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Exterior Stokes problem in the half-space

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Abstract The purpose of this work is to solve the exterior Stokes problem in the half-space $\mathbb{R}^n_+$. We study the existence and the uniqueness of generalized solutions in weighted L^p theory with $1 < p < \infty$. Moreover, we consider the case of strong solutions and very weak solutions. This paper extends the studies done in [3] for an exterior Stokes problem in the whole space and in [5] for the study of the Laplace equation in the same geometry as here.
Key words Weighted Sobolev spaces ; Stokes operator ; Dirichlet boundary conditions ; Exterior problem ; Half-space.
AMS Classification 35D05 ; 35D10 ; 35J50 ; 35J55 ; 35Q30 ; 76D07 ; 76N10.

1 Introduction and preliminaries

Consider ω_0 a compact region of $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n,\ x_n > 0\}$, Γ_0 the boundary of ω_0 and Ω the complement of ω_0 in $\mathbb{R}^n_+$. This paper is devoted to the resolution of the Stokes system

$$(\mathcal{S}_{\mathcal{D}}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \\ \boldsymbol{u} = \boldsymbol{g_1} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

We notice that this problem is an extension of the Stokes system for an exterior domain in the whole space, studied in several works where some of them introduce the homogeneous Sobolev spaces $\boldsymbol{H}^{1,p}_0({}^c\omega_0)$ (where ${}^c\omega_0$ is the complement in $\mathbb{R}^n$ of ω_0) obtained as the closure of $\boldsymbol{\mathcal{D}}({}^c\omega_0)$ with respect to the norm $\|\nabla \cdot\|_{\boldsymbol{L}^p({}^c\omega_0)}$. The existence and the uniqueness of a solution of such a problem with homogeneous boundary conditions in $\boldsymbol{H}^{1,p}_0({}^c\omega_0) \times L^p({}^c\omega_0)$ has been studied by Kozono and Sohr ([22], [23]) and Galdi and Simader ([18]). Another point of view, which is ours, is to search a solution in weighted Sobolev spaces $\boldsymbol{W}^{m,p}_\alpha({}^c\omega_0)$ (see definition below). These spaces are well-adapted to the Laplace and Stokes equations because they satisfy an optimal Poincaré-type inequality. They also provide some precise information on the behaviour of the functions at infinity, which is not obvious from the definition of $\boldsymbol{H}^{1,p}_0({}^c\omega_0)$. For this approach, we refer to Girault and Sequeira [19] (when $n = 2$ or $n = 3$, $p = 2$

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and $\alpha = 0$), Specovius-Neugebauer ([26] when $n \geq 3$ and $\frac{n}{p} + \alpha \notin \mathbb{Z}$ for strong solutions and when $n = 2$ and $\frac{2}{p} + \alpha \notin \mathbb{Z}$ for weak solutions in [27]) and to Alliot and Amrouche [3].

Here, our originality is to solve the exterior Stokes problem in the half-space and not anymore in the whole space. That implies an additional difficulty due to the nature of the boundary which is not bounded since it contains $\mathbb{R}^{n-1}$. So, we have to introduce weights even in the spaces of traces. We can cite Hanouzet [21] who has given the first results for such spaces in 1971 and Amrouche, Nečasovà [9] who have extended these results in 2001 to weighted Sobolev spaces which possess logarithmic weights (we just recall that logarithmic weights allow us to have a Poincaré-type inequality even in the "critical" cases ; see below for more details). We remind the works of some authors which have studied the Stokes problem in the half-space. The first ones are due to Cattabriga [14] who have chosen the setting of homogeneous Sobolev spaces. Similar results are given by Farwig and Sohr [16] and Galdi [17]. On the other hand Maz'ya, Plamenevskii and Stupyalis [24] have studied the problem in weighted Sobolev spaces, but only in the dimension 3. Finally, we cite the works of Amrouche, Nečasová and Raudin [10] who consider weak solutions for any dimension. Nevertheless, we notice that all these works concern only the Stokes system in the half-space whereas in this paper, we deal with the exterior Stokes problem in the half-space. We can summarize our work saying that it is an extension of the exterior problem in the whole space and of the problem in the half-space.

We state that, here, we will concentrate only on the basic weights for the sake of simplicity and because they are the most usual. The paper is organized as follows. Sections 2 and 3 are devoted to the case of generalized solutions respectively when $p = 2$ and $p \neq 2$. In Section 4, we consider strong solutions and give regularity results according to the data. Finally, in Section 5, we find very weak solutions to the homogeneous problem with singular boundary conditions. The main results of this work are Theorems 2.5 and 3.6 for generalized solutions, Theorems 4.2 and 4.4 for strong solutions and Corollary 5.4 and Theorem 5.5 for very weak solutions.

We complete this introduction with a short review of the weighted Sobolev spaces and their trace spaces. For any integer q we denote by $\mathcal{P}_q$ the space of polynomials in n variables, of degree less than or equal to q, with the convention that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative.
For any real number $p \in]1, +\infty[$, we denote by p' the dual exponent of p :

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a typical point of $\mathbb{R}^n$, $\boldsymbol{x'} = (x_1, \ldots, x_{n-1})$ and let $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote its distance to the origin. We shall use two basic weights :

$$\rho(r) = (1 + r^2)^{1/2} \quad \text{and} \quad lg\ r = ln(2 + r^2).$$

As usual, $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support, $\mathcal{D}'(\Omega)$ its dual space, called the space of distributions and $\mathcal{D}(\overline{\Omega})$ the

space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$.

Then, for any nonnegative integers n and m and real numbers $p > 1$ and α, setting

$$k = k(m,n,p,\alpha) = \begin{cases} -1 & \text{if } \dfrac{n}{p} + \alpha \notin \{1,\dots,m\}, \\ m - \dfrac{n}{p} - \alpha & \text{if } \dfrac{n}{p} + \alpha \in \{1,\dots,m\}, \end{cases}$$

we define the following space :

$$\begin{aligned} W_\alpha^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega);& \\ &\forall \lambda \in \mathbb{N}^n : 0 \leqslant |\lambda| \leqslant k,\ \rho^{\alpha-m+|\lambda|}(lg\ r)^{-1} D^\lambda u \in L^p(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : k+1 \leqslant |\lambda| \leqslant m,\ \rho^{\alpha-m+|\lambda|} D^\lambda u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm :

$$\begin{aligned} \|u\|_{W_\alpha^{m,p}(\Omega)} = (&\sum_{0\leqslant|\lambda|\leqslant k} \|\rho^{\alpha-m+|\lambda|}(lg\ r)^{-1} D^\lambda u\|^p_{L^p(\Omega)} \\ &+ \sum_{k+1\leqslant|\lambda|\leqslant m} \|\rho^{\alpha-m+|\lambda|} D^\lambda u\|^p_{L^p(\Omega)})^{1/p}. \end{aligned}$$

We also define the semi-norm :

$$|u|_{W_\alpha^{m,p}(\Omega)} = (\sum_{|\lambda|=m} \|\rho^\alpha D^\lambda u\|^p_{L^p(\Omega)})^{1/p}.$$

The weights defined previously are chosen so that the space $\mathcal{D}(\overline{\Omega})$ is dense in $W_\alpha^{m,p}(\Omega)$ and so that the following Poincaré-type inequality holds in the following spaces : let α be a real number, $m \geq 1$ an integer and $q' = \min(q, m-1)$, where q is the highest degree of the polynomials contained in $W_\alpha^{m,p}(\Omega)$. Then :

$$\forall u \in W_\alpha^{m,p}(\Omega), \quad \inf_{k\in\mathcal{P}_{q'}} \|u+k\|_{W_\alpha^{m,p}(\Omega)} \leq C\ |u|_{W_\alpha^{m,p}(\Omega)},$$

and

$$\forall u \in \overset{\circ}{W}{}_\alpha^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_\alpha^{m,p}(\Omega)}}, \quad \|u\|_{W_\alpha^{m,p}(\Omega)} \leq C\ |u|_{W_\alpha^{m,p}(\Omega)}.$$

This theorem is proved by Amrouche, Girault and Giroire [8] in an exterior domain and by Amrouche and Nečasovà [9] in the half-space. It is extended to this domain by an adequate partition of unity. We denote by $W_{-\alpha}^{-m,p'}(\Omega)$ the dual space of $\overset{\circ}{W}{}_\alpha^{m,p}(\Omega)$ and we notice that it is a space of distributions.

