Hamiltonians with purely discrete spectrum
Vladimir Georgescu

To cite this version:
Vladimir Georgescu. Hamiltonians with purely discrete spectrum. 2014. hal-00335549v2

HAL Id: hal-00335549
https://hal.science/hal-00335549v2
Preprint submitted on 11 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
HAMILTONIANS WITH PURELY DISCRETE SPECTRUM

VLADIMIR GEORGESCU

Abstract. We discuss criteria for a self-adjoint operator on $L^2(X)$ to have empty essential spectrum. We state a general result for the case of a locally compact abelian group $X$ and give examples for $X = \mathbb{R}^n$.

1. Let $\Delta$ be the positive Laplacian on $\mathbb{R}^n$. We set $B_a(r) = \{ x \in \mathbb{R}^n \mid |x-a| \leq r \}$ and $B_a = B_a(1)$.

Proposition 1. Let $V$ be a real locally integrable function on $\mathbb{R}^n$ such that:

(i) if $\lambda > 0$ then the measure $\omega_\lambda(a)$ of the set $\{ x \in B_a \mid V(x) < \lambda \}$ satisfies $\lim_{a \to \infty} \omega_\lambda(a) = 0$.

(ii) the negative part of $V$ satisfies $V_- \leq \mu \Delta + \nu$ for some positive real numbers $\mu, \nu$ with $0 < \mu < 1$.

Then the spectrum of the self-adjoint operator $H$ associated to the form sum $\Delta + V$ is purely discrete.

Remark 2. Let $V_a = \max\{ \pm V, 0 \}$ and for each $\lambda > 0$ let $\Omega_\lambda = \{ x \mid V_+(x) < \lambda \}$. Then $\omega_\lambda(a)$ is the measure of the set $B_a \cap \Omega_\lambda$. From Lemma 5 it follows that the condition (i) is equivalent to

$$\lim_{a \to \infty} \int_{B_a} \frac{dx}{1 + V_+(x)} = 0.$$  \hfill (1)

Remark 3. From Lemma 7 we get $\lim_{a \to \infty} \omega_\lambda(a) = 0$ if $\int_{\Omega_\lambda} \omega^2 dx < \infty$ for some $p > 0$. Thus Theorems 1 and 3 from [S] are consequences of Proposition 1. In the case $V \geq 0$ Proposition 1 is a consequence of Theorem 2.2 from [MS]. More general results will be obtained below. Note, however, that our techniques are not applicable in the framework considered in Theorem 2 from [S] and in [WW].

Proposition 1 is very easy to prove if condition (1) is replaced by $\lim_{a \to \infty} V_a(x) = \infty$. In fact, let us consider an arbitrary locally compact space $X$ and let $\mathcal{H}$ be a Hilbert $X$-module, i.e. $\mathcal{H}$ is a Hilbert space and a nondegenerate $\ast$-morphism $\phi \mapsto \phi(Q)$ of $C_0(X)$ into $B(\mathcal{H})$ is given. For example, one may take $\mathcal{H} = L^2(X, \mu)$ for some Radon measure $\mu$. Then we have the following simple compactness criterion: if $R$ is a bounded self-adjoint operator on $\mathcal{H}$ such that (i) if $\phi \in C_0(X)$ then $\phi(Q)R$ is a compact operator, (ii) one has $\pm R \leq \theta(Q)$ for some $\theta \in C_0(X)$, then $R$ is a compact operator. Indeed, note first that the operator $R\phi \equiv \phi(Q)$ will also be compact for all $\phi \in C_0(X)$. Let $\varepsilon > 0$ and let us choose $\phi$ such that $0 \leq \phi \leq 1$ and $\theta \phi^- \leq \varepsilon$, where $\phi^\perp = 1 - \phi$. Then $\pm \phi^\perp R\phi^\perp \leq \phi^\perp \theta \phi^\perp \leq \varepsilon$ which implies $\| \phi^\perp R\phi^\perp \| \leq \varepsilon$. So we have $\| R - \phi R - \phi^\perp R\phi \| \leq \varepsilon$ and $\phi R + \phi^\perp R\phi$ is a compact operator. Now let us say that a self-adjoint operator $H$ on $\mathcal{H}$ is locally compact if $\phi(Q)(H+i)^{-1}$ is compact for all $\phi \in C_0(X)$. Then we get: If $H$ is a locally compact self-adjoint operator on $\mathcal{H}$ and if there is a continuous function $\Theta : X \to \mathbb{R}$ such that $\lim_{x \to \infty} \Theta(x) = +\infty$ and $H \geq \Theta(Q)$, then the spectrum of $H$ is purely discrete (the nondegeneracy of the morphism is needed for the definition of $\Theta(Q)$ for unbounded $\Theta$).

