, see [**16**, §2]. It suffices to establish the estimation $\widehat{\mu}_{\max}(\pi_*(\mathscr{L}^{\otimes n}), g_n') \ll n$. Let Σ be a generic family (i.e., Σ is dense in X) of algebraic points in X. Each point P in Σ extends in a unique way to a $\mathcal{O}_{K(P)}$ point of $\mathscr{X}$, where $K(P)$ is the field of definition of P. Therefore we may consider elements in Σ as points of $\mathscr{X}$ valued in algebraic integer rings. Now consider the evaluation map $\pi_*(\mathscr{L}^{\otimes n}) \longrightarrow \bigoplus_{P\in\Sigma} P^*\mathscr{L}$. It is generically injective since Σ is dense in X. Therefore, there exists a subset Σ_n of Σ whose cardinal is $\mathrm{rk}(\pi_*(\mathscr{L}^{\otimes n}))$ and such that the evaluation map

$$\phi_n : \pi_*(\mathscr{L}^{\otimes n}) \longrightarrow \bigoplus_{P\in\Sigma_n} P^*\mathscr{L}$$

is still generically injective. Therefore, after suitable extension of the ground field, the slope inequality asserts that

$$\widehat{\mu}_{\max}(\pi_*(\mathscr{L}^{\otimes n}), g_n') \leqslant \sup_{P\in\Sigma_n} nh_{\overline{\mathscr{L}}}(P) + h(\phi_n) \leqslant n \sup_{P\in\Sigma} h_{\overline{\mathscr{L}}}(P).$$

Since Σ is arbitrary, we obtain that $\frac{1}{n}\widehat{\mu}_{\max}(\pi_*(\mathscr{L}^{\otimes n}), g_n')$ is bounded from above by the essential minimum of $\overline{\mathscr{L}}$ (see [**32**, §5] for definition. Attention, in [**32**], the author denoted it as $e_1(\overline{\mathscr{L}})$).

The equality (23) comes from the following Lemma.

Lemma 4.7. — *Let $(E, g = (\|\cdot\|_\sigma))$ be a metrized vector bundle and $(E, g' = (\|\cdot\|_\sigma'))$ be a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Assume that $r := \mathrm{rk}(E) > 0$. Let*

$$D = \max_{\sigma:K\to\mathbb{C}} \sup_{0\neq s\in E_{\sigma,\mathbb{C}}} \big|\log\|s\|_\sigma - \log\|s\|_\sigma'\big|.$$

Denote by ν the Borel probability measure associated to the Harder-Narasimhan filtration of $\overline{E} := (E, g')$. Then there exists a function $C_0 : \mathbb{N}_* \to \mathbb{R}_+$, independent of all data above, satisfying $C_0(n) \ll n\log n$, and such that

$$\left|\delta_K r \int_{\mathbb{R}} \max\{x, 0\}\,\nu(\mathrm{d}x) - \widehat{h}^0(E, g)\right| \leqslant (\delta_K D + \log|\Delta_K|)r + C_0(r).$$

Proof of the Lemma. — Denote by $\overline{M} = (\mathcal{O}_K, (\|\cdot\|_\sigma^M))$ the Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ such that $\|\mathbf{1}\|_\sigma^M = e^{-D}$, where $\mathbf{1}$ is the unit element in $\mathcal{O}_K$. By définition, one has $\widehat{h}^0(\overline{E}\otimes\overline{M}^\vee) \leqslant \widehat{h}^0(\overline{E}, g) \leqslant \widehat{h}^0(\overline{E}\otimes\overline{M})$. Moreover, the Borel probability measures associated to the Harder-Narasimhan filtrations of $\overline{E}\otimes\overline{M}$ and $\overline{E}\otimes\overline{M}^\vee$ are respectively $\tau_D\nu$ and $\tau_{-D}\nu$. By [**12**, Lemma 7.1 and Proposition 3.3], there exists a function $C_0 : \mathbb{N}_* \to \mathbb{R}_+$, independent of $\overline{E}$, satisfying the estimation $C_n(n) \ll n\log n$, and such that

$$\widehat{h}^0(\overline{E}\otimes\overline{M}^\vee) \geqslant \delta_K r \int_{\mathbb{R}} \max\{x, 0\}\,\tau_{-D}\nu(\mathrm{d}x) - r\log|\Delta_K| - C_0(r)$$
$$\widehat{h}^0(\overline{E}\otimes\overline{M}) \leqslant \delta_K r \int_{\mathbb{R}} \max\{x, 0\}\,\tau_D\nu(\mathrm{d}x) + r\log|\Delta_K| + C_0(r)$$

Since $\max\{x+D, 0\} \leqslant \max\{x, 0\} + D$ and $\max\{x-D, 0\} \geqslant \max\{x, 0\} - D$, we obtain the desired inequality. $\square$