Now, we want to define the traces of functions of $W_\alpha^{m,p}(\Omega)$. These traces have a component on Γ_0 and another component on $\mathbb{R}^{n-1}$. For the traces on Γ_0, we return to Adams [1] or Nečas [25] for the definition of $W^{m-j-\frac{1}{p},p}(\Gamma_0)$ with $j = 0, ..., m-1$ and for the usual trace theorems. In order to define the traces

of functions on $\mathbb{R}^{n-1}$, we intoduce, for any $\sigma \in]0,1[$, the space

$$W_0^{\sigma,p}(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n), \omega^{-\sigma}u \in L^p(\mathbb{R}^n),$$
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(\boldsymbol{x})-u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x}d\boldsymbol{y} < \infty\},$$

where

$$\omega = \begin{cases} \rho & \text{if } \dfrac{n}{p} \neq \sigma, \\ \rho(lg\rho)^{1/\sigma} & \text{if } \dfrac{n}{p} = \sigma. \end{cases}$$

It is a reflexive Banach space equipped with its natural norm

$$(\|\frac{u}{\omega^\sigma}\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(\boldsymbol{x})-u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x}d\boldsymbol{y})^{1/p}.$$

For any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we set

$$W_\alpha^{s,p}(\mathbb{R}^n) = \{u \in W^{[s],p}_{[s]+\alpha-s}(\mathbb{R}^n),\ \forall|\lambda| = [s],\ \rho^\alpha D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^n)\}.$$

It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_\alpha^{s,p}(\mathbb{R}^n)} = \|u\|_{W^{[s],p}_{[s]+\alpha-s}(\mathbb{R}^n)} + \sum_{|\lambda|=s} \|\rho^\alpha D^\lambda u\|_{W_0^{s-[s],p}(\mathbb{R}^n)}.$$

We notice that this definition coincides with the definition given at the beginning of this paper when $s = m$ is a nonnegative integer. As in [9], we have the following lemma :

Lemma 1.1. *For any integer $m \geq 1$ and real number α, we define the mapping*

$$\gamma : \mathcal{D}(\overline{\mathbb{R}^n_+}) \to\ (\mathcal{D}(\mathbb{R}^{n-1}))^m$$
$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1}u),$$

where for any $k = 0, \ldots, m-1$, $\gamma_k u = \dfrac{\partial^k u}{\partial x_n^k}$. Then, γ can be extended by continuity to a linear and continuous mapping still denoted by γ from $W_\alpha^{m,p}(\mathbb{R}^n_+)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Moreover, γ is onto and

$$Ker\ \gamma = \overset{\circ}{W}{}_\alpha^{m,p}(\mathbb{R}^n_+).$$

In all this article, we suppose that Γ_0 is of class $C^{1,1}$, except when $p = 2$, where Γ_0 can be considered to be Lipschitz-continuous only.

We will denote by C a positive and real constant which may vary from line to line and we set $\boldsymbol{E} = E^n$ for any space E.

2 Study of the problem $(\mathcal{S}_{\mathcal{D}})$ when $p = 2$.

First, we notice that it is equivalent to solve the problem with homogeneous boundary conditions. Indeed, the function $\boldsymbol{g_1}$ is in $\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$, so, thanks to Lemma 1.1, there exists $\boldsymbol{u_1} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)$ such that $\boldsymbol{u_1} = \boldsymbol{g_1}$ on $\mathbb{R}^{n-1}$ and

$$\|\boldsymbol{u_1}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} \leq \ C \ \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}.$$

Now, let $\boldsymbol{\eta}$ be the trace of $\boldsymbol{u_1}$ on Γ_0, $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{\eta} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and let $R > 0$ be such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$. It is clear that the function $\boldsymbol{h_0}$ defined by

$$\boldsymbol{h_0} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{h_0} = \boldsymbol{0} \text{ on } \partial B_R,$$

belongs to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0 \cup \partial B_R)$. We know that there exists an extension $\boldsymbol{u_{h_0}} \in \boldsymbol{H}^1(\Omega_R)$, where $\Omega_R = \Omega \cap B_R$, such that $\boldsymbol{u_{h_0}} = \boldsymbol{h_0}$ on $\Gamma_0 \cup \partial B_R$ and such that $\|\boldsymbol{u_{h_0}}\|_{\boldsymbol{H}^1(\Omega_R)} \leq \ C \ \|\boldsymbol{h_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0 \cup \partial B_R)}$. We set

$$\boldsymbol{u_0} = \boldsymbol{u_{h_0}} \text{ in } \Omega_R, \quad \boldsymbol{u_0} = \boldsymbol{0} \text{ in } \Omega \setminus \Omega_R.$$

We have $\boldsymbol{u_0} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u_0} = \boldsymbol{g}$ on Γ_0, $\boldsymbol{u_0} = \boldsymbol{0}$ on $\mathbb{R}^{n-1}$ and

$$\|\boldsymbol{u_0}\|_{\boldsymbol{H}^1(\Omega)} \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)}.$$

Thus the function $\boldsymbol{u_0} + \boldsymbol{u_{1|\Omega}}$ is in $\boldsymbol{W}_0^{1,2}(\Omega)$ and its traces are $\boldsymbol{g_0}$ on Γ_0 and $\boldsymbol{g_1}$ on Γ_1. This allows us to solve only the following problem : let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,2}(\Omega)$ and h be in $L^2(\Omega)$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_0) \left\{ \begin{array}{lll} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, & \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_0, & \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{array} \right.$$

Now, we want to establish Lemma 2.2 to have a data for the divergence reduced to zero. For this, we use this preliminary lemma :

Lemma 2.1. *There exists a real constant $C > 0$ depending only on ω_0 such that the following holds. For any $h \in L^2(\Omega)$, there exists a unique $\varphi \in W_0^{2,2}(\Omega)/\mathbb{R}$ solution of*

$$\Delta \varphi = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}.$$

Moreover, φ satisfies

$$\|\varphi\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leq \ C \ \|h\|_{L^2(\Omega)}.$$

Proof. First, we define Ω' the symmetric region of Ω with respect to $\mathbb{R}^{n-1}$, $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \partial \widetilde{\Omega}$. Let h be in $L^2(\Omega)$ and let the function h_* be defined, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, by

$$h_*(\boldsymbol{x'}, x_n) = \left\{ \begin{array}{ll} h(\boldsymbol{x'}, x_n) & \text{if } x_n \geq 0, \\ h(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{array} \right.$$

Then, we set in $\mathbb{R}^n$ the function

$$\widetilde{h} = h_* \text{ in } \widetilde{\Omega}, \quad \widetilde{h} = 0 \text{ in } \mathbb{R}^n \setminus \widetilde{\Omega}.$$

So, $\widetilde{h} \in L^2(\mathbb{R}^n)$ and, supposing first that $n > 2$, as [7] allows us to say that

$$\Delta : W_0^{2,2}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

is onto, we deduce that there exists $\widetilde{u} \in W_0^{2,2}(\mathbb{R}^n)$ such that $\Delta\widetilde{u} = \widetilde{h}$ in $\mathbb{R}^n$ and $\|\widetilde{u}\|_{W_0^{2,2}(\mathbb{R}^n)} \leq C \, \|h\|_{L^2(\Omega)}$. We denote by $u \in W_0^{2,2}(\widetilde{\Omega})$ the restriction of $\widetilde{u}$ to $\widetilde{\Omega}$. We notice that we have $\Delta u = h_*$ in $\widetilde{\Omega}$ and that $\dfrac{\partial u}{\partial \boldsymbol{n}} \in H^{\frac{1}{2}}(\widetilde{\Gamma}_0)$. Thanks to Proposition 3.12 in [8], (there is no condition of compatibility because $n > 2$), there exists $z \in W_1^{2,2}(\widetilde{\Omega}) \subset W_0^{2,2}(\widetilde{\Omega})$ such that

$$\Delta z = 0 \text{ in } \widetilde{\Omega} \quad \text{and} \quad \frac{\partial z}{\partial \boldsymbol{n}} = \frac{\partial u}{\partial \boldsymbol{n}} \text{ on } \widetilde{\Gamma}_0,$$

checking

$$\|z\|_{W_0^{2,2}(\widetilde{\Omega})} \leq C \, \|u\|_{W_0^{2,2}(\widetilde{\Omega})}.$$

