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A NOTE ON TALAGRAND'S TRANSPORTATION INEQUALITY AND LOGARITHMIC SOBOLEV INEQUALITY

PATRICK CATTIAUX ♠ , **ARNAUD GUILLIN** ◇, AND **LI-MING WU** ◇

♠ Université de Toulouse

◇ Université Blaise Pascal

Abstract. We give by simple arguments sufficient conditions, so called Lyapunov conditions, for Talagrand's transportation information inequality and for the logarithmic Sobolev inequality. Those sufficient conditions work even in the case where the Bakry-Emery curvature is not lower bounded. Several new examples are provided.

Key words : Lyapunov functions, Talagrand transportation information inequality, logarithmic Sobolev inequality.

MSC 2000 : 26D10, 47D07, 60G10, 60J60.

1. Introduction and main results.

Transportation cost information inequalities have been recently deeply studied, especially for their connection with the concentration of measure phenomenon, or for deviation inequalities for Markov processes (see [24, 22]). In particular, Talagrand [29] establishes the so-called T_2 inequality (or Talagrand's transportation inequality, or W_2H inequality) for the Gaussian measure, establishing thus Gaussian dimension free concentration of measure. But before going further in the numerous results around these inequalities, let us present the object under study.

Given a metric space (E, d) equipped with its Borel σ field, and $1 \leq p < +\infty$, the L^p Wasserstein distance between two probability measures μ and ν on E is defined as

$$W_p(\mu, \nu) := \left(\inf_{\pi} \int_{E \times E} d^p(x, y) \ \pi(dx, dy) \right)^{1/p} \tag{1.1}$$

where the infimum runs over all coupling π of (μ, ν), see Villani [31] for an extensive study of such quantities.

A probability measure μ is then said to satisfy the transportation-entropy inequality $W_pH(C)$, where $C > 0$ is some constant, if for all probability measure ν

$$W_p(\nu, \mu) \leq \sqrt{2C \ H(\nu|\mu)} \tag{1.2}$$

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where $H(\nu|\mu)$ is the Kullback-Leibler information, or relative entropy, of ν with respect to μ:

$$H(\nu|\mu) := \begin{cases} \int \log\left(\frac{d\nu}{d\mu}\right) d\nu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

Marton [25] has first shown how W_1H inequality implies Gaussian concentration of measure and Talagrand, via a tensorization argument, established that the standard Gaussian measure, in any dimension, satisfies $W_2H(C)$ with the sharp constant $C = 1$.

However, if W_1H is completely characterized via a practical Gaussian integrability criterion (see [14, 9]), W_2H is much more difficult to describe. Nevertheless several equivalent beautiful conditions are known.

Theorem 1.4. *The following conditions are equivalent*

(1) *μ satisfies $W_2H(C)$ for some constant $C > 0$.*
(2) *For any bounded and measurable function f with $\mu(f) = 0$, defining the inf-convolution*

$$Qf(x) = \inf_{y\in E}\{f(y) + d^2(x, y)\},$$

we have

$$\int e^{\frac{1}{2C}Qf} d\mu \leq 1. \tag{1.5}$$

(3) *There exist a, r_0, b such that for all n all measurable $A \subset E^n$, with $\mu^{\otimes n}(A) \geq 1/2$, the probability measure $\mu^{\otimes n}$ satisfies*

$$\mu^{\otimes n}(A^r) \geq 1 - b\ e^{-a(r-r_0)^2} \tag{1.6}$$

where $A^r = \{x \in E^n;\ \exists y \in A, \sum_1^n d^2(x_i, y_i) \leq r^2\}$.

(1) $\Leftrightarrow$ (2) was proved in the seminal paper by Bobkov-Götze [8], and (1) $\Leftrightarrow$ (3) very recently by Gozlan [20]. Hence we have the beautiful characterization, W_2H is nothing else than a dimension free Gaussian concentration for the product measure. Note also that Gozlan-Léonard [21] established another criterion as a large deviation upper bound. One point is however important to remark: if these various characterizations have nice implications (concentration, deviation,...), it is rather difficult to directly use them to prove a W_2H inequality.

The first step towards practical criterion was done by Otto-Villani [28], soon followed by Bobkov-Gentil-Ledoux [7], who established that if μ satisfies a logarithmic Sobolev inequality, then μ satisfies W_2H (note that many explicit sufficient conditions for log-Sobolev inequalities are now known). To be more precise, let us present our framework.

Throughout this paper E is a complete and connected Riemannian manifold of finite dimension, d the geodesic distance, and dx the volume measure. $\mu(dx) = e^{-V(x)}dx/Z$ is the Boltzmann measure with $V \in C^2$ and $Z = \int e^{-V}dx < +\infty$. If the logarithmic Sobolev inequality $LSI(C)$ is verified, i.e. for all locally lispchitz g

$$\mathrm{Ent}_\mu(g^2) := \int g^2 \log\left(\frac{g^2}{\int g^2 d\mu}\right) d\mu \leq 2C \int |\nabla g|^2 d\mu \tag{1.7}$$

then μ satisfies also $W_2H(C)$. The proof of Otto-Villani [28] relies on a dynamical approach, namely to derive the Wasserstein distance between ν_t and ν_{t+s} when ν_t is the dynamical

transport leading from ν to μ, whereas Bobkov-Gentil-Ledoux [7] apply the hypercontractivity of the Hamilton semigroup, leading to an Herbst's like argument to derive W_2H.

It is only a few years ago that the two first authors [13] succeeded in proving that W_2H is strictly weaker than LSI, providing an example in one dimension of a measure (with unbounded curvature) satisfying W_2H but not LSI. Their method is a refinement of the argument of Bobkov-Gentil-Ledoux [7]: indeed, a full LSI is too strong to give W_2H, a LSI for a restricted class of functions is sufficient. They were however only able to give an explicit sufficient condition in dimension one for this restricted inequality. We will give here a Lyapunov condition ensuring that this restricted logarithmic Sobolev inequality holds, and thus W_2H too. We will also show that if the Bakry-Emery curvature $Ric + Hess_V$ is lower bounded then the same condition implies LSI.