By using Lemma 4.7, we obtain

$$\left|\widehat{h}^0(\mathscr{B}_n, g_n) - nr_n\delta_K \int_{\mathbb{R}} \max\{x, 0\}\,\nu_n'(\mathrm{d}x)\right| \leqslant \frac{3}{2}\delta_K r_n\log(r_n) + r_n\log|\Delta_K| + C_0(r_n),$$

where $r_n = \mathrm{rk}(B_n)$, and ν'_n is the Borel probability measure associated to $(\mathscr{B}_n, g'_n)$ (here we still use g'_n to denote the metrics on $\mathscr{B}_n$ induced from $(\pi_*(\mathscr{L}^{\otimes n}), g'_n)$). We have shown that $(\nu'_n)_{n\geqslant 1}$ also converge vaguely to ν_B. Furthermore, Proposition 2.4 show that $r_n = \mathrm{vol}(B)n^{d-1}/(d-1)!+o(n^{d-1})$. By passing to limit, we obtain (23). □

4.3. Examples of approximable graded linear series. — In this subsection, we give some examples of approximable graded linear series of L which come from the arithmetic.

Denote by $B=\bigoplus_{n\geqslant 0}H^0(X,L^{\otimes n})$ the sectional algebra of L. For any real number λ, let $B^{[\lambda]}$ be the graded sub-K-module of B defined as follows:

$$B_0^{[\lambda]} := K, \qquad B_n^{[\lambda]} := \mathrm{Vect}_K\big(\{s\in B_n \,|\, \forall\,\sigma : K\rightarrow\mathbb{C},\ \|s\|_{\sigma,\sup}\leqslant e^{-\lambda n}\}\big). \tag{25}$$

The following property is straightforward from the definition.

Proposition 4.8. — *For any $\lambda\in\mathbb{R}$, $B^{[\lambda]}$ is a graded linear series of L.*

Note that $B^{[0]}$ is noting but the graded linear series generated by effective sections. For any integer $n\geqslant 0$ and any real number λ, denote by $\mathscr{B}_n=\pi_*(\mathscr{L}^{\otimes n})$ and by $\mathscr{B}_n^{[\lambda]}$ the saturation of $B_n^{[\lambda]}$ in $\mathscr{B}_n$. We shall use the symbol g_n to denote the family of sup-norms on $\mathscr{B}_n$ or on $\mathscr{B}_n^{[\lambda]}$. By definition, for any integer $n\geqslant 1$ and any $\lambda\in\mathbb{R}$, one has

$$B_n^{[\lambda]}=\mathcal{F}^M_{n\lambda}B_n,$$

where $\mathcal{F}^M$ is the minimum filtration of $(\mathscr{B}_n, g_n)$.

Since we have assumed $\overline{\mathscr{L}}$ to be arithmetically big, the line bundle L is also big (see [**22**, Introduction] and [**29**, Corollary 2.4]). Hence by Theorem 4.4, the total graded linear series B is approximable. By Corollary 3.13, we obtain that the sequence $(\frac{1}{n}e_{\max}(\mathscr{B}_n,g_n))_{n\geqslant 1}$ converges to a real number which we denote by $\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}})$. Note that, if $\overline{M}$ is a Hermitian line bundle on $\mathrm{Spec}\,\mathcal{O}_K$, then

$$\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}}\otimes\pi^*(\overline{M}))=\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}})+\delta_K^{-1}\widehat{\deg}(\overline{M}).$$

For any real number λ, denote by $\overline{\mathcal{O}}_\lambda$ the Hermitian line bundle on $\mathrm{Spec}\,\mathcal{O}_K$ whose underlying $\mathcal{O}_K$-module is trivial, and such that $\|\mathbf{1}\|_\sigma=e^{-\lambda}$ for any σ. Note that the Arakelov degree of $\overline{\mathcal{O}}_\lambda$ is $\widehat{\deg}(\overline{\mathcal{O}}_\lambda)=\delta_K\lambda$.

Proposition 4.9. — *Let λ be a real number such that $\lambda<\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}})$. Then the graded linear series $B^{[\lambda]}$ contains an ample divisor, and for sufficiently large n, one has $B_n^{[\lambda]}\neq 0$.*

Proof. — Note that $\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda})>0$. Since L is big, by [**12**, Theorem 5.4], $\overline{\mathscr{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda}$ is arithmetically big. Therefore, for sufficiently large n, $\mathscr{L}^{\otimes n}$ has a non-zero global section s_n such that $\|s_n\|_{\sigma,\sup}\leqslant e^{-\lambda n}$ for any $\sigma : K\rightarrow\mathbb{C}$, which proves that $B_n^{[\lambda]}\neq 0$. Furthermore, since $\overline{\mathscr{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda}$ is arithmetically big, by [**29**, Corollary 2.4], there exists an integer $p\geqslant 1$ and two Hermitian line bundles $\overline{\mathscr{A}}$ and $\overline{\mathscr{M}}$, such that $\overline{\mathscr{A}}$ is ample in the sense of Zhang [**32**], $\overline{\mathscr{M}}$ has a non-zero effective global section s, and that $(\overline{\mathscr{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda})^{\otimes p}\cong\overline{\mathscr{A}}\otimes\overline{\mathscr{M}}$. By taking p sufficiently divisible, we may assume that the graded K-algebra $\bigoplus_{n\geqslant 0} H^0(X,\mathscr{A}_K^{\otimes n})$ is generated by effective sections of $\mathscr{A}$. These sections, viewed as sections of $\overline{\mathscr{A}}\otimes\pi^*\overline{\mathcal{O}}_\lambda^{\otimes p}$, have sup-norms $\leqslant e^{-p\lambda}$. Therefore the homomorphism