Now, we set $w = u - z$. Then $w \in W_0^{2,2}(\widetilde{\Omega})$ satisfies

$$\Delta w = h_* \text{ in } \widetilde{\Omega} \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0, \tag{1}$$

and we have

$$\|w\|_{W_0^{2,2}(\widetilde{\Omega})} \leq C \, \|h\|_{L^2(\Omega)}.$$

If $n = 2$, we can not apply this reasoning because a condition of compatibility appears when we want to use Proposition 3.12 of [8]. Nevertheless, we can find directly $w \in W_0^{2,2}(\widetilde{\Omega})$, solution of (1), without needing the space $W_1^{2,2}(\widetilde{\Omega})$ (see Theorem 7.13 in [20]). Then, we set, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$,

$$v(\boldsymbol{x'}, x_n) = w(\boldsymbol{x'}, -x_n).$$

As h_* is even with respect to x_n, we easily check that v is solution of the same problem that w satisfies. So, noticing that the kernel of this problem is $\mathbb{R}$, we deduce that $v = w + c$ in $\widetilde{\Omega}$, with $c \in \mathbb{R}$, and consequently, $\dfrac{\partial w}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. Thus, the function $w_{|\Omega} \in W_0^{2,2}(\Omega)$ is solution of our problem. Moreover, this solution is unique up to a real constant. Indeed, if $z \in W_0^{2,2}(\Omega)$ is in the kernel of this problem, $z_* \in W_0^{2,2}(\widetilde{\Omega})$ is in $\mathbb{R}$, the kernel of the problem (1), so $z \in \mathbb{R}$. $\square$

Lemma 2.2. *There exists a real constant $C > 0$ depending only on ω_0 such that for any $h \in L^2(\Omega)$, there exists $\boldsymbol{w} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$ checking*

$$\operatorname{div} \boldsymbol{w} = h \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \, \|h\|_{L^2(\Omega)}.$$

Proof. Let h be in $L^2(\Omega)$. We know, thanks to the previous lemma, that there exists a unique $\varphi \in W_0^{2,2}(\Omega)/\mathbb{R}$ satisfying

$$\Delta\varphi = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1},$$

with

$$\|\varphi\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leq C \, \|h\|_{L^2(\Omega)}.$$

We set $\boldsymbol{v} = \nabla\varphi \in \boldsymbol{W}_0^{1,2}(\Omega)$. So $\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C\ \|h\|_{L^2(\Omega)}$. Moreover, we set $\boldsymbol{g_0} = \boldsymbol{v}_{|\Gamma_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and $\boldsymbol{g_1} = \boldsymbol{v}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$. Thanks to Theorem 4.2 in [10], there exists $(\boldsymbol{z},\theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ solution of

$$-\Delta\boldsymbol{z} + \nabla\theta = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1},$$

satisfying

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} \leq\ C\ \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}.$$

We denote again by $\boldsymbol{z}$ the restriction of $\boldsymbol{z}$ to Ω and $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{z}_{|\Gamma_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$. We observe that

$$\int_{\Gamma_0} \boldsymbol{g}\cdot\boldsymbol{n}\ d\sigma = \int_{\Gamma_0} \boldsymbol{v}\cdot\boldsymbol{n}\ d\sigma - \int_{\Gamma_0} \boldsymbol{z}\cdot\boldsymbol{n}\ d\sigma = \int_{\Gamma_0} \frac{\partial\varphi}{\partial\boldsymbol{n}}\ d\sigma - \int_{\omega_0} \text{div } \boldsymbol{z}\ d\boldsymbol{x} = 0.$$

Now, let $R > 0$ be such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$ and $\Omega_R = B_R \cap \Omega$. Then, the previous condition being checked, we have the following result (see [6]) : there exists $\boldsymbol{y} \in \boldsymbol{H}^1(\Omega_R)$ such that

$$\text{div } \boldsymbol{y} = 0 \text{ in } \Omega_R, \quad \boldsymbol{y} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \text{ on } \partial B_R,$$

and

$$\|\boldsymbol{y}\|_{\boldsymbol{H}^1(\Omega_R)} \leq\ C_R\ (\|\boldsymbol{g_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}).$$

We denote again by $\boldsymbol{y}$ its extension by $\boldsymbol{0}$ in Ω. So $\boldsymbol{y} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and

$$\text{div } \boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

Finally, we set $\boldsymbol{u} = \boldsymbol{z}_{|\Omega} + \boldsymbol{y} \in \boldsymbol{W}_0^{1,2}(\Omega)$. The function $\boldsymbol{u}$ satifies

$$\text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g_0} \text{ on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1},$$

and the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C\ \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)}.$$

Finally the function $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ is solution of the setted problem. $\square$

So to solve $(\mathcal{S}_0)$, it is sufficient to solve the following problem $(\mathcal{S}_{00})$: find $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_{00}) \left\{ \begin{array}{lll} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, & \text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_0, & \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{array} \right.$$

For this, as an immediate consequence of the previous lemma, we derive first the following Babuška-Brezzi condition (see [12] and [13]).

Corollary 2.3. *There exists a real constant $\beta > 0$, depending only on ω_0, such that*

$$\inf_{h\in L^2(\Omega)} \sup_{\boldsymbol{w}\in\ \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega h\ div\ \boldsymbol{w}\ d\boldsymbol{x}}{\|\boldsymbol{w}\|_{\mathring{\boldsymbol{W}}_0^{1,2}(\Omega)}\|h\|_{L^2(\Omega)}} \ \geq\ \frac{1}{\beta}. \tag{2}$$

We introduce the continuous bilinear form defined on $\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ by

$$b(\boldsymbol{w}, q) = -\int_\Omega q \text{ div } \boldsymbol{w} \, d\boldsymbol{x}.$$

Let $B \in \mathcal{L}\ (\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega), L^2(\Omega))$ be the associated linear operator and let $B' \in \mathcal{L}\ (L^2(\Omega), \boldsymbol{W}_0^{-1,2}(\Omega))$ the dual operator of B, *i.e*

$$b(\boldsymbol{w}, q) = < B\boldsymbol{w}, q >_{L^2(\Omega)\times L^2(\Omega)} = < \boldsymbol{w}, B'q >_{\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega), \boldsymbol{W}_0^{-1,2}(\Omega)} .$$

It is clear that $B = -$ div and that $B' = \nabla$. As a consequence of the "inf-sup" condition (2), we know that B is an isomorphism from $\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)/\boldsymbol{V}$ onto $L^2(\Omega)$ and B' is an isomorphism from $L^2(\Omega)$ onto $\boldsymbol{V}^\circ$ with

$$\boldsymbol{V} = \{\boldsymbol{v} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega),\ \text{div } \boldsymbol{v} = 0 \text{ in } \Omega\},$$

which is an Hilbert space and

$$\boldsymbol{V}^\circ = \{\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega),\ \forall \boldsymbol{w} \in \boldsymbol{V},\ < \boldsymbol{f}, \boldsymbol{w} >_{\boldsymbol{W}_0^{-1,2}(\Omega), \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)} = 0\}.$$

Thus, we have the following De Rham's theorem :

Corollary 2.4. *The operator ∇ is an isomorphism from $L^2(\Omega)$ to $\boldsymbol{V}^\circ$.*

Now, we define the problem : find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$(\mathcal{FV}) \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{v} >_{\boldsymbol{W}_0^{-1,2}(\Omega), \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)} .$$

Using the second Poincaré-type inequality given in the introduction for the equivalence of the norm and the semi-norm in $\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$ and applying Lax-Milgram theorem, we check that $(\mathcal{FV})$ has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Finally, we notice that problems $(\mathcal{S}_{00})$ and $(\mathcal{FV})$ are equivalent, obtaining the pressure thanks to Corollary 2.4. Thus, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of $(\mathcal{S}_{00})$.

In consequence, we have the following theorem :

Theorem 2.5. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of the problem*

$$(\mathcal{S}_\mathcal{D}) \left\{ \begin{array}{lll} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, & \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, & \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{array} \right.$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq\ & C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} \\ & + \|\boldsymbol{g_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}), \end{aligned}$$

where C is a real positive constant which depends only on ω_0.

