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Limit Behaviour of Sequential Empirical Measure Processes

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Abstract

In this paper we obtain some uniform laws of large numbers and functional central limit theorems for sequential empirical measure processes indexed by classes of product functions satisfying appropriate Vapnik-Červonenkis properties.

Key words: Functional Central Limit Theorems, Sequential Empirical Measure, Uniform Laws of Large Numbers.

AMS Classifications: 62G20 60F17 60F25 60G50

1 Introduction

Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables of law ν , defined on a probability space (Ω, \mathcal{A}, P) and taking values in some measurable space (U, \mathcal{U}) . Let \mathcal{Q} be a class of measurable functions

$$q : ([0, 1] \times U, \mathcal{B}([0, 1]) \otimes \mathcal{U}) \rightarrow (\mathfrak{R}, \mathcal{B}(\mathfrak{R})),$$

where $\mathcal{B}([0, 1])$ and $\mathcal{B}(\mathfrak{R})$ denote the class of Borel sets of $[0, 1]$ and \mathfrak{R} , respectively. The stochastic process

$$\mathbb{P}_n(q) \triangleq \frac{1}{n} \sum_{i=1}^n q(i/n, X_i) = \frac{1}{n} \sum_{i=1}^n \delta_{(i/n, X_i)}(q), \quad q \in \mathcal{Q}, \quad (1)$$

is called *the sequential empirical measure process (s.e.m.p.) indexed by \mathcal{Q}* .

It is known that any symmetric statistic can be seen as a functional of the classical empirical measure. The fact that the sequential empirical measure

$$\mathbb{P}_n \triangleq \frac{1}{n} \sum_{i=1}^n \delta_{(i/n, X_i)}$$

enables to reconstruct the whole sequence X_1, \dots, X_n (rather than the sample up to a permutation) makes of \mathbb{P}_n a very flexible tool to represent complex and highly non-symmetric statistics. One can, for instance think of two-sample sequential rank statistics, V -statistics or fractional ARIMA processes as treated in Barbe and Broniatowski [1997], [1998a] and [1998b]. In these references, sequential empirical measure representations allowed to obtain functional large deviation principles for classes of complex statistics as the ones evoked. Weak convergence in the case in which $\mathcal{Q} = \{\mathbf{1}_{[0,t]} \cdot f : 0 < t \leq 1, f \in \mathcal{F}\}$, where \mathcal{F} is a Donsker class, has been treated in van der Vaart and Wellner [1996]. In that case, it was found that weak convergence occurred in $\ell^\infty([0, 1] \times \mathcal{F})$, the limiting process being the standard Kiefer-Müller process indexed by $[0, 1] \times \mathcal{F}$.

The aim of this paper is to obtain uniform laws of large numbers (U.L.L.N's) and functional central limit theorems (F.C.L.T's) for sequential empirical measure processes indexed by classes of product functions satisfying certain properties. Our results generalize those in van der Vaart and Wellner [1996] and allow

to obtain uniform results for classes of non-symmetric statistics of independent random variables whose laws depend non linearly on time. More precisely, let $\mathcal{H} \subset \mathfrak{R}^{[0,1]}$ and $\mathcal{G} \subset \mathfrak{R}^U$ be two classes of functions. Define the class

$$\mathcal{F}_\pi \triangleq \mathcal{H} \cdot \mathcal{G} \triangleq \{f = hg : h \in \mathcal{H}, g \in \mathcal{G}\}.$$

In the sequel, we will consider combinations of the following three conditions:

- (H1) \mathcal{H} is a uniformly bounded Vapnik-Červonenkis graph class (V.C.G.C.) of almost everywhere continuous functions;
- (H2) \mathcal{G} is a V.C.G.C. with envelope $G \in L_1(\nu)$ such that, for all $x \in U$, $G(x) < +\infty$.
- (H3) \mathcal{G} is a V.C.G.C. with envelope $G \in L_2(\nu)$ such that, for all $x \in U$, $G(x) < +\infty$.

For an extensive account on Vapnik-Červonenkis Theory, the reader is referred to Chapter 4 of Dudley [1999]. To avoid any measurability issues, all function classes considered in this paper will be assumed to be *permissible* in the sense of Appendix C of Pollard [1984]. We will now introduce some notation to be used throughout the paper. First note that, since ν is a finite measure, condition (H3) implies condition (H2) but the converse fails in general. We will write $\mathcal{F}_\pi \in \pi(\nu G, J\text{-}VC)$ if conditions (H1) and (H2) are verified and if \mathcal{F}_π is jointly a V.C.G.C. (i.e. as a function of two variables). We will write $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$ if conditions (H1) and (H3) are verified and if \mathcal{F}_π (as a function of two variables) is a V.C.G.C.. Finally, we will write $\mathcal{F}_\pi \in \pi(UB, M\text{-}VC)$ if condition (H1) is verified and condition (H2) (or, equivalently, condition (H3)) is verified with G a constant function, i.e. \mathcal{G} is uniformly bounded as well. In this respect, it is to be noticed that V.C. properties of two function classes are not inherited by the class of their pointwise products. Similarly, a V.C.G.C. of functions of two variables does not transmit its V.C. properties to the two classes of one variable obtained by marginalization or projection. For more details, the reader is referred to Dudley [1999] and Stelgle ad Yukich [1989]. In the sequel, whenever a uniformly bounded class of function is considered, it will be tacitly assumed, without loss of generality, that the envelope is identically equal to 1.

Since condition (H1) ensures Riemann integrability of every $h \in \mathcal{H}$ (see Ash [1972], Theorem 1.7.1), we have that, for all $h \in \mathcal{H}$,

$$\lim_{n \rightarrow +\infty} \lambda_n(h) \triangleq \int_{[0,1]} h(s) \lambda_n(ds) = \frac{1}{n} \sum_{i=1}^n h(i/n) = \int_{[0,1]} h(s) \lambda(ds) \triangleq \lambda(h),$$

where

$$\lambda_n \triangleq n^{-1} \sum_{i=1}^n \delta_{i/n},$$

will be referred to as the *uniform discrete measure on* $[0, 1]$ and where λ denotes the trace of Lebesgue measure on $[0, 1]$.

The paper is organized as follows: in Section 2 the main results uniform laws of large numbers are presented and discussed. The same is done in Section 3 for functional central limit theorems. All the proofs are contained in the last section.

2 Uniform laws of large numbers

We start with some classical results. Let (Υ, \mathcal{E}) be a measurable space and let \mathcal{F} be a class of \mathcal{E} -measurable functions $f : (\Upsilon, \mathcal{E}) \rightarrow (\mathfrak{R}, \mathcal{B}(\mathfrak{R}))$. Let F be an \mathcal{E} -measurable envelope of \mathcal{F} . Let $(w_{ni})_{n \geq 1, 1 \leq i \leq i(n)}$, with $\lim_n i(n) = +\infty$, be a triangular array of random probability measures on (Υ, \mathcal{E}) , and let $(\xi_{ni})_{n \geq 1, 1 \leq i \leq i(n)}$ be a triangular array of real-valued random variables defined on (Υ, \mathcal{E}) . The \mathcal{F} -indexed stochastic process

$$S_n(f) \triangleq \sum_{i \leq i(n)} w_{ni}(f) \xi_{ni}, \tag{2}$$

is called *random measure process*. General random measure processes have been introduced and studied in Gaenssler [1992, 1993], Gaenssler and Ziegler [1994b], Gaenssler, Rost and Ziegler [1998] and Ziegler [1997]. This treatment allowed to obtain uniform laws of large numbers and functional central limit theorems for smoothed empirical processes, partial-sum processes with fixed and random locations, empirical versions of the non parametric regression function and empirical-type estimators of the intensity measure of spatial Poisson processes.

To see that sequential empirical measure processes are special cases of random measure processes, it suffices to take $\Upsilon = [0, 1] \times U$, $\mathcal{E} = \mathcal{B}([0, 1]) \otimes \mathcal{U}$, $i(n) = n$, $\xi_{ni} \equiv n^{-1}$ and $w_{ni} = \delta_{(i/n, X_i)}$. In Gaenssler, Rost and Ziegler [1998], the following result is shown:

Theorem 1 (Gaenssler, Rost and Ziegler 1998). *Assume the following conditions hold:*

(i) *There exists $p \geq 1$ such that, for all $\delta > 0$,*

$$\lim_{n \rightarrow +\infty} \sum_{1 \leq i \leq i(n)} E^{\frac{1}{p}} \left(w_{ni}(F)^p \cdot |\xi_{ni}|^p \cdot \mathbf{1}_{(\delta, +\infty)}(w_{ni}(F) |\xi_{ni}|) \right) = 0;$$

(ii) *There exists $\delta_1 > 0$ such that*

$$\sup_{n \geq 1} \sum_{1 \leq i \leq i(n)} E \left(w_{ni}(F) \cdot |\xi_{ni}| \cdot \mathbf{1}_{[0, \delta_1]}(w_{ni}(F) |\xi_{ni}|) \right) < +\infty;$$

(iii) *For all $\tau > 0$, there exists $\delta = \delta(\tau)$ such that the sequence*

$$\left\{ N \left(\tau \mu_{n\delta}(F), \mathcal{F}, d_{\mu_{n\delta}}^{(1)} \right) : n \geq 1 \right\},$$

of random covering numbers is stochastically bounded, where the random measure $\mu_{n\delta}$ is defined by

$$\mu_{n\delta} \triangleq \sum_{1 \leq i \leq i(n)} w_{ni} \cdot |\xi_{ni}| \cdot \mathbf{1}_{[0, \delta]}(w_{ni}(F) |\xi_{ni}|),$$

and the random pseudo-metric $d_{\mu_{n\delta}}^{(1)}$ is defined on \mathcal{F} by

$$d_{\mu_{n\delta}}^{(1)}(f, g) \triangleq \sum_{1 \leq i \leq i(n)} |w_{ni}(f) - w_{ni}(g)| \cdot |\xi_{ni}| \cdot \mathbf{1}_{[0, \delta]}(w_{ni}(F) |\xi_{ni}|), \quad f, g \in \mathcal{F}.$$

Then, the sequence $\{S_n(f) : f \in \mathcal{F}\}$, $n \geq 1$, of random measure processes as defined in (2) verifies the following version of the L_p -uniform law of large numbers:

$$\sup_{f \in \mathcal{F}} |S_n(f) - E(S_n(f))| \xrightarrow{L_p} 0.$$

The reader is referred to Section 4.1 for the definition of covering numbers. As a consequence of Theorem 1, we have the following U.L.L.N. for \mathcal{F}_π -indexed sequential empirical measure processes

Corollary 2 *If $\mathcal{F}_\pi \in \pi(\nu G^1, J\text{-VC})$, then*

$$\sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda_n \otimes \nu)(f)| \xrightarrow{L_1} 0. \quad (3)$$

