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Precession of a planet with a satellite.

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Abstract

The contribution of a satellite in the precession motion of the axis of an oblate planet has been previously studied in the approximation of a distant satellite, or in the approximation of a very close satellite. Here we study the general problem for an arbitrary semimajor axis for the satellite, without performing the usual gyroscopic approximation. We present precessional equations valid in a very general setting, and we demonstrate that this problem, after the classical expansion of the satellite potential, and averaging over the fast angles, is indeed integrable. We provide here the complete solution of this problem by quadrature, as well as some explicit approximate solutions. We also demonstrate that after averaging over the nutation motion, the pole of the spin axis, the pole of the satellite orbit, and the pole of the planet orbit remain coplanar with the total angular momentum and precess uniformly around the total angular momentum.

Keywords : CELESTIAL MECHANICS, PLANETARY DYNAMICS, ROTATIONAL DYNAMICS, SATELLITES DYNAMICS, MOON

1 Introduction

We are considering here a relatively simple system composed of a central star, a planet orbiting the star, and a satellite orbiting the planet. We increase the complexity of the problem by considering that our planet is a solid non-spherical body. The most obvious system of this kind is the Sun-Earth-Moon system, but some triple star systems will fit in our study as well. We focus here on the precessional motion of the spin axis of the planet, and in a lesser degree on the precessional motion of the orbital plane of the satellite and of the planet.

The computation of the precession of the spin axis of

a planet in presence of a distant satellite is well known (see Murray, 1983). In this approximation, the precession torque, and thus the precession frequency, increases as $1/r^3$ when the distance r of the satellite to the planet decreases. It is thus clear that these formulas are no longer valid for a close satellite.

The understanding of the contribution of a close satellite was first motivated by the study of the Martian satellites, Phobos and Deimos. Goldreich (1965) investigated first the interaction of a close satellite with a precessing planet, and demonstrated that a close satellite will follow the planet with a nearly constant inclination to the equator. This work was followed by the contributions of (Kinoshita, 1993) who analyzed the motion of the Uranian satellites under the secular change of the obliquity of the planet, and (Efroimsky, 2004) who consider non uniform precessions.

In his beautiful study of the Lunar orbit, (Goldreich, 1966) extended his work to the Sun-Earth-Moon system, but assumed that the planet orbit is fixed and circular orbits for both satellite and planet. This work was extended by (Touma and Wisdom, 1994a) using a non averaged Hamiltonian, and equations of motion expressed in the mobile frame linked to the planet.

Explicit analytical expressions for the contribution of a close satellite to the precessional motion of a planet were derived by Ward (1975), using the equations of Goldreich (1966), with zero eccentricities, zero inclinations, and the gyroscopic approximation (i.e. one assumes that the axis of rotation is the axis of figure of the planet). These computations were improved by Tremaine (1991) who considered the inclinations, and corrected the mass factors of Goldreich (1966).

In section 2, we consider the general problem with an oblate planet and a satellite. Contrarily to many of the previous study, we do not make the gyroscopic approximation, thus allowing for an axis of figure different from the axis of rotation. Nevertheless, in the present work,

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we simplify the equations of motion by averaging over the rotational motion of the planet, providing some precession (and nutation) equations that can be used in a very general setting.

In a second stage (section 3), we derive the secular equations that are obtained by averaging over the orbital motion of the satellite and the planet, and over the argument of perihelion of the satellite. We obtain a set of secular equations that describe in a closed way, the evolution of the spin of the planet, the orbital plane of the planet, and the orbital plane of the satellite in a very general setting.

We then demonstrate that, quite surprisingly, this differential system of order 9 with 7 integrals is integrable, and can be decomposed as a relative periodic motion (the nutation) and a general precessional motion. The two periods can be derived by quadratures. We obtain thus some general formulas (although not explicit) that provide the precession formulas for the axis of the planet, in all cases, for a distant or a close satellite, but also in the intermediate regime where none of the previous approximations is valid (section 4). In this section, we also demonstrate that after averaging over the nutation motion, the pole of the spin axis, the pole of the satellite orbit, and the pole of the planet orbit remain coplanar with the total angular momentum (section 4.3). This is in some sense a generalization of the Cassini Laws.

After some discussion of numerical examples (section 5), in section 6, we proceed to an additional approximation that allows to obtain a completely explicit solution of this problem, for arbitrary values of the eccentricities, inclination, and semi major axis of the planet and satellite, whenever the averaging is possible. We can then compare this approximation with our rigorous expression, and with the results of non averaged numerical integrations in different settings. In particular, we use as a test model a Lunar motion where the Moon distance to the Earth is varied from the surface of the Earth to some limit distance where, due to the solar perturbation, the Moon escapes, and no longer remains a satellite of the Earth. We make also comparison of our results with the classical computation in the case of a distant satellite, and the previous expressions of Ward (1975) and Tremaine (1991), for close satellites (section 7).

2 Fundamental equations

We are considering here a three body problem with a central star, an oblate planet, and a satellite orbiting the planet, with respective barycentric coordinates $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ and masses m_0, m_1, m_2 . The full Hamiltonian of this

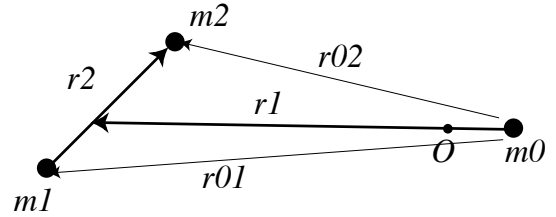


Figure 1: Jacobi coordinates.

problem can be expressed as

$$\mathcal{H} = H_N + H_E + H_I \quad (1)$$

where H_N is the Hamiltonian of three mass points, H_E describes the free rigid body motion and H_I contains the gravitational interaction between the bulge and the other two mass points. In such a satellite problem, the orbital Hamiltonian is naturally expressed in Jacobi coordinates,

$$\begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 - \delta & \delta \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \quad (2)$$

where $\delta = m_2/(m_1 + m_2)$. With this choice, $\mathbf{r}_0 = \mathbf{u}_0$ is the barycentric position vector of the Sun, \mathbf{r}_1 the position vector of the planet-satellite barycenter relative to the Sun, and \mathbf{r}_2 the position vector of the satellite relative to the planet. The symplectic structure is preserved using the conjugate momentum

$$\begin{pmatrix} \tilde{\mathbf{r}}_0 \\ \tilde{\mathbf{r}}_1 \\ \tilde{\mathbf{r}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \delta & 1 - \delta \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_0 \\ \tilde{\mathbf{u}}_1 \\ \tilde{\mathbf{u}}_2 \end{pmatrix} \quad (3)$$

where $\tilde{\mathbf{u}}_i = m_i \dot{\mathbf{u}}_i$ ($i = 0, 2$) are the barycentric momentum. In these coordinates, the Newtonian interaction is

$$H_N(\mathbf{r}, \tilde{\mathbf{r}}) = \frac{\tilde{\mathbf{r}}_1^2}{2\beta_1} + \frac{\tilde{\mathbf{r}}_2^2}{2\beta_2} - \frac{\mu_2 \beta_2}{r_2} - \frac{\mathcal{G}m_0 m_1}{r_{01}} - \frac{\mathcal{G}m_0 m_2}{r_{02}} \quad (4)$$

where $\mu_2 = \mathcal{G}(m_1 + m_2)$, $\beta_1 = M_{12}m_0/(M_{12} + m_0)$, $\beta_2 = m_1 m_2/(m_1 + m_2)$ are the reduced masses with $M_{12} = m_1 + m_2$, and r_{01}, r_{02} are the modulus of the position vectors $\mathbf{r}_{01}, \mathbf{r}_{02}$ from the Sun to the planet and satellite, expressed as

$$\mathbf{r}_{01} = \tilde{\mathbf{r}}_1 - \delta \tilde{\mathbf{r}}_2; \quad \mathbf{r}_{02} = \tilde{\mathbf{r}}_1 + (1 - \delta)\tilde{\mathbf{r}}_2. \quad (5)$$

The Hamiltonian of the free motion of a rigid body is

$$H_E = \frac{1}{2} {}^t \mathbf{G} \mathbf{I}^{-1} \mathbf{G} \quad (6)$$

where \mathbf{G} is the angular momentum of the rigid body and \mathcal{I} its inertia tensor. Let $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ be the principal frame where \mathcal{I} is diagonal ($\mathcal{I} = \text{diag}(A, B, C)$). For sake of clarity, we present here the case of an axisymmetric planet ($A = B$). The general case can be treated in the same way providing some additional averaging, and will be outlined in section 2.6. In the present case, we have then,

$$\mathcal{I}^{-1} = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & 1/A & 0 \\ 0 & 0 & 1/C \end{pmatrix}_{(\dots, \mathbf{K})} = \frac{1}{A} Id + \left(\frac{1}{C} - \frac{1}{A} \right) \mathbf{K}^t \mathbf{K} \quad (7)$$

which gives

$$H_E = \frac{\mathbf{G}^2}{2A} + \left(\frac{1}{C} - \frac{1}{A} \right) \frac{(\mathbf{K} \cdot \mathbf{G})^2}{2} \quad (8)$$

where A and C are the moments of inertia of the planet.

The interaction between the bulge and the other two mass points is expanded in terms of Legendre polynomials as

$$H_I = -\frac{\mathcal{G}(C-A)m_0}{2r_{01}^3} \left[1 - 3 \left(\frac{\mathbf{r}_{01} \cdot \mathbf{K}}{r_{01}} \right)^2 \right] - \frac{\mathcal{G}(C-A)m_2}{2r_2^3} \left[1 - 3 \left(\frac{\mathbf{r}_2 \cdot \mathbf{K}}{r_2} \right)^2 \right] \quad (9)$$

2.1 Equations of motion

The Hamiltonian is written in terms of non canonical coordinates $(\mathbf{r}, \tilde{\mathbf{r}}, \mathbf{K}, \mathbf{G})$. The equations of motion are

$$\begin{aligned} \dot{y} &= \{\mathcal{H}, y\} \\ &= -B(y) \nabla_y \mathcal{H} \end{aligned}$$

where y is any kind of coordinate and $B(y)$ the matrix of Poisson brackets $\{y_i, y_j\}^1$. For the components of \mathbf{r}_k and $\tilde{\mathbf{r}}_k$, the standard symplectic structure holds, $\{r_{ki}, \tilde{r}_{kj}\} = -\delta_{ij}$. For (\mathbf{K}, \mathbf{G}) , the Euler-Poisson structure holds, $\{K_i, K_j\} = 0$, $\{K_i, \mathbf{G}_j\} = -\varepsilon_{ijk} K_k$ and $\{\mathbf{G}_i, \mathbf{G}_j\} = -\varepsilon_{ijk} \mathbf{G}_k$ (Borisov and Mamaev, 2005, see also Dullin, 2004)². All other fundamental Poisson brackets

¹With position (q_i) and associated momentum (p_i) , we define the Poisson bracket as $\{f, g\} = \sum_i \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}$.

²The Levi-Civita symbol ε_{ijk} is zero if two indices i, j, k are equal, and is the signature of the permutation (i, j, k) otherwise.

ets are identically equal to zero. $B(\mathbf{r}, \tilde{\mathbf{r}}, \mathbf{K}, \mathbf{G})$ is then

$$B(y) = \begin{pmatrix} 0 & -I & & 0 & & 0 \\ I & 0 & & 0 & & 0 \\ \hline 0 & 0 & & 0 & 0 & -K_3 \\ 0 & 0 & & K_3 & 0 & -K_1 \\ \hline 0 & 0 & & -K_2 & K_1 & 0 \\ 0 & 0 & & 0 & -K_3 & K_2 \\ \hline 0 & 0 & & K_3 & 0 & -K_1 \\ 0 & 0 & & -K_2 & K_1 & 0 \\ \hline 0 & -\mathbf{G}_3 & & \mathbf{G}_3 & 0 & -\mathbf{G}_1 \\ -\mathbf{G}_2 & \mathbf{G}_1 & & -\mathbf{G}_2 & \mathbf{G}_1 & 0 \end{pmatrix} \quad (10)$$

And the equations of motion are

$$\begin{aligned} \dot{\mathbf{r}} &= \nabla_{\tilde{\mathbf{r}}} \mathcal{H} & \dot{\mathbf{K}} &= \nabla_{\mathbf{G}} \mathcal{H} \wedge \mathbf{K} \\ \dot{\tilde{\mathbf{r}}} &= -\nabla_{\mathbf{r}} \mathcal{H} & \dot{\mathbf{G}} &= \nabla_{\mathbf{K}} \mathcal{H} \wedge \mathbf{K} + \nabla_{\mathbf{G}} \mathcal{H} \wedge \mathbf{G} \end{aligned} \quad (11)$$

Now, the advantage of taking \mathbf{K} and \mathbf{G} as coordinates to define the orientation of the rigid body is obvious : the study can be done in an heliocentric frame and equations in \mathbf{K} and \mathbf{G} look like equations of precession.

2.2 First simplification

We have assumed that the rigid body is axisymmetric. This is why \mathbf{I} and \mathbf{J} do not appear in the equations of motion (11). Because of this symmetry, the rotational angle of the planet l (see Fig.2) will not appear as well in the Hamiltonian. It is easy to verify that $\mathbf{G} \cdot \mathbf{K}$ is constant and thus any function $h(\mathbf{G} \cdot \mathbf{K})$ in the Hamiltonian will not contribute to the equations of motion (11).

The complete Hamiltonian can then be expressed on the form

$$\mathcal{H} = \mathcal{H}_a(\mathbf{r}, \tilde{\mathbf{r}}) + \mathcal{H}_b(\mathbf{r}, K, \mathbf{G}) \quad (12)$$

with

$$\mathcal{H}_a(\mathbf{r}, \tilde{\mathbf{r}}) = H_N - \frac{\mathcal{G}(C-A)}{2} \left(\frac{m_0}{r_{01}^3} + \frac{m_2}{r_2^3} \right) \quad (13)$$

and

$$\mathcal{H}_b(\mathbf{r}, K, \mathbf{G}) = \frac{\mathbf{G}^2}{2A} + (\mathbf{u}_1 \cdot \mathbf{K})^2 + (\mathbf{u}_2 \cdot \mathbf{K})^2 \quad (14)$$

where

$$\begin{aligned} \mathbf{u}_1 &= \left[\frac{3\mathcal{G}(C-A)m_0}{2r_{01}^5} \right]^{\frac{1}{2}} \mathbf{r}_{01} ; \\ \mathbf{u}_2 &= \left[\frac{3\mathcal{G}(C-A)m_2}{2r_2^5} \right]^{\frac{1}{2}} \mathbf{r}_2 . \end{aligned} \quad (15)$$

As a consequence, $\nabla_{\mathbf{G}} \mathcal{H}$ and \mathbf{G} are collinear and the equation of motion for (\mathbf{K}, \mathbf{G}) in (11) simplifies to

$$\begin{aligned} \dot{\mathbf{K}} &= \mathbf{G}/A \wedge \mathbf{K} , \\ \dot{\mathbf{G}} &= 2(\mathbf{u}_1 \cdot \mathbf{K})\mathbf{u}_1 \wedge \mathbf{K} + 2(\mathbf{u}_2 \cdot \mathbf{K})\mathbf{u}_2 \wedge \mathbf{K} . \end{aligned} \quad (16)$$

2.3 Averaging

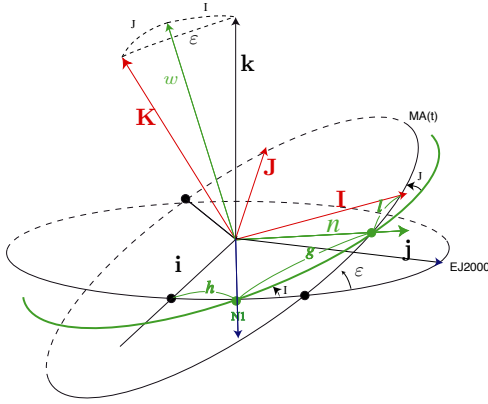


Figure 2: Definition of Andoyer's coordinates. $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a fixed reference frame, and $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ the reference frame of the principal axis of inertia of the solid body. The Andoyer action variables are $(G, H = \mathbf{G} \cdot \mathbf{k}, L = \mathbf{G} \cdot \mathbf{K})$ with the associated angles (g, h, l) (Andoyer, 1923).