3 Study of the problem $(\mathcal{S}_{\mathcal{D}})$ when $p \neq 2$.

First, we suppose that $p > 2$ and we want to study the kernel of the Stokes system. We set :

$$\mathcal{D}_0^p(\Omega) = \{(\boldsymbol{z}, \eta) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega),\ -\Delta \boldsymbol{z} + \nabla \eta = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega\}.$$

To characterize this space, it is useful to show the following lemma :

Lemma 3.1. *Let $p > 2$, $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and h be in $L^p(\mathbb{R}^n_+)$, both with compact support in $\mathbb{R}^n_+$, and $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ the unique solution of*

$$(\mathcal{S}_+) \begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \mathbb{R}^n_+, \\ \text{div } \boldsymbol{v} = h & \text{in } \mathbb{R}^n_+, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Then, we have $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ and $(\boldsymbol{v}, \eta)$ satisfies

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)} + \|\eta\|_{L^p(\mathbb{R}^n_+)} + \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} + \|\eta\|_{L^2(\mathbb{R}^n_+)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)} + \|h\|_{L^p(\mathbb{R}^n_+)}),$$

where C is a real positive constant which depends only on p, ω_0 and the support of $\boldsymbol{f}$ and h.

Proof. Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and h in $L^p(\mathbb{R}^n_+)$ with compact support in $\mathbb{R}^n_+$; we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\mathbb{R}^n_+)$ and $h \in L^2(\mathbb{R}^n_+)$ because $p > 2$ and let $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ be the solution of $(\mathcal{S}_+)$ satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} + \|\eta\|_{L^2(\mathbb{R}^n_+)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^n_+)} + \|h\|_{L^2(\mathbb{R}^n_+)}). \tag{3}$$

Thanks to [10], there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ solution of $(\mathcal{S}_+)$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)} + \|\pi\|_{L^p(\mathbb{R}^n_+)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)} + \|h\|_{L^p(\mathbb{R}^n_+)}). \tag{4}$$

We set $(\boldsymbol{w}, \tau) = (\boldsymbol{u} - \boldsymbol{v}, \pi - \eta)$ which satisfies

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and we want to prove that $(\boldsymbol{w}, \tau) = (\boldsymbol{0}, 0)$. We easily show (see Proposition 4.1 in [10]) that w_n, the nth component of $\boldsymbol{w}$, which is in $W_0^{1,p}(\mathbb{R}^n_+) + W_0^{1,2}(\mathbb{R}^n_+)$, satisfies

$$\Delta^2 w_n = 0 \text{ in } \mathbb{R}^n_+, \quad w_n = 0 \text{ on } \mathbb{R}^{n-1}, \quad \frac{\partial w_n}{\partial x_n} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Here, the discussion splits into three steps : first, if $p \neq n$ and $n \neq 2$, then $w_n \in W_{-1}^{0,p}(\mathbb{R}^n_+) + W_{-1}^{0,2}(\mathbb{R}^n_+)$. For almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$, we set

$$\widetilde{w}_n(\boldsymbol{x'}, x_n) = \begin{cases} w_n(\boldsymbol{x'}, x_n) & \text{if } x_n \geq 0, \\ (-w_n - 2x_n \dfrac{\partial w_n}{\partial x_n} - x_n^2 \Delta w_n)(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we check (see [11], [15]) that $\widetilde{w}_n$ is the unique extension of w_n such that $\Delta^2\widetilde{w}_n = 0$ in $\mathbb{R}^n$. Moreover, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$< \widetilde{w}_n, \varphi >_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} w_n[\varphi - 5\psi - 6x_n\frac{\partial\psi}{\partial x_n} - x_n^2\Delta\psi]\ d\boldsymbol{x}$$

where $\psi \in \mathcal{D}(\mathbb{R}^n)$ is defined by $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, which allows us to prove that $\widetilde{w}_n$ is in $W^{-2,p}_{-3}(\mathbb{R}^n) + W^{-2,2}_{-3}(\mathbb{R}^n)$. So $\widetilde{w}_n$ is a biharmonic tempered distribution and consequently a biharmonic polynomial. Finally, as the space $W^{-2,p}_{-3}(\mathbb{R}^n) + W^{-2,2}_{-3}(\mathbb{R}^n)$ does not contain polynomial, we deduce from this that $\widetilde{w}_n = 0$ in $\mathbb{R}^n$ and so $w_n = 0$ in $\mathbb{R}^n_+$. Now, if $n = p$, we have $W^{1,p}_0(\mathbb{R}^n_+) \subset W^{0,p}_{-1,-1}(\mathbb{R}^n_+)$, and we may proceed with the same reasoning since the logarithmic factor does not change the proof. When $n = 2$, we have $W^{1,2}_0(\mathbb{R}^n_+) \subset W^{0,2}_{-1,-1}(\mathbb{R}^n_+)$ and get the same result with the same arguments, simply noticing that w_n could be equal to a constant in $\mathbb{R}^n_+$ but that this constant would be necessary equal to zero because $w_n = 0$ on $\mathbb{R}^{n-1}$.

Consequently, in any case, we have $w_n = 0$ in $\mathbb{R}^n_+$. We deduce from this (see Proposition 4.1, [10]) that $\tau \in L^p(\mathbb{R}^n_+) + L^2(\mathbb{R}^n_+)$ satisfies

$$\Delta\tau = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial\tau}{\partial\boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Now, we set for almost any $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$,

$$\tau_*(\boldsymbol{x'}, x_n) = \begin{cases} \tau(\boldsymbol{x'}, x_n) & \text{if } x_n \geq 0, \\ \tau(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we easily check that τ_* is a harmonic tempered distribution, so a harmonic polynomial, included in $L^p(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, a space which does not contain polynomial. Thus, we conclude that $\tau = 0$ in $\mathbb{R}^n_+$. Then, we show that $\boldsymbol{w'} = (w_1, \dots w_{n-1}) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+) + \boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+)$ satisfies

$$\Delta\boldsymbol{w'} = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w'} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}.$$

We set for almost any $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$,

$$\boldsymbol{w'}^*(\boldsymbol{x'}, x_n) = \begin{cases} \boldsymbol{w'}(\boldsymbol{x'}, x_n) & \text{if } x_n \geq 0, \\ -\boldsymbol{w'}(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we easily check that $\boldsymbol{w'}^* \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n) + \boldsymbol{W}^{1,2}_0(\mathbb{R}^n)$ is a harmonic tempered distribution, so a harmonic polynomial in $\mathbb{R}^n$. Thus, $\boldsymbol{w'}$ is a harmonic polynomial in $\mathbb{R}^n_+$ and $\nabla\boldsymbol{w'}$ is an harmonic polynomial in $\boldsymbol{L}^p(\mathbb{R}^n_+) + \boldsymbol{L}^2(\mathbb{R}^n_+)$, a space which does not contain polynomial. So $\nabla\boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^n_+$ and like $\boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^{n-1}$, we have $\boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^n_+$. Finally, we deduce from this that $(\boldsymbol{w}, \tau) = (\boldsymbol{0}, 0)$. $\square$

Now, we have the following theorem :

Theorem 3.2. *The kernel $\mathcal{D}^p_0(\Omega)$ is reduced to $\{(\boldsymbol{0}, 0)\}$ when $p > 2$.*

Proof. Let $(\boldsymbol{z}, \pi)$ be in $\mathcal{D}^p_0(\Omega)$. We denote by $\widetilde{\boldsymbol{z}}$ and $\widetilde{\pi}$ the extensions by 0 of $\boldsymbol{z}$ and π in $\mathbb{R}^n_+$. We have $\widetilde{\boldsymbol{z}} \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+)$ and $\widetilde{\pi} \in L^p(\mathbb{R}^n_+)$. We set $\widetilde{\boldsymbol{h}} =$

$-\Delta\widetilde{\boldsymbol{z}}+\nabla\widetilde{\pi}\in\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and we easily check that $\widetilde{\boldsymbol{h}}$ has a compact support in $\mathbb{R}^n_+$. Thus, we can apply the previous lemma which assures us that there exists a unique $(\boldsymbol{v},\eta)\in(\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)\cap\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+))\times(L^p(\mathbb{R}^n_+)\cap L^2(\mathbb{R}^n_+))$ solution of

$$-\Delta\boldsymbol{v}+\nabla\eta=\widetilde{\boldsymbol{h}}\text{ in }\mathbb{R}^n_+,\quad \text{div }\boldsymbol{v}=0\text{ in }\mathbb{R}^n_+,\quad \boldsymbol{v}=\boldsymbol{0}\text{ on }\mathbb{R}^{n-1}.$$