Consider the μ-symmetric operator $\mathcal{L} = \Delta - \nabla V.\nabla$ on E. A Lyapunov condition is of the form: there exists $W \geq 1$ and $r, b > 0$ such that for some positive function ϕ

$$\mathcal{L}W \leq -\phi W + b\mathbb{1}_{B(x_0,r)}. \tag{1.8}$$

Such Lyapunov conditions have been used a lot both in discrete and continuous time case to study the speed of convergence towards the invariant measure of the associated semigroup under various norms, see [27, 16, 15]. The deep connection between such conditions and various form of functional inequalities have been recently studied by the authors (and coauthors). For example, if ϕ is constant, it is shown in [6] that the Lyapunov condition implies both a Poincaré inequality and a Cheeger inequality (with some slight additional assumptions on W). If $\phi := \phi(W)$ and ϕ is sub-linear then optimal weak Poincaré or isoperimetric inequalities can be established, see [3, 11]. Finally if $\phi := \phi(W)$ is super-linear, then it is shown to imply super Poincaré inequalities [12], and thus various F-Sobolev inequalities including logarithmic Sobolev inequalities.

Their implications in transportation cost inequalities were up to now not explored. It is the purpose of this short note.

Here is our main result:

Theorem 1.9. *Let μ be a Boltzmann measure.*
1) Suppose that there exists a C^2-function $W : E \to [1, \infty[$, some point x_0 and constants $b, c > 0$ such that

$$\mathcal{L}W \leq (-cd^2(x, x_0) + b)\ W,\ x \in E \tag{1.10}$$

or more generally there exists some nonnegative locally Lipschitzian function U $(= \log W)$ such that in the distribution sense (see the remark below),

$$\mathcal{L}U + |\nabla U|^2 \leq -cd^2(x, x_0) + b \tag{1.11}$$

then $W_2H(C)$ holds for some constant $C > 0$.
2) Under the Lyapunov condition (1.10), suppose moreover that $Hess(V) + Ric \geq KId$ for some $K \leq 0$ (in the sense of matrix). Then the logarithmic Sobolev inequality (1.7) holds.

Remark 1.12. (1) In both cases, it is of course possible to track all the constants involved to get an upper bound of the constant of $W_2H(C)$ inequality and of the logarithmic Sobolev inequality, as will be seen from the proof. One will also remark that contrary to [3, 11, 12], we will not use localization technique, constants are thus easier to derive.

(2) If $U = \log W \in C^2$, then $\mathcal{L}U + |\nabla U|^2 = -\mathcal{L}W/W$ so that (1.10) and (1.11) are equivalent. The condition (1.11) in the distribution sense means that for any $h \in C_0^\infty(E)$ (the space of infinitely differentiable functions with compact support) such that $h \geq 0$,

$$\int (\mathcal{L}U + |\nabla U|^2) h dx := \int U \Delta h dx + \int \left(-\nabla V \cdot \nabla U + |\nabla U|^2\right) h dx$$
$$\leq \int (-cd^2(x, x_0) + b) h d\mu.$$

(3) The Lyapunov condition (1.10) implies that there exists $r_0 > 0$ and $b', \lambda > 0$, such that

$$\mathcal{L}W \leq -\lambda W + b' \mathbb{1}_{B(x_0, r_0)}$$

so that, by [6], μ satisfies a Poincaré inequality.

This paper is organized as follows. In the next section we present several corollaries and examples for showing the usefulness and sharpness of the Lyapunov condition (1.11). The very simple proof of Theorem 1.9 is given in Section 3. And in the last section we combine the above-tangent lemma and the Lyapunov function method to yield the LSI in the unbounded curvature case.

2. Corollaries and examples

Some practical conditions. From Theorem 1.9, one easily deduces

Corollary 2.1. *Suppose that μ is a Boltzmann measure on $E = \mathbb{R}^d$. Let $x \cdot y$ and $|x| = \sqrt{x \cdot x}$ be the Euclidean inner product and norm, respectively.*
1) If one of the following conditions

$$\exists a < 1, R, c > 0, \text{such that if } \ |x| > R, \qquad (1-a)|\nabla V|^2 - \Delta V \geq c\ |x|^2 \tag{2.2}$$

or

$$\exists R, c > 0, \text{such that } \forall |x| > R, \qquad x \cdot \nabla V(x) \geq c\ |x|^2 \tag{2.3}$$

is satisfied, then W_2H holds.
2) Under the same conditions, suppose moreover that $Hess(V) \geq KId$ then a logarithmic Sobolev inequality (LSI in short) holds.

Proof. Under (2.2), one takes $W = e^{aV}$; and under (2.3) one choose $W = e^{a|x|^2}$ with $0 < a < c/2$. One sees that condition (1.10) is satisfied in both case. □

Remark 2.4. (1) Condition (2.2) is of course reminiscent to the Kusuoka-Stroock condition for logarithmic Sobolev inequality (replace d^2 by V). On the real line, it implies the condition of [13, Prop. 5.5].

(2) Gozlan [20, Prop. 3.9 and Theorem 4.8] proves W_2H on $\mathbb{R}^d$ under the condition

$$\liminf_{|x| \to \infty} \sum_{i=1}^d \left[\frac{1}{4} \left(\frac{\partial V}{\partial x_i} \right)^2 - \frac{\partial^2 V}{\partial x_i^2} \right] \frac{1}{1 + x_i^2} \geq c$$

for some positive c, using weighted Poincaré inequality. Note that this condition is in general not comparable to ours, for the terms in the sum can be negative, and also

for we have more freedom with the choice of a (limited to 3/4 in Gozlan's method). Whether this condition can be retrieved from a right choice of W in (1.10) seems unlikely. We will however simply show how to retrieve (and generalize) Gozlan's like conditions in the last section.

(3) Condition (2.3) may also be compared with condition (1.7) in [6]: $x \cdot \nabla V(x) \geq c\, d(x, x_0)$ which implies Poincaré inequality.

Comparison with Wang's criterion. Wang's criterion for LSI says the following: if $Hess_V + Ric \geq KId$ with $K \leq 0$ and

$$\int e^{(|K|/2+\varepsilon)d^2(x,x_0)} d\mu(x) < +\infty,$$

then $\mu = e^{-V}dx/C$ satisfies the LSI. We give now an example for which the previous criterion does not apply, but ours does.