The vector \mathbf{K} precesses around the unit vector $\mathbf{w} = \mathbf{G}/G$ with nearly the rotation rate of the planet \mathbf{G}/A . The unit vector $\mathbf{n} = \mathbf{w} \wedge \mathbf{K} / \|\mathbf{w} \wedge \mathbf{K}\|$, is thus rotating in the orthogonal plane to \mathbf{w} (Fig.2). We want now to average over this fast motion. If we use Andoyer variables (G, H, L, g, h, l) (Fig.2), in eq. (16), \mathbf{K} do not depend on l and we have

$$\mathbf{K}(G, H, L, g, h) = (\cos J)\mathbf{w} + (\sin J)\mathbf{n} \wedge \mathbf{w} \quad (17)$$

with

$$\cos J = L/G \quad (18)$$

In the fixed reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, the coordinates of \mathbf{w} are

$$\mathbf{w}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})} = \begin{pmatrix} \sin I \sin h \\ -\sin I \cos h \\ \cos I \end{pmatrix} \quad (19)$$

with $\cos I = H/G$. Moreover, only \mathbf{n} depends on the fast

angle g with an averaged value

$$\langle \mathbf{n} \rangle_g = \mathbf{0} . \quad (20)$$

In order to average \mathcal{H}_b over g , we write in matrix form, for $i = 1, 2$

$$(\mathbf{u}_i \cdot \mathbf{K})^2 = {}^t \mathbf{u}_i \mathbf{K}^t \mathbf{K} \mathbf{u}_i \quad (21)$$

All terms of degree 1 in \mathbf{n} will average to $\mathbf{0}$. Using (17), and after a circular permutation in the triple product, we have

$$\langle (\mathbf{u}_i \cdot \mathbf{K})^2 \rangle_g = (\mathbf{u}_i \cdot \mathbf{w})^2 \cos^2 J + \langle (\mathbf{n} \cdot (\mathbf{w} \wedge \mathbf{u}_i))^2 \rangle_g \sin^2 J \quad (22)$$

We thus need to compute the average $\langle \mathbf{n}^t \mathbf{n} \rangle_g$. In the intermediary basis $(\mathbf{w}, \mathbf{w}', \mathbf{w}'')$, we have

$$\mathbf{n}_{(\mathbf{w}, \mathbf{w}', \mathbf{w}'')} = \begin{pmatrix} 0 \\ \cos g \\ \sin g \end{pmatrix} \quad (23)$$

and thus

$$\langle \mathbf{n}^t \mathbf{n} \rangle_g = \frac{1}{2} (\text{Id} - \mathbf{w}^t \mathbf{w}) \quad (24)$$

The $\mathbf{w}^t \mathbf{w}$ part will cancel, and remains only

$$\begin{aligned} \langle (\mathbf{u}_i \cdot \mathbf{K})^2 \rangle_g &= (\mathbf{u}_i \cdot \mathbf{w})^2 \cos^2 J + \frac{1}{2} (\mathbf{u}_i \wedge \mathbf{w})^2 \sin^2 J \\ &= \frac{\mathbf{u}_i^2}{2} \sin^2 J + (\mathbf{u}_i \cdot \mathbf{w})^2 (1 - \frac{3}{2} \sin^2 J) . \end{aligned} \quad (25)$$

The averaged Hamiltonian $\langle \mathcal{H}_b \rangle_g$ is thus

$$\langle \mathcal{H}_b \rangle_g = \frac{G^2}{2A} + \sum_{i=1,2} \frac{\mathbf{u}_i^2}{2} \sin^2 J + (\mathbf{u}_i \cdot \mathbf{w})^2 (1 - \frac{3}{2} \sin^2 J) \quad (26)$$

As $\langle \mathcal{H}_b \rangle_g$ does not depend on g and l , G and L are constant, and so is J , as $\cos J = L/G$. As in section 2.1, we have $\{w_i, w_j\} = -\epsilon_{ijk} w_k / G$, and the equations in \mathbf{w} become thus

$$\dot{\mathbf{w}} = \frac{1}{G} \nabla_{\mathbf{w}} \langle \mathcal{H} \rangle_g \wedge \mathbf{w} , \quad (27)$$

that is

$$\dot{\mathbf{w}} = \frac{2 - 3 \sin^2 J}{G} ((\mathbf{u}_1 \cdot \mathbf{w}) \mathbf{u}_1 \wedge \mathbf{w} + (\mathbf{u}_2 \cdot \mathbf{w}) \mathbf{u}_2 \wedge \mathbf{w}) . \quad (28)$$

Remark. We have not proceeded here to the gyroscopic approximation that consists to assume that the axis of figure (\mathbf{K}) is the same as the angular momentum axis (\mathbf{w}), but we have simply averaged the Hamiltonian over the fast rotation angle g . Although for a fast rotating (and non rigid) planet, the angle J is small ($J = 10^{-7}$ radians for the Earth), we prefer the present formulation that is less confusing. One should nevertheless notice that

if the terms of order $\sin^2 J$ are neglected, the rotational Hamiltonian \mathcal{H}_b (26) becomes

$$\mathcal{H}_1 = \frac{\mathcal{G}^2}{2A} + (\mathbf{u}_1 \cdot \mathbf{w})^2 + (\mathbf{u}_2 \cdot \mathbf{w})^2 \quad (29)$$

which corresponds to the gyroscopic approximation. It should be noted that usually, in the gyroscopic approximation, one merely replaces \mathbf{K} by \mathbf{w} , which corresponds to neglect terms in $O(\sin J)$. Here, by using the averaged equations, we show that we obtain in fact a better approximation, as $\mathcal{H}_1 = \mathcal{H}_b + O(\sin^2 J)$.

2.4 Hamiltonian in $(\mathbf{r}, \tilde{\mathbf{r}}, \mathbf{w})$

We can now gather the parts of the Hamiltonian that will drive the evolution of the orbital variables $(\mathbf{r}, \tilde{\mathbf{r}})$ and spin axis (\mathbf{w}) . From now on, we will call spin axis the axis of the rotational angular momentum of the planet, with unit vector \mathbf{w} . We have

$$\mathcal{H}(\mathbf{r}, \tilde{\mathbf{r}}, \mathbf{w}) = \mathcal{H}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_{01} \quad (30)$$

with

$$\begin{aligned} \mathcal{H}_0 &= \frac{\tilde{\mathbf{r}}_1^2}{2\beta_1} + \frac{\tilde{\mathbf{r}}_2^2}{2\beta_2} - \frac{\mu_1\beta_1}{r_1} - \frac{\mu_2\beta_2}{r_2}, \\ \mathcal{U}_1 &= \frac{\mu_1\beta_1}{r_1} - \frac{\mathcal{G}m_0m_1}{r_{01}} - \frac{\mathcal{G}m_0m_2}{r_{02}}, \\ \mathcal{U}_2 &= -\mathcal{C}_1m_2 \left(\frac{1}{r_2^3} - 3\frac{(\mathbf{r}_2 \cdot \mathbf{w})^2}{r_2^5} \right), \\ \mathcal{U}_{01} &= -\mathcal{C}_1m_0 \left(\frac{1}{r_{01}^3} - 3\frac{(\mathbf{r}_{01} \cdot \mathbf{w})^2}{r_{01}^5} \right), \end{aligned} \quad (31)$$

with

$$\mathcal{C}_1 = \frac{\mathcal{G}(C - A)}{2} \left(1 - \frac{3}{2} \sin^2 J \right). \quad (32)$$

2.5 Averaging over the orbital motion

The unperturbed part \mathcal{H}_0 is the Hamiltonian of two distinct Keplerian problems and

$$\mathcal{H}_0 = -\frac{\mu_1\beta_1}{2a_1} - \frac{\mu_2\beta_2}{2a_2}, \quad (33)$$

while $\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_{01}$ is a perturbation of this Keplerian problem. We will now average the Hamiltonian over the mean anomalies of the orbital motion of the planet and the satellite, using the relations detailed in annex 1. We will use also the orthonormal basis $(\mathbf{w}_1, \mathbf{w}'_1, \mathbf{w}''_1)$ and $(\mathbf{w}_2, \mathbf{w}'_2, \mathbf{w}''_2)$ where \mathbf{w}'_i is in the direction of the perihelion of the orbit defined by \mathbf{r}_i , and \mathbf{w}_i is the unit vector in the direction of the orbital angular momentum $\mathbf{G}_i = \beta_i \mathbf{r}_i \wedge \dot{\mathbf{r}}_i$.

By averaging over the mean anomaly M_2 of the satellite motion, we have

$$\langle \mathcal{U}_2 \rangle_{M_2} = \frac{\mathcal{C}_1 m_2}{2a_2^3(1 - e_2^2)^{3/2}} (1 - 3(\mathbf{w} \cdot \mathbf{w}_2)^2). \quad (34)$$

In the expansion of \mathcal{U}_{01} in terms of $\rho = r_2/r_1$ and δ , we will neglect all terms of order higher than ρ^2 . We will thus neglect terms of order $\delta\rho^2, \delta^2\rho^2, \dots$. We have thus

$$\frac{1}{r_{01}^3} \approx \frac{1}{r_1^3} \left(1 + 3\delta \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1^2} \right) \quad (35)$$

and

$$\begin{aligned} \frac{(\mathbf{r}_{01} \cdot \mathbf{w})^2}{r_{01}^5} &\approx \frac{(\mathbf{r}_1 \cdot \mathbf{w})^2}{r_1^5} \\ -2\delta \frac{(\mathbf{r}_1 \cdot \mathbf{w})(\mathbf{r}_2 \cdot \mathbf{w})}{r_1^5} &+ 5\delta \frac{(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{r}_1 \cdot \mathbf{w})^2}{r_1^7} \end{aligned} \quad (36)$$

In the computation of $\langle \mathcal{U}_{01} \rangle_{M_1, M_2}$ (see annex 1), all the terms of order δ are in fact of order at least $e_1 e_2 \delta \rho$. These terms are usually very small, but in order to allow for large eccentricities, we will average over the argument of perihelion of the satellite ω_2 . If we notice that

$$\langle \mathbf{r}_2 \rangle_{M_2, \omega_2} = 0 \quad (37)$$

then we see that all terms in δ disappear from $\langle \mathcal{U}_{01} \rangle_{M_1, M_2, \omega_2}$ and we are then left with

$$\langle \mathcal{U}_{01} \rangle_{M_1, M_2, \omega_2} \approx \frac{\mathcal{C}_1 m_0}{2a_1^3(1 - e_1^2)^{3/2}} (1 - 3(\mathbf{w} \cdot \mathbf{w}_1)^2). \quad (38)$$

After expanding up to second order in ρ , we have

$$\mathcal{U}_1 \approx \frac{\mathcal{G}m_0\beta_2}{2} \left(\frac{r_2^2}{r_1^3} - 3\frac{(\mathbf{r}_1 \cdot \mathbf{r}_2)^2}{r_1^5} \right) \quad (39)$$

Averaging over M_2 leads to

$$\begin{aligned} \langle \mathcal{U}_1 \rangle_{M_2} &= -\frac{\mathcal{G}m_0\beta_2}{4} \frac{a_2^2}{r_1^3} \\ &\left(1 - 3\frac{(\mathbf{r}_1 \cdot \mathbf{w}_2)^2}{r_1^2} - 3e_2^2 \left[1 + \frac{(\mathbf{r}_1 \cdot \mathbf{w}_2'')^2}{r_1^2} - 4\frac{(\mathbf{r}_1 \cdot \mathbf{w}_2')^2}{r_1^2} \right] \right) \end{aligned} \quad (40)$$

and after averaging over M_1

$$\begin{aligned} \langle \mathcal{U}_1 \rangle_{M_1, M_2} &= \frac{\mathcal{G}m_0\beta_2}{8} \frac{a_2^2}{a_1^3(1 - e_1^2)^{3/2}} \\ &\left(1 - 3(\mathbf{w}_1 \cdot \mathbf{w}_2)^2 - 3e_2^2(1 + (\mathbf{w}_1 \cdot \mathbf{w}_2'')^2 - 4(\mathbf{w}_1 \cdot \mathbf{w}_2')^2) \right) \end{aligned} \quad (41)$$

We will also average over the argument of perihelion ω_2 of the satellite. We have

$$\langle (\mathbf{w}_1 \cdot \mathbf{w}'_2)^2 \rangle_{\omega_2} = \langle (\mathbf{w}_1 \cdot \mathbf{w}''_2)^2 \rangle_{\omega_2} = \frac{1}{2} (1 - (\mathbf{w}_1 \cdot \mathbf{w}_2)^2) , \quad (42)$$

and thus

$$\langle \mathcal{U}_1 \rangle_{M1, M2, \omega_2} = \frac{\mathcal{G} m_0 \beta_2 a_2^2 (1 + \frac{3}{2} e_2^2)}{8 a_1^3 (1 - e_1^2)^{3/2}} (1 - 3(\mathbf{w}_1 \cdot \mathbf{w}_2)^2) \quad (43)$$

With these approximations, in the averaged Hamiltonian, the semimajor axis of the planet and satellite are constant, as well as their eccentricity, while their orbital plane will precess and change its inclination. If we do not consider the constant terms in the averaged Hamiltonian, we are left with a Hamiltonian \mathcal{H}_s that will describe the evolution of $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$

$$\mathcal{H}_s = -\frac{\mathbf{a}}{2} (\mathbf{w} \cdot \mathbf{w}_1)^2 - \frac{\mathbf{b}}{2} (\mathbf{w} \cdot \mathbf{w}_2)^2 - \frac{\mathbf{c}}{2} (\mathbf{w}_1 \cdot \mathbf{w}_2)^2 \quad (44)$$

with

$$\begin{aligned} \mathbf{a} &= \frac{3\mathcal{C}_1 m_0}{a_1^3 (1 - e_1^2)^{3/2}} \\ \mathbf{b} &= \frac{3\mathcal{C}_1 m_2}{a_2^3 (1 - e_2^2)^{3/2}} \\ \mathbf{c} &= \frac{3\mathcal{G} m_0 \beta_2 a_2^2}{4a_1^3 (1 - e_1^2)^{3/2}} (1 + \frac{3}{2} e_2^2) \end{aligned} \quad (45)$$

2.6 Non axisymmetric case ($A \neq B$)

For sake of simplicity, we have presented above the case of an axisymmetric planet ($A = B$). In fact, the general case can be treated in the same way if we average also over the Andoyer rotational angle l . Indeed, for any unit vector \mathbf{u} , the potential generated at $\mathbf{r} = r\mathbf{u}$ by the solid body is

$$\begin{aligned} V &= -\frac{\mathcal{G}}{2r^3} [(B + C - 2A) \\ &\quad + 3(A - B)(\mathbf{u} \cdot \mathbf{J})^2 - 3(C - A)(\mathbf{u} \cdot \mathbf{K})^2] \end{aligned} , \quad (46)$$

with the average over l, g

$$\langle (\mathbf{u} \cdot \mathbf{J})^2 \rangle_{g, l} = \frac{\mathbf{u}^2}{2} (1 - \frac{\sin^2 J}{2}) + \frac{(\mathbf{u} \cdot \mathbf{w})^2}{2} \left(\frac{3}{2} \sin^2 J - 1 \right) . \quad (47)$$

It is then easy to show that the only change induced in the averaged equations (28) is to replace $(C - A)$ by $(C - A/2 - B/2)$ in the expression of \mathbf{u}_1 and \mathbf{u}_2 (Eq.15). In the same way, the only change in the secular Hamiltonian \mathcal{H}_s (44) will be to replace \mathcal{C}_1 (32) by

$$\mathcal{C}'_1 = \frac{\mathcal{G}(2C - A - B)}{4} (1 - \frac{3}{2} \sin^2 J) , \quad (48)$$

and as well in the expressions of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (45).