Noticing that div $\widetilde{\boldsymbol{z}}=0$ in $\mathbb{R}^n_+$, we see that $(\widetilde{\boldsymbol{z}},\widetilde{\pi})$ and $(\boldsymbol{v},\eta)$ are solutions of the same problem, which, thanks to [10], has a unique solution in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)\times L^p(\mathbb{R}^n_+)$. So $(\widetilde{\boldsymbol{z}},\widetilde{\pi})=(\boldsymbol{v},\eta)$ in $\mathbb{R}^n_+$ and, setting again $\boldsymbol{v}$ and η the restrictions of $\boldsymbol{v}$ and η to Ω, we deduce that

$$\boldsymbol{v}=\boldsymbol{z},\quad \eta=\pi\quad\text{in }\Omega.$$

So, $(\boldsymbol{v},\eta)\in\boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ satisfies

$$-\Delta\boldsymbol{v}+\nabla\eta=0\text{ in }\Omega,\quad \text{div }\boldsymbol{v}=0\text{ in }\Omega,\quad \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_0\cup\mathbb{R}^{n-1}.$$

But, $(\boldsymbol{v},\eta)\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2(\Omega)$ and in this space, there is, thanks to Theorem 2.5, a unique solution to the above problem, which is $(\boldsymbol{0},0)$. Thus, $\mathcal{D}_0^p(\Omega)=\{(\boldsymbol{0},0)\}$. $\square$

Now, supposing that $p>2$, we want to solve the Stokes system with homogemeous boundary conditions, that is to say : let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$, we want to find $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ solution of the problem

$$(\mathcal{S}_0)\left\{\begin{array}{lll} -\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f} & \text{in }\Omega, & \text{div }\boldsymbol{u}=h \quad \text{in }\Omega,\\ \boldsymbol{u}=\boldsymbol{0} & \text{on }\Gamma_0, & \boldsymbol{u}=\boldsymbol{0}\quad\quad\text{on }\mathbb{R}^{n-1}.\end{array}\right.$$

First, we establish the following lemma :

Lemma 3.3. *For each $p>2$ and for any $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,p}(\Omega)$ and $h\in L^p(\Omega)$, both with compact support in Ω, there exists a unique $(\boldsymbol{u},\pi)\in(\boldsymbol{W}_0^{1,p}(\Omega)\cap\boldsymbol{W}_0^{1,2}(\Omega))\times(L^p(\Omega)\cap L^2(\Omega))$ solution of $(\mathcal{S}_0)$.*

Proof. Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$ with compact support in Ω. Then, like $p>2$, we easily checks that $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,2}(\Omega)$ and $h\in L^2(\Omega)$ and that

$$\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)}+\|h\|_{L^2(\Omega)}\leq\ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)}+\|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p, ω_0 and the supports of $\boldsymbol{f}$ and h. We deduce from Theorem 2.5 that there exists a unique $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2(\Omega)$ solution of $(\mathcal{S}_0)$. It stays to show that $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$. We denote by $\widetilde{\boldsymbol{u}}\in\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)$ and $\widetilde{\pi}\in L^2(\mathbb{R}^n_+)$ the extensions by 0 in $\mathbb{R}^n_+$ of $\boldsymbol{u}$ and π and we set

$$\widetilde{\boldsymbol{f}}=-\Delta\widetilde{\boldsymbol{u}}+\nabla\widetilde{\pi}\ \text{ and }\ \widetilde{h}=\ \text{div }\widetilde{\boldsymbol{u}}.$$

Let us show now that $\widetilde{\boldsymbol{f}}\in\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and $\widetilde{h}\in L^p(\mathbb{R}^n_+)$. We define the function $\chi\in\mathcal{D}(\Omega)$ such that $\chi=1$ in θ where θ is an open bounded subset of Ω such that supp $\boldsymbol{f}\subset\theta$. We denote by $\widetilde{\chi}$ the extension of χ by 0 in $\mathbb{R}^n_+$. For $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)$, we have

$$<\widetilde{\boldsymbol{f}},\boldsymbol{\varphi}>_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_+),\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)}=<\widetilde{\boldsymbol{f}},\widetilde{\chi}\boldsymbol{\varphi}>_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_+),\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)}=<\boldsymbol{f},\chi\boldsymbol{\varphi}>_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}$$

and for $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$, we have

$$< \widetilde{h}, \varphi >_{\mathcal{D}'(\mathbb{R}^n_+),\mathcal{D}(\mathbb{R}^n_+)} = \int_\Omega h\varphi \, d\boldsymbol{x}.$$

So, $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and $\widetilde{h} \in L^p(\mathbb{R}^n_+)$. Finally, we can apply Lemma 3.1 to conclude that $(\widetilde{\boldsymbol{u}}, \widetilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. Thus, by restriction, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. $\square$

Now, we establish the following theorem :

Theorem 3.4. *For each $p > 2$, there exists a real constant $C > 0$ depending only on ω_0 and p such that the following holds. For any $\boldsymbol{g_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of*

$$(\mathcal{S}') \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \, (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}).$$

Proof. The uniqueness comes from Theorem 3.2. Then, thanks to Proposition 4.1 of [10], there exists a unique $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1}.$$

We denote again by $\boldsymbol{w}$ and τ the restrictions of $\boldsymbol{w}$ and τ to Ω and we set $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{w}_{|\Gamma_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$. Thus, it remains to show that there exists $(\boldsymbol{y}, \lambda) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}'')$

$$(\mathcal{S}'') \begin{cases} -\Delta \boldsymbol{y} + \nabla \lambda = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{y} = 0 \quad \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

For this, let $R > 0$ be such that $\boldsymbol{w}_0 \subset B_R \subset \mathbb{R}^n_+$, $\Omega_R = B_R \cap \Omega$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ with support included in Ω_R such that

$$\int_{\Omega_R} \psi(\boldsymbol{x}) \, d\boldsymbol{x} + \int_{\Gamma_0} \boldsymbol{g} \cdot \boldsymbol{n} \, d\boldsymbol{\sigma} = 0.$$

Thanks to results in bounded domains (see [6]), there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}^{1,p}(\Omega_R) \times L^p(\Omega_R)$ such that

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{0} & \text{in } \Omega_R, \quad \text{div } \boldsymbol{v} = \psi \quad \text{in } \Omega_R, \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial B_R. \end{cases}$$

Next, we extend $(\boldsymbol{v}, \eta)$ by $(\boldsymbol{0}, 0)$ in Ω and we denote by $(\widetilde{\boldsymbol{v}}, \widetilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ this extension which satisfies

$$\begin{cases} -\Delta \widetilde{\boldsymbol{v}} + \nabla \widetilde{\eta} = \boldsymbol{\xi} & \text{in } \Omega, \quad \text{div } \widetilde{\boldsymbol{v}} = \psi \quad \text{in } \Omega, \\ \widetilde{\boldsymbol{v}} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \widetilde{\boldsymbol{v}} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

where $\boldsymbol{\xi} \in \boldsymbol{W}_0^{-1,p}(\Omega)$. We notice that $\boldsymbol{\xi}$ and ψ have a compact support in Ω_R so that by the previous lemma, there exists $(\boldsymbol{z}, \nu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{z} + \nabla \nu = -\boldsymbol{\xi} & \text{in } \Omega, \quad \text{div } \boldsymbol{z} = -\psi \quad \text{in } \Omega, \\ \boldsymbol{z} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{z} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Finally, $(\boldsymbol{y}, \lambda) = (\widetilde{\boldsymbol{v}} + \boldsymbol{z}, \widetilde{\eta} + \nu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}'')$, so $(\boldsymbol{u}, \pi) = (\boldsymbol{w} + \boldsymbol{y}, \mu + \lambda) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}')$ and the estimate follows immediately. $\square$

Now, we can solve the problem with homogeneous boundary conditions in the case $p > 2$.