Example 2.5. Let $E = \mathbb{R}^2$ and $V(x, y) = r^2 g(\theta)$ for all $r := \sqrt{x^2 + y^2} \geq 1$ (and $V \in C^\infty(\mathbb{R}^2)$), where (r, θ) is the polar coordinates system and $g(\theta) = 2 + \sin(k\theta)$ ($k \in \mathbb{N}^*$) for all $\theta \in S^1 \equiv [0, 2\pi]$. We have for $r > 1$,

$$(x, y) \cdot \nabla V(x, y) = r\partial_r V = 2r^2 g(\theta) \geq 2r^2$$

i.e., the condition (2.3) is satisfied. Moreover $Hess_V$ is bounded. Thus by Corollary 2.1, $\mu = e^{-V}dxdy/C$ satisfies the LSI.

However Wang's integrability condition is not satisfied for large k. Indeed $\Delta V = 4g(\theta) + g''(\theta) = 8 + (4 - k^2)\sin\theta$, then the smallest eigenvalue $\lambda_{\min}$ of $Hess_V$ satisfies

$$\lambda_{\min} \leq \frac{1}{2} tr(Hess_V) = \frac{1}{2}\Delta V = 4 + (2 - k^2/2)\sin(k\theta).$$

Then the largest constant K so that $Hess_V \geq KId$ in the case $k \geq 2$ satisfies

$$K \leq 6 - k^2/2.$$

When $k \geq 4$, $K/2 \leq 3 - k^2/4 \leq -1$ and Wang's integrability condition is not satisfied for $\int e^{r^2} d\mu = +\infty$. In other words Wang's criteria does not apply for this example once $k \geq 4$.

Riemannian manifold with unbounded curvature. Let E be a $d-$dimensional ($d \geq 2$) connected complete Riemannian manifold with

$$Ric_x \geq -(c + \sigma^2 d^2(x, x_0)),\ x \in E \tag{2.6}$$

for some constants $c, \sigma > 0$, where x_0 is some fixed point x_0. Let $V \in C^2(E)$ such that

$$\langle \nabla d(x, x_0), \nabla V\rangle \geq \delta d(x, x_0) - k \ \text{ outside of } cut(x_0) \text{ for some constants } \delta, k > 0. \tag{2.7}$$

Here $cut(x_0)$ denotes the the cut-locus of x_0.

Corollary 2.8. *Assume (2.6) and (2.7). If $\delta > \sigma\sqrt{d-1}$, then $\mu = e^{-V}dx/C$ satisfies $W_2H(C)$.*

Remark 2.9. Assume that $Hess_V \geq \delta$. Pick some $x \notin cut(x_0)$, and denote by U the unit tangent vector along the minimal geodesic $(x_s)_{0\leq s\leq d(x,x_0)}$ from x_0 to x, we have

$$\langle \nabla d(x, x_0), \nabla V\rangle = \langle \nabla V, U\rangle(x_0) + \int_0^{d(x,x_0)} Hess_V(U, U)(x_s)ds \geq \delta d(x, x_0) - c_1.$$

So condition (2.7) holds. Furthermore if $Hess_V \geq \delta > (1+\sqrt{2})\sigma\sqrt{d-1}$, Wang [33] proves the LSI for μ. When $\sigma\sqrt{d-1} < \delta \leq (1+\sqrt{2})\sigma\sqrt{d-1}$, the LSI is actually unknown. Also see [2] for the Harnack type inequality on this type of manifold.

One main feature of our condition (2.7) is: it demands only on the radial derivative of V, NOT on $Hess_V$.

Proof. At first we borrow the proof of [33, Lemma 2.1] for controlling $\Delta\rho$ where $\rho(x) = d(x, x_0)$. By (2.6) and the Laplacian comparison theorem, we have for $x \notin cut(x_0)$ different from x_0

$$\Delta\rho \leq \sqrt{(c+\sigma^2\rho^2)(d-1)}\coth\left(\rho\sqrt{(c+\sigma^2\rho^2)/(d-1)}\right).$$

Then outside of $cut(x_0)$ we get

$$\begin{aligned}\Delta\rho^2 &= 2\rho\Delta\rho + 2\\ &\leq 2\rho\sqrt{(c+\sigma^2\rho^2)(d-1)}\coth\left(\rho\sqrt{(c+\sigma^2\rho^2)/(d-1)}\right) + 2\\ &\leq 2d + 2\rho\sqrt{(c+\sigma^2\rho^2)(d-1)}\end{aligned}$$

where the last inequality follows by $r\cosh r \leq (1+r)\sinh r$ $(r \geq 0)$. It is well known that $\Delta\rho$ in the distribution sense gives a non-positive measure on $cut(x_0)$, the above inequality holds in the distribution sense over E.

Hence under the condition that $\delta > \sigma\sqrt{d-1}$, for $U = \lambda\rho^2$ where $0 < \lambda < \frac{1}{2}(\delta - \sigma\sqrt{d-1})$, we have in the sense of distribution

$$\begin{aligned}\mathcal{L}U + |\nabla U|^2 &\leq 2\lambda[2d + 2\rho\sqrt{(c+\sigma^2\rho^2)(d-1)}] - 2\lambda\rho\langle\nabla\rho, \nabla V\rangle + 4\lambda^2\rho^2\\ &\leq -c\rho^2 + b\end{aligned}$$

for some positive constants b, c, i.e. condition (1.11) is satisfied. So the W_2H inequality follows by Theorem 1.9(1). □

Our condition "$\delta > \sigma\sqrt{d-1}$" for W_2H is sharp as shown by the following example taken from [33].

Example 2.10. Let $E = \mathbb{R}^2$ be equipped with the following Riemannian metric

$$ds^2 = dr^2 + (re^{kr^2})d\theta^2$$

under the polar coordinates (r, θ), where $k > 0$ is constant. Then $Ric_{(r,\theta)} = -4k - 4k^2r^2$. Then (2.6) holds with $\sigma = 2k$. Let $V := \frac{\delta}{2}r^2$, which satisfies (2.7). If $\delta > \sigma\sqrt{d-1} = 2k$, we have W_2H. But if $\delta \leq \sigma\sqrt{d-1} = 2k$, $e^{-V}dx = re^{kr^2-\delta r^2/2}drd\theta$ is infinite measure, so that W_2H does not hold.