3 Secular equations

The Hamiltonian (44) depends only on the unit vectors $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ of the rotational angular momentum of the planet \mathbf{G} , and of the orbital angular momentum $\mathbf{G}_1, \mathbf{G}_2$ of the planet and of the satellite. The equations of motion can be derived easily as in section 2.1, and we have

$$\begin{aligned} \dot{\mathbf{G}} &= \nabla_{\mathbf{G}} \mathcal{H}_s \wedge \mathbf{G} , \\ \dot{\mathbf{G}}_1 &= \nabla_{\mathbf{G}_1} \mathcal{H}_s \wedge \mathbf{G}_1 , \\ \dot{\mathbf{G}}_2 &= \nabla_{\mathbf{G}_2} \mathcal{H}_s \wedge \mathbf{G}_2 . \end{aligned} \quad (49)$$

As $\mathbf{G} \cdot \dot{\mathbf{G}} = \mathbf{G}_1 \cdot \dot{\mathbf{G}}_1 = \mathbf{G}_2 \cdot \dot{\mathbf{G}}_2 = 0$, the norms γ, β, α of $\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2$ are constant ($\gamma = G$), and we obtain the equations in $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$

$$\begin{aligned} \dot{\mathbf{w}} &= \frac{1}{\gamma} \nabla_{\mathbf{w}} \mathcal{H}_s \wedge \mathbf{w} , \\ \dot{\mathbf{w}}_1 &= \frac{1}{\beta} \nabla_{\mathbf{w}_1} \mathcal{H}_s \wedge \mathbf{w}_1 , \\ \dot{\mathbf{w}}_2 &= \frac{1}{\alpha} \nabla_{\mathbf{w}_2} \mathcal{H}_s \wedge \mathbf{w}_2 . \end{aligned} \quad (50)$$

That is

$$\begin{aligned} \dot{\mathbf{w}} &= -\frac{\mathbf{a}}{\gamma} (\mathbf{w}_1 \cdot \mathbf{w}) \mathbf{w}_1 \wedge \mathbf{w} - \frac{\mathbf{b}}{\gamma} (\mathbf{w}_2 \cdot \mathbf{w}) \mathbf{w}_2 \wedge \mathbf{w} , \\ \dot{\mathbf{w}}_1 &= -\frac{\mathbf{c}}{\beta} (\mathbf{w}_2 \cdot \mathbf{w}_1) \mathbf{w}_2 \wedge \mathbf{w}_1 - \frac{\mathbf{a}}{\beta} (\mathbf{w} \cdot \mathbf{w}_1) \mathbf{w} \wedge \mathbf{w}_1 , \\ \dot{\mathbf{w}}_2 &= -\frac{\mathbf{b}}{\alpha} (\mathbf{w} \cdot \mathbf{w}_2) \mathbf{w} \wedge \mathbf{w}_2 - \frac{\mathbf{c}}{\alpha} (\mathbf{w}_1 \cdot \mathbf{w}_2) \mathbf{w}_1 \wedge \mathbf{w}_2 . \end{aligned} \quad (51)$$

These equations express the fact that each angular momentum is precessing in space around the other two. This system of equations is a priori of order 9 but we will show that it is in fact integrable.

3.1 Integrals

We have the integrals

$$\begin{aligned} \|\mathbf{w}\| &= 1 \\ \|\mathbf{w}_1\| &= 1 \\ \|\mathbf{w}_2\| &= 1 \\ -\mathbf{a}(\mathbf{w} \cdot \mathbf{w}_1)^2 - \mathbf{b}(\mathbf{w} \cdot \mathbf{w}_2)^2 - \mathbf{c}(\mathbf{w}_1 \cdot \mathbf{w}_2)^2 &= 2\mathcal{H}_s \\ \gamma\mathbf{w} + \beta\mathbf{w}_1 + \alpha\mathbf{w}_2 &= \mathbf{W}_0 = Cte \end{aligned} \quad (52)$$

where \mathbf{W}_0 is the total angular momentum of the system. We have 7 independent integrals in our system of order 9. We are thus missing one integral for a complete integration of the system.

3.2 Single planet case

When there is no satellite, the equations (51) simplifies to the system of order 6

$$\begin{aligned}\dot{\mathbf{w}} &= -\frac{\mathbf{a}}{\gamma}(\mathbf{w}_1 \cdot \mathbf{w})\mathbf{w}_1 \wedge \mathbf{w} , \\ \dot{\mathbf{w}}_1 &= -\frac{\mathbf{a}}{\beta}(\mathbf{w}_1 \cdot \mathbf{w})\mathbf{w} \wedge \mathbf{w}_1 ,\end{aligned}\quad (53)$$

With the 5 independent integrals given by

$$\begin{aligned}\|\mathbf{w}\| &= 1 \\ \|\mathbf{w}_1\| &= 1 \\ \gamma\mathbf{w} + \beta\mathbf{w}_1 &= \mathbf{W}_0 = Cte .\end{aligned}\quad (54)$$

As $x = \mathbf{w} \cdot \mathbf{w}_1$ is constant, the system is trivially integrable. We have indeed

$$\dot{\mathbf{w}} = \Omega_0 \mathbf{w}_0 \wedge \mathbf{w} , \quad \dot{\mathbf{w}}_1 = \Omega_0 \mathbf{w}_0 \wedge \mathbf{w}_1 ; \quad (55)$$

where $\mathbf{w}_0 = \mathbf{W}_0 / \|\mathbf{W}_0\|$ is the unit vector in the direction of the total angular momentum \mathbf{W}_0 , and

$$\Omega_0 = -\frac{\mathbf{a}x}{\gamma} \sqrt{1 + \frac{\gamma^2}{\beta^2} + 2\frac{\gamma}{\beta}x} . \quad (56)$$

Both vectors \mathbf{w}, \mathbf{w}_1 thus precess uniformly around the total angular momentum direction \mathbf{w}_0 with constant precession rate Ω_0 .

Remark. In the same way, the system (51) is also trivially integrable when the planet is reduced to a point mass.

3.3 Reduction

The general case (51) is more difficult, and in order to reduce the order of the differential system, we will consider the relative position of the vectors $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$, and forget about their absolute position in space. More precisely, let

$$x = \mathbf{w} \cdot \mathbf{w}_1 ; \quad y = \mathbf{w} \cdot \mathbf{w}_2 ; \quad z = \mathbf{w}_1 \cdot \mathbf{w}_2 . \quad (57)$$

and

$$v = (\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2) = \mathbf{w} \cdot (\mathbf{w}_1 \wedge \mathbf{w}_2) . \quad (58)$$

The evolution of x, y, z, v is then given by the differential equations

$$\begin{aligned}\dot{x} &= \left(\frac{\mathbf{c}}{\beta}z - \frac{\mathbf{b}}{\gamma}y \right) v \\ \dot{y} &= \left(\frac{\mathbf{a}}{\gamma}x - \frac{\mathbf{c}}{\alpha}z \right) v \\ \dot{z} &= \left(\frac{\mathbf{b}}{\alpha}y - \frac{\mathbf{a}}{\beta}x \right) v\end{aligned}\quad (59)$$

where the expression of v is given by the Gram determinant

$$v^2 = \begin{vmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{vmatrix} = 1 - x^2 - y^2 - z^2 + 2xyz . \quad (60)$$

We still have the two integrals

$$\begin{aligned}\mathbf{a}x^2 + \mathbf{b}y^2 + \mathbf{c}z^2 &= -2\mathcal{H}_s \\ \gamma\beta x + \gamma\alpha y + \beta\alpha z &= K ,\end{aligned}\quad (61)$$

the second being easily obtained as $2K = \mathbf{W}_0^2 - (\gamma^2 + \beta^2 + \alpha^2)$. The motion in (x, y, z) is thus integrable, and limited to the interior of the berlingot³ shaped surface \mathcal{B} determined by $v^2(x, y, z) = 0$. We can also notice that

$$\begin{aligned}\dot{v} &= -\left(\frac{\mathbf{c}}{\beta}z - \frac{\mathbf{b}}{\gamma}y \right) (x - yz) \\ &\quad - \left(\frac{\mathbf{a}}{\gamma}x - \frac{\mathbf{c}}{\alpha}z \right) (y - xz) \\ &\quad - \left(\frac{\mathbf{b}}{\alpha}y - \frac{\mathbf{a}}{\beta}x \right) (z - xy) ,\end{aligned}\quad (62)$$

so \dot{v} is a function of only (x, y, z) .

3.3.1 Remark

It can be noticed that the line of initial condition with direction vector $(1/\alpha\mathbf{a}, 1/\beta\mathbf{b}, 1/\gamma\mathbf{c})$ is a line of fixed points for the differential system (59).

3.4 Integration

The motion in the (x, y, z) space evolves on elliptic curves, intersections of the ellipsoid of energy with the plane of angular momentum (61). Indeed, with a change of scale and a change of time, we can actually integrate this system. Indeed, let

$$X = \sqrt{\alpha}x ; \quad Y = \sqrt{\beta}y ; \quad Z = \sqrt{\mathbf{c}}z ; \quad (63)$$

³A berlingot is a famous tetrahedron hard candy with rounded edges.

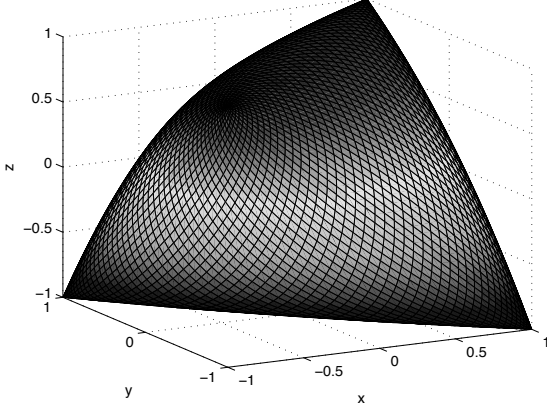


Figure 3: The surface $v^2(x, y, z) = 0$. As $v^2 \geq 0$, the allowed space is the interior of this berlingot shaped volume.

and

$$d\tau = v dt \quad (64)$$

The system (59) can then be written as

$$\frac{d}{d\tau} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{\Omega} \wedge \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (65)$$

with

$$\boldsymbol{\Omega} = \left(\frac{\sqrt{bc}}{\alpha}, \frac{\sqrt{ca}}{\beta}, \frac{\sqrt{ab}}{\gamma} \right). \quad (66)$$

We have thus reduced the problem to a simple rotation around the fixed vector $\boldsymbol{\Omega}$, with angular velocity $\omega = \|\boldsymbol{\Omega}\|$.

Remark. In the new variables, with $\mathbf{V} = (X, Y, Z)$, the integrals (61) are expressed as

$$\mathbf{V}^2 = -2\mathcal{H}_s; \quad \mathbf{V} \cdot \boldsymbol{\Omega} = \frac{\sqrt{abc}}{\gamma\beta\alpha} K. \quad (67)$$

The solution is expressed in terms of τ . If the volume $v = (\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ does not vanish, the relation with the usual time t is obtained through

$$t = \int_0^\tau \frac{d\tau}{v(\tau)}. \quad (68)$$

3.4.1 Remark

With $\mathbf{M} = (x, y, z)$, we have

$$\dot{v} = \frac{1}{2} \frac{d\mathbf{M}}{d\tau} \cdot \nabla_{\mathbf{M}} v^2. \quad (69)$$

As $d\mathbf{M}/d\tau$ is a tangent vector to the trajectory, one can see that, for a point on the berlingot \mathcal{B} (i.e. $v = 0$), $\dot{v} = 0$ is equivalent to the tangency of the trajectory with the berlingot \mathcal{B} .

3.4.2 Special solutions

It is easy to see that the sphere $B_{(0, \sqrt{3}/2)}$ centered on the origin, with radius $\sqrt{3}/2$, is included in the interior of the berlingot \mathcal{B} . From the expression of the integral of energy (61), one can deduce that for any initial condition (x, y, z) inside a sphere $B_{(0, \rho_0)}$, with

$$\rho_0 < \frac{\sqrt{3}}{2} \sqrt{\frac{\min(\mathbf{a}, \mathbf{b}, \mathbf{c})}{\mathbf{a} + \mathbf{b} + \mathbf{c}}}, \quad (70)$$

the motion will evolve on an ellipse in the (x, y, z) space, that remains included in $B_{(0, \sqrt{3}/2)}$. We have thus a lower bound for v^2 ($v^2 > v_0^2 > 0$, with $v_0 > 0$), and, with a positive orientation for our initial conditions $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$, the volume v is bounded from below ($v > v_0 > 0$). The time t is a monotonic function of τ and τ goes to infinity as t goes to infinity. The motion in the (X, Y, Z) space is a circle described uniformly with τ with period T_τ . In the (x, y, z) space, the motion will thus be on an ellipse with the same period T_τ . The motion with respect to time will still be periodic, but with a period T given by

$$T = \int_0^{T_\tau} \frac{d\tau}{v(\tau)}. \quad (71)$$

Indeed,

$$\begin{aligned} t + T &= \int_0^\tau \frac{d\tau}{v(\tau)} + \int_0^{T_\tau} \frac{d\tau}{v(\tau)} \\ &= \int_0^\tau \frac{d\tau}{v(\tau)} + \int_\tau^{\tau+T_\tau} \frac{d\tau}{v(\tau)} \\ &= \int_0^{\tau+T_\tau} \frac{d\tau}{v(\tau)} \end{aligned} \quad (72)$$

and, as for y, z , we have $x(t + T) = x(\tau + T_\tau) = x(\tau) = x(t)$. These solutions, with non vanishing volume will be called *special* solutions. Among them, we have the singular solution for which $x = y = z = 0$ at the origin. This solution is a fixed point in the (x, y, z) space, and we have for all time $v = 1$. In this solution, the three angular momentum vectors $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ remain orthogonal for all time, and all torques vanish.