$$\bigoplus_{n\geqslant 0} H^0(X,\mathscr{A}_K)\longrightarrow\bigoplus_{n\geqslant 0} H^0(X,L^{\otimes np})$$

induced by s factors through $\bigoplus_{n\geqslant 0} B_{np}^{[\lambda]}$. □

Corollary 4.10. — *For any real number λ such that $\lambda<\widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})$, the graded linear series $B^{[\lambda]}$ of L is approximable.*

Proof. — This is a direct consequence of Proposition 4.9 and Theorem 4.4. □

5. Arithmetic Fujita approximation

In this section, we establish the conjecture of Moriwaki on the arithmetic analogue of Fujita approximation. Let $\pi:\mathscr{X}\rightarrow\operatorname{Spec}\mathcal{O}_K$ be an arithmetic variety of total dimension d and $\overline{\mathscr{L}}$ be a Hermitian line bundle on $\mathscr{X}$ which is arithmetically big.

Write $L=\mathscr{L}_K$ and denote by $B:=\bigoplus_{n\geqslant 0} H^0(X,L^{\otimes n})$ the total graded linear series of L. For any integer $n\geqslant 1$, let $\overline{\mathscr{B}}_n$ be the $\mathcal{O}_K$-module $\pi_*(\mathscr{L}^{\otimes n})$ equipped with sup-norms. Define by convention $\overline{\mathscr{B}}_0$ as the trivial Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$. Denote by $\widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})=\lim\limits_{n\rightarrow\infty}\frac{1}{n}e_{\max}(\overline{\mathscr{B}}_n)$.

For any real number λ, let $B^{[\lambda]}$ be the graded linear series of L defined in (25). For any integer $n\geqslant 0$, let $\overline{\mathscr{B}}_n^{[\lambda]}$ be the saturation of $B_n^{[\lambda]}$ in B_n equipped with induced metrics. For any integer $p\geqslant 1$ such that $B_p^{[0]}\neq 0$, let $B^{(p)}$ be the graded sub-K-algebra of B generated by $B_p^{[0]}$. For any integer $n\geqslant 1$, let $\overline{\mathscr{B}}_{np}^{(p)}$ be the saturated Hermitian vector subbundle of $\overline{\mathscr{B}}_{np}$ such that $\mathscr{B}_{np,K}^{(p)}=B_{np}^{(p)}$.

Theorem 5.1. — *The following equality holds:*

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}})=\sup_p\widehat{\mathrm{vol}}\big(B^{(p)}\big),$$

where $B^{(p)}$ is the graded linear series of L generated by $B_p^{[0]}$ defined above.

Proof. — For any integer $n\geqslant 1$, let $\nu_n=T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F}^M)}$, where $\mathcal{F}^M$ is the minimum filtration of $\overline{\mathscr{B}}_n$. We have shown in Proposition 4.6 that the sequence $(\nu_n)_{n\geqslant 1}$ converges vaguely to a Borel probability measure which we denote by ν. Similarly, for any integer $n\geqslant 1$, let $\nu_n^{(p)}=T_{\frac{1}{np}}\nu_{(B_{np}^{(p)},\mathcal{F}^M)}$. The sequence $(\nu_n^{(p)})_{n\geqslant 1}$ also converges vaguely to a Borel probability measure which we denote by $\nu^{(p)}$.

For any subdivision $D:0=t_0<t_1<\cdots<t_m<\widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})$ of the interval $[0,\widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})[$ such that

$$\nu(\{t_1,\cdots,t_m\})=0, \tag{26}$$

denote by $h_D:\mathbb{R}\rightarrow\mathbb{R}$ the function such that

$$h_D(x)=\sum_{i=0}^{m-1}t_i\,1\!\!1_{[t_i,t_{i+1}[}(x)+t_m\,1\!\!1_{[t_m,\infty[}(x).$$

After Corollary 4.10, for any $\varepsilon>0$, there exists a sufficiently large integer $p=p(\varepsilon,D)\geqslant 1$ such that $B^{(p)}$ approximates simultaneously all algebras $B^{[t_i]}$ ($i\in\{0,\cdots,m\}$). That is, there exists $N_0\in\mathbb{N}$ such that, for any $n\geqslant N_0$, one has

$$\inf_{0\leqslant i\leqslant m}\frac{\operatorname{rk}\big(\operatorname{Im}(S^nB_p^{[t_i]}\rightarrow B_{np}^{[t_i]})\big)}{\operatorname{rk}(B_{np}^{[t_i]})}\geqslant 1-\varepsilon.$$