If, moreover, the class \mathcal{H} in the definition of \mathcal{F}_π is such that

$$\sup_{h \in \mathcal{H}} |\lambda_n(h) - \lambda(h)| \rightarrow 0, \quad (4)$$

then

$$\sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda \otimes \nu)(f)| \xrightarrow{L^1} 0. \quad (5)$$

On the other hand, if $\mathcal{F}_\pi \in \pi(UB, M-VC)$, then, for all $p \geq 1$,

$$\sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda_n \otimes \nu)(f)| \xrightarrow{L^p} 0. \quad (6)$$

If, moreover, condition (4) holds, then, for all $p \geq 1$,

$$\sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda \otimes \nu)(f)| \xrightarrow{L^p} 0. \quad (7)$$

Proof. If $\mathcal{F}_\pi \in \pi(UB, M-VC)$ or $\mathcal{F}_\pi \in \pi(\nu G^1, J-VC)$, and taking $f = hg$, then

$$\begin{aligned} E(\mathbb{P}_n(f)) &= \frac{1}{n} \sum_{i=1}^n h(i/n) \nu(g) \\ &= \lambda_n(h) \nu(g) \\ &= (\lambda_n \otimes \nu)(f). \end{aligned}$$

Now, let $\mathcal{F}_\pi \in \pi(UB, M-VC)$. Since $F \equiv 1$ is an envelope of \mathcal{F}_π , then conditions (i) and (ii) of Theorem 1 are easily verified. In particular, condition (i) holds for all $p \geq 1$. Moreover, Proposition 19 implies condition (iii) of Theorem 1 thus completing the proof of (6). To show that (4) implies (7), note that, for all $p \geq 1$,

$$\begin{aligned} \left\| \sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda \otimes \nu)(f)| \right\|_p &\leq \left\| \sup_{f \in \mathcal{F}_\pi} |\mathbb{P}_n(f) - (\lambda_n \otimes \nu)(f)| \right\|_p \\ &\quad + \left\| \sup_{f \in \mathcal{F}_\pi} |(\lambda_n \otimes \nu)(f) - (\lambda \otimes \nu)(f)| \right\|_p \\ &\triangleq A(n) + B(n), \end{aligned}$$

where $\|\cdot\|_p$ denotes the $L_p(\nu)$ norm. The term $A(n)$ converges to 0 by (6); as for the term $B(n)$, noting that $G \equiv 1$ is an envelope of \mathcal{G} ,

$$\begin{aligned} B(n) &\leq \left\| \sup_{g \in \mathcal{G}} |\nu(g)| \right\|_p \left\| \sup_{h \in \mathcal{H}} |\lambda_n(h) - \lambda(h)| \right\|_p \\ &\leq \left\| \sup_{h \in \mathcal{H}} |\lambda_n(h) - \lambda(h)| \right\|_p, \end{aligned}$$

Since we have set $\nu(G) = 1$, by convention. Convergence to 0 of $B(n)$ follows from (4). The case $\mathcal{F}_\pi \in \pi(\nu G, J-VC)$ can be treated analogously. Conditions (i) and (ii) are verified immediately and condition (iii) follows from the fact that \mathcal{F}_π is a V.C.G.C. ■

If \mathcal{H} verifies (4), we will say that \mathcal{H} is *uniformly Riemann-integrable*. Here is an example of a uniformly Riemann-integrable class.

Example 3 Let $\mathcal{H}(T, C, \beta)$, $T, C, \beta > 0$, the class of all Hölder functions of parameters C and β such that $|h(0)| \leq T$. In other words, for all $x_1, x_2 \in [0, 1]$,

$$|h(x_1) - h(x_2)| \leq C |x_1 - x_2|^\beta.$$

First of all, note that $\mathcal{H}(T, C, \beta)$ is uniformly bounded. In fact, for all $h \in \mathcal{H}(T, C, \beta)$ and all $x \in [0, 1]$,

$$|h(x)| \leq C|x|^\beta + T \leq C + T.$$

To show uniform Riemann-integrability, define the sequence of functions

$$h_n(x) \triangleq \sum_{i=1}^n h(i/n) \mathbf{1}_{(\frac{i-1}{n}, \frac{i}{n}]}(x), \quad n \geq 1.$$

Clearly, $\lambda_n(h) = \lambda(h_n)$, so that

$$\begin{aligned} |\lambda_n(h) - \lambda(h)| &\leq \int_0^1 |h(x) - h_n(x)| dx \\ &\leq \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} |h(x) - h(i/n)| dx \\ &\leq \frac{C}{n^\beta}. \end{aligned}$$

Uniform Riemann-integrability follows since $\delta_n = C/n^\beta$ does not depend on h .

Remark 4 Note that continuity is not necessary to achieve uniform Riemann-integrability. It suffices, in fact, to observe that the class

$$\mathcal{H}_{[0,1]} \triangleq \{\mathbf{1}_{[0,t]} : 0 < t \leq 1\},$$

is uniformly Riemann-integrable.

The remaining part of this section is devoted to the presentation of a U.L.L.N. in which convergence occurs almost-surely. Let $\mathcal{B} \subset \mathcal{B}([0, 1])$ be the class of all regular Borel-sets. Recall that a Borel set B is called *regular* if (i) $\lambda(B) > 0$ and (ii) $\lambda(\partial B) = 0$, where ∂B is the the boudary of B . By Lemma 2 in Dedecker [2001], if B is a regular Borel set of $[0, 1]$, then

$$\lim_{n \rightarrow +\infty} \lambda_n(B) = \lambda(B), \quad (8)$$

where λ_n and λ denote, respectively, the discrete uniform measure and the trace of Lebesgue measure on $[0, 1]$. Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of random variables of law ν defined on a probability space (Ω, \mathcal{A}, P) and taking values in some measurable space (U, \mathcal{U}) . For all $B \in \mathcal{B}$, define the *B-empirical measure* on (U, \mathcal{U}) by

$$\nu_{n,B} \triangleq \frac{1}{\text{card}(B \cap \{\frac{1}{n}, \dots, 1\})} \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}, \quad (9)$$

with the convention that $\nu_{n,B} \equiv 0$ if $B \cap \{\frac{1}{n}, \dots, 1\} = \emptyset$.

With $\mathbb{P}_n \triangleq n^{-1} \sum_{i=1}^n \delta_{(i/n, X_i)}$ the sequential empirical measure, note that for $B \in \mathcal{B}$ and $A \in \mathcal{U}$,

$$\begin{aligned} \mathbb{P}_n(B \times A) &= \frac{1}{n} \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}(A) \\ &= \lambda_n(B) \nu_{n,B}(A), \end{aligned}$$

so that

$$E(\mathbb{P}_n(B \times A)) = \lambda_n(B) \nu(A).$$

It turns out that the B -empirical measure verifies a Glivenko-Cantelli-type result (see Lemma 21). This fact will be used later to prove the following version of the U.L.L.N. for $\mathcal{F}_{\mathcal{B}_\#, \mathcal{W}}$ -indexed sequential empirical measure processes where $\mathcal{F}_{\mathcal{B}_\#, \mathcal{W}} \triangleq \{\mathbf{1}_B \mathbf{1}_W : B \in \mathcal{B}_\#, W \in \mathcal{W}\}$, and $\mathcal{B}_\# \subset \mathcal{B}$ is a C.V.C. of regular Borel subsets of $[0, 1]$.

Theorem 5 *Let the notations of this section prevail. If $\mathcal{W} \subset \mathcal{U}$ is a V.C.C. and if $\mathcal{B}_\#$ is a V.C.C. of regular Borel sets of $[0, 1]$, then*

$$P \left(\lim_{n \rightarrow +\infty} \sup_{B \in \mathcal{B}_\#} \sup_{W \in \mathcal{W}} |\mathbb{P}_n(B \times W) - \lambda_n(B) \nu(W)| = 0 \right) = 1. \quad (10)$$

If, moreover, $\mathcal{B}_\#$ is such that

$$\lim_{n \rightarrow +\infty} \sup_{B \in \mathcal{B}_\#} |\lambda_n(B) - \lambda(B)| = 0, \quad (11)$$

then

$$P \left(\lim_{n \rightarrow +\infty} \sup_{B \in \mathcal{B}_\#} \sup_{W \in \mathcal{W}} |\mathbb{P}_n(B \times W) - \lambda(B) \nu(W)| = 0 \right) = 1. \quad (12)$$

We end this section with some comments and an open problem. First, in the following two examples, we present special cases of V.C.C.'s of regular Borel sets satisfying (11). In the third example we present a class of regular Borel-sets that is not a V.C.C. and for which (11) does not hold.

Example 6 *Fix $j \in \mathbb{N}$ and a vector $\mathbf{t}_{2j+1} = (t_0, \dots, t_{2j}) \in [0, 1]^{2j+1}$, with $t_{2k} \leq t_{2k+1}$, for all $k = 0, \dots, j-1$ and $t_{2k-1} < t_{2k}$, for all $k = 1, \dots, j$. We will adopt the following convention: for $j = 0$, $\mathbf{t}_{2j+1} = \mathbf{t}_1 = t_0 \triangleq t \in (0, 1]$. Let \mathfrak{T}_{2j+1} be the set of all such vectors. For all $\mathbf{t}_{2j+1} \in \mathfrak{T}_{2j+1}$, let $B_{\mathbf{t}_{2j+1}} \in \mathcal{B}([0, 1])$ be defined by*

$$B_{\mathbf{t}_{2j+1}} \triangleq \langle 0, t_0 \rangle \cup \langle t_1, t_2 \rangle \cup \dots \cup \langle t_{2j-1}, t_{2j} \rangle,$$

where $\langle \alpha, \beta \rangle$ denotes an open, closed or semi-open interval with extremes $\alpha < \beta$. Define the class

$$\mathcal{B}(2j+1) \triangleq \{B_{\mathbf{t}_{2j+1}} : \mathbf{t}_{2j+1} \in \mathfrak{T}_{2j+1}\}.$$

In particular, $\mathcal{B}(1) = \{\langle 0, t \rangle : 0 < t \leq 1\}$. It can be shown that $\mathcal{B}(2j+1)$ is a V.C.C. of dimension $2j+1$ (see Dudley [1999], Problem 4.11). On the other hand, for all $B_{\mathbf{t}_{2j+1}} \in \mathcal{B}(2j+1)$, there exists a vector $\mathbf{t}_{2j+1} \in \mathfrak{T}_{2j+1}$ such that

$$\lambda_n(B_{\mathbf{t}_{2j+1}}) = \frac{\lfloor nt_0 \rfloor}{n} + \sum_{k=1}^j \left(\frac{\lfloor nt_{2k} \rfloor}{n} - \frac{\lfloor nt_{2k-1} \rfloor}{n} \right),$$

and

$$\lambda(B_{\mathbf{t}_{2j+1}}) = t_0 + \sum_{k=1}^j (t_{2k} - t_{2k-1}).$$

It is easily seen that

$$|\lambda_n(B_{\mathbf{t}_{2j+1}}) - \lambda(B_{\mathbf{t}_{2j+1}})| \leq \frac{2(2j+1)}{n}.$$