One can also notice that for these *special* solutions, the average volume $v = (\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ is not zero. Indeed for the present choice of orientation,

$$\langle v \rangle = \frac{1}{T} \int_0^T v(t) dt = \frac{1}{T} \int_0^{T_\tau} d\tau = \frac{T_\tau}{T} > 0 \quad (73)$$

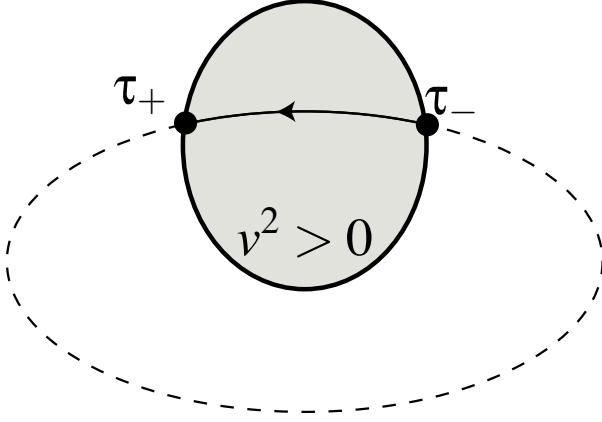


Figure 4: The shaded area correspond to the region where $v^2 > 0$, inside the berlingot \mathcal{B} . The orbit in τ intersects the berlingot \mathcal{B} in τ_+ and τ_- .

3.4.3 General solutions

In fact, in most cases of astronomical importance, the angular momentum vectors are far from orthogonal, and planar configuration will occur, with a cancellation of the volume v . We will call these solutions the *general* solutions. In such a solution, an orbit of the equations in τ , starting from inside \mathcal{B} , will intersect \mathcal{B} in positive time at τ_+ , and in negative time at τ_- (Fig.4). Starting at $\tau = 0$ (and $t = 0$) and with a positive volume v (by convention), the orbit in τ is a uniform rotation in (X, Y, Z) , until the orbit reaches \mathcal{B} at time t_+

$$t_+ = \int_0^{\tau_+} \frac{d\tau}{|v(\tau)|}. \quad (74)$$

As the volume $v = (\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ is a smooth function of t , if $\dot{v} \neq 0$, the volume becomes then negative, and

$$t = t_+ - \int_{\tau_+}^{\tau} \frac{d\tau}{|v(\tau)|}. \quad (75)$$

until the orbit bounces again on \mathcal{B} at τ_- at time

$$t_- = t_+ - \int_{\tau_+}^{\tau_-} \frac{d\tau}{|v(\tau)|}. \quad (76)$$

We will reach again the initial point at $t = T$ for $\tau = 0$ such that

$$T = t_- + \int_{\tau_-}^0 \frac{d\tau}{|v(\tau)|}; \quad (77)$$

The motion of (x, y, z) is periodic in t , with period T

$$T = 2 \int_{\tau_-}^{\tau_+} \frac{d\tau}{|v(\tau)|}. \quad (78)$$

An important question that arise is the convergence of the integral

$$t_+ = \int_0^{\tau_+} \frac{d\tau}{|v(\tau)|}. \quad (79)$$

In the vicinity of τ_+ , as $v(\tau_+) = 0$, we have

$$v^2(\tau) = 2v \frac{dv}{d\tau}(\xi)(\tau - \tau_+), \quad (80)$$

with $\xi \in]\tau, \tau_+[$, that is, as $d\tau = vdt$,

$$v^2(\tau) = 2 \frac{dv}{dt}(A)(\tau - \tau_+) \quad (81)$$

where $A(x, y, z)$ is a point in the vicinity of A_+ , different from A_+ (the intersection point of the orbit with \mathcal{B}). We have thus, in the vicinity of τ_+ ,

$$v^2(\tau) \approx 2\dot{v}(A_+)(\tau - \tau_+) \quad (82)$$

where $\dot{v}(A_+) \neq 0$. Thus $v(\tau) \approx \sqrt{2\dot{v}(A_+)(\tau - \tau_+)}$, and thus the above integral converges. The point A_+ is thus reached in finite time.

Remark. For a *general* solution, as after one period T , $\tau(t+T) = \tau(t)$, we have then

$$\langle v \rangle = \frac{1}{T} \int_0^T v(t)dt = \frac{\tau(T) - \tau(0)}{T} = 0 \quad (83)$$

and thus, the average volume $v = (\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ over one period T is zero.

3.4.4 Tangency case and Cassini states

We have not yet consider the tangency case, when the orbit in (x, y, z) becomes tangent to the berlingot \mathcal{B} . At the tangency point P_0 , we have

$$v = 0; \quad \dot{v} = 0. \quad (84)$$

P_0 is thus a fixed point. We will call these critical orbits 'Cassini states', (Colombo 1966, Peale, 1969, Ward, 1975) where the three vectors $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ remain in a plane that precess in time. On the other hand, an orbit starting with initial condition inside (strictly) \mathcal{B} will thus never reach the surface \mathcal{B} with $\dot{v} = 0$. Indeed, in the tangency case, we have $\dot{v}(A_+) = 0$, and thus

$$v^2(\tau) = O((\tau - \tau_+)^2) \quad (85)$$

and the integral is divergent. This is expected, as the tangency point is an equilibrium. It cannot be reached in finite time.

3.5 Relative solution

In the variables $\mathbf{V} = (X, Y, Z), \tau$, the motion is a simple rotation around $\boldsymbol{\Omega}$ (66) with angular velocity $\omega = \|\boldsymbol{\Omega}\|$. We have thus

$$\begin{aligned} \mathbf{V}(\tau) &= \frac{\mathbf{V}_0 \cdot \boldsymbol{\Omega}}{\omega^2} \boldsymbol{\Omega} + \\ &+ \left(\mathbf{V}_0 - \frac{\mathbf{V}_0 \cdot \boldsymbol{\Omega}}{\omega^2} \boldsymbol{\Omega} \right) \cos \omega\tau + \frac{\boldsymbol{\Omega} \wedge \mathbf{V}_0}{\omega} \sin \omega\tau \end{aligned} \quad (86)$$

from which we obtain easily (x, y, z) and v^2 . $x(\tau), y(\tau), z(\tau)$ are of degree 1 in $\cos(\omega\tau)$ and $\sin(\omega\tau)$. Thus $v^2(\tau)$ is a polynomial expression of total degree 3 in $\cos(\omega\tau), \sin(\omega\tau)$.

$$\begin{aligned} v^2(\tau) &= a_0 + a_1 \sin(\omega\tau) + a_2 \sin^2(\omega\tau) + a_3 \sin^3(\omega\tau) \\ &- \cos(\omega\tau)[b_0 + b_1 \sin(\omega\tau) + b_2 \sin^2(\omega\tau)] \end{aligned} \quad (87)$$

The solutions of $v^2(\tau) = 0$ are obtained by the resolution of the polynomial equation of degree 6 in $\phi = \sin(\omega\tau)$

$$(a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3)^2 = (1 - \phi^2)[b_0 + b_1\phi + b_2\phi^2]^2. \quad (88)$$

For each real solution ϕ_0 of (88) in the interval $[-1, +1]$, $\tau_0^+ = \arcsin(\phi_0)/\omega$ or $\tau_0^- = (\pi - \arcsin(\phi_0))/\omega$ will be a solution of $v^2(\tau) = 0$. In the non tangency case, τ_- will be the largest negative solution, while τ_+ is the smallest positive solution. The period T (and $\omega = 2\pi/T$) can then be computed through (78).

4 Global solution

We assume here that the vectors $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ are non planar ($v \neq 0$). Let \mathcal{W} be the matrix $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ and V the Gram matrix of the basis $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$

$$V = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}, \quad (89)$$

Using the expression of the vector product in the basis $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ (see Annex 2), one can transform the system (51) as

$$\dot{\mathcal{W}} = \mathcal{W}\mathcal{B} \quad (90)$$

where

$$\mathcal{B} = vV^{-1}\mathcal{A} \quad (91)$$

is a matrix depending only on (x, y, z) . Indeed

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{cz}{\beta} & -\frac{cz}{\alpha} \\ -\frac{by}{\gamma} & 0 & \frac{by}{\alpha} \\ \frac{ax}{\gamma} & -\frac{ax}{\beta} & 0 \end{pmatrix} \quad (92)$$

and

$$V^{-1} = \frac{1}{v^2} \begin{pmatrix} 1 - z^2 & yz - x & xz - y \\ yz - x & 1 - y^2 & xy - z \\ xz - y & xy - z & 1 - x^2 \end{pmatrix}. \quad (93)$$

As (x, y, z) are periodic functions of period T , the system (90) is a linear differential system with periodic coefficients of period T . If $\mathcal{W}(t)$ is a solution of (90), then $\mathcal{W}(t+T)$ is also a solution. Following Floquet theory, one can deduce that

$$\mathcal{R}_T(t) = \mathcal{W}(t+T)\mathcal{W}(t)^{-1} \quad (94)$$

is constant with t . Indeed we have then

$$\mathcal{W}(t+T) = \mathcal{R}_T(t)\mathcal{W}(t) \quad (95)$$

and

$$\begin{aligned} \dot{\mathcal{W}}(t+T) &= \mathcal{W}(t+T)\mathcal{B}(t) \\ &= \mathcal{R}_T(t)\mathcal{W}(t)\mathcal{B}(t) \\ &= \dot{\mathcal{R}}_T(t)\mathcal{W}(t) + \mathcal{R}_T(t)\dot{\mathcal{W}}(t) \\ &= \dot{\mathcal{R}}_T(t)\mathcal{W}(t) + \mathcal{R}_T(t)\mathcal{W}(t)\mathcal{B}(t). \end{aligned} \quad (96)$$

Thus $\dot{\mathcal{R}}_T(t) = 0$. \mathcal{R}_T is thus a constant matrix. As the Gram matrix V of the vectors $(\mathbf{w}(t), \mathbf{w}_1(t), \mathbf{w}_2(t))$ is conserved over one period T , the norm is conserved by \mathcal{R}_T (see Annex 2), and \mathcal{R} is an isometry of \mathbb{R}^3 . Moreover, this isometry is positive, as the volume v is conserved over a full period T (see section 3.4). The invariance of the total angular momentum \mathbf{W}_0 (52) also implies that the vector (γ, β, α) is invariant by \mathcal{R}_T . \mathcal{R}_T is thus a rotation matrix of axis \mathbf{W}_0 and angle θ_T . Let us denote $\mathcal{R}(t)$ the rotation of axis \mathbf{W}_0 and angle $t\theta_T/T$ (i.e. $\mathcal{R}(T) = \mathcal{R}_T$). Let

$$\tilde{\mathcal{W}}(t) = \mathcal{R}^{-1}(t)\mathcal{W}(t). \quad (97)$$

Proposition 1. $\tilde{\mathcal{W}}(t)$ is periodic with period T .

Indeed, as for all t, t' , $\mathcal{R}(t+t') = \mathcal{R}(t)\mathcal{R}(t')$,

$$\begin{aligned} \tilde{\mathcal{W}}(t+T) &= \mathcal{R}^{-1}(t+T)\mathcal{W}(t+T) \\ &= \mathcal{R}^{-1}(t)\mathcal{R}^{-1}(T)\mathcal{W}(t+T) \\ &= \mathcal{R}^{-1}(t)\mathcal{W}(t) \\ &= \tilde{\mathcal{W}}(t). \end{aligned} \quad (98)$$

The complete solution $\mathcal{W}(t)$ can thus be expressed on the form

$$\mathcal{W}(t) = \mathcal{R}(t)\tilde{\mathcal{W}}(t), \quad (99)$$

where $\tilde{\mathcal{W}}(t)$ is periodic with period T , and $\mathcal{R}(t)$ a rotation of axis \mathbf{W}_0 and angle $t\theta_T/T$. The motion has thus two periods : the (usually) short period T and the precession period

$$T' = \frac{2\pi}{\theta_T}T. \quad (100)$$

Remark. As $\mathcal{R}(t)$ is a rotation, we have $\det(\tilde{\mathcal{W}}(t)) = \det(\mathcal{W}(t))$, and thus, from (83), for a general solution,

$$\int_0^T \det(\tilde{\mathcal{W}}(t)) dt = 0. \quad (101)$$

The three unit vectors $\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2$, defined as the column vectors of $\tilde{\mathcal{W}}(t)$ have thus an averaged volume equal to zero over one period T . As these vectors describe loops of period T , this result can be interpreted by stating that the origin and center of the three loops generated by $\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1$, and $\tilde{\mathbf{w}}_2$ are nearly coplanar. We have indeed demonstrated that the averaged value $\langle (\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2) \rangle$ is null, and not that the determinant of the averaged vectors $\langle (\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2) \rangle$ is null, which is the condition stating that the *centers* of the loops generated by $\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2$ are coplanar with the origin. We will see in section 4.3 that this is indeed the case.

4.1 Complete solution

Once the form of the general solution (99) is obtained, we have an elementary way to compute the precession period T' . Indeed, one can first compute the short period T through the quadratures of section 3.4, and then numerically integrate the full system of equation (51) over a full period T . The angle θ_T is then the angle of the rotation

$$\mathcal{R}(T) = \mathcal{W}(t+T)\mathcal{W}(t)^{-1} \quad (102)$$

and the precession period is obtained through (100). As the solution $\mathcal{W}(t)$ is known over the interval of length T , $[t_0, t_0 + T]$, the T -periodic function $\tilde{\mathcal{W}}(t)$ is also known over the full period $[t_0, t_0 + T]$, and the solution for all time is obtained through (99), as for all integer n ,

$$\mathcal{W}(t+nT) = \mathcal{R}(t+nT)\tilde{\mathcal{W}}(t). \quad (103)$$

4.2 Computation of the precession period by quadratures

In the previous section, we have seen that the precession period can be obtained by numerical integration of the full equations (51), but we will derive also here some formulas for the direct computation of the precession period. Let $W_0 = \|\mathbf{W}_0\|$ and $\mathbf{w}_0 = \mathbf{W}_0/W_0$ be a unit vector along the total angular momentum \mathbf{W}_0 . With

$$p = \mathbf{w} \cdot \mathbf{w}_0, \quad (104)$$

the projection \mathbf{S} of \mathbf{w} on the plane orthogonal to \mathbf{w}_0 will be

$$\mathbf{S} = \mathbf{w} - p\mathbf{w}_0. \quad (105)$$

Assuming $\mathbf{w} \neq \mathbf{w}_0$, we will have $p < 1$. With $S = \|\mathbf{S}\|$, and $\mathbf{s} = \mathbf{S}/S$, we have

$$\mathbf{S} = S\mathbf{s}; \quad \dot{\mathbf{S}} = \dot{S}\mathbf{s} + \dot{\theta}(\mathbf{w}_0 \wedge \mathbf{S}) \quad (106)$$

and

$$\begin{aligned} \dot{\mathbf{S}} &= \dot{\mathbf{w}} - \dot{p}\mathbf{w}_0, \\ \dot{\mathbf{S}}^2 &= \dot{\mathbf{w}}^2 - \dot{p}^2; \end{aligned} \quad (107)$$

we have

$$\mathbf{S} \cdot \dot{\mathbf{S}} = S\dot{S} = -p\dot{p} \quad (108)$$

and

$$S = \sqrt{1-p^2}; \quad \dot{S} = -\frac{p\dot{p}}{\sqrt{1-p^2}}. \quad (109)$$

As

$$\begin{aligned} \dot{\mathbf{S}}^2 &= \dot{S}^2 + \dot{\theta}^2(\mathbf{w}_0 \wedge \mathbf{S})^2 \\ &= \dot{S}^2 + \dot{\theta}^2(1-p^2) \end{aligned} \quad (110)$$

we have

$$\dot{\theta}^2 = \frac{\dot{\mathbf{w}}^2 - \dot{p}^2/(1-p^2)}{1-p^2}. \quad (111)$$

With (52), one has

$$p = \frac{1}{W_0}(\gamma + \beta x + \alpha y). \quad (112)$$

and thus

$$\dot{p} = \frac{v}{W_0} \left(\frac{\alpha\mathbf{a}}{\gamma}x - \frac{\beta\mathbf{b}}{\gamma}y \right) \quad (113)$$

We have also from (51)

$$\dot{\mathbf{w}}^2 = \frac{1}{\gamma^2} (\mathbf{a}^2x^2 + \mathbf{b}^2y^2 + 2\mathbf{a}\mathbf{b}xyz - (\mathbf{a}x^2 + \mathbf{b}y^2)^2) \quad (114)$$

With these expressions, equation (111) can be written on the form

$$\dot{\theta}^2 = \Theta(x, y, z). \quad (115)$$

The sign of $\dot{\theta}$ ($\varepsilon_{\dot{\theta}}$) can be determined through (106). Indeed $\dot{\theta}$ is a function of $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$, but its sign can only change when $\dot{\theta} = 0$, that is from (111), when

$$\dot{\mathbf{w}}^2(1-p^2) = \dot{p}^2. \quad (116)$$

This equation is a polynomial equation in (x, y, z) , of valuation 2 and total degree 6 in (x, y, z) , and total degree 2 in z . It determines an algebraic surface \mathcal{S} of the (x, y, z) space and thus θ is obtained by quadrature

$$\theta(t) - \theta(0) = \int_0^t \varepsilon_{\dot{\theta}} \sqrt{\Theta(x, y, z)} dt. \quad (117)$$

The computation of θ_T is obtained by the integration of the above expression over a full period T . As in the discussion of section 3.4, one has to be careful for the change of signs of $\dot{\theta}$.