We then obtain that

$$\operatorname{rk}(\mathcal{F}^M_{npt_i}B^{(p)}_{np})\geqslant\operatorname{rk}\big(\operatorname{Im}(S^nB_p^{[t_i]}\rightarrow B_{np}^{[t_i]})\big)\geqslant(1-\varepsilon)\operatorname{rk}(B_{np}^{[t_i]}).$$

Note that

$$\begin{aligned}np\operatorname{rk}(B^{(p)}_{np})\int_{\mathbb{R}}\max\{t,0\}\,\nu^{(p)}_{np}(\mathrm{d}t)&=-\int_{\mathbb{R}}\max\{t,0\}\,\mathrm{d}\operatorname{rk}(\mathcal{F}^M_tB^{(p)}_{np})\\ \geqslant\sum_{i=0}^{m-1}npt_i\Big(\operatorname{rk}(\mathcal{F}^M_{npt_i}B^{(p)}_{np})-\operatorname{rk}(\mathcal{F}^M_{npt_{i+1}}B^{(p)}_{np})\Big)&+npt_m\operatorname{rk}(\mathcal{F}^M_{npt_m}B^{(p)}_{np}).\end{aligned}$$

By Abel summation formula, one obtains

$$\begin{aligned}\operatorname{rk}(B^{(p)}_{np})\int_{\mathbb{R}}\max\{t,0\}\,\nu^{(p)}_{np}(\mathrm{d}t)&\geqslant\sum_{i=1}^m(t_i-t_{i-1})\operatorname{rk}(\mathcal{F}^M_{npt_i}B^{(p)}_{np})\\ &\geqslant(1-\varepsilon)\sum_{i=1}^m(t_i-t_{i-1})\operatorname{rk}(B^{[t_i]}_{np}).\end{aligned}$$

Still by Abel summation formula, one gets

$$\operatorname{rk}(B^{(p)}_{np})\int_{\mathbb{R}}\max\{t,0\}\,\nu^{(p)}_{np}(\mathrm{d}t)\geqslant(1-\varepsilon)\operatorname{rk}_K(B_{np})\int h_D(x)\,\nu_{np}(\mathrm{d}t).$$

By (23), one has

$$\begin{aligned}&\lim_{n\rightarrow\infty}\frac{\delta_Kd}{(np)^{d-1}/(d-1)!}\operatorname{rk}(B^{(p)}_{np})\int_{\mathbb{R}}\max\{t,0\}\nu^{(p)}_{np}(\mathrm{d}t)\\ &=\delta_Kd\operatorname{vol}(B^{(p)})\int_{\mathbb{R}}\max\{t,0\}\,\nu^{(p)}(\mathrm{d}t)=\widehat{\operatorname{vol}}(B^{(p)}).\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\operatorname{vol}}(B^{(p)})&\geqslant\lim_{n\rightarrow\infty}\frac{\delta_Kd}{(np)^{d-1}/(d-1)!}(1-\varepsilon)\operatorname{rk}(B_{np})\int_{\mathbb{R}}h_D\,\mathrm{d}\nu_{np}\\ &=\delta_Kd(1-\varepsilon)\operatorname{vol}(L)\int_{\mathbb{R}}h_D\,\mathrm{d}\nu,\end{aligned}$$

where the equality follows from [**8**, IV §5 $n°12$ Proposition 22]. Choose a sequence of subdivisions $(D_j)_{j\in\mathbb{N}}$ verifying the condition (26) and such that $h_{D_j}(t)$ converges uniformly to $\max\{t,0\}-\max\{t-\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}}),0\}$ when $j\to\infty$, one obtains

$$\widehat{\mathrm{vol}}(B^{(p)})\geqslant\delta_K d(1-\varepsilon)\mathrm{vol}(L)\int_{\mathbb{R}}\max\{t,0\}\,\nu(\mathrm{d}t)=(1-\varepsilon)\widehat{\mathrm{vol}}(\overline{\mathscr{L}}),$$

thanks to (23). The theorem is thus proved. □

In the following, we explain why Theorem 5.1 implies the Fujita's arithmetic approximation theorem in the form conjectured by Moriwaki. Our strategy is quite similar to Corollary 4.5, except that the choice of metrics on the approximating invertible sheaf requires rather subtle analysis on the superadditivity of probability measures associated to a filtered graded algebra, which we put in the appendix.

Theorem 5.2 **(Arithmetic Fujita approximation)**. — *For any $\varepsilon>0$, there exists a birational morphism $\nu:\mathscr{X}'\to\mathscr{X}$, an integer $p\geqslant 1$ together with an decomposition $\nu^*\overline{\mathscr{L}}^{\otimes p}\cong\overline{\mathscr{A}}\otimes\overline{\mathscr{M}}$ such that*

1) $\overline{\mathscr{M}}$ is effective and $\overline{\mathscr{A}}$ is arithmetically ample;

2) one has $p^{-d}\widehat{\mathrm{vol}}(\overline{\mathscr{A}})\geqslant\widehat{\mathrm{vol}}(\overline{\mathscr{L}})-\varepsilon$.