Since $\delta_n = \frac{2(2j+1)}{n}$ converges to 0 and does not depend on the vector \mathbf{t}_{2j+1} , we have that, for all $j \geq 1$,

$$\lim_n \sup_{B_{\mathbf{t}_{2j+1}} \in \mathcal{B}(2j+1)} |\lambda_n(B_{\mathbf{t}_{2j+1}}) - \lambda(B_{\mathbf{t}_{2j+1}})| = 0.$$

Example 7 *Fix $j \in \mathbb{N}^+$ and a vector $\mathbf{t}_{2j} = (t_1, \dots, t_{2j}) \in [0, 1]^{2j}$ with $t_{2k} \leq t_{2k+1}$ for all $k = 1, \dots, j-1$ and $t_{2k-1} < t_{2k}$ for all $k = 1, \dots, j$. Let \mathfrak{T}_{2j} be the set of all such vectors. For all $\mathbf{t}_{2j} \in \mathfrak{T}_{2j}$, let $B_{\mathbf{t}_{2j}} \in \mathcal{B}([0, 1])$ be defined by*

$$B_{\mathbf{t}_{2j}} = \langle t_1, t_2 \rangle \cup \dots \cup \langle t_{2j-1}, t_{2j} \rangle,$$

where the notation $\langle \alpha, \beta \rangle$ is defined in the previous Example. Define the class

$$\mathcal{B}(2j) \triangleq \{B_{\mathbf{t}_{2j}} : \mathbf{t}_{2j} \in \mathfrak{T}_{2j}\}.$$

Again, it can be shown that $\mathcal{B}(2j)$ is a V.C.C. of dimension $2j$ and

$$\sup_{B_{\mathbf{t}_{2j}} \in \mathcal{B}(2j)} |\lambda_n(B_{\mathbf{t}_{2j}}) - \lambda(B_{\mathbf{t}_{2j}})| \leq \frac{4j}{n} \rightarrow 0 \quad (n \rightarrow +\infty).$$

Here is a class of regular Borel sets, closely related to those treated in Examples 6 and 7, for which (11) does not hold

Example 8 Define the class

$$\mathcal{B}_\infty \triangleq \bigcup_{n \geq 1} \mathcal{B}(n),$$

where $\mathcal{B}(n)$ is defined in Example 6 or 7 according to whether n is odd or even, respectively. To see that (11) does not hold for \mathcal{B}_∞ , consider the sequence $(B_n)_{n \geq 1}$ of Borel sets defined by

$$B_n = \left(0, \frac{1}{n} - \varepsilon_n\right] \cup \left(\frac{1}{n}, \frac{2}{n} - \varepsilon_n\right] \cup \cdots \cup \left(\frac{n-1}{n}, 1 - \varepsilon_n\right], \quad n \geq 1,$$

where $\varepsilon_n = \frac{1}{n2^n}$, $n \geq 1$. Clearly, for all $n \geq 1$, $B_n \in \mathcal{B}(n)$, $\lambda_n(B_n) = 0$ and $\lambda(B_n) = 1 - n\varepsilon_n$. Consequently,

$$\lim_{n \rightarrow +\infty} |\lambda_n(B_n) - \lambda(B_n)| = \lim_{n \rightarrow +\infty} (1 - n\varepsilon_n) = 1.$$

Finally, note that \mathcal{B}_∞ is clearly not a V.C.C.

Here is the statement of the announced open problem:

Problem. Is it possible to establish a U.L.L.N. along the lines of Theorem 5 for sequential empirical measure processes indexed by the class

$$\mathcal{F}_{\mathcal{B}^*, \mathcal{W}} \triangleq \{f = \mathbf{1}_B \mathbf{1}_W : B \in \mathcal{B}^*, W \in \mathcal{W}\},$$

where $\mathcal{W} \subset \mathcal{U}$ is a V.C.C. and where \mathcal{B}^* is an arbitrary class of regular Borel sets (not necessarily a C.V.C.)?

Clearly, if \mathcal{B}^* is not a V.C.C., Theorem 5 does not apply. Moreover, our techniques are not conclusive. In fact, rehearsing the proof in section 4.2, one is confronted with studying the convergence of the double series

$$S(D, c) = \sum_{n \geq 1} \sum_{k=1}^n k^D \binom{n}{k} \exp(-cn),$$

where D is a strictly positive natural number and c is a positive real number. Now, if $c > \log 2$,

$$S(D, c) \leq \sum_{n \geq 1} n^D \left(\frac{2}{e^c}\right)^n < +\infty,$$

whereas, if $c \leq \log 2$,

$$S(D, c) \geq \sum_{n \geq 1} \left(\frac{2}{e^c}\right)^n = +\infty.$$

3 Functional central limit theorems

This section is devoted to obtaining functional central limit theorems for sequences of \mathcal{F}_π -indexed sequential empirical measure processes, when $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$ and $\mathcal{F}_\pi \in \pi(UB, M\text{-}VC)$. Obtaining F.C.L.T's amounts essentially to proving two facts: (i) convergence of finite dimensional distributions of the sequence to those of a centered Gaussian process and (ii) asymptotic equicontinuity of the sequence of processes. We start by showing convergence of finite-dimensional laws. Note that this will be done for a much larger class than those mainly considered in this paper. The reason for this is to exhibit the limiting Gaussian process one needs to consider in view of possible generalizations of our results. Let \mathcal{Q} be the class of all $(\mathcal{B}([0, 1]) \otimes \mathcal{U})$ -measurable functions $q \in \mathbb{R}^{[0,1] \times U}$ such that the following two conditions hold:

(a) for every $q \in \mathcal{Q}$, there exists a function $g_q \in L_2(\nu)$ such that, for all $(s, x) \in [0, 1] \times U$,

$$|q(s, x)| \leq g_q(x);$$

(b) for all $q \in \mathcal{Q}$ and all $x \in U$, the function

$$s \mapsto q(s, x) : [0, 1] \rightarrow \mathbb{R},$$

is λ -almost everywhere continuous.

Consider the sequence of \mathcal{Q} -indexed s.e.m.p.'s $\{Z_n(q) : q \in \mathcal{Q}\}$, $n \geq 1$, where, for each $q \in \mathcal{Q}$ and $n \geq 1$, $Z_n(q)$ is defined by

$$Z_n(q) \triangleq \sqrt{n}(\mathbb{P}_n(q) - (\lambda_n \otimes \nu)(q)),$$

the quantity $\mathbb{P}_n(q)$ being defined in (1) and λ_n denoting the discrete uniform measure on $[0, 1]$. Let $\{Z(q) : q \in \mathcal{Q}\}$ be a centered Gaussian process whose covariance structure is given by

$$\text{Cov}(Z(q_1), Z(q_2)) = \int_{[0,1]} [\nu(q_1 \cdot q_2)(s) - \nu(q_1)(s) \cdot \nu(q_2)(s)] \lambda(ds), \quad (13)$$

where λ is the trace of Lebesgue measure on $[0, 1]$, where

$$\nu(q_1 \cdot q_2)(s) \triangleq \int_U q_1(s, x) q_2(s, x) \nu(dx),$$

and where

$$\nu(q_p)(s) \triangleq \int_U q_p(s, x) \nu(dx), \quad p = 1, 2.$$

Note that if $q_1 = \mathbf{1}_{[0,s]}$, $0 \leq s \leq 1$ and $q_2 = \mathbf{1}_{[0,x]}$, $0 \leq x \leq 1$, then the covariance structure described in (13) defines that of the classical Kiefer process on $[0, 1]^2$. In the next section, we will prove the following

Proposition 9 *The finite dimensional laws of the sequence of \mathcal{Q} -indexed s.e.m.p.'s $\{Z_n(q) : q \in \mathcal{Q}\}$, $n \geq 1$, converge to those of the centered Gaussian process $\{Z(q) : q \in \mathcal{Q}\}$ whose covariance structure is given by (13).*

We will first use the Lindeberg central limit theorem to prove that, for all $q \in \mathcal{Q}$, the sequence of random variables $Z_n(q)$, $n \geq 1$, converges in law to $Z(q)$. The proof of Proposition 9 will be then completed by an application of the Cramér-Wold device. In what follows, we analyze more closely the defining conditions of the class \mathcal{Q} . First of all, we observe that for all $q \in \mathcal{Q}$, the function

$$\nu(q^2)(s) \triangleq \int_U q^2(s, x) \nu(dx),$$

is Riemann-integrable. In fact, condition (b) implies that, for all $1 \leq p < +\infty$, the function $s \mapsto q^p(s, x)$ is λ -almost everywhere continuous for all $x \in U$. This, in turn, implies the λ -almost everywhere continuity of

$$\nu(q^p)(s) \triangleq \int_U q^p(s, x) \nu(dx),$$

for all $1 \leq p < +\infty$ and in particular for $p = 2$. Now, condition (a) implies the boundedness of $\nu(q^2) : [0, 1] \rightarrow \mathbb{R}$, since this conditions ensures the existence of a function $g_q : U \rightarrow \mathbb{R}$ such that, for all $s \in [0, 1]$,

$$\int_U q^2(s, x) \nu(dx) \leq \int_U g_q^2(x) \nu(dx) = \nu(g_q^2) < +\infty.$$

Riemann-integrability of $\nu(q^2)(\cdot)$ follows by Theorem 1.7.1 in Ash [1972]. This fact has an important consequence for our purposes. Namely

$$\lim_{n \rightarrow +\infty} (\lambda_n \otimes \nu)(q^2) = \int_{[0,1] \times U} q^2(s, x) (\lambda \otimes \nu)(ds, dx) \triangleq (\lambda \otimes \nu)(q^2), \quad (14)$$

a condition without which applying Lindeberg central limit theorem would be impossible. Moreover, \mathcal{Q} is a linear space. In other words, for all $(a_1, \dots, a_K) \in \mathbb{R}^K$ and all $(g_1, \dots, g_K) \in \mathcal{Q}^K$, $K = 1, 2, \dots$, the function $q_\Sigma \triangleq \sum_{j=1}^K a_j q_j(s, x)$ is an element of \mathcal{Q} . In fact, it is immediately seen that q_Σ verifies condition (b). As for condition (a), observe that, for all $(s, x) \in [0, 1] \times U$,

$$\left| \sum_{j=1}^K a_j q_j(s, x) \right| \leq M_\Sigma \sum_{j=1}^K |g_{q_j}(x)|,$$

where $M_\Sigma \triangleq \max_{1 \leq j \leq K} |a_j|$ is a positive (finite) constant and where g_{q_1}, \dots, g_{q_K} are elements $L_2(\nu)$ whose existence is guaranteed by condition (b). Cauchy-Schwartz inequality now gives

$$\nu(g_\Sigma^2) = \sum_{j=1}^K \nu(g_j^2) + \sum_{1 \leq l \neq m \leq K} \nu(|g_{q_l}| |g_{q_m}|) < +\infty,$$

proving that q_Σ verifies condition (a). Once more, this is key to employing the Cramér-Wold device.