4.3 Symmetry in the loop

It is now possible to prove a more precise result on the periodic loops generated by $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ in the precessing frame.

Proposition 2. In the frame rotating uniformly with the precession period, the three vectors $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ describe periodic loops $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ that are all symmetric with respect to the same plane \mathcal{S} containing \mathbf{w}_0 .

Consequence. Let us call $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ the averages of $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ over the nutation angle. $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ are respectively the poles of the spin axis, the pole of the planet orbit, and the pole of satellite orbit. Due to the symmetry of the orbits, the three poles $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ remain in the symmetry plane \mathcal{S} containing \mathbf{w}_0 , and precessing uniformly around \mathbf{w}_0 . Each vector $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ nutates around its pole, respectively $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$.

Proof. We will consider uniquely \mathbf{w} , the other cases being similar. We consider here a general solution (section 3.4.3). We choose here the origin of time in τ_+ which corresponds to a spin vector \mathbf{w}_+ . At $t = -T/2$, the orbit in the (x, y, z) space is in τ_- , corresponding to $\mathbf{w} = \mathbf{w}_-$. In the (x, y, z) space, the orbit describes an elliptic arc (τ_-, τ_+) over the time interval $[-T/2, 0]$, and the same arc in the reverse way (τ_+, τ_-) over $[0, T/2]$ (Fig. 4). Moreover, as the motion is a pure rotation in the scaled (X, Y, Z) coordinates (section 3.4), over the interval $[-T/2, T/2]$, the orbit of $\mathbf{M} = (x, y, z)$ is even, that is $\mathbf{M}(-t) = \mathbf{M}(t)$.

Next, we can remark that as the differential system (51) is polynomial, the solutions $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ are analytical in time t , and so will be the coordinate angle $\theta(t)$ of \mathbf{w} . On the other hand, we have the following lemma

Lemma. Let $f(t)$ be an analytic function over an interval $[-A, A]$, such that $f^2 = g$, where $g(t)$ is even over $[-A, A]$ ($A > 0$). Then $f(t)$ is either odd or even. If $f(0) \neq 0$, $f(t)$ is even.

The proof is easily obtained with analytic continuation. Moreover, we have (115)

$$\dot{\theta}^2(t) = \Theta(x, y, z), \quad (118)$$

thus $\dot{\theta}(t)$ is odd or even. If $\dot{\theta}(t)$ is even on $[-T/2, T/2]$, for all $h \in [0, T/2]$, we have thus $\theta(h) - \theta(0) = \theta(0) - \theta(-h)$. As the cosine p of the angle from \mathbf{w} to \mathbf{w}_0 (104) depends only on x, y (112), we have $p(h) = p(-h)$, and $\mathbf{w}(h)$ and $\mathbf{w}(-h)$ are symmetrical with respect to the $(\mathbf{w}_0, \mathbf{w}_+)$ plane. It will still be the same in the rotating frame with the precession period. In this rotating frame, the periodic loop generated by \mathbf{w} is thus symmetric with respect to the plane $(\mathbf{w}_0, \mathbf{w}_+)$.

Moreover, at $t = 0$ (τ_+), the volume v is null, and thus $\mathbf{w}_0, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ are coplanar. In the rotating frame, all

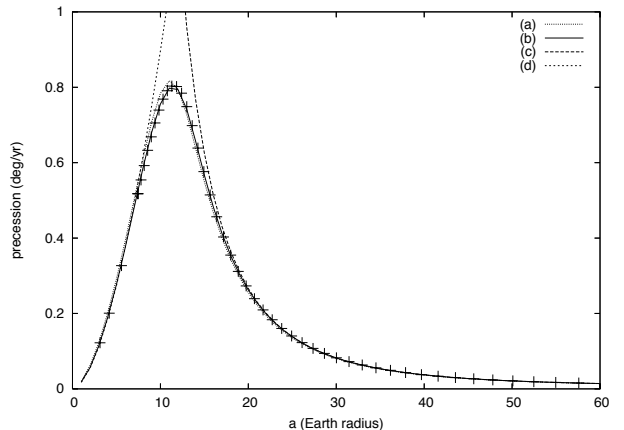


Figure 5: Different precession frequency computations. (c) is the classical computation in the approximation of a far satellite, while (d) correspond to the close satellite approximation (Tremaine, 1991). These formulas are no longer valid for a close (resp. far) satellite, or in the intermediary region (between 10 and 20 R_E for the Earth-Moon System). (a) and (b) are computed with the formula (129) of the present paper, using either the raw initial conditions of the integration (a), or the averaging obtained after a first iteration (b) (Eq.135). The crosses correspond to numerical experiments with the complete Sun-Earth-Moon problem, without averaging.

three orbits generated by $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ are thus symmetrical with respect to the same plane $(\mathbf{w}_0, \mathbf{w}_+)$.

The only case when $\dot{\theta}(t)$ is odd, occurs when $\dot{\theta}(0) = 0$. As $v(0) = 0$, we have $\dot{p}(0) = 0$ (113) and $\dot{\mathbf{w}}(0) = 0$ (111). In the same way, we will have $\dot{\mathbf{w}}_1(0) = \dot{\mathbf{w}}_2(0) = 0$, and the vector field (51) vanished at $t = 0$. The three vectors $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ are thus stationary and coplanar.

This is a special Cassini state (section 3.4.4) where the precession frequency is zero.

5 Description of the solutions

In order to better visualize the solutions, we have plotted in Figure 6 the projections of the three vector $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ in the plane (\mathbf{i}, \mathbf{j}) . More precisely, as we know the general form of the solution (Eq.99), we have plotted this projection in a framework in rotation with the computed precession period T' . According to (Eq.99), we thus obtain the projection of $\dot{\mathcal{W}}(t)$ in the fixed reference plane (\mathbf{i}, \mathbf{j}) . We thus expect to obtain for each vector $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$, a periodic smooth curve. In all our examples, the curves described by $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ are in fact very close to circles. It should be noted that in Figure 6, we have plotted the output of the non averaged equations (Eq.31). This is intentional as this allows to check at the same time the relevance of the averaging made in section 2.5. This explains why instead of exactly thin loops, we have thick

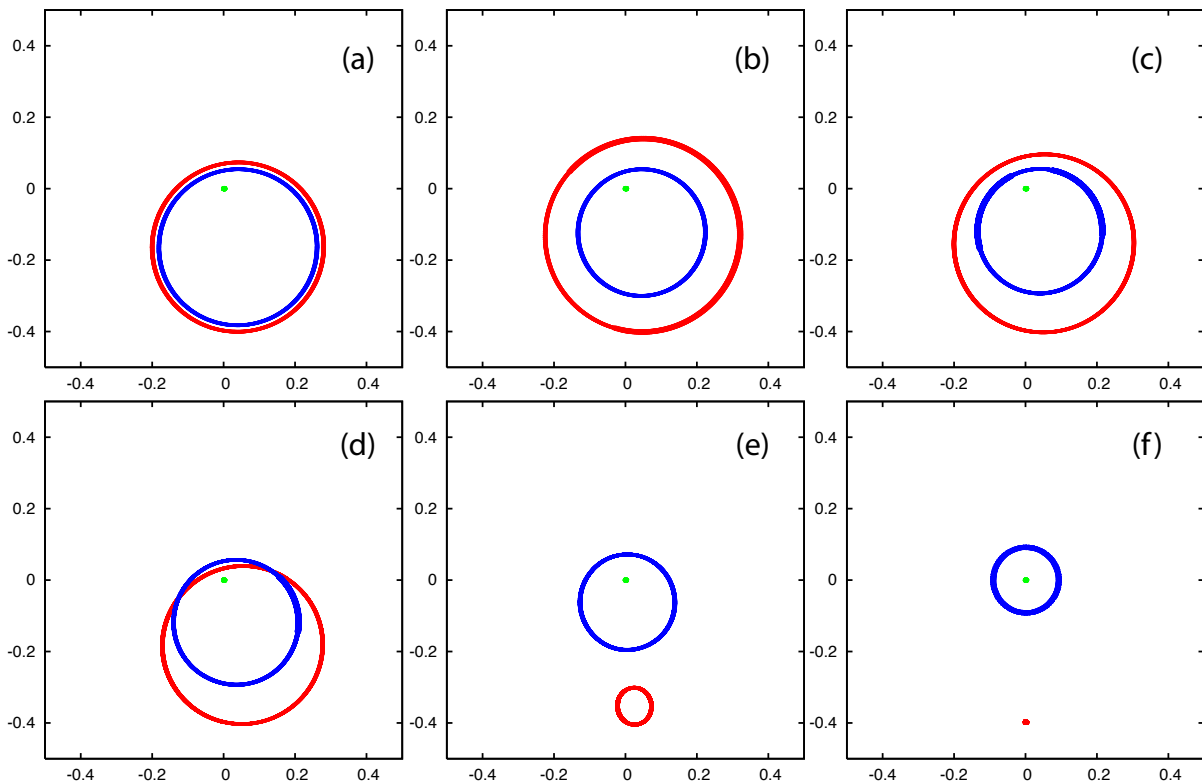


Figure 6: Precession for different Earth-Moon distances, starting with $3 R_E$ (a); $7 R_E$ (b); $9 R_E$ (c); $10 R_E$ (d); $14 R_E$ (e); $60.1 R_E$ (f); The last one corresponding to the actual Earth-Moon distance (expressed in Earth radius). The projection of the poles \mathbf{w} (red), \mathbf{w}_1 (green), \mathbf{w}_2 (blue) are plotted in the (i, j) plane in a rotating frame with the precession frequency Ω . Scales are in radians. The pole of the planet orbit (\mathbf{w}_1 , in green) almost coincide with the origin, while the pole of the planet \mathbf{w} and the pole of the satellite orbit \mathbf{w}_2 describe a large variety of configurations, smoothly evolving from a configuration where the axis of the planet and the pole of the satellite are concentric (a), to the present configuration (f).

lines, which occurs for the contribution of the orbital short period terms.

In these examples, we have taken a fictitious Moon around the Earth. The Earth-Moon distance is then varied from a very close position (up to 2 Earth radius) to 100 Earth radius, close to the distance where the Moon is no longer a satellite of the Earth.

In all cases, the pole of the orbit of the planet \mathbf{w}_1 is very close to the origin, as \mathbf{w}_1 is very close to the constant angular momentum vector \mathbf{w}_0 . The vectors \mathbf{w} and \mathbf{w}_2 describe circles with varying center position and radius.

5.1 Far solutions

When the Moon is far from the Earth (as it is at present), in the precessing rotating frame, \mathbf{w} is nearly fixed and \mathbf{w}_2 is circulating around the pole of the orbit, that is around the origin (Fig. 6.f). In this case, the obliquity of the planet and the inclination of the Moon on the ecliptic

are nearly constant. The orbit of the Moon is precessing uniformly around the pole of the ecliptic with the fast period T (18.6 years for the present Moon). In fact there is still a small variation of the obliquity, with the same period T , but with a very small amplitude (about 9 arcsec for the present Moon). This motion is the principal term in the nutation that was discovered observationally by Bradley in 1748 and computed by d'Alembert in 1749 (see the introduction of M. Chapront and J Souchay (2006) of d'Alembert complete work for a detailed account of these discoveries).

5.2 Intermediate solutions

When the Moon is closer (Fig.6.e), the amplitude of the nutation of the Earth axis becomes much larger. The precession of the plane of the orbit of the Moon is no longer centered around the pole of the ecliptic, and the inclination of the Moon with respect to the ecliptic is

not constant. But as the pole of the Moon orbit still describes a circle, the inclination remains constant with respect to a plane orthogonal to the center of this circle. This plane is often called the Laplace plane of the satellite. It should be noted that the pole of the Laplace plane with this definition will precess around the total angular momentum with the slow precession period.

5.3 Close solutions

When the Moon is very close to the planet (6.a,b), The satellite precession and the planet nutation are both roughly around the same pole, and the inclination of the satellite on the planet equator is nearly constant (Fig 7 (c3)).

5.4 General case

In the general case (Fig.6), the planet nutation occurs around the nutation pole, the satellite orbit precesses around the Laplacian pole that is different from the nutation pole and from the ecliptic pole, but all three poles precess slowly with the same frequency around the total angular momentum.

In the solution of the averaged equations (51), for any distance of the satellite from the planet, the general motion of the planet orbit, of the satellite orbit, and of the planet rotation can be described as follows (with the notations of section 4.3) :

The planet orbit precesses around a planetary pole \mathcal{P}_1 (usually with very small amplitude). The satellite orbit precesses around a satellite pole \mathcal{P}_2 (that can be called the Laplacian pole (see Burns, 1986)). The axis of rotation of the planet nutates around the rotational pole \mathcal{P} , all with the same period T , usually called the period of precession of the satellite, but here we will reserve the name precession for the long period, and we will call this short period the nutation period. This motion is periodic (each axis described a closed loop with period T). In addition, all three poles $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ precess uniformly with the same period (that we will call the precession period) around the total angular momentum of the system \mathcal{W}_0 .