Proof. — By [**22**, Theorem 4.3], we may assume that $\mathscr{X}$ is generically smooth. For any integer $p\geqslant 1$ such that $B_p^{[0]}\neq 0$, let $\phi_p:\mathscr{X}_p\to\mathscr{X}$ be the blow up (twisted by $\mathscr{L}$) of $\mathscr{X}$ along the base locus of $\mathscr{B}_p^{[0]}$. In other words, $\mathscr{X}_p$ is defined as

$$\mathscr{X}_p=\mathrm{Proj}\Big(\mathrm{Im}\Big(\bigoplus_{n\geqslant 0}\pi^*\mathscr{B}_p^{(np)}\longrightarrow\bigoplus_{n\geqslant 0}\mathscr{L}^{\otimes np}\Big)\Big).$$

Let $\mathscr{A}_p=\mathcal{O}_{\mathscr{X}_p}(1)$ and $\mathscr{M}_p$ be the invertible sheaf defined by the exceptional divisor. Let s be the global section of $\mathscr{M}_p$ which trivializes $\mathscr{M}_p$ outside the exceptional divisor. By definition, one has $\phi_p^*\mathscr{L}^{\otimes p}\cong\mathscr{A}_p\otimes\mathscr{M}_p$. On the other hand, the canonical homomorphism $\phi_p^*\pi^*\mathscr{B}_p^{[0]}\to\mathscr{A}_p$ induces a morphism $i_p:\mathscr{X}_p\to\mathbb{P}(\mathscr{B}_p^{[0]})$ such that $i_p^*(\mathcal{L}_p)\cong\mathscr{A}_p$, where $\mathcal{L}_p=\mathcal{O}_{\mathbb{P}(\mathscr{B}_p^{[0]})}(1)$. The restriction of global sections of $\mathcal{L}_p^{\otimes n}$ gives an injective homomorphism

$$\mathrm{Im}(S^n\mathscr{B}_p^{[0]}\to\mathscr{B}_{np})=\mathscr{B}_{np}^{(p)}\longrightarrow H^0(\mathscr{X}_p,\mathscr{A}_p^{\otimes n}),\tag{27}$$

where the last $\mathcal{O}_K$-module is considered as a submodule of $H^0(\mathscr{X}_p,\phi_p^*\mathscr{L}^{\otimes p})$ via s.

For any integer $n\geqslant 1$ and any embedding $\sigma:K\to\mathbb{C}$, denote by $\|\cdot\|_{\sigma,n}$ the quotient Hermitian norm on $\mathscr{A}_{p,\sigma}$ induced by the surjective homomorphism $\phi_p^*\pi^*\mathscr{B}_{np}^{(p)}\to\mathscr{A}_p^{\otimes n}$, where on $\mathscr{B}_{np}^{(n)}$ we have chosen the John norm $\|\cdot\|_{\sigma,\mathrm{John}}$ associated to the sup-norm $\|\cdot\|_{\sigma,\sup}$ (see [**16**, §4.2] for details). Thus the Hermitian norms on $\mathscr{A}_p$ are positive and smooth. Now let $\sigma:K\to\mathbb{C}$ be an embedding and x be a complex point of $\mathscr{X}_p$ outside the exceptional divisor. It corresponds to an one-dimensional quotient of $B_{p,\sigma}^{[0]}$, which induces, for any integer $n\geqslant 1$, an one-dimensional quotient $l_{n,x}$ of $B_{np,\sigma}^{(p)}$. By classical result on convex bodies in Banach space, there exists an affine hyperplane parallel to the $\mathrm{Ker}(B_{np,\sigma}^{(p)}\to l_{n,x})$ and tangent to the closed unit ball of $B_{np,\sigma}^{(p)}$. In other words, there exists $v \in B^{(p)}_{np,\sigma}$ whose image in $\mathscr{A}^{\otimes n}_{p,\sigma}(x)$ has norm $\|v\|_{\sigma,\mathrm{John}} \geqslant \|v\|_{\sigma,\sup}$. Note that, as a section of $L^{\otimes n}_\sigma$ over $X_\sigma(\mathbb{C})$, one has $\|v_x\|_\sigma \leqslant \|v\|_{\sigma,\sup}$. Hence, for any section u of $\mathscr{A}_{p,\sigma}$ over a neighbourhood of x, one has $\|u_x\|_{\sigma,n} \geqslant \|u_x \otimes s_x\|_\sigma$. Therefore, if we equip $\mathscr{A}_p$ with metrics $\alpha_n = (\|\cdot\|_{\sigma,n})_{\sigma:K\to\mathbb{C}}$ and define $(\mathscr{M}_p, \beta_n) := \phi_p^*\overline{\mathscr{L}} \otimes (\mathscr{A}_p, \alpha_n)^\vee$. Then the section s of $\mathscr{M}_p$ is an effective section. For any integer $n \geqslant 1$, one has

$$p^{-d}\widehat{\mathrm{vol}}(\mathscr{A}_p, \alpha_n) \geqslant \widehat{\mathrm{vol}}(B^{(p)}, \alpha_n).$$