Let us return to \mathcal{F}_π -indexed s.e.m.p.'s with $\mathcal{F}_\pi \in \pi(\nu G^2, J-VC)$ or $\mathcal{F}_\pi \in \pi(UB, M-VC)$. It is trivial to see that, in both cases, $\mathcal{F}_\pi \subset \mathcal{Q}$. Now for all $f_1, f_2 \in \mathcal{F}_\pi = \mathcal{H} \cdot \mathcal{G}$, write $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$, with $h_1, h_2 \in \mathcal{H} \subset \mathfrak{R}^{[0,1]}$ and $g_1, g_2 \in \mathcal{G} \subset \mathfrak{R}^U$, to see that the covariance structure of the \mathcal{F}_π -indexed limiting centered Gaussian process $\{Z(f) : f \in \mathcal{F}_\pi\}$ is given by

$$Cov(Z(f_1), Z(f_2)) = \lambda(h_1 h_2) [\nu(g_1 g_2) - \nu(g_1) \nu(g_2)]. \quad (15)$$

In the next section we will prove weak convergence of the sequence of s.e.m.p.'s $\{Z_n(f) : f \in \mathcal{F}_\pi\}$, $n \geq 1$, to a centered Gaussian process with uniformly bounded and uniformly continuous sample paths and covariance function given by (15), both when $\mathcal{F}_\pi \in \pi(\nu G^2, J-VC)$ and $\mathcal{F}_\pi \in \pi(UB, M-VC)$. By "weak convergence" we mean in the sense of Hoffman-Jørgensen \mathcal{L} -convergence (see Hoffmann Jørgensen [1991]). Observe, in fact, that the sequence of processes Z_n , $n \geq 1$, can be seen as a sequence of random quantities (i.e. not necessarily measurable) with sample paths in the pseudo-metric space $(\ell^\infty(\mathcal{F}_\pi), \|\cdot\|_{\mathcal{F}_\pi})$, where $\|\cdot\|_{\mathcal{F}_\pi}$ denotes the sup norm defined by $\|\cdot\|_{\mathcal{F}_\pi} \triangleq \sup_{f \in \mathcal{F}_\pi} |\cdot|$. Towards our aim, we need to show that there exists a version of the limiting Gaussian process with uniformly bounded and uniformly continuous sample paths, a fact for which one only needs that the sub-space $(UB^b(\mathcal{F}_\pi, d), \|\cdot\|_{\mathcal{F}_\pi})$ of uniformly bounded and uniformly continuous functions (with respect to some metric d) to be a separable subset of $(\ell^\infty(\mathcal{F}_\pi), \|\cdot\|_{\mathcal{F}_\pi})$. For this, it suffices that (\mathcal{F}_π, d) be a totally bounded pseudometric space. That is

why, from now on, \mathcal{F}_π will be endowed with the totally bounded pseudometric (see the section 4.1 for the proof)

$$d(f_1, f_2) = d(h_1 g_1, h_2 g_2) \triangleq d_\lambda^{(2)}(h_1, h_2) + d_\nu^{(2)}(g_1, g_2),$$

where $d_\lambda^{(2)}$ and $d_\nu^{(2)}$ denote, respectively, the $L_2(\lambda)$ pseudometric on \mathcal{H} and the $L_2(\nu)$ pseudo metric on \mathcal{G} . Once this result is at hand, we will need to prove asymptotic (d -)equicontinuity of the sequence Z_n , $n \geq 1$. We will denote the fact that Z_n converges weakly to a separable centred Gaussian process Z with uniformly bounded and uniformly continuous sample paths by writing, as in Ziegler [1997], $Z_n \xrightarrow[\text{sep}]{\mathcal{L}} Z$ in $\ell^\infty(\mathcal{F}_\pi, d)$. In the next section we will prove the following results:

Theorem 10 *If $\mathcal{H} \cdot \mathcal{G} \triangleq \mathcal{F}_\pi \in \pi(\nu G^2, J\text{-VC})$ and if \mathcal{H} is such that*

$$\lim_{n \rightarrow +\infty} \sup_{h_1, h_2 \in \mathcal{H}} \left| \lambda_n \left((h_1 - h_2)^2 \right) - \lambda \left((h_1 - h_2)^2 \right) \right| = 0, \quad (16)$$

then

$$Z_n \xrightarrow[\text{sep}]{\mathcal{L}} Z \text{ in } \ell^\infty(\mathcal{F}_\pi, d).$$

Theorem 11 *If $\mathcal{H} \cdot \mathcal{G} \triangleq \mathcal{F}_\pi \in \pi(UB, M\text{-VC})$ and if \mathcal{H} verifies (16), then*

$$Z_n \xrightarrow[\text{sep}]{\mathcal{L}} Z \text{ in } \ell^\infty(\mathcal{F}_\pi, d).$$

As will be seen in the next section, Theorem 10 is in fact a corollary of the findings in Section 4.2 of Ziegler [1997]. As for the proof of Theorem 11, we will use a maximal inequality stated in Theorem 3.1 of the same reference. We end this section presenting two examples of classes $\mathcal{H} \subset \mathfrak{R}^{[0,1]}$ for which (16) holds.

Example 12 *It is easily seen that if \mathcal{H} is (1) uniformly Riemann-integrable, (2) such that*

$$\mathcal{H}^2 \triangleq \{h^2 : h \in \mathcal{H}\}$$

is uniformly Riemann-interable and (3) such that for all $h_1, h_2 \in \mathcal{H}$, $h \triangleq h_1 h_2$ is an element of \mathcal{H} , then condition (16) holds.

Example 13 *We show that the class $\mathcal{H}(T, C, \beta)$ defined in Example 3 satisfies (16). Incidentally, observe that condition (3) of Example 12 does not hold for $\mathcal{H}(T, C, \beta)$. Now, for all $h_1, h_2 \in \mathcal{H}(T, C, \beta)$, define the two respective function sequences*

$$h_{1,n}(x) \triangleq \sum_{i=1}^n h_1(i/n) \mathbf{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]}(x), \quad n \geq 1,$$

$$h_{2,n}(x) \triangleq \sum_{i=1}^n h_2(i/n) \mathbf{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]}(x), \quad n \geq 1.$$

Then, for all $n \geq 1$,

$$(h_{1,n}(x) - h_{2,n}(x)) = \sum_{i=1}^n [h_1(i/n) - h_2(i/n)] \mathbf{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]}(x),$$

and

$$(h_{1,n}(x) - h_{2,n}(x))^2 = \sum_{i=1}^n [h_1(i/n) - h_2(i/n)]^2 \mathbf{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]}(x).$$

It follows that

$$\lambda_n \left((h_1 - h_2)^2 \right) = \frac{1}{n} \sum_{i=1}^n [h_1(i/n) - h_2(i/n)]^2 = \lambda \left((h_{1,n}(x) - h_{1,n}(x))^2 \right).$$

Note that

$$\begin{aligned} \sup_{h_1, h_2 \in \mathcal{H}(T, C, \beta)} \left| \lambda \left((h_1 - h_2)^2 - (h_{1,n}(x) - h_{1,n}(x))^2 \right) \right| &\leq 2 \sup_{h \in \mathcal{H}(T, C, \beta)} |\lambda(h_n^2 - h^2)| \\ &\quad + 2 \sup_{h_1, h_2 \in \mathcal{H}(T, C, \beta)} |\lambda(h_{1,n}h_{2,n} - h_1h_2)|. \end{aligned}$$

Since, by Cauchy-Schwartz inequality and uniform boundedness of $\mathcal{H}(T, C, \beta)$ (see Example 3),

$$\begin{aligned} |\lambda(h_n^2 - h^2)| &\leq \sqrt{\lambda(h_n^2) \lambda((h_n - h)^2)} + \sqrt{\lambda(h^2) \lambda((h_n - h)^2)} \\ &\leq 2(C + T) \sqrt{\lambda((h_n - h)^2)}, \end{aligned}$$

and since, for all $h \in \mathcal{H}(T, C, \beta)$ and all $x_1, x_2 \in [0, 1]$,

$$|h(x_1) - h(x_2)|^2 \leq C^2 |x_1 - x_2|^{2\beta},$$

proceeding exactly as in Example 3, one has

$$\lambda((h_n - h)^2) \leq \frac{C^2}{n^{2\beta}},$$

so that

$$\lim_{n \rightarrow +\infty} \sup_{h \in \mathcal{H}(T, C, \beta)} |\lambda(h_n^2 - h^2)| = 0.$$

Analogously, it is possible to show that

$$\begin{aligned} |\lambda(h_{1,n}h_{2,n} - h_1h_2)| &\leq \sqrt{\lambda(h_{2,n}^2) \lambda((h_{1,n} - h_1)^2)} + \sqrt{\lambda(h_2^2) \lambda((h_{2,n} - h_2)^2)} \\ &\leq \frac{2C(C + T)}{n^\beta}, \end{aligned}$$

thus proving

$$\lim_{n \rightarrow +\infty} \sup_{h_1, h_2 \in \mathcal{H}(T, C, \beta)} |\lambda(h_{1,n}h_{2,n} - h_1h_2)| = 0.$$

We have shown that (16) holds for $\mathcal{H}(T, C, \beta)$.

4 Proofs

4.1 Ancillary lemmas

Here are the definition of covering numbers of a pseudo-metric space and the statements of four elementary facts regarding such quantities.

Definition 14 Let (M, d) be a pseudo-metric space. Given $u > 0$, a set

$$M^{(n)} = \{m_1, \dots, m_n\} \subset M$$

is called a u -net in (M, d) if, for all $m \in M$, there exists $m_i \in \{m_1, \dots, m_n\}$ such that $d(m, m_i) < u$. The number

$$N(u, M, d) \triangleq \inf \left\{ n \geq 1 : \text{there exists a } u\text{-net } M^{(n)} \text{ in } (M, d) \right\},$$

is called u -covering number of (M, d) .