6 Analytical approximation

In section 4, we have obtained the complete solution of the averaged equations (51), but although these solutions can be computed by quadrature, they are not explicit. Nevertheless, the rigorous expression (99) allows us to give a general description of the solutions, valid in all cases. In the present section, we will make some additional approximations in order to provide an explicit form of the

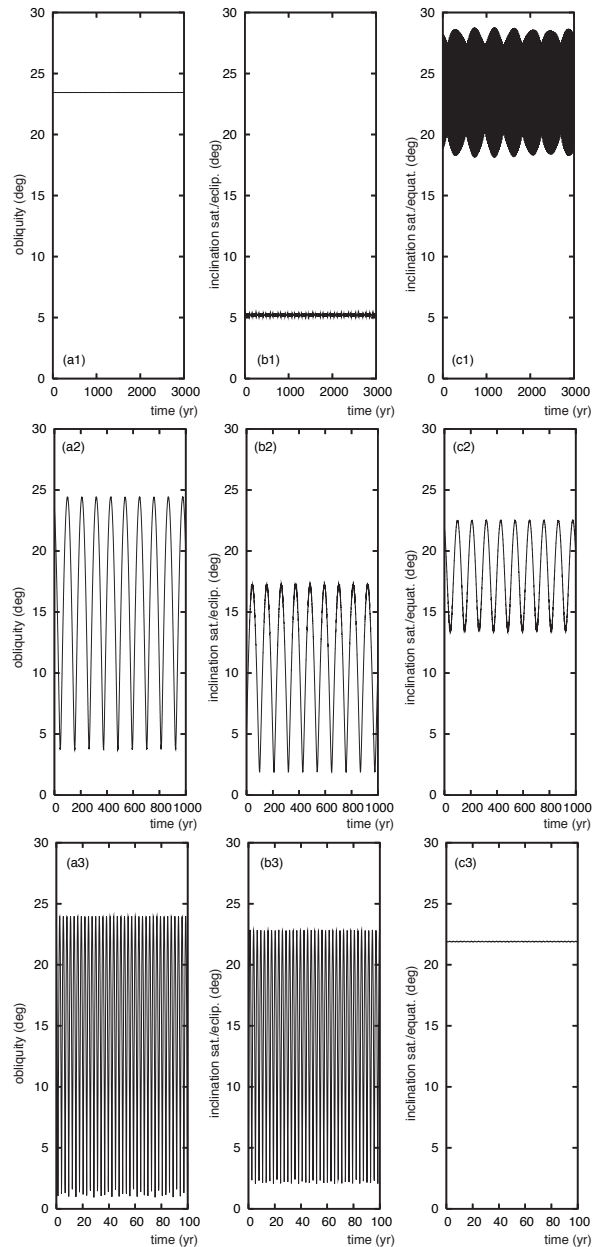


Figure 7: Evolution of the obliquity of the Earth (a_1, a_2, a_3), inclination of the orbit of the Moon with respect to the ecliptic (b_1, b_2, b_3), and with respect to the equator (c_1, c_2, c_3), for different Earth-Moon semi-major axis. The solution (1) (a_1, b_1, c_1) correspond to the actual Earth-Moon system (Fig.6f), while the Earth-Moon distance is about $10R_E$ (Fig.6d) in (a_2, b_2, c_2), and about $4R_E$ in (a_3, b_3, c_3). This latter case is similar to (Fig.6a).

solutions. More precisely, as we realize that the periodic loops generated by $\hat{\mathcal{W}}(t)$ in (99) are very close to circular uniform motion, we will search for approximate solutions expressed on the form of a composition of periodic terms.

For simplicity, we will assume here that x, y, z are positive. Other cases, as for Neptune – Triton ($y < 0, z < 0$), can be treated in the same way.

6.1 Equations

As the angular momentum of the system is essentially contained in the orbital motion of the planet, we will consider that

$$\mathbf{w}_1 \approx \mathbf{w}_0. \quad (119)$$

With a fixed reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ with $\mathbf{k} = \mathbf{w}_0$, we will thus have

$$\mathbf{w}_1 \approx \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (120)$$

and as $x = \mathbf{w} \cdot \mathbf{w}_1, z = \mathbf{w}_1 \cdot \mathbf{w}_2$, let the coordinates of \mathbf{w} and \mathbf{w}_2 in this basis be

$$\mathbf{w} = \begin{pmatrix} \xi \\ \eta \\ x \end{pmatrix}; \quad \mathbf{w}_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \\ z \end{pmatrix}; \quad (121)$$

As we are considering now the projections of \mathbf{w}, \mathbf{w}_2 in the (\mathbf{i}, \mathbf{j}) plane, we will use complex coordinates in this plane. Let

$$\mathfrak{z} = \xi + i\eta; \quad \mathfrak{z}_2 = \xi_2 + i\eta_2; \quad (122)$$

with these notations, the secular equations (51) become

$$\frac{d}{dt} \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}_2 \end{pmatrix} = iM \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}_2 \end{pmatrix} \quad (123)$$

with

$$M(x, y, z) = \begin{pmatrix} -\frac{\mathbf{a}x}{\gamma} - \frac{\mathbf{b}yz}{\gamma} & \frac{\mathbf{b}xy}{\gamma} \\ \frac{\mathbf{b}yz}{\alpha} & -\frac{\mathbf{c}z}{\alpha} - \frac{\mathbf{b}xy}{\alpha} \end{pmatrix}. \quad (124)$$

M is thus a real matrix with periodic coefficients of period T . As we are not searching for the exact solution of the problem, we will make here a crude approximation by averaging this matrix over the fast period T . We will even replace the three varying quantities x, y, z by some averaged quantities $\tilde{x}, \tilde{y}, \tilde{z}$. The matrix M is then transformed into a real matrix \tilde{M} with constant coefficients

$$\tilde{M} = M(\tilde{x}, \tilde{y}, \tilde{z}), \quad (125)$$

and the solution of (123) becomes straightforward. Let T and D be the trace and determinant of \tilde{M} . The eigenvalues of \tilde{M} are given by the second degree equation

$$\lambda^2 - \lambda T + D = 0 \quad (126)$$

with, when $\tilde{x}, \tilde{y}, \tilde{z}$ are positive (the other cases can be treated in the same way),

$$T = - \left(\frac{\mathbf{a}\tilde{x}}{\gamma} + \frac{\mathbf{b}\tilde{y}\tilde{z}}{\gamma} + \frac{\mathbf{c}\tilde{z}}{\alpha} + \frac{\mathbf{b}\tilde{x}\tilde{y}}{\alpha} \right) < 0 \quad (127)$$

and discriminant

$$\Delta = T^2 - 4D = \left(\frac{\mathbf{a}\tilde{x}}{\gamma} + \frac{\mathbf{b}\tilde{y}\tilde{z}}{\gamma} - \frac{\mathbf{c}\tilde{z}}{\alpha} - \frac{\mathbf{b}\tilde{x}\tilde{y}}{\alpha} \right)^2 + 4 \frac{\mathbf{b}^2 \tilde{x} \tilde{y}^2 \tilde{z}}{\gamma \alpha} > 0. \quad (128)$$

We have thus always two distinct eigenvalues Ω , and $\Omega + \nu$ (we consider here that Ω is the slow precession frequency and ν , the nutation frequency), with

$$\Omega = \frac{T + \sqrt{\Delta}}{2}; \quad \nu = -\sqrt{\Delta}. \quad (129)$$

After diagonalization, we obtain two eigenmodes

$$\mathfrak{z} = \mathbf{r}e^{i(\Omega t + \Phi)}; \quad \mathfrak{z}' = \mathbf{s}e^{i((\Omega + \nu)t + \Phi + \phi)}; \quad (130)$$

where $\mathbf{r}, \mathbf{s}, \Phi, \phi$ are real numbers. A basis of eigenvector is then $(\mathbf{e}_1, \mathbf{e}_2)$, with

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}; \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ \lambda' \end{pmatrix}; \quad (131)$$

and

$$\lambda = \frac{\mathbf{a}\tilde{x} + \mathbf{b}\tilde{y}\tilde{z} + \gamma\Omega}{\mathbf{b}\tilde{x}\tilde{y}}; \quad \lambda' = \frac{\mathbf{a}\tilde{x} + \mathbf{b}\tilde{y}\tilde{z} + \gamma(\Omega + \nu)}{\mathbf{b}\tilde{x}\tilde{y}}. \quad (132)$$

The solution in $\mathfrak{z}, \mathfrak{z}_2$ becomes

$$\begin{aligned} \mathfrak{z} &= e^{i(\Omega t + \Phi)} (\mathbf{r} + \mathbf{s}e^{i(\nu t + \phi)}) \\ \mathfrak{z}_2 &= e^{i(\Omega t + \Phi)} (\lambda \mathbf{r} + \lambda' \mathbf{s}e^{i(\nu t + \phi)}) \end{aligned} \quad (133)$$

where $\mathbf{r}, \mathbf{s}, \lambda, \lambda'$ are real numbers. Moreover, it is easy to show that $\lambda > 0, \lambda' < 0$. This results from the diagonalization of a general 2×2 matrix (M_{ij}) with real coefficients and positive product of the antidiagonal terms ($M_{12}M_{21} > 0$). We should notice here that \mathfrak{z} and \mathfrak{z}_2 have the same phase Φ in the precession motion, and opposite phase ϕ and $\phi + \pi$ for the nutation motion. We obtain here thus an additional general result :

Proposition 3. Within the present approximations (119, 125), the pole of precession of the axis and the pole of precession of the satellite orbit (the Laplace pole of the satellite) are always aligned with the total angular momentum, and on the same side of the total angular momentum.

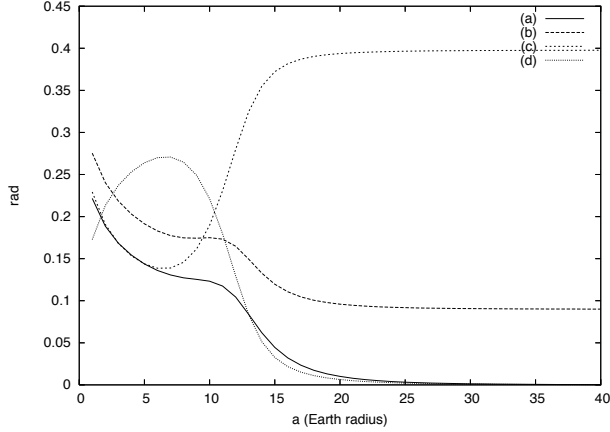


Figure 8: The approximate analytical solutions for the evolution of the planet axis and satellite orbit are expressed with only two periodic terms (133). The values of the different involved radius are given here in term of the Earth-Moon distance (in Earth radius) with the correspondence : $\lambda \mathbf{r}$ (a); $-\lambda' \mathbf{s}$ (b); \mathbf{r} (c); \mathbf{s} (d).

6.2 Parameters of the solution

As we know the general form of the solution (133), we can compute now the averaged quantities $\tilde{x}, \tilde{y}, \tilde{z}$. From the definition of $\mathfrak{z}, \mathfrak{z}_2$, we have

$$\begin{aligned} x &= \sqrt{1 - |\mathfrak{z}|^2} \\ z &= \sqrt{1 - |\mathfrak{z}_2|^2} \\ y &= \frac{1}{2}(\mathfrak{z}\bar{\mathfrak{z}}_2 + \bar{\mathfrak{z}}\mathfrak{z}_2) + xz \end{aligned} \quad (134)$$

As it would be unnecessary complicated to obtain explicit averaged values over the fast period ν in the complete expression of the frequencies (129), and in the sake of simplicity, here we will average under the radical (that is average x^2, z^2 instead of x, z) over the fast frequency ν . One thus obtain

$$\begin{aligned} \tilde{x} &= \sqrt{1 - \mathbf{r}^2 - \mathbf{s}^2} \\ \tilde{z} &= \sqrt{1 - \lambda^2 \mathbf{r}^2 - \lambda'^2 \mathbf{s}^2} \\ \tilde{y} &= \lambda \mathbf{r}^2 + \lambda' \mathbf{s}^2 + \tilde{x} \tilde{z} \end{aligned} \quad (135)$$

6.3 Initial conditions

With equations (129, 132, 135), the solutions (133) depend only on the four real numbers $\mathbf{r}, \mathbf{s}, \Phi, \phi$. At the origin of time ($t = 0$), we have

$$\begin{aligned} \mathfrak{z}_0 &= e^{i\Phi}(\mathbf{r} + \mathbf{s}e^{i\phi}) \\ \mathfrak{z}_{20} &= e^{i\Phi}(\lambda \mathbf{r} + \lambda' \mathbf{s}e^{i\phi}) \end{aligned} \quad (136)$$

| planet | Ω_0 ("/yr) |
|---------|-------------------|
| Earth | -15.948799 |
| Mars | -7.581155 |
| Jupiter | -0.908216 |
| Saturn | -0.189667 |
| Uranus | 0.001102 |
| Neptune | -0.001652 |

Table 1: Precession rate for a single planet with no satellite given by equation (56).

This system is solved easily as

$$\begin{aligned} \mathbf{r}e^{i\Phi} &= \frac{\lambda' \mathfrak{z}_0 - \mathfrak{z}_{20}}{\lambda' - \lambda} \\ \mathbf{s}e^{i(\Phi+\phi)} &= \frac{\lambda \mathfrak{z}_0 - \mathfrak{z}_{20}}{\lambda - \lambda'} \end{aligned} \quad (137)$$

The computation of λ, λ' requires to know the averaged values $\tilde{x}, \tilde{y}, \tilde{z}$, but can easily be done by iteration, starting with the initial values, that is, for the first iteration

$$\tilde{x} = x(t=0); \quad \tilde{y} = y(t=0); \quad \tilde{z} = z(t=0). \quad (138)$$

In all our computations, a single iteration after this first try with the initial conditions was sufficient.

6.4 Numerical applications

The computation of the precession frequency in the Earth-Moon system was provided in figure 5. With the analytical approximation (128), we have also computed the evolution of the radius and location of the precession circles of Fig.6 with respect to the Earth-Moon distance (Fig.8). It should be stressed that this computation is made with a fictitious Moon with initial obliquity and precession equal to the present one. In particular, we have not attempted here to follow a realistic evolution of the Earth-Moon system under tidal evolution as in (Goldreich, 1966, Touma and Wisdom, 1994b).

We have applied the computation of the precession motion for a variety of examples in the Solar System (Table 4, 2). In each case, only the system Sun-Planet-Satellite is taken into account, without trying to take into account mutual perturbations, or accumulated effects of multiple satellites. These examples are used to compare the results of the numerical integration of the averaged equations (51) (Table 2) to the results obtained using either the exact solution of section 4, the quadrature formulas of section 4.2, or the approximate solutions computed with the explicit formulas of section 6.

| satellite | $(\Omega_c - \Omega_0)/\Omega_0$ | ν_c (deg/yr) | $(\Omega - \Omega_0)/\Omega_0$ | ν (deg/yr) | $\mathcal{A}(\tau)$ (deg) | $\mathcal{A}(\mathfrak{s})$ (") | $\mathcal{A}(\lambda\tau)$ (deg) | $\mathcal{A}(\lambda'\mathfrak{s})$ (deg) |
|-----------|----------------------------------|---------------------|--------------------------------|-------------------|------------------------------|------------------------------------|-------------------------------------|--|
| Moon | 2.18 | -20.128 | 2.18 | -20.128 | 23.438 | 7.447 | 2.25×10^{-3} | 5.158 |
| Phobos | 3.17×10^{-5} | -158.851 | 3.16×10^{-5} | -158.851 | 25.191 | 3.24×10^{-3} | 25.179 | 0.798 |
| Deimos | 1.98×10^{-4} | -6.599 | 1.98×10^{-4} | -6.599 | 25.191 | 1.02×10^{-2} | 24.290 | 1.645 |
| Io | 5.43×10^{-2} | -47.372 | 5.42×10^{-2} | -47.373 | 3.128 | 0.198 | 3.127 | 0.036 |
| Europa | 7.45×10^{-2} | -9.301 | 7.45×10^{-2} | -9.301 | 3.127 | 1.697 | 3.121 | 0.462 |
| Ganymede | 5.77×10^{-1} | -1.854 | 5.77×10^{-1} | -1.854 | 3.127 | 2.653 | 3.064 | 0.188 |
| Callisto | 9.87×10^{-1} | -0.338 | 9.87×10^{-1} | -0.338 | 3.129 | 6.312 | 2.320 | 0.615 |
| Mimas | 1.78×10^{-5} | -360.483 | 1.78×10^{-5} | -360.483 | 26.728 | 7.76×10^{-3} | 26.717 | 1.539 |
| Enceladus | 5.83×10^{-5} | -150.642 | 5.82×10^{-5} | -150.642 | 26.728 | 1.64×10^{-4} | 26.728 | 0.015 |
| Tethys | 7.71×10^{-4} | -71.349 | 7.71×10^{-4} | -71.349 | 26.728 | 1.09×10^{-1} | 26.723 | 1.035 |
| Dione | 2.17×10^{-3} | -30.014 | 2.17×10^{-3} | -30.013 | 26.728 | 3.34×10^{-3} | 26.727 | 0.017 |
| Rhea | 9.36×10^{-3} | -9.328 | 9.36×10^{-3} | -9.328 | 26.728 | 0.164 | 26.719 | 0.315 |
| Titan | 2.88 | -0.508 | 2.88 | -0.508 | 26.732 | 12.512 | 26.113 | 0.276 |
| Iapetus | 4.31×10^{-2} | -0.075 | 4.31×10^{-2} | -0.075 | 26.729 | 1.286 | 3.481 | 12.051 |
| Miranda | 2.72×10^{-3} | 18.967 | 2.72×10^{-3} | 18.967 | 82.147 | 0.365 | 81.082 | 2.408 |
| Ariel | 1.23×10^{-1} | 4.908 | 1.23×10^{-1} | 4.908 | 82.147 | 0.162 | 82.146 | 0.043 |
| Umbriel | 2.07×10^{-1} | 1.545 | 2.07×10^{-1} | 1.545 | 82.147 | 0.345 | 82.143 | 0.089 |
| Titania | 1.67 | 0.275 | 1.67 | 0.275 | 82.147 | 1.269 | 82.120 | 0.085 |
| Oberon | 2.59 | 0.099 | 2.59 | 0.099 | 82.147 | 0.992 | 82.035 | 0.067 |
| Triton | 3.60 | 0.487 | 3.60 | 0.487 | 28.912 | 2016.063 | 24.273 | 29.561 |