Theorem 3.5. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_0)$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. The uniqueness comes from Theorem 3.2. Then, as a consequence of the second Poincaré-type inequality given in the introduction, we know there exists a tensor of second order $F \in L^p(\Omega)$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} = \text{div } \widetilde{F}$ and we notice that $\widetilde{\boldsymbol{f}}_{|\Omega} = \boldsymbol{f}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. Thanks to [2], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega)$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{v}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, thus, thanks to Theorem 3.4, there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{w} = -\boldsymbol{v}_{|\mathbb{R}^{n-1}} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, \eta + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}_0)$ and the estimate follows immediately. $\square$

Now, we suppose that $1 < p < 2$. Thanks to the previous theorem,

$$\begin{aligned} T : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) &\longrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega), \\ (\boldsymbol{u}, \pi) &\longrightarrow (-\Delta \boldsymbol{u} + \nabla \pi, \text{div } \boldsymbol{u}), \end{aligned}$$

is an isomorphism. So, by duality,

$$T^* : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega),$$

is also an isomorphism and, as it is standard to check that $T^*(\boldsymbol{u}, \pi) = (-\Delta \boldsymbol{u} - \nabla \pi, -\text{div } \boldsymbol{u})$, we have Theorem 3.5 for any $p < 2$. $\square$

Finally, it remains to return to the general problem with $p \neq 2$ and nonhomogeneous boundary conditions. For this, like for the case $p = 2$, we show that there exists a function $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{g_0}$ in Γ_0 and $\boldsymbol{w} = \boldsymbol{g_1}$ in $\mathbb{R}^{n-1}$. Then, we have just seen that there exists a unique $(\boldsymbol{v}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{w} & \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_0, \\ \text{div } \boldsymbol{v} = h - \text{div } \boldsymbol{w} & \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

In consequence, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is a solution of the problem $(\mathcal{S}_\mathcal{D})$ and we have the following theorem :

Theorem 3.6. *For any $p \neq 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of the problem $(\mathcal{S}_\mathcal{D})$*

$$(\mathcal{S}_\mathcal{D}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

4 Strong solutions and regularity for the Stokes system $(\mathcal{S}_\mathcal{D})$.

In this section, we are interested in the existence of strong solutions of the Stokes system $(\mathcal{S}_\mathcal{D})$, *i.e.* of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{\ell+1}^{2,p}(\Omega) \times W_{\ell+1}^{1,p}(\Omega)$. Here, we limit ourselves to the two cases $\ell = 0$ and $\ell = -1$.

First, we give results for the case $\ell = 0$. We notice that in this case, we have the continuous injections $\boldsymbol{W}_1^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p}(\Omega)$ and $W_1^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. So, the two theorems which follow show that generalized solutions of Theorems 2.5 and 3.6, with a stronger hypothesis on the data, are in fact strong solutions.

Theorem 4.1. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$, $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$ and $h \in W_1^{1,p}(\Omega)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ solution of $(\mathcal{S}_0)$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\Omega)} + \|\pi\|_{W_1^{1,p}(\Omega)} \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\Omega)} + \|h\|_{W_1^{1,p}(\Omega)}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. First, we notice that we have the continuous injections $\boldsymbol{W}_1^{0,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$ because $\dfrac{n}{p'} \neq 1$ and $W_1^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Thus, thanks to Theorems

2.5 ($p=2$) and 3.6 ($p\neq 2$), there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ solution of $(\mathcal{S}_0)$. It remains to show that $(\boldsymbol{u},\pi)\in \boldsymbol{W}_1^{2,p}(\Omega)\times W_1^{1,p}(\Omega)$. For this, we introduce the following partition of unity :

$$\psi_1,\ \psi_2\in C^\infty(\mathbb{R}^n),\ 0\leq \psi_1,\psi_2\leq 1,\ \psi_1+\psi_2=1 \text{ in } \mathbb{R}^n,$$
$$\psi_1=1 \text{ in } B_R,\quad \text{supp } \psi_1\subset B_{R+1},$$

whith $0<R<R'<\infty$ such that $\omega_0\subset B_R\subset B_{R'}\subset \mathbb{R}^n_+$. We set $\Omega_R=\Omega\cap B_R$, $\Omega_{R+1}=\Omega\cap B_{R+1}$, $\boldsymbol{u_i}=\psi_i\boldsymbol{u}\in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega)$ and $\pi_i=\psi_i\pi\in L^p(\Omega)$ for $i=1$ or 2. We notice that supp $(\boldsymbol{u_1},\pi_1)\subset\Omega_{R+1}$ and we denote by $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}},\widetilde{\pi}_1)$ the extension by $(\boldsymbol{0},0)$ of $(\boldsymbol{u_1},\pi_1)$ in ${}^c\omega_0$. Finally, we set

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{1}}=-\Delta\widetilde{\boldsymbol{u}}_{\boldsymbol{1}}+\nabla\widetilde{\pi}_1,\quad \widetilde{h}_1=\ \text{div } \widetilde{\boldsymbol{u}}_{\boldsymbol{1}}$$

and $(\boldsymbol{f_1},h_1)$ their restriction to Ω. We have in Ω :

$$\boldsymbol{f_1}=-\Delta\boldsymbol{u_1}+\nabla\pi_1=\psi_1\boldsymbol{f}-2\nabla\psi_1\cdot\nabla\boldsymbol{u}-\Delta\psi_1\ \boldsymbol{u}+\pi\nabla\psi_1$$

and

$$h_1=\text{div } \boldsymbol{u_1}=\psi_1 h+\ \text{div } \psi_1\boldsymbol{u}.$$

As $\boldsymbol{u}\in \boldsymbol{W}_0^{1,p}(\Omega)$ and supp $\psi_1\subset\Omega_{R+1}$, then $\boldsymbol{f_1}\in \boldsymbol{W}_1^{0,p}(\Omega)$ and $h_1\in W_1^{1,p}(\Omega)$. Thus $\widetilde{\boldsymbol{f}}_{\boldsymbol{1}}\in \boldsymbol{W}_1^{0,p}({}^c\omega_0)$, $\widetilde{h}_1\in W_1^{1,p}({}^c\omega_0)$ and $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}},\widetilde{\pi}_1)$ satisfies

$$\begin{cases} -\Delta\widetilde{\boldsymbol{u}}_{\boldsymbol{1}}+\nabla\widetilde{\pi}_1=\widetilde{\boldsymbol{f}}_{\boldsymbol{1}} & \text{in } {}^c\omega_0,\\ \text{div } \widetilde{\boldsymbol{u}}_{\boldsymbol{1}}=\widetilde{h}_1 & \text{in } {}^c\omega_0,\\ \widetilde{\boldsymbol{u}}_{\boldsymbol{1}}=\boldsymbol{0} & \text{on } \Gamma_0.\end{cases}$$

So, thanks to regularity results in a "classical" exterior domain (see [3]), we have $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}},\widetilde{\pi}_1)\in \boldsymbol{W}_1^{2,p}({}^c\omega_0)\times W_1^{1,p}({}^c\omega_0)$ and consequently $(\boldsymbol{u_1},\pi_1)\in \boldsymbol{W}_1^{2,p}(\Omega)\times W_1^{1,p}(\Omega)$.

Now, we denote by $(\widetilde{\boldsymbol{u}}_{\boldsymbol{2}},\widetilde{\pi}_2)$ the extension by $(\boldsymbol{0},0)$ of $(\boldsymbol{u_2},\pi_2)$ in $\mathbb{R}^n_+$ and

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{2}}=-\Delta\widetilde{\boldsymbol{u}}_{\boldsymbol{2}}+\nabla\widetilde{\pi}_2,\quad \widetilde{h}_2=\ \text{div } \widetilde{\boldsymbol{u}}_{\boldsymbol{2}}.$$

As supp $(\widetilde{\boldsymbol{f}}_{\boldsymbol{2}},\widetilde{h}_2)\subset\Omega$ and as $\widetilde{\boldsymbol{f}}_{\boldsymbol{2}|\Omega}=\boldsymbol{f}-\boldsymbol{f_1}\in \boldsymbol{W}_1^{0,p}(\Omega)$ and $\widetilde{h}_{2|\Omega}=h-h_1\in W_1^{1,p}(\Omega)$, we have

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{2}}\in \boldsymbol{W}_1^{0,p}(\mathbb{R}^n_+),\quad \text{and}\quad \widetilde{h}_2\in W_1^{1,p}(\mathbb{R}^n_+).$$

Thus, thanks to Theorem 5.2 of [10], we deduce from this that $\widetilde{\boldsymbol{u}}_{\boldsymbol{2}}\in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+)$, and $\widetilde{\pi}_2\in W_1^{1,p}(\mathbb{R}^n_+)$. By restriction, we have $\boldsymbol{u_2}\in \boldsymbol{W}_1^{2,p}(\Omega)$, $\pi_2\in W_1^{1,p}(\Omega)$ and so $(\boldsymbol{u},\pi)\in \boldsymbol{W}_1^{2,p}(\Omega)\times W_1^{1,p}(\Omega)$. The estimate follows immediately. $\square$

Now, as at the end of the previous section, we can solve the problem with nonhomogeneous boundary conditions.

Theorem 4.2. *For any $p>1$ satisfying $\dfrac{n}{p'}\neq 1$, $\boldsymbol{f}\in \boldsymbol{W}_1^{0,p}(\Omega)$, $h\in W_1^{1,p}(\Omega)$, $\boldsymbol{g_0}\in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1}\in \boldsymbol{W}_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u},\pi)\in$*

$\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ *solution of the problem* $(\mathcal{S}_{\mathcal{D}})$. *Moreover,* $(\boldsymbol{u}, \pi)$ *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\Omega)} + \|\pi\|_{W_1^{1,p}(\Omega)} \leq C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\Omega)} + \|h\|_{W_1^{1,p}(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C *is a real positive constant which depends only on* p *and* ω_0.