Lemma 15 Let (M, d) be a metric space and let $M' \subset M$. Then, for all $u > 0$,

$$N(u, M', d) \leq N(u, M, d).$$

Lemma 16 Let d and d' be two pseudo-metrics on some set M and suppose that, for all $m_1, m_2 \in M$, $d(m_1, m_2) \leq d'(m_1, m_2)$. Then, for all $u > 0$,

$$N(u, M, d) \leq N(u, M, d').$$

Lemma 17 Let (M_1, d_1) and (M_2, d_2) be two pseudo-metric spaces. Define the pseudo metric space (M, d) by $M \triangleq M_1 \times M_2$ and $d \triangleq d_1 + d_2$. Then, for all $u > 0$ and all $t \in (0, 1)$,

$$N(u, M, d) \leq N(tu, M_1, d_1) N((1-t)u, M_2, d_2).$$

Lemma 18 Let (M, d) and (M', d') be two pseudo-metric spaces. Let $b : M \rightarrow M'$ be a bijection such that for all $m'_1, m'_2 \in M'$,

$$d'(m'_1, m'_2) = d(b^{-1}(m'_1), b^{-1}(m'_2)).$$

Then, for all $u > 0$,

$$N(u, M, d) = N(u, M', d').$$

The remaining part of this section is devoted to the proof of some entropy properties of classes $\mathcal{F}_\pi \in \pi(UB, M-VC)$ and $\mathcal{F}_\pi \in \pi(\nu G^2, J-VC)$. We start by proving stochastic boundedness of some random covering numbers when $\mathcal{F}_\pi \in \pi(UB, M-VC)$. This fact is used to show that \mathcal{F}_π -indexed s.e.m.p. verifies an L_p -uniform law of large numbers (see Corollary 2). Let $f, f_1, f_2 \in \mathcal{F}_\pi \in \pi(UB, M-VC)$ and write $f = hg$, $f_i = h_i g_i$, $i = 1, 2$, for some $h, h_1, h_2 \in \mathcal{H}$ and $g, g_1, g_2 \in \mathcal{G}$. Define the quantities

$$\|f\|_{\mathbb{P}_n} \triangleq \mathbb{P}_n(|f|) = \frac{1}{n} \sum_{i=1}^n |h(i/n) g(X_i)|,$$

and

$$\begin{aligned} d_{\mathbb{P}_n}^{(1)}(f_1, f_2) &\triangleq \|f_1 - f_2\|_{\mathbb{P}_n} \\ &= \frac{1}{n} \sum_{i=1}^n |h_1(i/n) g_1(X_i) - h_2(i/n) g_2(X_i)|. \end{aligned}$$

Proposition 19 If $\mathcal{F}_\pi \in \pi(UB, M-VC)$, then, for all $\tau > 0$, there exists a constant C , depending only on τ and the V.C. dimensions of classes \mathcal{H} and \mathcal{G} , such that, for all $\omega \in \Omega$ and all $n \geq 1$,

$$\sup_{\omega \in \Omega} \sup_{n \geq 1} N\left(\tau, \mathcal{F}_\pi, d_{\mathbb{P}_n}^{(1)}\right) \leq C$$

Proof. Let ν_n and λ_n denote, respectively, the (classical) empirical measure defined by $\nu_n \triangleq n^{-1} \sum_{i=1}^n \delta_{X_i}$ and the discrete uniform measure on $[0, 1]$ defined by $\lambda_n \triangleq n^{-1} \sum_{i=1}^n \delta_{\{i/n\}}$. For all $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$ in $\mathcal{F}_\pi \in \pi(UB, M-VC)$, by uniform boundedness of classes \mathcal{H} and \mathcal{G} , it is immediately seen that, for all $\omega \in \Omega$ and all $n \geq 1$,

$$d_{\mathbb{P}_n}^{(1)}(f_1, f_2) \leq d_{\nu_n}^{(1)}(g_1, g_2) + d_{\lambda_n}^{(1)}(h_1, h_2),$$

where $d_{\nu_n}^{(1)}(g_1, g_2) \triangleq \nu_n(|g_1 - g_2|)$ and $d_{\lambda_n}^{(1)}(h_1, h_2) \triangleq \lambda_n(|h_1 - h_2|)$. By Lemma 16, we have that, for all $\tau > 0$, all $\omega \in \Omega$ and all $n \geq 1$,

$$N\left(\tau, \mathcal{F}_\pi, d_{\mathbb{P}_n}^{(1)}\right) \leq N\left(\tau, \mathcal{F}_\pi, d_{\nu_n}^{(1)} + d_{\lambda_n}^{(1)}\right). \quad (17)$$

Equip $\mathcal{H} \times \mathcal{G}$ with the equivalence relation \sim defined by

$$(h, g) \sim (h', g') \quad \Leftrightarrow \quad hg \equiv h'g',$$

and let $\widetilde{\mathcal{H} \times \mathcal{G}}$ be the subset of $\mathcal{H} \times \mathcal{G}$ obtained by choosing exactly one element from each equivalence class. Now the application $b : \widetilde{\mathcal{H} \times \mathcal{G}} \rightarrow \mathcal{F}_\pi : (h, g) \mapsto hg$ is a bijection so that, applying Lemma 18 then Lemma 15, one obtains that, for all $\tau > 0$, all $\omega \in \Omega$ and all $n \geq 1$,

$$N\left(\tau, \mathcal{F}_\pi, d_{\nu_n}^{(1)} + d_{\lambda_n}^{(1)}\right) = N\left(\tau, \widetilde{\mathcal{H} \times \mathcal{G}}, d_{\nu_n}^{(1)} + d_{\lambda_n}^{(1)}\right) \leq N\left(\tau, \mathcal{H} \times \mathcal{G}, d_{\nu_n}^{(1)} + d_{\lambda_n}^{(1)}\right) \quad (18)$$

Lemma 17 with $t = 1/2$ now gives for all $\omega \in \Omega$ and all $n \geq 1$,

$$N\left(\tau, \mathcal{H} \times \mathcal{G}, d_{\nu_n}^{(1)} + d_{\lambda_n}^{(1)}\right) \leq N\left(\tau/2, \mathcal{H}, d_{\lambda_n}^{(1)}\right) N\left(\tau/2, \mathcal{G}, d_{\nu_n}^{(1)}\right). \quad (19)$$

Since \mathcal{H} is a V.C.G.C. and λ_n is a finite measure for all $n \geq 1$, for all $\varepsilon > 0$, there exists a constant C_1 depending only on ε and on the V.C. dimension of \mathcal{H} such that, for all $n \geq 1$, $N\left(\varepsilon, \mathcal{H}, d_{\lambda_n}^{(1)}\right) \leq C_1$. Analogously, since \mathcal{G} is a V.C.G.C. and ν_n is a finite measure for all $\omega \in \Omega$ and all $n \geq 1$, for all $\varepsilon > 0$, there exists a constant C_2 depending only on ε and the V.C. dimension of \mathcal{G} such that, for all $\omega \in \Omega$ and $n \geq 1$, $N\left(\varepsilon, \mathcal{G}, d_{\nu_n}^{(1)}\right) \leq C_2$. Take $\tau = 2\varepsilon$. Combining inequalities (17), (18) and (19) gives

$$\sup_{\omega \in \Omega} \sup_{n \geq 1} N\left(\tau, \mathcal{F}_\pi, d_{\mathbb{P}_n}^{(1)}\right) \leq C_1 C_2.$$

The proof is complete with $C = C_1 C_2$. ■

We now turn to the case $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$. First of all, we show \mathcal{F}_π can be made into a totally bounded pseudo-metric space. Let $d_\lambda^{(2)}$ and $d_\nu^{(2)}$ be the $L_2(\lambda)$ and the $L_2(\nu)$ pseudo-metrics on \mathcal{H} and \mathcal{G} , respectively. Endow \mathcal{F}_π with the pseudo-metric

$$d(f_1, f_2) \triangleq d_\lambda^{(2)}(h_1, h_2) + d_\nu^{(2)}(g_1, g_2), \quad (20)$$

where $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$ are any two elements of \mathcal{F}_π . It suffices to rehearse the arguments in the proof of Proposition 19 to show that, for all $\varepsilon > 0$,

$$N(\varepsilon, \mathcal{F}_\pi, d) \leq N\left(\varepsilon/2, \mathcal{H}, d_\lambda^{(2)}\right) N\left(\varepsilon/2, \mathcal{G}, d_\nu^{(2)}\right) < +\infty,$$

and, therefore, that (\mathcal{F}_π, d) is totally bounded. This fact and the one presented in the following proposition are key to proving uniform central limit theorems for sequences of \mathcal{F}_π -indexed sequential empirical measure processes.

Proposition 20 *Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables defined on a probability space (Ω, \mathcal{A}, P) and taking values in some measurable space (U, \mathcal{U}) . Let ν be the law of X_1 and let \mathcal{F}_π be a class of product functions $f = hg$, where $h \in \mathcal{H} \subset \mathfrak{R}^{[0,1]}$ and $g \in \mathcal{G} \subset \mathfrak{R}^U$. Suppose \mathcal{H} and \mathcal{G} verify conditions (H1) and (H3), respectively. Let $\|\cdot\|_{(\lambda_n \otimes \nu)}$ be the semi-norm on \mathcal{F}_π such that*

$$\|f\|_{(\lambda_n \otimes \nu)}^2 = (\lambda_n \otimes \nu)(f^2), \quad f \in \mathcal{F}_\pi,$$

where λ_n is the discrete uniform measure on $[0, 1]$, and let $d_{(\lambda_n \otimes \nu)}^{(2)}$ be the pseudo-metric on \mathcal{F}_π defined by

$$d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) \triangleq \|f_1 - f_2\|_{(\lambda_n \otimes \nu)}, \quad f_1, f_2 \in \mathcal{F}_\pi.$$

If

$$\lim_{n \rightarrow +\infty} \sup_{h_1, h_2 \in \mathcal{H}} \left| \lambda_n (h_1 - h_2)^2 - \lambda (h_1 - h_2)^2 \right| = 0, \quad (21)$$

then

$$\lim_{\alpha \downarrow 0} \limsup_{n \rightarrow +\infty} \sup_{f_1, f_2 \in \mathcal{F}_\pi^\alpha} d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) = 0,$$

where $\mathcal{F}_\pi^\alpha \triangleq \{f_1, f_2 \in \mathcal{F}_\pi : d(f_1, f_2) \leq \alpha\}$ and d is defined in (20).