Table 2: Numerical solution for various satellites. In each case, the system Sun-Planet-Satellite is considered. Ω_0 is the planet precession rate in absence of satellite. Ω_c and ν_c are the precession and nutation frequencies computed by quadrature, while Ω and ν are the same quantities obtained numerically using frequency analysis (Laskar, 1990, 2005). $\mathcal{A}(\tau)$ and $\mathcal{A}(\mathfrak{s})$ are the precession and nutation amplitude for the axis of the planet, $\mathcal{A}(\lambda\tau)$ and $\mathcal{A}(\lambda'\mathfrak{s})$ are the same quantities for the satellite orbit. These quantities are obtained numerically through frequency analysis. $\mathcal{A}(\lambda\tau)$ is thus the inclination of the Laplace pole of the satellite with respect to the pole of the orbit of the planet.

The numerical integrations are performed only over a few (about 20) nutation periods. The precession frequency is then determined by iteration with great accuracy, searching for a uniform rotating frame where the motion is periodic (see section 4). The nutation frequency and amplitudes $\mathcal{A}(\tau)$, $\mathcal{A}(\mathfrak{s})$, $\mathcal{A}(\lambda\tau)$, $\mathcal{A}(\lambda'\mathfrak{s})$ are then determined using frequency analysis (Laskar, 1990, 2005). The results are displayed in Table 2 together with the frequencies obtained by quadrature. We can verify that the quadrature formulas (section 4.2) give virtually identical results as the numerical integration.

In table 4 are displayed the results obtained with our analytical approximate formulas (129). It can be seen that these explicit formulas provide in a simple way both the frequencies and amplitude of the terms in most situations. We have not attempted (although it should be possible to do it following the lines of section 6) to derive approximate formulas for Uranus satellites, when the obliquity of the planet is very large, and thus the projection on the plane of the orbit questionable.

Remark. It should be noted that although in our approximate formulas (129) the solutions are given with a single periodic term, the nutation motion in the rotating frame with precession frequency is not exactly a pure ro-

| ν_i | a_i |
|-----------|---------------------|
| 0.00000 | 0 0.397753940 |
| -20.12804 | ν 0.000036105 |
| 20.12804 | $-\nu$ 0.000003746 |
| -40.25608 | -2ν 0.000000390 |

Table 3: Quasiperiodic decomposition of the motion of the projection of the Earth spin axis (\mathfrak{j}) in the orthogonal plane to the total angular momentum \mathbf{W}_0 ($\mathfrak{j} = \sum a_i \exp i(\nu_i t + \phi_i)$). All frequencies ν_i given by frequency analysis are easily recognized as integer multiples of the main nutation frequency ν (column 2).

tation but a more general periodic motion. It can thus be decomposed into several periodic terms with frequencies that are harmonics of the nutation frequency. The amplitude of these harmonics are usually small compared with the main periodic term. As an example, the quasiperiodic decomposition of the motion of the Earth spin axis in the Sun-Earth-Moon system is given in Table 3.

| satellite | $(\Omega_a - \Omega_0)/\Omega_0$ | ν_a (deg/yr) | $\mathcal{A}(\boldsymbol{\tau})$ (deg) | $\mathcal{A}(\boldsymbol{s})$ (") | $\mathcal{A}(\lambda\boldsymbol{\tau})$ (deg) | $\mathcal{A}(\lambda'\boldsymbol{s})$ (deg) |
|-----------|----------------------------------|---------------------|---|--------------------------------------|--|--|
| Moon | 2.19 | -20.128 | 23.438 | 8.220 | 2.25×10^{-3} | 5.158 |
| Phobos | 3.16×10^{-5} | -143.761 | 25.191 | 3.22×10^{-3} | 25.179 | 0.794 |
| Deimos | 1.98×10^{-4} | -6.036 | 25.191 | 9.68×10^{-3} | 24.292 | 1.570 |
| Io | 5.42×10^{-2} | -47.302 | 3.128 | 0.196 | 3.127 | 0.036 |
| Europa | 7.44×10^{-2} | -9.288 | 3.127 | 1.697 | 3.121 | 0.462 |
| Ganymede | 5.77×10^{-1} | -1.851 | 3.127 | 2.655 | 3.064 | 0.189 |
| Callisto | 9.87×10^{-1} | -0.338 | 3.129 | 6.307 | 2.320 | 0.615 |
| Mimas | 1.17×10^{-5} | -321.974 | 26.728 | 7.88×10^{-3} | 26.718 | 1.563 |
| Enceladus | 5.22×10^{-5} | -134.546 | 26.728 | 1.73×10^{-4} | 26.728 | 0.016 |
| Tethys | 7.65×10^{-4} | -63.727 | 26.728 | 1.07×10^{-1} | 26.723 | 1.019 |
| Dione | 2.16×10^{-3} | -26.807 | 26.728 | 3.19×10^{-3} | 26.727 | 0.016 |
| Rhea | 9.36×10^{-3} | -8.333 | 26.728 | 0.169 | 26.719 | 0.323 |
| Titan | 2.88 | -0.457 | 26.732 | 11.970 | 26.113 | 0.266 |
| Iapetus | 4.41×10^{-2} | -0.076 | 26.729 | 1.383 | 3.481 | 12.067 |
| Triton | 4.70 | 0.439 | 28.927 | 1932.238 | 24.990 | 28.221 |

Table 4: Approximate solution for various satellites using the analytical formulas of section 6.1. In each case, the system Sun-Planet-Satellite is considered. Ω_0 is the planet precession rate in absence of satellite. Ω_a and ν_a are the precession and nutation frequencies computed with the approximate formulas (129). $\mathcal{A}(\boldsymbol{\tau})$ and $\mathcal{A}(\boldsymbol{s})$ are the precession and nutation amplitude for the axis of the planet, $\mathcal{A}(\lambda\boldsymbol{\tau})$ and $\mathcal{A}(\lambda'\boldsymbol{s})$ are the same quantities for the satellite orbit. $\mathcal{A}(\lambda\boldsymbol{\tau})$ is thus the inclination of the Laplace pole of the satellite with respect to the pole of the orbit of the planet.

7 Comparison with previous work

The complete solutions we have derived here in section 4 and 6 are different from the previous approximations of (Goldreich, 1966, Ward, 1975) or more recently (Tremaine, 1991). Nevertheless, starting from equations (51), we can recover the already known approximations in a more general setting, as we consider non zero inclinations, and the constants $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are computed with non zero eccentricity (45), without the gyroscopic approximation (32), and in the non axisymmetric case (section 2.6). In all the following approximations, \mathbf{w}_1 is still considered as constant as it is very close to the unit vector of the total angular momentum, \mathbf{w}_0 .

7.1 Far satellite

For a far satellite, we have $\mathbf{c} \gg \mathbf{b}$. We have then

$$\dot{\mathbf{w}}_2 \approx -\frac{\mathbf{c}}{\alpha}(\mathbf{w}_1 \cdot \mathbf{w}_2)\mathbf{w}_1 \wedge \mathbf{w}_2. \quad (139)$$

Then $z = \mathbf{w}_1 \cdot \mathbf{w}_2$ is constant and \mathbf{w}_2 precesses uniformly around \mathbf{w}_1 (Fig.6.f) with angle ψ and frequency $\nu + \Omega = -cz/\alpha$. We have thus in a base with third vector \mathbf{w}_1

$$\mathbf{w}_2 = \begin{pmatrix} \sqrt{1-z^2} \cos \psi \\ \sqrt{1-z^2} \sin \psi \\ z \end{pmatrix}_{(-, -, \mathbf{w}_1)}. \quad (140)$$

For the axis of the planet, we have

$$\dot{\mathbf{w}} = -\frac{\mathbf{a}}{\gamma}(\mathbf{w}_1 \cdot \mathbf{w})\mathbf{w}_1 \wedge \mathbf{w} + \frac{\mathbf{b}}{\gamma}\mathbf{w} \wedge {}^t\mathbf{w}_2\mathbf{w}_2\mathbf{w}. \quad (141)$$

Averaging over ψ give then

$$\langle {}^t\mathbf{w}_2\mathbf{w}_2 \rangle_\psi = \frac{1-z^2}{2}Id + \left(-\frac{1}{2} + \frac{3}{2}z^2\right){}^t\mathbf{w}_1\mathbf{w}_1. \quad (142)$$

The spin vector of the planet \mathbf{w} thus precesses around \mathbf{w}_1 with constant obliquity ($x = \mathbf{w} \cdot \mathbf{w}_1$), and constant precession rate

$$\Omega = -\frac{\alpha x}{\gamma} \left[1 + \frac{\mathbf{b}}{\mathbf{a}} \left(-\frac{1}{2} + \frac{3}{2}z^2 \right) \right]. \quad (143)$$

We recover here the classical formula, the only novelty here being the treatment of the gyroscopic approximation (32).

7.2 Close satellite

The most advanced previous computation of the precession rate for a close satellite was obtained by Tremaine (1991), based on the equations of Goldreich (1966). In this case, $\mathbf{b} \gg \mathbf{a}$ and $\mathbf{b} \gg \mathbf{c}$. Following Tremaine, one can see easily from equations (51) that the total angular momentum of the planet-satellite system $\tilde{\mathbf{G}} = \mathbf{G} + \mathbf{G}_2$ is

nearly constant with norm $\tilde{G} = \sqrt{\gamma^2 + \alpha^2 + 2\gamma\alpha y}$. We have then with this approximation, from (51)

$$\dot{\mathbf{w}} \approx -\frac{\mathbf{b}}{\gamma\alpha}(\mathbf{w}_2 \cdot \mathbf{w})\tilde{\mathbf{G}} \wedge \mathbf{w} , \quad (144)$$

and

$$\dot{\mathbf{w}}_2 \approx -\frac{\mathbf{b}}{\gamma\alpha}(\mathbf{w}_2 \cdot \mathbf{w})\tilde{\mathbf{G}} \wedge \mathbf{w}_2 , \quad (145)$$

$\mathbf{w} \cdot \mathbf{w}_2$ is thus nearly constant, and the two vectors \mathbf{w} and \mathbf{w}_2 thus precess around $\tilde{\mathbf{G}}$ with the same nutation frequency

$$\nu = -\frac{\mathbf{b}y}{\gamma\alpha}\tilde{G} = -\frac{\mathbf{b}y}{\gamma\alpha}\sqrt{\gamma^2 + \alpha^2 + 2\gamma\alpha y} . \quad (146)$$

This corresponds to figures 6.a,b, where \mathbf{w} and \mathbf{w}_2 precess on concentric circles with opposite nutation phases. The computation of the precession frequency can then be treated as previously in section (7.1). Indeed, from (51), we have

$$\frac{d\tilde{\mathbf{G}}}{dt} = -\mathbf{a}\mathbf{w}_1 \wedge {}^t\mathbf{w}\mathbf{w}\mathbf{w}_1 - \mathbf{c}\mathbf{w}_1 \wedge {}^t\mathbf{w}_2\mathbf{w}_2\mathbf{w}_1 . \quad (147)$$

If $\tilde{\mathbf{w}}$ is the unit vector of $\tilde{\mathbf{G}}$ and using the notations of (Tremaine, 1991)

$$\begin{aligned} \cos \theta &= \mathbf{w} \cdot \tilde{\mathbf{w}} = (\gamma + \alpha y)/\tilde{G} ; \\ \cos \theta_1 &= \mathbf{w}_1 \cdot \tilde{\mathbf{w}} = (\gamma x + \alpha z)/\tilde{G} ; \\ \cos \theta_2 &= \mathbf{w}_2 \cdot \tilde{\mathbf{w}} = (\gamma y + \alpha)/\tilde{G} . \end{aligned} \quad (148)$$

We have $\tilde{G} = \gamma \cos \theta + \alpha \cos \theta_2$, and we obtain after averaging over the nutation angle

$$\frac{d\tilde{\mathbf{w}}}{dt} = -\frac{\mathfrak{A}}{\tilde{G}}(\mathbf{w}_1 \cdot \tilde{\mathbf{w}})\mathbf{w}_1 \wedge \tilde{\mathbf{w}} \quad (149)$$

with

$$\mathfrak{A} = \mathbf{a} \left(-\frac{1}{2} + \frac{3}{2} \cos^2 \theta \right) + \mathbf{c} \left(-\frac{1}{2} + \frac{3}{2} \cos^2 \theta_2 \right) . \quad (150)$$

The angular momentum unit vector $\tilde{\mathbf{w}}$ thus precesses around \mathbf{w}_1 with precession frequency

$$\Omega = -\frac{\mathfrak{A} \cos \theta_1}{\tilde{G}} . \quad (151)$$

7.3 Comparison with our analytical expressions

With the expression of \mathbf{T} (127) and Δ (128), we have obtained in section (6.1) the approximate expressions for

the precession (Ω) and nutation (ν) frequencies (Eq.129). These formulas are valid for any value of the planet-satellite distance. In the above above approximations of close or far satellite, these expressions will simplify as follow :

7.3.1 Close satellite

In the case of a close satellite, we have $\mathbf{b} \gg \mathbf{a}$ and $\mathbf{b} \gg \mathbf{c}$. Neglecting terms in \mathbf{a}, \mathbf{c} in front of \mathbf{b} in (129), we obtain

$$\Omega \approx -\frac{\mathbf{a}x^2 + \mathbf{c}z^2}{\alpha z + \gamma x} ; \quad \nu \approx -\frac{\mathbf{b}y}{\gamma\alpha}(\alpha z + \gamma x) . \quad (152)$$

That is, with $\alpha z + \gamma x = \tilde{G} \cos \theta_1$,

$$\nu \approx -\frac{\mathbf{b}y}{\gamma\alpha}\tilde{G} \cos \theta_1 . \quad (153)$$