Now, we examine the basic case $\ell = -1$, corresponding to $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. First, we study the kernel of such a problem. We set

$$\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{z}, \pi) \in \ \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega), \ -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega, \\ \text{div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \boldsymbol{z} = \boldsymbol{0} \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}$$

The characterization of this kernel is given by this proposition :

Proposition 4.3. *For each* $p > 1$ *such that* $\dfrac{n}{p'} \neq 1$, *we have the following statements : i) If* $p < n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{0}, 0)\}$.
ii) If $p \geq n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu), \quad \boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}, \ \mu \in \mathbb{R}\}$ *where* $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda})) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ *is the unique solution of*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{\lambda} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Proof. Let $(\boldsymbol{z}, \pi) \in \mathcal{S}_0^p(\Omega)$. We easily show that there exists $(\widetilde{\boldsymbol{z}}, \widetilde{\pi}) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$ such that $(\widetilde{\boldsymbol{z}}, \widetilde{\pi})_{|\Omega} = (\boldsymbol{z}, \pi)$. We set

$$\boldsymbol{\xi} = -\Delta \widetilde{\boldsymbol{z}} + \nabla \widetilde{\pi} \quad \text{and} \quad \sigma = \text{div } \widetilde{\boldsymbol{z}} \quad \text{in } \mathbb{R}_+^n.$$

Then, $\boldsymbol{\xi} \in \boldsymbol{L}^p(\mathbb{R}_+^n)$, $\sigma \in W_0^{1,p}(\mathbb{R}_+^n)$ and $(\widetilde{\boldsymbol{z}}, \widetilde{\pi}) \in \ \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$ satisfies

$$(\mathcal{S}_+) \begin{cases} -\Delta \widetilde{\boldsymbol{z}} + \nabla \widetilde{\pi} = \boldsymbol{\xi} & \text{in } \mathbb{R}_+^n, \\ \text{div } \widetilde{\boldsymbol{z}} = \sigma & \text{in } \mathbb{R}_+^n, \\ \widetilde{\boldsymbol{z}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $\boldsymbol{\xi}$ and σ have a compact support, so $\boldsymbol{\xi} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^n)$, $\sigma \in W_1^{1,p}(\mathbb{R}_+^n)$, and thanks to Theorem 5.2 of [10], there exists $(\boldsymbol{v}, \eta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^n) \times W_1^{1,p}(\mathbb{R}_+^n)) \subset (\boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n))$ solution of $(\mathcal{S}_+)$. Thus, (see Theorem 5.6 in [10]), if $p < n$, we deduce from this that

$$\widetilde{\boldsymbol{z}} = \boldsymbol{v} \quad \text{and} \quad \widetilde{\pi} = \eta \quad \text{in } \mathbb{R}_+^n,$$

and if $p \geq n$, there exists $\boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}$ and $\mu \in \mathbb{R}$ such that

$$\boldsymbol{v} - \widetilde{\boldsymbol{z}} = \boldsymbol{\lambda} \quad \text{and} \quad \eta - \widetilde{\pi} = \mu \quad \text{in } \mathbb{R}_+^n.$$

So, if $p < n$, we have $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and thanks to the uniqueness of the solution of the problem of Theorem 4.1, we conclude that $(\boldsymbol{z}, \pi) = \{(\boldsymbol{0}, 0)\}$ and if $p \geq n$, we have the characterization we were looking for. $\square$

We have the following result, corresponding to Theorem 4.2 :

Theorem 4.4. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u},\pi) \in (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))/\mathcal{S}_0^p(\Omega)$ solution of the problem $(\mathcal{S}_\mathcal{D})$. Moreover, $(\boldsymbol{u},\pi)$ satisfies*

$$\inf_{(\boldsymbol{z},p)\in\mathcal{S}_0^p(\Omega)} (\|\boldsymbol{u}+\boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi+p\|_{W_0^{1,p}(\Omega)}) \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. We easily show that there exist extensions $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p(\mathbb{R}^n)$ of $\boldsymbol{f}$ and $\widetilde{h} \in W_0^{1,p}(\mathbb{R}^n)$ of h in $\mathbb{R}^n$ and, thanks to Theorem 3.10 of [3], there exists $(\boldsymbol{v},\eta) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n) \times W_0^{1,p}(\mathbb{R}^n)$ solution of

$$-\Delta\boldsymbol{v} + \nabla\eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n, \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

Now, it remains to solve the problem : find $(\boldsymbol{z},\mu) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\boldsymbol{z} + \nabla\mu = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} = \boldsymbol{g_0} - \boldsymbol{v}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{z} = \boldsymbol{g_1} - \boldsymbol{v}_{|\mathbb{R}^{n-1}} & \text{on } \mathbb{R}^{n-1}, \end{cases}$$

By Theorem 5.6 of [10], there exists $(\boldsymbol{w},\tau) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta\boldsymbol{w} + \nabla\tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{g_1} - \boldsymbol{v}_{|\mathbb{R}^{n-1}} \text{ on } \mathbb{R}^{n-1}.$$

Moreover, thanks to Theorem 4.2, there exists $(\boldsymbol{y},p) \in (\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)) \subset (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))$ solution of

$$\begin{cases} -\Delta\boldsymbol{y} + \nabla p = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{g_0} - \boldsymbol{v}_{|\Gamma_0} - \boldsymbol{w}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $(\boldsymbol{z},\mu) = (\boldsymbol{y}+\boldsymbol{w}, p+\tau) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and $(\boldsymbol{u},\pi) = (\boldsymbol{v}+\boldsymbol{z}, \eta+\mu)$ is solution to our problem. The estimate follows immediately. $\square$

5 Very weak solutions for the homogeneous Stokes system

The aim of this section is to study the system $(\mathcal{S}_\mathcal{D})$ with $\boldsymbol{f} = \boldsymbol{0}$, $h = 0$ and singular data on the boundary. For this, we must firstly give a meaning to singular data for this problem. More precisely, we want to show that boundary conditions of the form $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_{\ell-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ are meaningful. Here, we limit ourselves to the two cases $\ell = 0$ and $\ell = 1$. Our work is related to that of Amrouche, Nečasovà and Raudin for the half space ([10]) and of Amrouche and Girault for a bounded domain ([6]). We refer to these papers for the ideas of proofs for the first results of this section. Here, we suppose that $\dfrac{n}{p} \neq 1$.

We introduce the space :

$$\boldsymbol{M}_\ell(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ and div } \boldsymbol{u} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\},$$

and we show that we have the identity

$$\boldsymbol{M}_\ell(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ and } \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}.$$

Then, we define

$$\boldsymbol{X}_\ell(\Omega) = \{\boldsymbol{v} \in \overset{\circ}{\boldsymbol{W}}{}^{1,p'}_{-\ell}(\Omega),\ \text{div } \boldsymbol{v} \in \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\Omega)\},$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_\ell(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\Omega)} + \|\text{div } \boldsymbol{v}\|_{W^{1,p'}_{-\ell+1}(\Omega)}.$$

We check that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{X}_\ell(\Omega)$ and we denote by $\boldsymbol{X}'_\ell(\Omega)$ the dual space of $\boldsymbol{X}_\ell(\Omega)$. Now, we introduce the spaces

$$\boldsymbol{T}_\ell(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega),\ \Delta \boldsymbol{v} \in \boldsymbol{X}'_\ell(\Omega)\},$$

$$\boldsymbol{T}_{\ell,\sigma}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{T}_\ell(\Omega),\ \text{div } \boldsymbol{v} = 0 \text{ dans } \Omega\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega)} + \|\Delta \boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\Omega)},$$

where $\|\cdot\|_{\boldsymbol{X}'_\ell(\Omega)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\Omega)$. It can be shown that the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{T}_\ell(\Omega)$ and that the space $\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}),\ \text{div } \boldsymbol{v} = 0\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\Omega)$.