Proof. Take $f_1, g_1 \in \mathcal{F}_\pi$ with $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$. Then

$$\begin{aligned} d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) &= \|h_1 g_1 - h_2 g_2\|_{(\lambda_n \otimes \nu)} \\ &\leq \|h_1\|_{(\lambda_n \otimes \nu)} \|g_1 - g_2\|_{(\lambda_n \otimes \nu)} + \|g_2\|_{(\lambda_n \otimes \nu)} \|h_1 - h_2\|_{(\lambda_n \otimes \nu)} \\ &= \sqrt{\lambda_n (h_1^2) \nu \left((g_1 - g_2)^2 \right)} + \sqrt{\nu (g_2^2) \lambda_n \left((h_1 - h_2)^2 \right)} \\ &\leq \nu (G^2) \left[d_\nu^{(2)}(g_1, g_2) + d_{\lambda_n}^{(2)}(h_1, h_2) \right], \end{aligned}$$

where G is the envelope of \mathcal{G} . Then, assuming, without loss of generality, that $\nu(G^2) = 1$, we have

$$\begin{aligned} \sup_{f_1, f_2 \in \mathcal{F}_\pi^\alpha} d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) &\leq \sup_{g_1, g_2 \in \mathcal{G}^\alpha} d_\nu^{(2)}(g_1, g_2) + \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} d_{\lambda_n}^{(2)}(h_1, h_2) \\ &\leq \alpha + \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} d_{\lambda_n}^{(2)}(h_1, h_2), \end{aligned}$$

where

$$\mathcal{G}_\nu^\alpha \triangleq \left\{ g_1, g_2 \in \mathcal{G} : d_\nu^{(2)}(g_1, g_2) \leq \alpha \right\},$$

and

$$\mathcal{H}_\lambda^\alpha \triangleq \left\{ h_1, h_2 \in \mathcal{H} : d_\lambda^{(2)}(h_1, h_2) \leq \alpha \right\}.$$

Now,

$$\begin{aligned} \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} d_{\lambda_n}^{(2)}(h_1, h_2) &\leq \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} d_\lambda^{(2)}(h_1, h_2) \\ &\quad + \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} \sqrt{\left| \lambda_n (h_1 - h_2)^2 - \lambda (h_1 - h_2)^2 \right|} \\ &\leq \alpha + \sup_{h_1, h_2 \in \mathcal{H}_\lambda} \sqrt{\left| \lambda_n (h_1 - h_2)^2 - \lambda (h_1 - h_2)^2 \right|}. \end{aligned}$$

Taking limits, one has

$$\lim_{n \rightarrow +\infty} \sup_{h_1, h_2 \in \mathcal{H}_\lambda^\alpha} d_{\lambda_n}^{(2)}(h_1, h_2) \leq \alpha,$$

so that

$$\lim_{n \rightarrow +\infty} \sup_{f_1, f_2 \in \mathcal{F}_\pi^\alpha} d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) \leq 2\alpha.$$

Taking limits for $\alpha \downarrow 0$ completes the proof. ■

4.2 Proof of Theorem 5.

Here is the announced proof of the Glivenko-Cantelli type result for the B -indexed empirical measure

Lemma 21 *Let $\mathcal{W} \subset \mathcal{U}$ be a Vapnik-Červonenkis class of measurable subsets of U . Then, for any regular Borel subset, B , of $[0, 1]$, we have that*

$$P \left(\lim_{n \rightarrow +\infty} \sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| = 0 \right) = 1,$$

where $\nu_{n,B}$ is defined in 9.

Proof. The proof follows closely the lines of the arguments presented in Pollard [1984], Section II.3, pages 13-16. Almost sure convergence follows by Borel-Cantelli Lemma, once we show that, for all $\varepsilon > 0$,

$$\sum_{n \geq 1} P \left(\sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| > \varepsilon \right) < +\infty. \quad (22)$$

Define the sequence of natural numbers $(k_n^B)_{n \geq 1}$ by

$$k_n^B \triangleq \text{card} \left(B \cap \left\{ \frac{1}{n}, \frac{1}{n-1}, \dots, 1 \right\} \right),$$

so that the B -empirical measure can be written

$$\nu_{n,B} = \frac{1}{k_n^B} \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}.$$

As in Pollard [1984], it is possible to show that for all $n \geq 1$ such that $k_n^B \geq 8\varepsilon^{-2}$,

$$P \left(\sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| > \varepsilon \right) \leq 4P \left(\sup_{W \in \mathcal{W}} \left| \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \sigma_i \mathbf{1}_W(X_i) \right| > \frac{\varepsilon k_n^B}{4} \right),$$

where $(\sigma_n)_{n \geq 1}$ is a sequence of Rademacher random variables independent of the sequence $(X_n)_{n \geq 1}$. Now we deal with the conditional probability

$$P \left(\sup_{W \in \mathcal{W}} \left| \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \sigma_i \mathbf{1}_W(X_i) \right| > \frac{\varepsilon k_n^B}{4} \mid \left\{ X_i = x_i : \frac{i}{n} \in B \right\} \right).$$

Since \mathcal{W} is a V.C.C., once the x_i 's are fixed, there exist sets $W_1, \dots, W_{K_n^B}$, where K_n^B coincides with the k_n^B -th shatter coefficient of \mathcal{W} , not depending on the finite family $\{x_i : \frac{i}{n} \in B\}$ such that

$$\begin{aligned} & P \left(\sup_{W \in \mathcal{W}} \left| \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \sigma_i \mathbf{1}_W(X_i) \right| > \frac{\varepsilon k_n^B}{4} \mid \left\{ X_i = x_i : \frac{i}{n} \in B \right\} \right) \\ &= P \left(\max_{1 \leq j \leq K_n^B} \left| \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \sigma_i \mathbf{1}_{W_j}(X_i) \right| > \frac{\varepsilon k_n^B}{4} \mid \left\{ X_i = x_i : \frac{i}{n} \in B \right\} \right). \end{aligned}$$

Again, as in Pollard [1984], apply the union bound followed by Hoeffding inequality then integrate out to obtain that, for all $n \geq 1$ such that $k_n^B \geq 8\varepsilon^{-2}$,

$$P \left(\sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| > \varepsilon \right) \leq 8K_n^B \exp \left(-\frac{\varepsilon^2 k_n^B}{32} \right). \quad (23)$$

Since \mathcal{W} is a V.C.C., then $K_n^B \leq (k_n^B + 1)^{\mathcal{S}(\mathcal{W})}$, where $\mathcal{S}(\mathcal{W})$ is the V.C. dimension of \mathcal{W} . Combining (22) and (23), it is enough to show that

$$\sum_{n \geq 1} (k_n^B + 1)^{\mathcal{S}(\mathcal{W})} \exp \left(-\frac{\varepsilon^2 k_n^B}{32} \right) < +\infty. \quad (24)$$

Given B is a regular Borel set, then, by (8), $\lim_n \frac{k_n^B}{n} = \lambda(B)$, so that, for sufficiently large n , there exists $0 < \gamma < \lambda(B)$ such that

$$n(\lambda(B) - \gamma) \leq k_n^B \leq n(\lambda(B) + \gamma).$$

It follows that

$$\sum_{n \geq 1} (k_n^B + 1)^{\mathcal{S}(\mathcal{W})} \exp \left(-\frac{\varepsilon^2 k_n^B}{32} \right) \leq \sum_{n \geq 1} (n(\lambda(B) + \gamma) + 1)^{\mathcal{S}(\mathcal{W})} \exp \left(-\frac{n(\lambda(B) - \gamma)\varepsilon^2}{32} \right) < +\infty,$$

since $\lambda(B) - \gamma > 0$. ■

We are now ready to prove Theorem 5. To prove (10), it is enough to show that, for all $b > 0$,

$$\sum_{n \geq 1} P \left(\sup_{B \in \mathcal{B}_\#} \lambda_n(B) \sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| > b \right) < +\infty. \quad (25)$$

As before, we need to find a suitable bound for

$$\begin{aligned} & P \left(\sup_{B \in \mathcal{B}_\#} \lambda_n(B) \sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| > b \right) \\ &= P \left(\sup_{B \in \mathcal{B}_\#} \frac{k_n^B}{n} \sup_{W \in \mathcal{W}} \left| \frac{1}{k_n^B} \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}(W) - \nu(W) \right| > b \right). \end{aligned}$$

Let \mathcal{J}_n be the class of all subsets of $\{\frac{1}{n}, \dots, 1\}$. Write

$$\mathcal{J}_n = \bigcup_{k=0}^n \mathcal{J}_n^k,$$

where $\mathcal{J}_n^k \triangleq \{J_{n,j}^k, j = 1, \dots, \binom{n}{k}\}$, $k = 0, \dots, n$, is the class of all subsets of $\{\frac{1}{n}, \dots, 1\}$ of cardinality exactly equal to k . Define

$$\mathcal{J}_n(\mathcal{B}_\#) \triangleq \left\{ B \cap \left\{ \frac{1}{n}, \dots, 1 \right\} : B \in \mathcal{B}_\# \right\}.$$

Since $\mathcal{B}_\#$ is a V.C.C., $\text{card}(\mathcal{J}_n(\mathcal{B}_\#)) \leq m^{\mathcal{B}_\#}(n)$, where $m^{\mathcal{B}_\#}(n)$ is the n -th shatter coefficient of $\mathcal{B}_\#$. Finally, define

$$\mathcal{J}_n^k(\mathcal{B}_\#) \triangleq \mathcal{J}_n^k \cap \mathcal{J}_n(\mathcal{B}_\#),$$

and let

$$K_{n,K}^{\mathcal{B}\#} \triangleq \text{card} \left(\mathcal{J}_n^k(\mathcal{B}\#) \right).$$

Clearly,

$$\sum_{k=0}^n K_{n,k}^{\mathcal{B}\#} \leq m^{\mathcal{B}\#}(n).$$

It follows that

$$\begin{aligned} P \left(\sup_{B \in \mathcal{B}\#} \frac{k_n^B}{n} \sup_{W \in \mathcal{W}} \left| \frac{1}{k_n^B} \sum_{i \in B \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}(W) - \nu(W) \right| > b \right) \\ \leq P \left(\max_{0 \leq k \leq n} \frac{k}{n} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} \sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > b \right), \end{aligned}$$

where $\nu_J \triangleq \frac{1}{\text{card}(J)} \sum_{i \in J \cap \{\frac{1}{n}, \dots, 1\}} \delta_{X_i}$, with the convention that $\nu_J \equiv 0$ if $J = \emptyset$. Then,

$$P \left(\max_{0 \leq k \leq n} \frac{k}{n} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} \sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > b \right) = P \left(\max_{1 \leq k \leq n} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} \sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > \frac{bn}{k} \right).$$

Now,

$$\begin{aligned} & P \left(\max_{1 \leq k \leq n} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} \sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > \frac{bn}{k} \right) \\ & \leq \sum_{k=1}^n \sum_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} P \left(\sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > \frac{bn}{k} \right) \\ & \leq \sum_{k=1}^n K_{n,k}^{\mathcal{B}\#} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} P \left(\sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > \frac{bn}{k} \right) \\ & \leq \sum_{k=1}^n m^{\mathcal{B}\#}(n) P \left(\sup_{W \in \mathcal{W}} |\nu_k(W) - \nu(W)| > \frac{bn}{k} \right), \end{aligned}$$

where $\nu_k \triangleq k^{-1} \sum_{i=1}^k \delta_{X_i}$, and where the last inequality follows from independence and identity in distribution of the X_i 's. Now, apply (23) for $B = [0, 1]$ to obtain

$$P \left(\sup_{W \in \mathcal{W}} |\nu_k(W) - \nu(W)| > \frac{bn}{k} \right) \leq 8m^{\mathcal{W}}(k) \exp \left(-\frac{b^2 n^2}{32k} \right),$$

where $m^{\mathcal{W}}(k)$ is the k -th shatter coefficient of \mathcal{W} . Consequently,

$$P \left(\max_{1 \leq k \leq n} \max_{J \in \mathcal{J}_n^k(\mathcal{B}\#)} \sup_{W \in \mathcal{W}} |\nu_J(W) - \nu(W)| > \frac{bn}{k} \right) \leq 8 \sum_{k=1}^n m^{\mathcal{B}\#}(n) m^{\mathcal{W}}(k) \exp \left(-\frac{b^2 n^2}{32k} \right).$$