3. Proof of Theorem 1.9

3.1. **Several lemmas.** As was recalled in a previous remark, we may assume without loss of generality that μ verifies a Poincaré inequality with constant C_P, i.e. $\int g^2 d\mu \leq C_P \int |\nabla g|^2 d\mu$ for all smooth g with $\mu(g) = 0$.

We begin with the following

Lemma 3.1. ([13, Theorem 1.13]) *If μ satisfies the Poincaré inequality with constant C_P, then for all smooth and bounded g,*

$$Ent_\mu(g^2) \leq 2C_P \left(2\log 2 + \frac{1}{2}\log\frac{\|g^2\|_\infty}{\mu(g^2)}\right)\int |\nabla g|^2 d\mu. \tag{3.2}$$

Conversely, if the preceding restricted logarithmic Sobolev is true then μ satisfies a Poincaré inequality with constant $4C_P \log 2$.

Lemma 3.3. *Assume that the following restricted logarithmic Sobolev inequality holds: there exist constants $\eta, C_\eta > 0$ such that*

$$Ent_\mu(g^2) \leq 2C_\eta \int |\nabla g|^2 d\mu$$

for all smooth and bounded functions g satisfying

$$g^2 \leq \left(\int g^2 d\mu\right) e^{2\eta(d^2(x,x_0)+\int d^2(y,x_0)d\mu(y))}. \tag{3.4}$$

Then μ satisfies $W_2H(C)$ with $C = \max\{C_\eta; (2\eta)^{-1}\}$.

Proof. We recall the (short and simple) proof from [13, Theorem 1.17].
Given a fixed bounded f with $\mu(f) = 0$ consider for any $\lambda \in \mathbb{R}$, $g_\lambda^2 := e^{\tilde{\eta}Q(\lambda f)}$ where $\tilde{\eta} := \min\{1/(2C_\eta); \eta\} \in (0, \eta]$. By the definition of Q we easily get

$$Q(\lambda f)(x) \leq \int (\lambda f(y) + d^2(x,y))d\mu(y) \leq 2d^2(x,x_0) + 2\int d^2(y,x_0)\mu(dy).$$

Let $G(\lambda) = \mu(g_\lambda^2)$. By Bobkov-Goetze's criterion (Theorem 1.4(2)), if $G(1) \leq 1$ (for all such f), then $W_2H(C)$ holds with $C = 1/(2\tilde{\eta}) = \max\{C_\eta; (2\eta)^{-1}\}$. Assume by absurd that $G(1) > 1$. Introduce $\lambda_0 = \inf\{\lambda \in [0,1];\ G(u) > 1, \forall u \geq \lambda\}$, and remark that $\lambda_0 < 1$, $G(\lambda_0) = 1$ as well as $G(0) = 1$ and that $G(\lambda) > 1$ as soon as $\lambda \in]\lambda_0, 1]$.
Note at first that if $G(\lambda) \geq 1$ then

$$g_\lambda^2 \leq e^{2\tilde{\eta}(d^2(x,x_0)+\int d^2(x,x_0)d\mu(x))} \leq G(\lambda)e^{2\eta(d^2(x,x_0)+\int d^2(x,x_0)d\mu(x))}$$

i.e., g_λ satisfies condition (3.4). Since $Q_t f(x) := \inf_{y\in E}(f(y) + \frac{1}{2t}d^2(x,y))$ is the Hopf-Lax solution of the Hamilton-Jacobi equation: $\partial_t Q_t f + \frac{1}{2}|\nabla Q_t f|^2 = 0$ ([7]) and $Q(\lambda f) = \lambda Q_{\lambda/2} f$, we have

$$\lambda G'(\lambda) = \int g_\lambda^2 \log g_\lambda^2 d\mu - \frac{1}{\tilde{\eta}}\int |\nabla g_\lambda|^2 d\mu.$$

Since $\tilde{\eta} = \min\{1/(2C_\eta); \eta\}$, the restricted logarithmic Sobolev inequality in Lemma 3.3 yields for $\lambda \in]\lambda_0, 1]$

$$\lambda G'(\lambda) \leq G(\lambda)\log G(\lambda)$$

which is nothing else than the differential inequality $(\lambda^{-1}\log G(\lambda))' \leq 0$. That implies that $\lambda^{-1}\log G(\lambda)$ is nonincreasing so that

$$\log G(1) \leq \frac{\log(G(\lambda_0))}{\lambda_0}$$

(taken as limit $\lim_{\lambda\to 0}\frac{\log(G(\lambda))}{\lambda} = 0$ if $\lambda_0 = 0$). It readily implies that $G(1) \leq 1$ which is the Bobkov-Goetze's condition. □

Remark 3.5. The fact that the restricted logarithmic Sobolev inequality implies W_2H inequality was proven in [13, Th. 1.17]. In addition a Hardy criterion for this inequality on the real line is given in [13, Prop. 5.5].

Let $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ be the Dirichlet form associated with $\mathcal{L}$ in $L^2(\mu)$. It is the closure of $\mathcal{E}(f,g) = \langle -\mathcal{L}f, g\rangle_{L^2(\mu)} = \int \nabla f \cdot \nabla g d\mu,\ f,g \in C_0^\infty(E)$ by the essential self-adjointness of $(\mathcal{L}, C_0^\infty(E))$.