2. On the other hand, Proposition 1 can be significantly generalized. For example, $\Delta$ may be replaced by a higher order operator with matrix valued coefficients and $V$ does not have to be a function. These results are consequences of the following “abstract” fact. We fix a locally compact abelian group $X$, choose a finite dimensional Hilbert space $E$, and define $\mathcal{H} = L^2(X) \otimes E$. For $a \in X$ and $k \in X^*$ (the dual locally compact abelian group) we denote $U_a$ and $V_k$ the unitary operators on $\mathcal{H}$ given by

$$(U_a f)(x) = f(x+a) \quad \text{and} \quad (V_k f)(x) = k(x)f(x).$$

We denote additively the operations both in $X$ and in $X^*$ and denote 0 their neutral elements.

Date: July 11, 2014.
Theorem 4. Let $H$ be a self-adjoint operator on $\mathcal{H}$ such that for some (hence for all) $z \in \mathbb{C}$ not in the spectrum of $H$ the operator $R = (H - z)^{-1}$ satisfies
\[
\lim_{k \to 0} ||V_k RV_k^* - R|| = 0, \quad \lim_{a \to 0} ||U_a - 1||R|| = 0.
\]
Then $H$ has purely discrete spectrum if and only if $w\text{-}\lim_{a \to \infty} U_a R U_a^* = 0$.

Proof: If the spectrum of $H$ is purely discrete then $R$ is compact so $w\text{-}\lim_{a \to \infty} U_a R U_a^* = 0$. The reciprocal assertion follows from Theorem 1.2 from [GI]. Indeed, with the terminology used there, all the localizations at infinity of $H$ will be equal to $\infty$ hence the essential spectrum of $H$ will be empty.

Some notations: if $\phi$ is a $B(E)$-valued Borel function on $X$ then $\phi(Q)$ is the operator of multiplication by $\phi$ on $\mathcal{H}$; if $\psi$ is a similar function on $X^*$ then $\psi(P) = \mathcal{F}^{-1}M_\psi \mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation and $M_\psi$ is the operator of multiplication by $\psi$ on $L^2(X^*) \otimes E$. Note that $V_k \psi(P) V_k^* = \psi(P + k)$.

If $\phi \in L^\infty(X)$ and $\phi \geq 0$ then it is easy to check that $w\text{-}\lim_{a \to \infty} U_a \phi(Q) U_a^* = 0$ if and only if $s\text{-}\lim_{a \to \infty} \phi(Q) U_a = 0$ and also if and only if there is a compact neighborhood of the origin $W$ such that $\lim_{a \to \infty} \int_{1 + W} \phi \, dx = 0$. Then we say that $\phi$ is weakly vanishing (at infinity). See Section 6 in [GG] for further properties of this class of functions. Below $W$ is a compact neighborhood of the origin, $W_a = a + W$, and we denote $|M|$ the Haar measure of a set $M$.

Lemma 5. A positive function $\phi \in L^\infty(X)$ is weakly vanishing if and only if for any number $\lambda > 0$ the set $\Omega^\lambda = \{ x \mid \phi(x) > \lambda \}$ has the property $\lim_{a \to \infty} |W_a \cap \Omega^\lambda| = 0$.

This follows from the estimates
\[
|W_a \cap \Omega^\lambda| \leq \int_{W_a} \phi \, dx \leq ||\phi||_{L^\infty}|W_a \cap \Omega^\lambda| + \lambda|W|.
\]

Proposition 6. Let $H$ be an invertible self-adjoint operator satisfying (2) and such that $\pm H^{-1} \leq \phi(Q)$ for some weakly vanishing function $\phi$. Then $H$ has purely discrete spectrum.