To show (25), it is therefore enough to show that

$$\sum_{n \geq 1} \sum_{k=1}^n m^{\mathcal{B}\#}(n) m^{\mathcal{W}}(k) \exp \left(-\frac{b^2 n^2}{32k} \right) = \sum_{k \geq 1} m^{\mathcal{W}}(k) \sum_{n \geq k} m^{\mathcal{B}\#}(n) \exp \left(-\frac{b^2 n^2}{32k} \right) < +\infty. \quad (26)$$

Since $n \geq k$, we have $\exp\left(-\frac{b^2 n^2}{32k}\right) \leq \exp\left(-\frac{b^2 n}{32}\right)$. Moreover, for all $n \geq k > \max\{\mathcal{S}(\mathcal{B}_\#), \mathcal{S}(\mathcal{W})\}$, where $\mathcal{S}(\mathcal{B}_\#)$ and $\mathcal{S}(\mathcal{W})$ denote the V.C. dimensions of $\mathcal{B}_\#$ and \mathcal{W} , respectively, we have that $m^{\mathcal{W}}(k) \leq (k+1)^{\mathcal{S}(\mathcal{W})} < M(\mathcal{W})k^{\mathcal{S}(\mathcal{W})}$ and that $m^{\mathcal{B}_\#}(n) \leq (n+1)^{\mathcal{S}(\mathcal{B}_\#)} < M(\mathcal{B}_\#)n^{\mathcal{S}(\mathcal{B}_\#)}$, $M(\mathcal{W})$ and $M(\mathcal{B}_\#)$ being constants depending only on classes \mathcal{W} and $\mathcal{B}_\#$, respectively. The convergence of the double series in the RHS of (26) follows if it holds that, for all $c > 0$,

$$\sum_{k \geq 1} k^{\mathcal{S}(\mathcal{W})} \sum_{n \geq k} n^{\mathcal{S}(\mathcal{B}_\#)} \exp(-cn) < +\infty,$$

which is, in turn, true if for all $c > 0$ and all $D_1, D_2 \in \mathbb{N}^+$,

$$I(c, D_1, D_2) \triangleq \int_1^\infty \int_y^{+\infty} y^{D_1} x^{D_2} \exp(-cx) dx dy < +\infty.$$

Elementary calculus gives

$$I(c, D_1, D_2) = e^{-c} \sum_{p=0}^{D_2} \sum_{l=0}^{D_1+D_2-p} c^{-(p+l+2)} \frac{D_2! (D_1 + D_2 - p)!}{(D_2 - p)! (D_1 + D_2 - p - l)!} < +\infty, \quad (27)$$

completing the proof of (10). As for (12), just observe that

$$\sup_{B \in \mathcal{B}_\#} \sup_{W \in \mathcal{W}} |\mathbb{P}_n(B \times W) - \lambda(B) \nu(W)| \leq \sup_{B \in \mathcal{B}_\#} \lambda_n(B) \sup_{W \in \mathcal{W}} |\nu_{n,B}(W) - \nu(W)| + \sup_{B \in \mathcal{B}_\#} |\lambda_n(B) - \lambda(B)|.$$

The first term of the RHS converges almost surely to 0 thanks to (27) so that the deterministic convergence to 0 described in (11) implies (12).

4.3 Functional central limit theorems

Let $\mathcal{Q} \subset \mathfrak{R}^{[0,1] \times U}$ be a class of $\mathcal{B}([0,1] \otimes \mathcal{U})$ -measurable functions defined by conditions (a) and (b) of Section 3. Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables of law ν defined on a probability space (Ω, \mathcal{A}, P) and taking values in some measurable space (U, \mathcal{U}) . As already announced in that Section, will prove convergence of finite dimensional laws of the sequence of \mathcal{Q} -indexed processes $\{Z_n(q) : q \in \mathcal{Q}\}$, $n \geq 1$, where

$$Z_n(q) \triangleq \sqrt{n} (\mathbb{P}_n(q) - (\lambda_n \otimes \nu)(q)),$$

λ_n being the discrete uniform measure and $\mathbb{P}_n(q)$ being defined in (1), to those of the \mathcal{Q} -indexed centered Gaussian process $\{Z(q) : q \in \mathcal{Q}\}$ whose covariance structure is given by (13). To begin with, we will prove that, for all $q \in \mathcal{Q}$, $Z_n(q) \xrightarrow{\mathcal{L}} Z(q)$. We first need to make some remarks and introduce some notation. Note that if $q \in \mathcal{Q}$ then $\tilde{q} \in \mathcal{Q}$ where \tilde{q} is defined, for all $(s, x) \in [0, 1] \times U$, by

$$\tilde{q}(s, x) \triangleq q(s, x) - \nu(q)(s). \quad (28)$$

Now, use the sequence $(X_n)_{n \geq 1}$ to construct the triangular array $(X_{in})_{1 \leq i \leq n, n \geq 1}$ of independent random variables as follows:

$$X_{in} \triangleq \frac{\tilde{q}(i/n, X_i)}{\sqrt{n}},$$

where \tilde{q} is defined in (28). Note that

$$Z_n(q) = \sum_{i=1}^n X_{in},$$

and that

$$Z(q) \stackrel{d}{=} N(0, (\lambda \otimes \nu)(\tilde{q}^2)).$$

Lemma 22 For all $q \in \mathcal{Q}$, $Z_n(q) \xrightarrow{\mathcal{L}} Z(q)$.

Proof. We first treat the case when $(\lambda \otimes \nu)(\tilde{q}^2) = 0$. In this case, $Z(q) \stackrel{d}{=} \delta_0$. Moreover, it is easily seen that $E\left((Z_n(q))^2\right) = (\lambda_n \otimes \nu)(\tilde{q}^2)$, so that, by (14), $Z_n(q) \xrightarrow{L_2} Z(q)$, and consequently $Z_n(q) \xrightarrow{\mathcal{L}} \delta_0$. In the case in which $(\lambda \otimes \nu)(\tilde{q}^2) > 0$, Lindeberg central limit theorem implies that $Z_n(q) \xrightarrow{\mathcal{L}} Z(q)$ if

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow +\infty} \frac{1}{n(\lambda_n \otimes \nu)(\tilde{q}^2)} \sum_{i=1}^n \int_{\{x: |\tilde{q}(i/n, X_i)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\}} \tilde{q}^2(i/n, X_i) \nu(dx) = 0. \quad (29)$$

By condition (a) in section 3, by (14) and by the fact that, for all $\varepsilon > 0$,

$$\left\{x: |\tilde{q}(i/n, X_i)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\right\} \subset \left\{x: |g_{\tilde{q}}(x)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\right\},$$

for sufficiently large n , there exists $0 < \gamma < (\lambda \otimes \nu)(\tilde{q}^2)$ such that, for all $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{n(\lambda_n \otimes \nu)(\tilde{q}^2)} \sum_{i=1}^n \int_{\{x: |\tilde{q}(i/n, X_i)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\}} \tilde{q}^2(i/n, X_i) \nu(dx) \\ \leq \frac{1}{(\lambda \otimes \nu)(\tilde{q}^2)} \int_{\{x: |g_{\tilde{q}}(x)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\}} g_{\tilde{q}}^2(x) \nu(dx). \end{aligned}$$

Since $g_{\tilde{q}} \in L_2(\nu)$,

$$\lim_{n \rightarrow +\infty} \int_{\{x: |g_{\tilde{q}}(x)| \geq \varepsilon \sqrt{n(\lambda_n \otimes \nu)(\tilde{q}^2)}\}} g_{\tilde{q}}^2(x) \nu(dx) = 0,$$

which implies (29). ■

To complete the proof of Proposition 9, we will employ the Cramér-Wold device. Fix a natural number $1 \leq K < +\infty$ and functions $g_1, \dots, g_K \in \mathcal{Q}$. Consider the random vector $Z_{nK} = (Z_n(q_1), \dots, Z_n(q_K))$. To show that the sequence $(Z_{nK})_{n \geq 1}$ converges in law to the vector $Z_{nK} = (Z(q_1), \dots, Z(q_K))$, it is enough to show that the sequence of random vectors $\tilde{Z}_{nK} = (Z_n(\tilde{q}_1), \dots, Z_n(\tilde{q}_K))$, $n \geq 1$, converges in law to the random vector $\tilde{Z}_{nK} = (Z(\tilde{q}_1), \dots, Z(\tilde{q}_K))$. This is equivalent to showing that, for all $(a_1, \dots, a_K) \in \mathfrak{R}^K$,

$$\sum_{i=1}^K a_i Z_n(\tilde{q}_i) \xrightarrow{\mathcal{L}} \sum_{i=1}^K a_i Z(\tilde{q}_i).$$

It is easily seen that

$$\sum_{i=1}^K a_i Z_n(\tilde{q}_i) = Z_n \left(\sum_{i=1}^K a_i \tilde{q}_i \right),$$

and, since \mathcal{Q} is a linear space, we have by Lemma 22,

$$Z_n \left(\sum_{i=1}^K a_i \tilde{q}_i \right) \xrightarrow{\mathcal{L}} Z \left(\sum_{i=1}^K a_i \tilde{q}_i \right).$$

Finally, it is easily seen that

$$\text{Var} \left(Z \left(\sum_{i=1}^K a_i \tilde{q}_i \right) \right) = \text{Var} \left(\sum_{i=1}^K a_i Z(\tilde{q}_i) \right) = (\lambda \otimes \nu) \left(\left(\sum_{i=1}^K a_i \tilde{q}_i \right)^2 \right),$$

proving Proposition 9.

Let's return back to \mathcal{F}_π -indexed s.e.m.p.'s with $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$ or $\mathcal{F}_\pi \in \pi(UB, M\text{-}VC)$. To prove weak convergence in the former case, we will use the findings of section 4.2 in Ziegler [1997]. The following Theorem summarizes the results needed.