Lemma 3.6. *Let U be a nonnegative locally Lipschitzian function such that $\mathcal{L}U + |\nabla U|^2 \le -\phi$ in the distribution sense, where ϕ is lower bounded, then for any $g \in \mathbb{D}(\mathcal{E})$,*

$$\int \phi g^2 d\mu \le \mathcal{E}(g,g). \tag{3.7}$$

Proof. As $\phi \wedge N$ satisfies also the condition, if (3.7) is true with $\phi \wedge N$, then it is true with ϕ by letting $N \to +\infty$. In other words we can and will assume that ϕ is bounded.
One can approach any $g \in \mathbb{D}(\mathcal{E})$ by $(g_n) \subset C_0^\infty(E) : \int (g_n - g)^2 d\mu + \mathcal{E}(g_n - g, g_n - g) \to 0$. Thus is enough to prove (3.7) for $g \in C_0^\infty(E)$. For $g \in C_0^\infty(E)$, we have

$$\int (-\mathcal{L}U) g^2 d\mu = \int U(-\mathcal{L}g^2) d\mu = \int \nabla U \cdot \nabla (g^2) d\mu$$

where the first equality comes from the definition of the distribution $-\mathcal{L}U$ and a direct calculus, the second one is true at first for $U \in C_0^\infty(E)$ and is extended at first to any Lipschitzian U with compact support, then to any locally Lipschitzian U.
Thus using $2g\nabla U \cdot \nabla g \le |\nabla U|^2 g^2 + |\nabla g|^2$, we get

$$\begin{aligned}\int \phi g^2 d\mu &\le \int (-\mathcal{L}U - |\nabla U|^2) g^2 d\mu \\ &= \int \left(2g\nabla U \cdot \nabla g - |\nabla U|^2 g^2\right) d\mu \le \int |\nabla g|^2 d\mu\end{aligned}$$

which is the desired result. □

We also require the consequence below of the Lyapunov condition (1.11).

Lemma 3.8. *If the Lyapunov condition (1.11) holds, then there exist $\delta > 0$, $x_0 \in E$ such that*

$$\int e^{\delta d^2(x,x_0)} d\mu < \infty. \tag{3.9}$$

Proof. Under the condition (1.11), $\mathcal{L}$ satisfies a spectral gap property in $L^2(\mu)$ and then by [22], the following W_1I-inequality holds:

$$W_1^2(\nu,\mu) \le 4C^2 I(\nu|\mu),\ \forall \nu \in M_1(E)$$

where

$$I(\nu|\mu) := \begin{cases} \mathcal{E}(\sqrt{h}, \sqrt{h}), & \text{if } \ \nu = h\mu,\ \sqrt{h} \in \mathbb{D}(\mathcal{E}) \\ +\infty, & \text{otherwise} \end{cases} \tag{3.10}$$

is the so called Fisher information. By [23], the above W_1I-inequality is stronger than $W_1H(C)$, which is equivalent to the gaussian integrability (3.9). □

It would be interesting to find a simple or direct argument leading to (3.9).

3.2. **Proof of Theorem 1.9(1).** Choose $\eta > 0$ such that $\eta < \min(1, \delta/2)$ where δ comes from the gaussian integrability condition (3.9) which holds by Lemma 3.8. We have only to prove the restricted LSI in Lemma 3.3 under the Lyapunov condition (1.10).
To simplify the notation, define $M = e^{2\eta \int d^2(x,x_0)d\mu(x)}$.
Let $h = g^2$ be positive and smooth with $\mu(h) = 1$ and $h \leq Me^{2\eta d^2(x,x_0)}$. By Lemma 3.8 and our choice of η, $\int h \log h d\mu$ is bounded by some constant, say $c(\eta, \mu)$. Take $K > e$, to be chosen later. We have

$$\begin{aligned}\int h \log h d\mu &= \int_{h\leq K} h \log h d\mu + \int_{h>K} h \log h d\mu \\ &\leq \int (h \wedge K) \log(h \wedge K) d\mu + (\log M) \int_{h>K} h d\mu + 2\eta \int_{h>K} h d^2(x, x_0) d\mu.\end{aligned} \tag{3.11}$$

As $\int_{h\leq K} h \log h d\mu \geq \int_{h\leq K} (h-1) d\mu \geq -\int_{h>K} h d\mu$, we have

$$\int h \log h d\mu \geq \int_{h>K} h \log h d\mu - \int_{h>K} h d\mu.$$

It yields

$$\int_{h>K} h d\mu \leq \frac{1}{\log K} \int_{h>K} h \log h d\mu \leq \frac{1}{\log K} \left(\int h \log h d\mu + \int_{h>K} h d\mu \right)$$

so that

$$\int_{h>K} h d\mu \leq \frac{1}{\log K - 1} \int h \log h d\mu \leq \frac{c(\eta, \mu)}{\log K - 1}. \tag{3.12}$$

(3.12) furnishes an immediate useful bound for the second term in the right hand side of (3.11). Indeed, if $3 \log M \leq \log K - 1$ then

$$\log M \int_{h>K} h d\mu \leq \frac{1}{3} \int h \log h d\mu.$$

Remark also that for $K > e$

$$1 \geq \int h \wedge K d\mu \geq 1 - \frac{c(\eta, \mu)}{\log K - 1}$$

so that for K large enough (independent of h), $\int h \wedge K d\mu \geq 1/2$ and thus by Lemma 3.1

$$\begin{aligned}\int (h \wedge K) \log(h \wedge K) d\mu &\leq \int (h \wedge K) \log \left(\frac{h \wedge K}{\int h \wedge K d\mu} \right) d\mu \\ &\leq C_P (2 \log 2 + \frac{1}{2} \log(2K)) \int |\nabla \sqrt{h}|^2 d\mu.\end{aligned}$$

We then only have to bound the last term in (3.11). Unfortunately, we cannot directly apply the Lyapunov condition due to a lack of regularity of $h\mathbb{1}_{h>K}$. So we first regularize this function. To this end, introduce the map ψ with

$$\psi(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq \sqrt{K/2} \\ \frac{\sqrt{2}}{\sqrt{2}-1}(u - \sqrt{K/2}) & \text{if } \sqrt{K/2} \leq u \leq \sqrt{K} \\ u & \text{if } \sqrt{K} \leq u. \end{cases}$$

Now using Lyapunov condition (1.11) and Lemma 3.6 (applicable for $\psi(\sqrt{h}) = \psi(g)$ is locally Lipschitzian), we have

$$\begin{aligned}
2\eta \int_{h>K} h d^2(x,x_0) d\mu &\leq 2\eta \int \psi^2(\sqrt{h}) d^2(x,x_0) d\mu \\
&\leq \frac{2\eta}{c} \int \psi^2(\sqrt{h})[cd^2(x,x_0) - b] d\mu + \frac{2\eta b}{c} \int \psi^2(\sqrt{h}) d\mu \\
&\leq \frac{2\eta}{c} \int |\nabla \psi(\sqrt{h})|^2 d\mu + \frac{2\eta b}{c} \int \psi^2(\sqrt{h}) d\mu \\
&\leq \frac{4\eta}{c(\sqrt{2}-1)^2} \int |\nabla \sqrt{h}|^2 d\mu + \frac{2\eta b}{c} \int \psi^2(\sqrt{h}) d\mu.
\end{aligned}$$

As $\psi^2(\sqrt{h}) \leq h 1_{h>K/2}$, the lat term above can be bounded by $(1/3)\int h \log h d\mu$ if K is large enough so that $2\eta b c^{-1} \leq (\log(K/2) - 1)/3$, by (3.12).
Plugging all those estimates into (3.11), we obtain the desired restricted LSI.