This formula thus differs from (146) by the factor $\cos \theta_1$. In the same way, if as stated in section (6.1), one replace the variables x and z by their averaged value over the nutation angle in the expression of Ω , that is

$$\tilde{x} = \cos \theta_1 \cos \theta ; \quad \tilde{z} = \cos \theta_1 \cos \theta_2 ; \quad (154)$$

we obtain

$$\Omega = -\frac{\mathfrak{A}' \cos \theta_1}{\tilde{G}} . \quad (155)$$

with

$$\mathfrak{A}' = \mathbf{a} \cos^2 \theta + \mathbf{c} \cos^2 \theta_2 . \quad (156)$$

Here again, we have a slight difference from formula (151) that remains very small for small values of the angles θ and θ_2 . This is due to the approximations that were performed in section 6.1, where we have averaged the matrix M . Doing this, we have exchanged the order of operation and averaging. This was necessary in order to obtain some simple expressions, valid for all values of the satellite-planet distance. As an example, if one average x^2 over the nutation angle, one obtains

$$\langle x^2 \rangle = \frac{1}{2} \sin^2 \theta + \left(-\frac{1}{2} + \frac{3}{2} \cos^2 \theta \right) \cos^2 \theta_1 \quad (157)$$

while

$$\langle x \rangle^2 = \cos^2 \theta \cos^2 \theta_1 . \quad (158)$$

It should be noted that these two quantities become very close when either θ or θ_1 is small. Indeed

$$\langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2} \sin^2 \theta \sin^2 \theta_1 . \quad (159)$$

7.3.2 Far satellite

In the case of a far satellite, we have $\mathbf{c} \gg \mathbf{a}$ and $\mathbf{c} \gg \mathbf{b}$. We then obtain from (129),

$$\Omega \approx -\frac{\mathbf{a}x}{\gamma} \left(1 + \frac{\mathbf{b}}{\mathbf{a}} \frac{yz}{x} \right) \quad (160)$$

and from section 7.1

$$\nu = -\frac{\mathbf{c}z}{\alpha} - \Omega . \quad (161)$$

In the case of a far satellite, x, z will be nearly constant, and the average value of y over the nutation angle will be

$$\langle y \rangle = xz . \quad (162)$$

The precession frequency then becomes

$$\Omega \approx -\frac{\mathbf{a}x}{\gamma} \left(1 + \frac{\mathbf{b}}{\mathbf{a}} z^2 \right) , \quad (163)$$

that is here again, very close, but different from the classical formula (143).

8 Conclusions

In this work, we have obtained a very general framework for the evolution of the spin axis of a planet with a satellite. The equations have been derived with minimal approximations. In particular, we do not require the planet to be axisymmetric (section 2.6). We do not perform neither the usual gyroscopic approximation, but we average over the rotational period of the planet (section 2). The precession equations (28) that we obtain are rigorously derived, and can be used for precise solutions of the evolution of the axis of the planets (Laskar *et al.*, 2004a,b). In this case, the equations (28) can be immediately generalized to the perturbation of multiple bodies.

For fast satellites, or for the analysis of the system evolution over very long time, averaging over the orbital motion is required (section 2.5). As we also average the equations over the argument of perihelion of the satellite, the averaging is performed without expansion in term of the elliptical elements, and can thus be used for large values of the eccentricity of the planet or satellite (45). Although these secular equations (51) can be developed for a large number of interacting bodies, we have concentrated in the present work on the case of a single planet orbiting the Sun with a single satellite (it can be noted that although here we choose a non spherical planet, the same study applies to a non spherical satellite). It is then remarkable that the system of equations (51) representing the evolution of the spin axis of the planet, the orbital plane of the

planet and the satellite plane is integrable, although the explicit integration is not trivial (section 3).

We believe that this integrable system should be used to clarify the terminology for satellite motions. In particular, we have demonstrated that there are only two frequencies in this system : a slow frequency Ω that we called the precession frequency and a fast frequency ν that we called the nutation frequency. In the frame precessing uniformly with the precession frequency Ω , the nutation motion is periodic. Moreover, if we denote $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ the averages of $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$ over the nutation angle, then, for a general solution (section 3.4.3), the planet orbit nutates around \mathcal{P}_1 , the satellite orbit nutates around \mathcal{P}_2 and the axis of rotation of the planet (or more precisely its angular momentum), nutates around the rotational pole \mathcal{P} , all with the same nutation frequency ν . Additionally, all three poles $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ are coplanar with the total angular momentum \mathbf{W}_0 and precess uniformly around \mathbf{W}_0 with the precession frequency Ω (section 4.3, proposition 2). Finally, in the rotating frame with Ω , the plane $\mathbf{W}_0, \mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ is a symmetry plane for the periodic orbits of $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2$.

We have provided here a quadrature procedure that allows to compute exactly (up to numerical accuracy) the precession and nutation frequency of the secular system (51) for all values of the planet satellite distance. Alternatively, the rigorous treatment of section 4 shows that these frequencies can also be obtained numerically by the numerical integration of the system (51) over a single cycle of the nutation period, the nutation period being computed through the quadrature procedure of section 3.4.

We have unified the computation of the precession frequency of a planet, with some approximate formulas (section 6) that can be used in a large variety of cases, and for all values of the planet satellite distance. In particular, these formulas are valid in the intermediate region, when none of the previously known formulas for far or close satellites (Goldreich, 1966, Tremaine, 1991) are valid. Our formulas provide also the amplitude of the nutation and precession terms with good accuracy (Table 4). Nevertheless, in the asymptotic case of a very far or vary close satellite, our formulas differs slightly from the known formulas, as we had to average over the nutation frequency in the computation process. We thus expect that the formula of Tremaine (1991) remains more precise in this asymptotic case. One should note that the theoretical results of section 4 probably allow a more explicit derivation of this formula than in the original paper of Tremaine (1991). Using our formalism, it was also simple to improve the formula of Goldreich and Tremaine as our derivation does not assume the gyroscopic approximation, is valid for non axisymmetric planet, and takes into account the contribution of the satellite eccentricity.

More important than the precise computation of the contribution of the precession frequency due to a satellite, we think that the full description of all cases of interactions provided by section 4 and Figure 6 will be of special interest for the understanding of the satellites orbits and planet spin evolution over long time intervals.

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Annex 1. Averaged quantities

For completeness, we gather here the formulae that are useful for the averaging over the orbital mean motions. Depending on the case, one will use either the eccentric anomaly (E) or true anomaly (ν) as an intermediate variables. We recall first the basic formulae

$$\begin{aligned} dM &= \frac{r}{a} dE &= \frac{r^2}{a^2 \sqrt{1-e^2}} d\nu \\ \mathcal{X} &= a(\cos E - e) &= r \cos \nu \\ \mathcal{Y} &= a\sqrt{1-e^2} \sin E &= r \sin \nu \\ r &= a(1 - e \cos E) &= \frac{a(1-e^2)}{1+e \cos \nu} \end{aligned}$$

where \mathcal{X} and \mathcal{Y} are the coordinates of a point on a Keplerian orbit in the reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ with \mathbf{i} in the direction of periapse, and (\mathbf{i}, \mathbf{j}) the orbital plane. We have then

$$\begin{aligned} \langle \mathcal{X}^2 \rangle &= a^2 \left(\frac{1}{2} + 2e^2 \right) \\ \langle \mathcal{Y}^2 \rangle &= a^2 \left(\frac{1}{2} - \frac{e^2}{2} \right) \\ \langle \mathcal{X}\mathcal{Y} \rangle &= 0 \\ \langle \mathbf{r}\mathbf{r}^t \rangle &= \frac{1}{2}a^2(Id - \mathbf{k}\mathbf{k}^t) + e^2a^2 \left(2\mathbf{i}\mathbf{i}^t - \frac{1}{2}\mathbf{j}\mathbf{j}^t \right) \\ \langle r^2 \rangle &= a^2 \left(1 + \frac{3}{2}e^2 \right) \\ \langle \mathbf{r} \rangle &= -\frac{3}{2}ea\mathbf{i} \\ \langle \frac{1}{r^3} \rangle &= \frac{1}{a^3(1-e^2)^{3/2}} \\ \langle \frac{\mathbf{r}\mathbf{r}^t}{r^5} \rangle &= \frac{1}{2a^3(1-e^2)^{3/2}}(Id - \mathbf{k}\mathbf{k}^t) \\ \langle \frac{\mathbf{r}}{r^5} \rangle &= -\frac{e}{a^4(1-e^2)^{5/2}}\mathbf{i} \\ \langle \frac{(\mathbf{r} \cdot \mathbf{u})^2 \mathbf{r}}{r^7} \rangle &= -\frac{e}{4a^4(1-e^2)^{5/2}} \times \\ &\quad \times [(3(\mathbf{u} \cdot \mathbf{i})^2 + (\mathbf{u} \cdot \mathbf{j})^2)\mathbf{i} + 2(\mathbf{u} \cdot \mathbf{i})(\mathbf{u} \cdot \mathbf{j})\mathbf{j}] \end{aligned}$$

Annex 2. Linear algebra

Let E be a vector space of dimension 3 over \mathbb{R} . Let $\mathcal{B}_0 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an orthonormal basis of E , and $\mathcal{B} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ a general basis of E . Let \mathbf{u}, \mathbf{v} be two vectors with coordinates

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix};$$

in \mathcal{B} . Let M be the matrix of the coordinates of $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ in the basis \mathcal{B}_0 . Let $G = {}^tMM$ be the Gram matrix of the scalar products $\langle \mathbf{f}_i, \mathbf{f}_j \rangle$. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = {}^tXGY.$$

and

$$\mathbf{u} \wedge \mathbf{v} |_{\mathcal{B}} = \det(M)G^{-1}X \times Y$$

where

$$X \times Y = \begin{pmatrix} x_2y_3 - y_2x_3 \\ y_1x_3 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

In particular,

$$\begin{aligned} \mathbf{f}_1 \wedge \mathbf{f}_2 &= \det(M)G^{-1}\mathbf{f}_3 \\ \mathbf{f}_2 \wedge \mathbf{f}_3 &= \det(M)G^{-1}\mathbf{f}_1 \\ \mathbf{f}_3 \wedge \mathbf{f}_1 &= \det(M)G^{-1}\mathbf{f}_2 \end{aligned}$$

Annex 3. Approximations in 3D

We give here a more detailed version of section 6.1. Let us consider a reference frame with the total angular momentum unit vector \mathbf{w}_0 as third axis, and with coordinates

$$\mathbf{w} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{pmatrix}$$

we have

$$\zeta = \frac{\gamma + \beta x + \alpha y}{W_0}, \quad \zeta_1 = \frac{\gamma x + \beta + \alpha z}{W_0}, \quad \zeta_2 = \frac{\gamma y + \beta z + \alpha}{W_0}.$$

Considering the projections on the plane orthogonal to \mathbf{w}_0 ,

$$\mathfrak{z} = \xi + i\eta, \quad \mathfrak{z}_1 = \xi_1 + i\eta_1, \quad \mathfrak{z}_2 = \xi_2 + i\eta_2,$$

we have

$$\frac{d}{dt} \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{pmatrix} = iM \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{pmatrix}.$$

with

$$M = \begin{pmatrix} -\frac{\alpha x}{\gamma} \zeta_1 - \frac{\beta y}{\gamma} \zeta_2 & \frac{\alpha x}{\gamma} \zeta & \frac{\beta y}{\gamma} \zeta \\ \frac{\alpha x}{\beta} \zeta_1 & -\frac{\alpha x}{\beta} \zeta - \frac{\epsilon z}{\beta} \zeta_2 & \frac{\epsilon z}{\beta} \zeta_1 \\ \frac{\beta y}{\alpha} \zeta_2 & \frac{\epsilon z}{\alpha} \zeta_2 & -\frac{\beta y}{\alpha} \zeta - \frac{\epsilon z}{\alpha} \zeta_1 \end{pmatrix}$$

It is easy to verify that $(\zeta, \zeta_1, \zeta_2)$ is an eigenvector with eigenvalue 0. The two other eigenvalues are the roots of the second degree equation

$$\lambda^2 - \mathbf{T}\lambda + \mathbf{P} = 0,$$

where \mathbf{T} is the trace of M and

$$\mathbf{P} = \left(\frac{\zeta}{\alpha\beta} + \frac{\zeta_1}{\alpha\gamma} + \frac{\zeta_2}{\beta\gamma} \right) (\mathbf{a}bxy\zeta + \mathbf{a}cxyz\zeta_1 + \mathbf{b}cyz\zeta_2).$$

The precession and nutation frequencies are then

$$\Omega = \frac{\mathbf{T} + \sqrt{\mathbf{T}^2 - 4\mathbf{P}}}{2}; \quad \nu = -\sqrt{\mathbf{T}^2 - 4\mathbf{P}}.$$

and the eigenmodes

$$\mathbf{u}e^{i\psi}; \quad \mathbf{r}e^{i(\Omega t + \Phi)}; \quad \mathbf{s}e^{i[(\Omega + \nu)t + \Phi + \phi]}$$

with eigenvectors

$$\mathbf{e}_0 = \begin{pmatrix} \zeta \\ \zeta_1 \\ \zeta_2 \end{pmatrix}; \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ \lambda \\ \mu \end{pmatrix}; \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ \lambda' \\ \mu' \end{pmatrix}$$

where $\lambda, \lambda', \mu, \mu'$ are real numbers. The solutions are then

$$\begin{aligned} \mathfrak{z} &= \zeta \mathbf{u}e^{i\psi} + e^{i(\Omega t + \Phi)} (\mathbf{r} + \mathbf{s}e^{i(\nu t + \phi)}) \\ \mathfrak{z}_1 &= \zeta_1 \mathbf{u}e^{i\psi} + e^{i(\Omega t + \Phi)} (\lambda \mathbf{r} + \lambda' \mathbf{s}e^{i(\nu t + \phi)}) \\ \mathfrak{z}_2 &= \zeta_2 \mathbf{u}e^{i\psi} + e^{i(\Omega t + \Phi)} (\mu \mathbf{r} + \mu' \mathbf{s}e^{i(\nu t + \phi)}) \end{aligned}$$

Moreover, $\gamma\mathfrak{z} + \beta\mathfrak{z}_1 + \alpha\mathfrak{z}_2 = 0$, as it is the projection of \mathbf{W}_0 on a plane orthogonal to \mathbf{W}_0 . This implies that its constant term $(\gamma\zeta + \beta\zeta_1 + \alpha\zeta_2)\mathbf{u}e^{i\psi}$ is also null, and as $\gamma\zeta + \beta\zeta_1 + \alpha\zeta_2 = W_0$, we have necessarily $\mathbf{u} = 0$. The solutions are thus

$$\begin{aligned} \mathfrak{z} &= e^{i(\Omega t + \Phi)} (\mathbf{r} + \mathbf{s}e^{i(\nu t + \phi)}) \\ \mathfrak{z}_1 &= e^{i(\Omega t + \Phi)} (\lambda \mathbf{r} + \lambda' \mathbf{s}e^{i(\nu t + \phi)}) \\ \mathfrak{z}_2 &= e^{i(\Omega t + \Phi)} (\mu \mathbf{r} + \mu' \mathbf{s}e^{i(\nu t + \phi)}) \end{aligned}$$

In this approximation, the three axis $(\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2)$ describe circular motion with nutation frequency ν around the three poles $(\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2)$ that precess uniformly with precession frequency Ω around the total angular momentum \mathbf{W}_0 . As in the general proposition 2 (section 4.3), the three poles $(\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2)$ remains always coplanar with \mathbf{W}_0 .

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