Theorem 23 (Ziegler, 1997, Paragraph 4.2) *Let (Υ, \mathcal{E}) be a measurable space and let $(\eta_{mi})_{1 \leq i \leq i(n), n \geq 1}$ be a triangular array of rowwise independent Υ -valued random variables with laws ν_{ni} , respectively. Let \mathcal{F} be a class of \mathcal{E} -measurable real valued functions defined on Υ with envelope F and assume that \mathcal{F} has a uniformly integrable L_2 -entropy (see Ziegler [1997] for the definition of uniformly integrable L_2 -entropy of a class \mathcal{F}). Assume also that there exists a metric d on \mathcal{F} such that (\mathcal{F}, d) is a totally bounded metric space. Consider the \mathcal{F} -indexed stochastic processes $\{S_n(f) : f \in \mathcal{F}\}$, $n \geq 1$, where, for all $n \geq 1$,*

$$S_n(f) \triangleq \frac{1}{\sqrt{i(n)}} \sum_{1 \leq i \leq i(n)} (f(\eta_{ni}) - \nu_{ni}(f)), \quad f \in \mathcal{F}.$$

define the probability measure

$$\tilde{\nu}_n \triangleq \frac{1}{i(n)} \sum_{1 \leq i \leq i(n)} \nu_{ni},$$

and the quantity

$$a_n(\alpha) \triangleq \sup_{\{f, g \in \mathcal{F} : d(f, g) \leq \alpha\}} \sqrt{\tilde{\nu}_n \left((f - g)^2 \right)}, \quad \alpha > 0.$$

If there exists a centered \mathcal{F} -indexed Gaussian process $\bar{G} = \{G(f) : f \in \mathcal{F}\}$ such that the finite dimensional distributions of the sequence of processes S_n converge to those of \bar{G} and if

- (i) $\sup_{n \geq 1} \tilde{\nu}_n(F^2) < +\infty$;
- (ii) $\lim_{\alpha \downarrow 0} \limsup_{n \rightarrow +\infty} a_n(\alpha) = 0$;
- (iii) for all $\delta > 0$,

$$\lim_{n \rightarrow +\infty} \sum_{1 \leq i \leq i(n)} E \left(F^2(\eta_{ni}) \cdot \mathbf{1}_{(\delta \sqrt{i(n)}, +\infty)}(F(\eta_{ni})) \right) = 0,$$

then there exists a version G of \bar{G} , with uniformly bounded and uniformly continuous sample paths such that

$$S_n \xrightarrow[\text{sep}]{\mathcal{L}} G.$$

Theorem 10 is a corollary of Theorem 23. To see this, define the triangular array of rowwise independent random variables

$$\eta_{mi} = (i/n, X_i), \quad 1 \leq i \leq n, \quad n \geq 1,$$

and observe that, for all $1 \leq i \leq n$ and all $n \geq 1$, the law of η_{mi} is $\delta_{i/n} \otimes \nu$. The sequence of stochastic processes $\{S_n(f) : f \in \mathcal{F}_\pi\}$ described in the statement Theorem 23 is nothing but the sequence of stochastic processes $\{Z_n(f) : f \in \mathcal{F}_\pi\}$. Now, (\mathcal{F}_π, d) , with $d = d_\lambda^{(2)} + d_\nu^{(2)}$ is a totally bounded and since any $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$ is by definition a V.C.G.C., then it possesses a uniformly integrable L_2 -entropy. As for convergence of finite dimensional laws, since $\mathcal{F}_\pi \subset \mathcal{Q}$, then the finite dimensional laws of the process Z_n converge to those of Z . We are, therefore, only left with the verification of conditions (i), (ii) and (iii) of Theorem 23. Condition (i) is immediate since it translates as follows:

$$\sup_{n \geq 1} (\lambda_n \otimes \nu)(G^2) < +\infty,$$

and $G \in L_2(\nu)$. Moreover, condition (iii) applied to our case becomes

$$\forall \delta > 0, \quad \frac{1}{n} \sum_{i=1}^n E \left(G^2(X_i) \cdot \mathbf{1}_{(\delta\sqrt{n}, +\infty)}(G(X_i)) \right) = 0,$$

which is true by dominated convergence, since $G \in L_2(\nu)$. Finally, condition (ii) is exactly the conclusion of Proposition 20.

We end this section with the proof of Theorem 11. All the other conditions being true as for the case of $\mathcal{F}_\pi \in \pi(\nu G^2, J\text{-}VC)$, we only need to prove d -equicontinuity of the sequence \mathcal{F}_π -indexed stochastic processes $\{Z_n(f) : f \in \mathcal{F}_\pi\}$, when $\mathcal{F}_\pi \in \pi(UB, M\text{-}VC)$. As argued in Ziegler [1997], it is enough to show that

$$\lim_{\alpha \downarrow 0} \limsup_{n \rightarrow +\infty} E^* \left(\sup_{\{f_1, f_2 \in \mathcal{F}_\pi : d(f_1, f_2) \leq \alpha\}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i [h_1(i/n) g_1(X_i) - h_2(i/n) g_2(X_i)] \right| \right) = 0,$$

where E^* denotes the outer expectation operator, where $(\varepsilon_n)_{n \geq 1}$ is a canonically formed Rademacher sequence and where we have written $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$. For all $\mathcal{F}_\pi \ni f = hg$, define the triangular array

$$\Phi_{ni}(f) \triangleq \frac{1}{\sqrt{n}} h(i/n) g(X_i),$$

the random seminorm $\|\cdot\|_{\rho_n}$ such that, for all $f \in \mathcal{F}_\pi$,

$$\|f\|_{\rho_n}^2 = \sum_{i=1}^n \Phi_{ni}^2(f),$$

and the random pseudo-metric

$$d_{\rho_n}^{(2)}(f_1, f_2) \triangleq \|f_1 - f_2\|_{\rho_n}.$$

With this notation, we need to prove that

$$\lim_{\alpha \downarrow 0} \limsup_{n \rightarrow +\infty} E^* \left(\sup_{\{f_1, f_2 \in \mathcal{F}_\pi : d(f_1, f_2) \leq \alpha\}} \left| \sum_{i=1}^n \varepsilon_i [\Phi_{ni}(f_1) - \Phi_{ni}(f_2)] \right| \right) = 0. \quad (30)$$

Note that

$$d_{(\lambda_n \otimes \nu)}^{(2)}(f_1, f_2) = \|f_1 - f_2\|_{(\lambda_n \otimes \nu)} = \sqrt{E \left(\|f_1 - f_2\|_{\rho_n}^2 \right)},$$

so that Theorem 3.1 in Ziegler [1997] yields the existence of universal constants K_1 and K_2 such that

$$E^* \left(\sup_{\{f_1, f_2 \in \mathcal{F}_\pi : d(f_1, f_2) \leq \alpha\}} \left| \sum_{i=1}^n \varepsilon_i [\Phi_{ni}(f_1) - \Phi_{ni}(f_2)] \right| \right) \leq K_1 A(n, \alpha) B(n) + K_2 C(n, \alpha),$$

with

$$\begin{aligned} A(n, \alpha) &= \sqrt{E^* \left(\max_{1 \leq i \leq n} \sup_{f_1, f_2 \in \mathcal{F}_\pi} \frac{1}{\sqrt{n}} |h_1(i/n) g_1(X_i) - h_2(i/n) g_2(X_i)| \cdot l_n(1) \right)} \\ &\leq \frac{\sqrt{2}}{\alpha \sqrt[4]{n}} \sqrt{E^*(l_n(1))}, \end{aligned}$$

$$B(n) = \sqrt{E^*(l_n(1)^2)},$$

$$C(n, \alpha) = E(l_n(\alpha)),$$

and

$$l_n(\alpha) = \int_0^\alpha \sqrt{\log N(\tau, \mathcal{F}_\pi, d_{\rho_n}^{(2)})} d\tau, \quad \alpha > 0.$$

Clearly, for all $f_1 = h_1 g_1$ and $f_2 = h_2 g_2$ in \mathcal{F}_π , all $\omega \in \Omega$ and all $n \geq 1$,

$$d_{\rho_n}^{(2)}(f_1, f_2) \leq d_{\nu_n}^{(2)}(g_1, g_2) + d_{\lambda_n}^{(2)}(h_1, h_2),$$

where

$$d_{\nu_n}^{(2)}(g_1, g_2) \triangleq \sqrt{\frac{1}{n} \sum_{i=1}^n (g_1(X_i) - g_2(X_i))^2},$$

and

$$d_{\lambda_n}^{(2)}(h_1, h_2) \triangleq \sqrt{\frac{1}{n} \sum_{i=1}^n (h_1(i/n) - h_2(i/n))^2}.$$

Rehearse the arguments of Proposition 19 to show that, for all $\omega \in \Omega$, all $\tau > 0$ and all $n \geq 1$,

$$N(\tau, \mathcal{F}_\pi, d_{\rho_n}^{(2)}) \leq N(\tau, \mathcal{H}, d_{\lambda_n}^{(2)}) N(\tau, \mathcal{G}, d_{\nu_n}^{(2)}).$$

It follows that, for all $\alpha > 0$, all $\omega \in \Omega$ and all $n \geq 1$,

$$\int_0^\alpha \sqrt{\log N(\tau, \mathcal{F}_\pi, d_{\rho_n}^{(2)})} d\tau \leq \int_0^\alpha \sqrt{\log N(\tau, \mathcal{H}, d_{\lambda_n}^{(2)})} d\tau + \int_0^\alpha \sqrt{\log N(\tau, \mathcal{G}, d_{\nu_n}^{(2)})} d\tau.$$

Since, for all $n \geq 1$, λ_n is a finite measure on $[0, 1]$ and since \mathcal{H} is a V.C.G.C., then there exists a function $\gamma_{\mathcal{H}}(\tau)$ such that, for all $n \geq 1$,

$$\int_0^\alpha \sqrt{\log N(\tau, \mathcal{H}, d_{\lambda_n}^{(2)})} d\tau \leq \int_0^\alpha \sqrt{\log \gamma_{\mathcal{H}}(\tau)} d\tau < +\infty.$$

Analogously, there exists a function $\gamma_{\mathcal{G}}(\tau)$ such that, for all $n \geq 1$ and all $\omega \in \Omega$,

$$\int_0^\alpha \sqrt{\log N(\tau, \mathcal{G}, d_{\nu_n}^{(2)})} d\tau \leq \int_0^\alpha \sqrt{\log \gamma_{\mathcal{G}}(\tau)} d\tau < +\infty.$$

Consequently,

$$E(l_n(1)^2) < +\infty,$$

which in turn implies that, for all $\alpha > 0$,

$$\limsup_{n \rightarrow +\infty} A(n, \alpha) B(n) = 0.$$

Moreover, since

$$\lim_{\alpha \downarrow 0} \int_0^\alpha \sqrt{\log \gamma_{\mathcal{H}}(\tau)} d\tau = \lim_{\alpha \downarrow 0} \int_0^\alpha \sqrt{\log \gamma_{\mathcal{G}}(\tau)} d\tau = 0,$$

then

$$\lim_{\alpha \downarrow 0} \limsup_{n \rightarrow +\infty} C(n, \alpha) = 0,$$

completing the proof of (30).

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