3.3. **Proof of Theorem 1.9(2).** Our argument will be a combination of the Lyapunov condition, leading to defective $W_2 I$ inequality and the HWI inequality of Otto-Villani.
We begin with the following fact ([31, Proposition 7.10]):

$$W_2^2(\nu,\mu) \leq 2\|d(\cdot,x_0)^2(\nu-\mu)\|_{TV}. \tag{3.13}$$

Now for every function g with $|g| \leq \phi(x) := cd(x,x_0)^2$, we have by (1.11) and Lemma 3.6,

$$\begin{aligned}
\int g d(\nu-\mu) &\leq \nu(\phi) + \mu(\phi) \\
&\leq \int \left(-cd^2(x,x_0) + b\right) d\nu(x) + \mu(\phi) \\
&\leq I(\nu|\mu) + b + \mu(\phi)
\end{aligned}$$

Taking the supremum over all such g, we get

$$\frac{c}{2} W_2^2(\nu,\mu) \leq c\|d(\cdot,x_0)^2(\nu-\mu)\|_{TV} \leq I(\nu|\mu) + b + \mu(\phi),$$

which yields thanks to (3.13)

$$W_2^2(\nu,\mu) \leq \frac{2}{c} I(\nu|\mu) + \frac{2}{c}[b + \mu(\phi)].$$

Substituting it into the HWI inequality of Otto-Villani [28] (or for its Riemannian version by Bobkov-Gentil-Ledoux [7]):

$$H(\nu|\mu) \leq 2\sqrt{I(\nu|\mu)} W_2(\nu,\mu) - \frac{K}{2} W_2^2(\nu,\mu), \tag{3.14}$$

and using $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ we finally get

$$\begin{aligned}
H(\nu|\mu) &\leq \varepsilon I(\nu|\mu) + \left(1 - \frac{K}{2} + \frac{1}{\varepsilon}\right) W_2(\nu,\mu)^2 \\
&\leq A I(\nu|\mu) + B
\end{aligned} \tag{3.15}$$

where

$$A = (1 - \frac{K}{2})\frac{2}{c} + \varepsilon, \quad B = \frac{2}{c}[b + \mu(\phi)]\left(1 - \frac{K}{2} + \frac{1}{\varepsilon}\right).$$

This inequality is sometimes called a defective log-Sobolev inequality. But it is well known by Rothaus' lemma, that a defective log-Sobolev inequality together with the spectral gap implies the (tight) log-Sobolev inequality

$$H(\nu|\mu) \leq [A + (B + 2)C_P)]I(\nu|\mu).$$

The proof is completed.

Remark 3.16. If for any $c > 0$, there are U, b such that the Lyap condition (1.11) holds, then the defective LSI (3.15) becomes the so called *super-LSI*, which is equivalent to the supercontractivity of the semigroup (P_t) generated by $\mathcal{L}$, i.e., $\|P_t\|_{L^p \to L^q} < +\infty$ for any $t > 0, q > p > 1$.

4. Some further remarks

4.1. A generalization of Corollary 2.8.

Corollary 4.1. *Assume that*

$$Ric_x \geq -\alpha(d(x, x_0))$$

where $\alpha(r)$ is some positive increasing function on $\mathbb{R}^+$, and

$$\langle \nabla d(x, x_0), \nabla V \rangle \geq \beta(d(x, x_0)) - b$$

for some constant $b > 0$ and some positive increasing function β on $\mathbb{R}^+$. If

$$\beta(r) - \sqrt{\alpha(r)} \geq \eta r, \ r > 0 \tag{4.2}$$

for some positive constant η, then μ satisfies W_2H.

Proof. As in the proof recalled in Corollary 2.8, for $\rho = d(x, x_0)$, by the Laplacian comparison theorem, there is some constant $c_1 > 0$ such that

$$\Delta\rho^2 \leq c_1(1 + \rho) + 2\rho\sqrt{\alpha(\rho)}$$

at first outside of $cut(x_0)$ then in distribution over E. Consequently by condition (4.2) there are positive constants $c_2 < 2\eta, c_3$ such that

$$\mathcal{L}\rho^2 = \Delta\rho^2 - 2\rho\langle \nabla\rho, \nabla V\rangle \leq c_1(1 + \rho) + 2\rho(\sqrt{\alpha(\rho)} - \beta(\rho) + b) \leq -c_2\rho^2 + c_3.$$

Now for $U = \lambda\rho^2$, it satisfies (1.11) when $\lambda > 0$ is small enough. Then the W_2H follows by Theorem 1.9. □

4.2. LSI in the unbounded curvature case.

We now generalize the LSI in Theorem 1.9 in the case where Bakry-Emery's curvature is not lower bounded, by means of the above-tangent lemma.

Proposition 4.3. *Assume that*

$$Ric_x + Hess_V \geq -\Phi(d(x, x_0)) \tag{4.4}$$

where Φ is some positive non-decreasing continuous function on $\mathbb{R}^+$, and there is some non-negative locally Lipschitzian function U such that for some constants $b, c > 0$

$$\mathcal{L}U + |\nabla U|^2 \leq -cd^2(x, x_0)\Phi(2d(x, x_0)) + b \tag{4.5}$$

in distribution, then μ satisfies the LSI.