Note that, for any $\sigma : K \to \mathbb{C}$ and any element $v \in B^{(p)}_{np,\sigma}$ considered as a section in $H^0(\mathscr{X}_{p,\sigma}(\mathbb{C}), \mathscr{A}^{\otimes n}_{p,\sigma})$ via (27), the sup-norms of v relatively to the metrics in α_n are bounded from above by the John norms of v considered as a section of L_σ corresponding to the sup-norms induced by the norms of $\overline{\mathscr{L}}$. Thus Corollary A.2 combined with (23) implies that

$$\sup_n \widehat{\mathrm{vol}}(B^{(p)}, \alpha_n) \geqslant \widehat{\mathrm{vol}}(B^{(p)}).$$

Therefore, by Theorem 5.1, for any $\varepsilon > 0$, there exist certain integers $p \geqslant 1$ and $n \geqslant 1$ such that $p^{-d}\widehat{\mathrm{vol}}(\mathscr{A}_p, \alpha_n) \geqslant \mathrm{vol}(\overline{\mathscr{L}}) - \varepsilon$. Here $(\mathscr{A}_p, \alpha_n)$ is rather nef and big since it is generated by effective global sections. However, a slight perturbation of $\overline{\mathscr{L}}$ permits to conclude. □

6. Approximating subalgebras

We keep the notation in §5. In this section, we show that if a positive finite generated subalgebra of $\overline{\mathscr{B}}$ approximates well the arithmetic volume of $\overline{\mathscr{L}}$, then it also approximates well the asymptotic measure of $\overline{\mathscr{L}}$ truncated at 0.

Let $p \geqslant 1$ be an integer. Assume that $\overline{\mathscr{L}}^{\otimes p}$ is decomposed as $\overline{\mathscr{A}} \otimes \overline{\mathscr{M}}$, where $\overline{\mathscr{A}}$ is arithmetically ample and $\overline{\mathscr{M}}$ has a non-zero effective section s. Through the section s we may consider the section algebra $\bigoplus_{n\geqslant 0} H^0(X, \mathscr{A}^{\otimes n}_K)$ as a graded sub-K-algebra of B. As $\overline{\mathscr{A}}$ is ample, for sufficiently large n, one has $H^0(X, \mathscr{A}^{\otimes n}_K) \subset \mathcal{F}^M_0(B_{np})$.

Proposition 6.1. — *Let $p \geqslant 1$ be an integer and S be a graded subalgebra of B generated by a subspace of B_p. For any integer $n \geqslant 1$, let $\overline{\mathscr{S}}_n$ be the saturated sub-$\mathcal{O}_K$-module of B_n, equipped with induced metrics, and such that $\mathscr{S}_{n,K} = S_n$; let $\nu_{\overline{\mathscr{S}}_n}$ be the measure associated to the minimum filtration of $\overline{\mathscr{S}}_n$. Denote by ν the vague limit of the measure sequence $(T_{\frac{1}{np}}\nu_{\overline{\mathscr{S}}_{np}})_{n\geqslant 1}$. The for any $x \in \mathbb{R}$, one has*

$$\mathrm{vol}(S)\nu([x, +\infty[) \leqslant \mathrm{vol}(L)\nu_{\overline{\mathscr{L}}}([x, +\infty[), \tag{28}$$

where $\nu_{\overline{\mathscr{L}}}$ is the vague limit of $(T_{\frac{1}{n}}\nu_{\overline{\mathscr{B}}_n})_{n\geqslant 1}$, $\nu_{\overline{\mathscr{B}}_n}$ being the measure associated to the minimum filtration of $\overline{\mathscr{B}}_n$. Furthermore, if $e_{\min}(\overline{\mathscr{S}}_{np}) > 0$ holds for sufficiently large n, then

$$\mathrm{vol}(S) \leqslant \mathrm{vol}(L)\nu_{\overline{\mathscr{L}}}([0, +\infty[) \tag{29}$$

Proof. — For any $x \in \mathbb{R}$, one has

$$\mathrm{rk}(S_{np})\nu_{\overline{\mathscr{S}}_{np}}([npx, +\infty[) \leqslant \mathrm{rk}(B_{np})\nu_{\overline{\mathscr{B}}_{np}}([npx, +\infty[),$$

since these two quantities are respectively the ranks of $\mathcal{F}^M_{npx}S_{np}$ and $\mathcal{F}^M_{npx}B_{np}$. By passing $n\to+\infty$, one obtains that, for any $x\in\mathbb{R}$,

$$\mathrm{vol}(S)\nu([x,+\infty[)\leqslant \mathrm{vol}(L)\nu_{\overline{\mathscr{L}}}([x,+\infty[).$$