Indeed, we may take $R = H^{-1}$ and then for any $f \in \mathcal{H}$ we have $|(f|U_a R U_a^*)| \leq |(f|U_a \phi(Q) U_a^*)|$. 

Proof of Proposition 1: Here $X = \mathbb{R}^n$ and we identify as usual $X$ with its dual by setting $k(x) = e^{ikx}$ for $x, k \in X$. Then if $P_j = -i\partial_j$ and $P = (P_1, \ldots, P_n)$ we get $V_k PV_k^* = P + k$. To simplify notations we write $H$ for $\Delta + V + 1 + \nu$, so that $H \geq (1 - \mu) \Delta + V + 1 \geq V + 1 \geq 1$. Then observe that the form domain of $H$ is $\mathcal{G} = D(H^{1/2}) = \{ f \in \mathcal{H} | V^{1/2} f \in L^2 \}$ where $\mathcal{H}$ is the first order Sobolev space. Thus $H = H^{-1} : L^2 \to \mathcal{H}$ is continuous and this implies the second part of condition (2). On the other hand, $H$ extends to a continuous bijective operator $\mathcal{G} \to \mathcal{G}^*$ whose inverse is an extension of $R$ to a continuous map $\mathcal{G}^* \to \mathcal{G}$. We keep the notations $H, R$ for these extensions. Clearly $V_k$ leaves invariant $\mathcal{G}$ hence extends to a continuous operator on $\mathcal{G}$ and the groups of operators $\{ V_k \}$ are of class $C_0$ in both spaces. Now $H_2 := V_k H V_k^* = (P + k)^2 + V = H + 2kP + k^2$ in $B(\mathcal{G}, \mathcal{G}^*)$ so if $H_2 := V_k RV_k^*$ then 
\[
R_k - R = R_k(H - H_2)R = -R_k(2kP + k^2)R
\]
in $B(\mathcal{G}^*, \mathcal{G})$. Now clearly the first part of (2) is fulfilled. Finally, it suffices to show that $H^{-1} \leq \phi(Q)$ for a weakly vanishing function $\phi$. But $H \geq 1 + V_a$ and we may take $\phi = (1 + V_a)^{-1}$ due to (1).

Remark 3 is a consequence of the next result.

Lemma 7. Let $\Omega \subset \mathbb{R}^n$ be a Borel set and let $\omega : \mathbb{R}^n \to \mathbb{R}$ be defined by $\omega(a) = |B_a \cap \Omega|$. If $\omega^p$ is integrable on $\Omega$ for some $p > 0$ then $\omega(a) \to 0$ as $a \to \infty$.

Proof: The main point is the following observation due to Hans Henrik Rugh: let $\nu$ be the minimal number of (closed) balls of radius $1/2$ needed to cover a ball of radius one; then for any $a$ there is a Borel set $A_a \subset B_a \subset \Omega$ with $|A_a| \geq \omega(a)/\nu$ such that $\omega(x) \geq \omega(a)/\nu$ if $x \in A_a$. Indeed, let $N$ be a set of $\nu$ points such that $B_a \subset \cup_{b \in N} B_b(1/2)$. If $D_b = B_a \setminus B_b(1/2)$ then $\omega(a) \leq \sum_b |D_b \cap \Omega|$ hence there is $b(a)$ such that $A_a = D_{b(a)} \cap \Omega$ satisfies $|A_a| \geq \omega(a)/\nu$. Since $A_a$ has diameter smaller than one, for
$x \in A_A$ we have $A_A \subset B_x \cap \Omega$ hence $\omega(x) \geq |A_A|$, which proves the remark. Now let us set $R = |a| - 1$ and denote $\Omega(R)$ the set of points $x \in \Omega$ such that $|x| \geq R$. Then we have
\[
\int_{\Omega(R)} \omega^p \, dx \geq \int_{A_A} \omega^p \, dx \geq [\omega(\alpha)/\nu]^{p+1}
\]
which clearly implies the assertion of the lemma.