Proof. Instead of the HWI in the proof of the LSI in Theorem 1.9, we go back to the above-tangent lemma (see [5, Theorem 7.1] and references therein) : for two probability measures $\nu = h\mu, \tilde{\nu} = \tilde{h}\mu$ with smooth and compactly supported densities $h, \tilde{h}$, let $T(x) := \exp_x(\nabla\theta)$ (where θ is some "convex" function) be the optimal transport pushing forward ν to $\tilde{\nu}$ and realizing $W_2^2(\nu, \tilde{\nu})$. Then

$$Ent_\mu(h) \leq Ent_\mu(\tilde{h}) - \int \langle \nabla\theta, \nabla h\rangle d\mu + \int \mathcal{D}_V(x, T(x)) h d\mu \tag{4.6}$$

where $\mathcal{D}_V(x, T(x))$ is the defect of the convexity of V, defined by

$$\mathcal{D}_V(x, T(x)) = -\int_0^1 (1-t)\left(Ric_{\gamma(t)} + Hess_{V,\gamma(t)}\right)(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

Here $\gamma(t) = \exp_x(t\nabla\theta)$ is the geodesic joining x to $T(x)$.
Choose a sequence of μ-probability measures $\mu_n := h_n\mu$ with $h_n \in C_0^\infty(E)$, such that $W_2(\mu_n, \mu) \to 0$ and $I(\mu_n|\mu) \to 0$ (recalling that the condition (4.5), stronger than (1.11), implies the Gaussian integrability of μ by Lemma 3.8). Below we apply the above-tangent lemma to $(\nu, \tilde{\nu} = \mu_n)$
The first term on the right hand of (4.6) is easy to control by Cauchy-Schwarz:

$$|\int \langle \nabla\theta, \nabla h\rangle d\mu| = |\int 2\sqrt{h}\langle \nabla\theta, \nabla\sqrt{h}\rangle d\mu| \leq 2\sqrt{\int |\nabla\theta|^2 h d\mu \int |\nabla\sqrt{h}|^2 d\mu} = 2W_2(\nu, \mu_n)\sqrt{I(\nu|\mu)}.$$

Now we treat the last term in (4.6). By our condition,

$$\mathcal{D}_V(x, T(x)) \leq \int_0^1 (1-t)\Phi(d(\gamma(t), x_0))|\nabla\theta|^2 dt.$$

Note that $|\nabla\theta| = d(x, T(x)) \leq 2\max\{d(x, x_0), d(T(x), x_0)\}$ and using $d(\gamma(t), x_0) \leq d(x, x_0) + td(x, T(x))$ for $t \in [0, 1/2]$ and $d(\gamma(t), x_0) \leq d(T(x), x_0) + (1-t)d(x, T(x))$ for $t \in [1/2, 1]$, $d(\gamma(t), x_0) \leq 2\max\{d(x, x_0), d(T(x), x_0)\}$. We thus obtain

$$\begin{aligned}\int \mathcal{D}_V(x, T(x)) h d\mu &\leq 2\int \Phi(2\max\{d(x, x_0), d(T(x), x_0)\})\max\{d(x, x_0)^2, d(T(x), x_0)^2\} h d\mu \\ &\leq 2\left(\int \Phi(2d(x, x_0))d(x, x_0)^2 h d\mu + \int \Phi(2d(T(x), x_0))d(T(x), x_0)^2 h d\mu\right)\end{aligned}$$

By Lemma 3.6 and our condition (4.5),

$$c\int \Phi(2d(x, x_0))d(x, x_0)^2 h d\mu \leq b + I(\nu|\mu)$$

$$c\int \Phi(2d(T(x), x_0))d(T(x), x_0)^2 h d\mu \leq b + I(\mu_n|\mu)$$

Plugging those estimates into (4.6) and letting $n \to \infty$, we get finally

$$H(\nu|\mu) \leq 2W_2(\nu, \mu)\sqrt{I(\nu|\mu)} + \frac{1}{c}(I(\nu|\mu) + 2b)$$

Again using Lemma 3.6 and our condition (4.5), we have

$$W_2^2(\nu,\mu) \le 2\left(\int d^2(x,x_0)d\mu + \int d^2(x,x_0)d\mu\right) \le \frac{2}{c\Phi(0)}(I(\nu|\mu)+2b).$$

Consequently we obtain the defective LSI:

$$H(\nu|\mu) \le 2\sqrt{\frac{2}{c\Phi(0)}}(I(\nu|\mu)+b) + \frac{1}{c}(I(\nu|\mu)+2b) \tag{4.7}$$

where the LSI follows for the spectral gap exists under (4.5). □

Remark 4.8. Under (4.4), if for any $c>0$ there are U,b such that the Lyapunov function condition (4.5) holds, the defective LSI (4.7) says that for any $\varepsilon>0$, there is some constant $B(\varepsilon)$ such that

$$H(\nu|\mu) \le \varepsilon I(\nu|\mu) + B(\varepsilon),\ \ \nu \in M_1(E)$$

which is well known to be equivalent to the supercontractivity of the semigroup (P_t) generated by $\mathcal{L}$, i.e., $\|P_t\|_{L^p\to L^q} < +\infty$ for any $t>0, q>p>1$.

Remark 4.9. Barthe and Kolesnikov [5] used the above-tangent lemma to derive modified LSI and isoperimetric inequalities. One aspect of their method consists in controlling the defective term $\int \mathcal{D}_V(x,T(x))hd\mu$ by $cEnt_\mu(h)+b$ for some positive constant $c<1$, by using some integrability condition on μ (as in Wang's criterion). Our method is to bound that defective term by $cI(\nu|\mu)+b$, by means of the Lyapunov function: the advantage here is that constant $c>0$ can be arbitrary.

Example 4.10. Let $E=\mathbb{R}^2$ equipped with the Euclidean metric. For any $p>2$ fixed, consider $V=r^p(2+\sin(k\theta))$, where (r,θ) is the polar coordinates system and $k\in\mathbb{N}^*$. Since

$$\Delta V = r^{p-2}[p^2(2+\sin(k\theta)) - k^2\sin(k\theta)]$$

Assume $k>\sqrt{3}p$. Then in the direction θ such that $\sin(k\theta)=1$, $Hess_V \le -\frac{1}{2}(k^2-3p^2)r^{p-2}$, i.e., the Bakry-Emery curvature is very negative and no known result exists in such case.