Since the positivity condition on last minima implies that $\nu([0,+\infty[)=1$, one obtains (29). ☐

Corollary 6.2. — *With the notation of Proposition 6.1, assume that*

$$\widehat{\mathrm{vol}}(S):=\lim_{n\to\infty}\frac{\widehat{\deg}(\overline{\mathscr{S}}_{np})}{(np)^d/d!}\geqslant(1-\varepsilon)\widehat{\mathrm{vol}}(\overline{\mathscr{L}}),$$

where $0<\varepsilon<1$ *is a constant. Then one has*

$$0\leqslant\delta_K d\int_0^{+\infty}\Big[\mathrm{vol}(L)\nu_{\overline{\mathscr{L}}}([x,+\infty[)-\mathrm{vol}(S)\nu([x,+\infty[)\Big]\,\mathrm{d}x\leqslant\varepsilon\widehat{\mathrm{vol}}(\overline{\mathscr{L}}).\tag{30}$$

Proof. — By (23), one obtains

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}})=\delta_K d\,\mathrm{vol}(L)\int_{\mathbb{R}}\max\{t,0\}\nu_{\overline{\mathscr{L}}}(\mathrm{d}t)=\delta_K d\,\mathrm{vol}(L)\int_0^{+\infty}\nu_{\overline{\mathscr{L}}}([x,+\infty[)\,\mathrm{d}x.$$

Similarly,

$$\widehat{\mathrm{vol}}(S)=\delta_K d\,\mathrm{vol}(S)\int_0^{+\infty}\nu([x,+\infty[)\,\mathrm{d}x.$$

Hence the inequality (30) results from (28). ☐

Appendix A
Comparison of filtered graded algebras

Let $B=\bigoplus_{n\geqslant 0}$ be an integral graded algebra of finite type over an infinite field K and $f:\mathbb{N}\to\mathbb{R}$ be a function such that $\lim\limits_{n\to+\infty}f(n)/n=0$. We suppose that $B_1\neq 0$ and that B is generated as K-algebra by B_1. Assume that each B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ such that B becomes a f-quasi-filtered graded algebra (see §2.3 for definition). For all integers $m,n\geqslant 0$, let $\mathcal{F}^{(m)}$ be another $\mathbb{R}$-filtration on B_n such that B equipped with $\mathbb{R}$-filtrations $\mathcal{F}^{(m)}$ is f-quasi-filtered. For all integers $m,n\geqslant 1$, let $\nu_n=T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F})}$ and $\nu_n^{(m)}=T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F}^{(m)})}$. Assume in addition that $\lambda_{\max}(B_n,\mathcal{F})\ll n$ and $\lambda_{\max}(B_n,\mathcal{F}^{(m)})\ll_m n$. By Theorem 2.11, the sequence of measures $(\nu_n^{(m)})_{n\geqslant 1}$ (resp. $(\nu_n)_{n\geqslant1}$) converges vaguely to a Borel probability which we denote by $\nu^{(m)}$ (resp. ν).

The purpose of this section is to establish the following comparison result:

Proposition A.1. — *Let φ be an increasing, concave and Lipschitz function on $\mathbb{R}$. Assume that, for any $m\geqslant 1$ and any $t\in\mathbb{R}$, one has $\mathcal{F}_tB_m\subset\mathcal{F}_t^{(m)}B_m$, then*

$$\limsup_{m\to+\infty}\int_{\mathbb{R}}\varphi\,\mathrm{d}\nu^{(m)}\geqslant\int_{\mathbb{R}}\varphi\,\mathrm{d}\nu.\tag{31}$$

Proof. — By Noether's normalization theorem, there exists a graded subalgebra A of B such that A is isomorphic to the polynomial algebra generated by A_1. We still use $\mathcal{F}^{(m)}$ (resp. $\mathcal{F}$) to denote the induced filtrations on A. Let $\widetilde{\nu}_n = T_{\frac{1}{n}}\nu_{(A_n,\mathcal{F})}$ and $\widetilde{\nu}_n^{(m)} = T_{\frac{1}{n}}\nu_{(A_n,\mathcal{F}^{(m)})}$. For any integer $m \geqslant 1$ and any $t \in \mathbb{R}$, one still has $\mathcal{F}_t A_m \subset \mathcal{F}_t^{(m)} A_m$. Furthermore, by [**11**, Proof of Theorem 3.4.3], the sequence of measures $(\widetilde{\nu}_n^{(m)})_{n\geqslant 1}$ (resp. $(\widetilde{\nu}_n)_{n\geqslant 1}$) converges vaguely to $\nu^{(m)}$ (resp. ν). Therefore, we may suppose that $B = A$ is a polynomial algebra. In this case, [**11**, Proposition 3.3.3] implies that