3. We present here some consequences of Proposition 6. We refer to [GI] for general classes of operators verifying condition (2) and consider here only some particular cases. We mention that if $H$ is a bounded from below operator satisfying (2) and if $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\theta(\lambda) \to +\infty$ when $\lambda \to +\infty$ the $\theta(H)$ also satisfies (2).

If $R \in B(\mathcal{H})$ satisfies the first part of (2) we say that $R$ is a regular operator (or $Q$-regular). The regularity of the resolvent of a differential operators on $\mathbb{R}^n$ is easy to check because $V_k PV_k^* = P + k$, cf. the proof of Proposition 1. The second part of (2) is equivalent to the existence of a factorization $R = \psi(P)S$ with $\psi \in C_c(X^*)$ and $S \in B(\mathcal{H})$. If $X = \mathbb{R}^n$ then it suffices that the domain of $H$ be included in some Sobolev space $\mathcal{H}^m$ with $m > 0$ real. We now give an extension of Proposition 1 which is proved in essentially the same way. We assume $X = \mathbb{R}^n$ and work with Sobolev spaces but a similar statement holds for an arbitrary $X$: it suffices to replace the function $(k)\nu^m$ which defines $\mathcal{H}^m$ by an arbitrary weight [GI] and the ball $B_k$ by $a + W$ where $W$ is a compact neighborhood of the origin.

**Proposition 8.** Let $H_0$ be a bounded from below self-adjoint operator on $\mathcal{H}$ with form domain equal to $\mathcal{H}^m$ for some real $m > 0$ and satisfying $\lim_{k \to 0} V_k H_0 V_k^* = H_0$ in norm in $B(\mathcal{H}^m, \mathcal{H}^{-m})$. Let $V$ be a positive locally integrable function such that $\lim_{k \to \infty} \{|x \in B_k | V(x) < \lambda\} = 0$ for each $\lambda > 0$. Then the self-adjoint operator $H$ associated to the form sum $H_0 + V$ has purely discrete spectrum.

Let $h : X \to B(E)$ be a continuous symmetric operator valued function with $c^i|p|^{2m} \leq h(p) \leq c^j|p|^{2m}$ (as operators on $E$) for some constants $c^i, c^j > 0$ and all large $p$. Let $W : \mathcal{H}^m \to \mathcal{H}^{-m}$ be a symmetric operator such that $W \geq -\mu h(P) - \nu$ with $\mu < 1$ and such that $V_k W V_k^* \to W$ in norm in $B(\mathcal{H}^m, \mathcal{H}^{-m})$ as $k \to 0$. Then the form sum $h(P) + W$ is bounded from below and closed on $\mathcal{H}^m$ and the self-adjoint operator $H_0$ associated to it satisfies the conditions of Proposition 8.

Assume that $m \geq 1$ is an integer and let $L = \sum_{\alpha, \beta} P^\alpha a_{\alpha, \beta}(Q) P^\beta : \mathcal{H}^m \to \mathcal{H}^{-m}$ where $\alpha, \beta$ are multi-indices of length $\leq m$ and $a_{\alpha, \beta}$ are functions $X \to B(E)$ such that $a_{\alpha, \beta}(Q)$ is a continuous map $\mathcal{H}^m \to \mathcal{H}^m$. If $\|f\|_{\mathcal{H}^m} \geq \mu \|f\|^2_{\mathcal{H}^m} - \nu\|f\|^2 |q|$ for some $\mu, \nu > 0$ then $L$ is a closed bounded from below on $\mathcal{H}^m$ and the self-adjoint operator $H_0$ associated to it verifies Proposition 8.

**Note added July 2014:** The theory can be extended to metric spaces by using the $C^*$-algebra introduced and studied in my paper “On the structure of the essential spectrum of elliptic operators on metric spaces”, J. Funct. Analysis 260, 1734–1765 (2011) and arXiv:1003.3454.

**Acknowledgment.** We thank Hans Henrik Rugh for the remark which made Lemma 7 obvious.

**References**


[S] B. Simon, Schrödinger operators with purely discrete spectrum, see preprint 08-191 at http://www.math.ucsd.edu/


CNRS and University of Cergy-Pontoise 95000 Cergy-Pontoise, France

E-mail address: vlad@math.cnrs.fr