It is easy to see that condition (4.4) is verified with $\Phi(r)=ar^{p-2}$ for some $a>0$. Taking $U=r^2$, we see that

$$\mathcal{L}U + |\nabla U|^2 = 4 - 2pr^p(2+\sin(k\theta)) + 4r^2$$

i.e., condition (4.5) is satisfied. We get thus the LSI for μ by Proposition 4.3.

4.3. **A Lyapunov condition for Gozlan's weighted Poincaré inequality.** As mentionned before, in a recent work, Gozlan [19] proved that W_2H inequality on $E=\mathbb{R}^d$ is implied by a weighted Poincaré inequality

$$\mathrm{Var}_\mu(f) \le c\int \sum_1^d \frac{1}{1+x_i^2}\left(\frac{\partial f}{\partial x_i}\right)^2 d\mu.$$

In dimension one, a Hardy criterion is available for this weighted Poincaré inequality which is not the same as the one from [13]. Note however that this weighted Poincaré inequality, as stronger than Poincaré inequality, can be shown to imply a converse weighted Poincaré inequality (the weight is now in the variance), by a simple change of function argument, and in dimension one a Hardy's criterion is also available for this inequality which is in fact the same as the one for the restricted logarithmic Sobolev inequality.

From this, we conclude that in fact, in the real line case, the restricted logarithmic Sobolev inequality is in fact implied by Gozlan's weighted Poincaré inequality. Whether it is the case in any dimension would have to be investigated.
It is however quite easy, following [6] to give a Lyapunov condition for Gozlan's weighted Poincaré inequality on $\mathbb{R}^d$.

Theorem 4.11. *Let $w_i = w_i(x_1, ..., x^d)$ be positive for all $(x_1, ..., x_d) \in \mathbb{R}^d$, and $\omega_i > \epsilon_r > 0$ on $B(0, r)$. Introduce the diffusion generator*

$$\widetilde{\mathcal{L}} = \sum_{i=1}^{d} \left(\omega_i \partial_i^2 + (\partial_i \omega_i - \omega_i \partial_i V)\partial_i\right),$$

where $\partial_i = \partial/\partial x_i$. Suppose now that there exists $W \geq 1$, $\lambda, b > 0$ and $R >$ such that

$$\widetilde{\mathcal{L}} W \leq -\lambda W + b \mathbb{1}_{B(0,R)} \tag{4.12}$$

then μ verifies a weighted Poincaré inequality with some constant $c > 0$

$$Var_\mu(f) \leq c \int \sum_{i=1}^{d} \omega_i \left(\partial_i f\right)^2 d\mu. \tag{4.13}$$

Proof. The proof follows exactly the line of the one of [6] once it has been remarked that

(1) $\widetilde{L}$ is associated to the Dirichlet form $\widetilde{\mathcal{E}}(f, g) = -\int f \widetilde{\mathcal{L}} g d\mu$ on $L^2(\mu)$, reversible w.r.t. μ and $\widetilde{\mathcal{E}}(f, f) = \int \sum_{i=1}^{d} \omega_i \left(\partial_i f\right)^2 d\mu$;
(2) a local weighted Poincaré inequality is valid for this Dirichlet form as $\omega_i > \epsilon_r > 0$ on $B(0, r)$ (as a local Poincaré inequality is available on balls).

□

Remark 4.14. Our setting is a little bit more general than Gozlan [19] concerning the assumption on ω but with the additional term $\partial_i \omega_i \partial_i W$ in the sum. Note once again that they are a little bit more difficult to handle than the one in Corollary 2.1 and still not comparable.
One of the major points of Gozlan's weighted Poincaré inequality is, in the case where $\omega_i(x_1, ..., x_d) = \omega(x_i)$, in fact equivalent to some transportation-information inequality (with an unusual distance function) when ω satisfies some conditions (namely, $\omega = \sqrt{\tilde{\omega}'}$ where $\tilde{\omega}$ is odd, at least linearly increasing). However, when $\omega_i = 1/(1 + x_i^2)$, this transportation inequality is stronger than W_2H.

We end up this note with some final conditions ensuring W_2H, similar to Gozlan's one (see Remark 2.4(2)).

Corollary 4.15. *In the setting of Corollary 2.1.*
1) If there are positive constants $a < 1, R, c > 0$, such that

$$\sum_{i=1}^{d} \left((1-a)\omega_i (\partial_i V)^2 - \partial_i \omega_i \partial_i V - \omega_i \partial_i^2 V\right) \geq c, \forall x : |x| > R \tag{4.16}$$

is verified then the weighted Poincaré inequality (4.13) is verified.
2) In particular, consider $\omega_i(x_1, ..., x_d) = (1 + x_i^2)^{-1}$, if there are positive constants $a <$

$1, R, c > 0$, *such that for all* $x \in \mathbb{R}^d$ *with* $|x| > R$, *one of*

$$\sum_{i=1}^{d} \left((1-a)(\partial_i V)^2 + \frac{2x_i}{1+x_i^2}\partial_i V - \partial_i^2 V \right) \frac{1}{1+x_i^2} \geq c \tag{4.17}$$

or

$$\sum_{i=1}^{d} \left(\frac{x_i \partial_i V}{1+x_i^2} - \frac{1-x_i^2}{(1+x_i^2)^2} \right) \geq c \tag{4.18}$$

is verified, then W_2H *holds.*

Proof. Part 1) is a particular case of Theorem 4.11 with $W = e^{aV}$, together with Gozlan's result. Condition (4.17) is just a particular version of part 1). The last case under condition (4.18) comes from Theorem 4.11 with $W = e^{a|x|^2}$ for sufficiently small a. □

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Patrick CATTIAUX,, Institut de Mathématiques de Toulouse. CNRS UMR 5219., Université Paul Sabatier, Laboratoire de Statistique et Probabilités,, 118 route de Narbonne, F-31062 Toulouse cedex 09.

E-mail address: `cattiaux@math.univ-toulouse.fr`

Arnaud GUILLIN,, Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal, avenue des Landais 63177 Aubière.

E-mail address: `guillin@math.univ-bpclermont.fr`

Li-Ming WU,, Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal, avenue des Landais 63177 Aubière.

E-mail address: `wuliming@math.univ-bpclermont.fr`