$$nm\int \varphi\,\mathrm{d}\nu_{nm}^{(m)} \geqslant nm\int \varphi\,\mathrm{d}\nu_m^{(m)} - n\|\varphi\|_{\mathrm{Lip}} f(m) \geqslant nm\int \varphi\,\mathrm{d}\nu_m - n\|\varphi\|_{\mathrm{Lip}} f(m).$$

since $\nu_m^{(m)} \succ \nu_m$. By passing $n \to \infty$, we obtain

$$\int \varphi\,\mathrm{d}\nu^{(m)} \geqslant \int \varphi\,\mathrm{d}\nu_m - n\|\varphi\|_{\mathrm{Lip}} \frac{f(m)}{m},$$

which implies (31). □

In the following, we apply Proposition A.1 to study algebras in metrized vector bundles. From now on, K denotes a number field. We assume given an $\mathcal{O}_K$-algebra $\mathscr{B} = \bigoplus_{n\geqslant 0}\mathscr{B}_n$, generated by $\mathscr{B}_1$, and such that

1) each $\mathscr{B}_n$ is a projective $\mathcal{O}_K$-module of finite type;
2) for any integer $n \geqslant 0$, $B_n = \mathscr{B}_{n,K}$;
3) the algebra structure of $\mathscr{B}$ is compatible to that of B.

For each integer $n \geqslant 1$, assume that g is a family of norms on $\mathscr{B}_n$ such that $(\mathscr{B}_n, g)$ becomes a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$. For all integers $n \geqslant 1$ and $m \geqslant 1$, let $g^{(m)}$ be another metric structure on $\mathscr{B}_n$ such that $(\mathscr{B}_n, g^{(m)})$ is also a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Denote by $\nu_{(B_n,g)}$ and $\nu_{(B_n,g^{(m)})}$ be the measure associated to the minimum filtration of (B_n, g) and of $(B_n, g^{(m)})$, respectively.

Corollary A.2. — *With the notation above, assume in addition that*

1) $(\mathscr{B}, g)$ and all $(\mathscr{B}, g^{(m)})$ verify the three conditions in Remark 3.14;
2) the identity homomorphism $\mathrm{Id} : (\mathscr{B}_m, g) \to (\mathscr{B}_m, g^{(m)})$ is effective (see §3.1).

Let ν and $\nu^{(m)}$ be respectively the limit measure of $(T_{\frac{1}{n}}\nu_{(B_n,g)})_{n\geqslant 1}$ and $(T_{\frac{1}{n}}\nu_{(\mathscr{B}_n,g^{(m)})})_{n\geqslant 1}$. Then for any increasing, concave and Lipschtiz function φ on $\mathbb{R}$, one has

$$\limsup_{m\to\infty}\int_{\mathbb{R}} \varphi\,\mathrm{d}\nu^{(m)} \geqslant \int_{\mathbb{R}} \varphi\,\mathrm{d}\nu.$$

In particular, if $\liminf_{n\to\infty}\frac{1}{n}e_{\min}(\mathscr{B}_n, g) \geqslant 0$ and if $\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{1}{n}e_{\min}(\mathscr{B}_n, g^{(m)}) \geqslant 0$, then

$$\limsup_{m\to\infty}\int_{\mathbb{R}} \max\{x,0\}\,\nu^{(m)}(\mathrm{d}x) \geqslant \int_{\mathbb{R}} \max\{x,0\}\,\nu(\mathrm{d}x). \tag{32}$$

Proof. — The first assertion is a direct consequence of Proposition A.1. In particular, one has

$$\limsup_{m\to\infty}\int_{\mathbb{R}}\max\{x,0\}\,\nu^{(m)}(\mathrm{d}x)\geqslant\int_{\mathbb{R}}\max\{x,0\}\,\nu(\mathrm{d}x).$$

The hypothèse $\liminf\limits_{n\to\infty}\frac{1}{n}e_{\min}(\mathscr{B}_n,g)\geqslant 0$ implies that the support of ν is bounded from below by 0, so $\int_{\mathbb{R}}\max\{x,0\}\,\nu(\mathrm{d}x)=\int_{\mathbb{R}}x\,\nu(\mathrm{d}x)$. For any integer $m\geqslant 1$, let $a_m=\liminf\limits_{n\to\infty}\frac{1}{n}e_{\min}(\mathscr{B}_n,g_m)$ and $b_m=\min(a_m,0)$. One has $\int_{\mathbb{R}}x\,\nu^{(m)}(\mathrm{d}x)=\int_{\mathbb{R}}\max(x,b_m)\,\nu^{(m)}(\mathrm{d}x)$. Note that

$$\left|\int_{\mathbb{R}}\max(x,b_m)\,\nu^{(m)}(\mathrm{d}x)-\int_{\mathbb{R}}\max(x,0)\,\nu^{(m)}(\mathrm{d}x)\right|\leqslant b_m,$$

which converges to 0 when $m\to\infty$, so we obtain (32). □

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