Finally, using exactly the same reasoning as in Lemma 6.4 and Remark 6.5 of [10] and Section 4.2 of [6], we conclude that for a function $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$, the trace of $\boldsymbol{u}$ on Γ_0 is in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and the trace of $\boldsymbol{u}$ on $\mathbb{R}^{n-1}$ is in $\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$. Moreover, we have for any $\boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\Omega)$ and for any $\boldsymbol{v} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$

$$\begin{aligned} <\Delta \boldsymbol{v}, \boldsymbol{\varphi}>_{\boldsymbol{X}'_\ell(\Omega),\boldsymbol{X}_\ell(\Omega)} &= <\boldsymbol{v}, \Delta\boldsymbol{\varphi}>_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega),\boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} \\ &- <\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - <\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}),\boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}. \end{aligned} \tag{5.0}$$

We remind that, for any $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, & (5.1) \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega, & (5.2) \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, & (5.3) \\ \boldsymbol{u} = \boldsymbol{g_1} & \text{on } \mathbb{R}^{n-1}. & (5.4) \end{cases}$$

First, we remark that if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ satisfies (5.1) and (5.2), then $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$ and thus (5.3) and (5.4) make sense. Indeed, the function $\boldsymbol{u}$ is in $\boldsymbol{W}^{0,p}_{\ell-1}(\Omega)$ and div $\boldsymbol{u} = 0$ in Ω. Moreover, because $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{X}_\ell(\Omega)$, we easily show that $\nabla \pi \in \boldsymbol{X}'_\ell(\Omega)$. Thus, thanks to (5.1), we have $\Delta \boldsymbol{u} \in \boldsymbol{X}'_\ell(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$. So, in this case, we have seen that $\boldsymbol{u}_{|\Gamma_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{u}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$.

Proposition 5.1. *For each $p > 1$ such that $\dfrac{n}{p} \neq 1$, we suppose that the functions $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$ satisfy*

$$\boldsymbol{g_0} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \boldsymbol{g_1} \cdot \boldsymbol{n} = 0 \quad \text{on } \mathbb{R}^{n-1}. \tag{5.5}$$

Then, problem (5.1)-(5.4) is equivalent to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{M}_\ell(\Omega)$ and for any $\eta \in W^{1,p'}_{-\ell+1}(\Omega)$, we have

$$(\mathcal{FV}) \quad < \boldsymbol{u}, -\Delta \boldsymbol{v} + \nabla \eta >_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega), \boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} - < \pi, \operatorname{div} \boldsymbol{v} >_{W^{-1,p}_{\ell-1}(\Omega), \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\Omega)} =$$

$$- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}.$$

Proof. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ be a solution of (5.1)-(5.4). Thanks to the previous remark, we have $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$. Let $\boldsymbol{v}$ be in $\boldsymbol{M}_\ell(\Omega)$. We deduce from (5.0) and (5.1) that

$$< \boldsymbol{u}, -\Delta \boldsymbol{v} >_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega), \boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} - < \pi, \operatorname{div} \boldsymbol{v} >_{W^{-1,p}_{\ell-1}(\Omega), \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\Omega)} =$$

$$- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}.$$

Moreover, like $\dfrac{n}{p} \neq 1$, the space $\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}),\ \operatorname{div} \boldsymbol{v} = 0\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\Omega)$ and using (5.2) and (5.5), we show that for any $\eta \in W^{1,p'}_{-\ell+1}(\Omega)$

$$< \boldsymbol{u}, \nabla \eta >_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega), \boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} = 0.$$

Thus, we conclude that $(\boldsymbol{u}, \pi)$ is solution of $(\mathcal{FV})$. Reciprocally, let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ be a solution of $(\mathcal{FV})$. With $\eta = 0$ and $\boldsymbol{v} \in \mathcal{D}(\Omega)$, we have

$$< -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{v} >_{\boldsymbol{\mathcal{D}}'(\Omega), \boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

and with $\boldsymbol{v} = \boldsymbol{0}$ and $\eta \in \mathcal{D}(\Omega)$, we have

$$< \operatorname{div} \boldsymbol{u}, \eta >_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = 0.$$

Thus, (5.1) and (5.2) hold. It remains to show (5.3) and (5.4). Let $\boldsymbol{v} \in \boldsymbol{M}_\ell(\Omega)$. Thanks to Green's formula (5.0) and $(\mathcal{FV})$, we have

$$< \boldsymbol{u}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{u}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})} =$$

$$< \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}.$$

Now, let $\boldsymbol{\mu}$ be in $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)$. We denote by $\boldsymbol{\mu_\tau}$ the tangential component of $\boldsymbol{\mu}$. It is defined by

$$\boldsymbol{\mu} = \boldsymbol{\mu_\tau} + (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}.$$

We easily show that there exists $\boldsymbol{w} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ such that

$$\begin{cases} \boldsymbol{w} = \boldsymbol{0} \text{ and } \dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} & \text{on } \Gamma_0, \\ \boldsymbol{w} = \dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $\boldsymbol{w} \in \boldsymbol{M}_\ell(\Omega)$ and

$$< \boldsymbol{u}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} .$$

Thus

$$< \boldsymbol{u}, \boldsymbol{\mu_\tau} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \boldsymbol{\mu_\tau} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} .$$

Finally, since $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_0 and by hypothesis $\boldsymbol{g_0} \cdot \boldsymbol{n} = 0$ on Γ_0, we conclude that

$$< \boldsymbol{u}, \boldsymbol{\mu} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \boldsymbol{\mu} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)},$$

i.e. $\boldsymbol{u} = \boldsymbol{g_0}$ on Γ_0. Now, let $\boldsymbol{\mu}$ be in $\boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})$. We know that there exists $\boldsymbol{s} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)$ such that

$$\boldsymbol{s} = \boldsymbol{0} \quad \text{and} \quad \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} \quad \text{on } \mathbb{R}^{n-1}.$$

Moreover, as above, we can find $\boldsymbol{y} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ such that

$$\begin{cases} \boldsymbol{y} = -\boldsymbol{s} \text{ and } \dfrac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} = -\dfrac{\partial \boldsymbol{s}}{\partial \boldsymbol{n}} & \text{on } \Gamma_0, \\ \boldsymbol{y} = \dfrac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $\boldsymbol{z} = \boldsymbol{s}_{|\Omega} + \boldsymbol{y} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ satisfies

$$\begin{cases} \boldsymbol{z} = \dfrac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \Gamma_0, \\ \boldsymbol{z} = \boldsymbol{0} \text{ and } \dfrac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Then, $\boldsymbol{z} \in \boldsymbol{M}_\ell(\Omega)$ and we easily conclude like above that $\boldsymbol{u} = \boldsymbol{g_1}$ on $\mathbb{R}^{n-1}$. Thus, we have the equivalence of the two problems. $\square$

Now, we can solve the homogeneous Stokes system (5.1)-(5.4) with singular boundary conditions. We will give separately the results for $\ell = 0$ and $\ell = 1$. Note that the first theorem (for the case $\ell = 0$) extends Theorems 2.5 and 3.6 (with $\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$) since $\boldsymbol{W}^{1,p}_0(\Omega) \subset \boldsymbol{W}^{0,p}_{-1}(\Omega)$ if $n \neq p$.

Theorem 5.2. *For any $p > 1$ such that $\dfrac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$ satisfying (5.5), there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of (5.1)-(5.4). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq C \left(\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}\right),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. In fact, we solve $(\mathcal{FV})$. For this, we argue by duality. Since $\dfrac{n}{p} \neq 1$, thanks to Theorem 4.1, we can say that for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p'}(\Omega)$ and $h \in \overset{\circ}{\boldsymbol{W}}{}_1^{1,p'}(\Omega)$, there exists a unique $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_1^{2,p'}(\Omega) \times W_1^{1,p'}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = h \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_1^{2,p'}(\Omega)} + \|\eta\|_{W_1^{1,p'}(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}(\Omega)} + \|h\|_{W_1^{1,p'}(\Omega)}).$$

Then,

$$\begin{aligned} &| < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_1^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ &\quad \leq C\ (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})})(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}(\Omega)} + \|h\|_{W_1^{1,p'}(\Omega)}). \end{aligned}$$

We can deduce from this that the linear mapping T defined by

$$T(\boldsymbol{f}, h) = < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} > + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >, \tag{5.6}$$

on $\boldsymbol{W}_1^{0,p'}(\Omega) \times \overset{\circ}{\boldsymbol{W}}{}_1^{1,p'}(\Omega)$ is continuous. So, according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\Omega) \times W_{-1}^{-1,p}(\Omega)$ such that

$$\begin{aligned} &< \boldsymbol{u}, \boldsymbol{f} >_{\boldsymbol{W}_{-1}^{0,p}(\Omega), \boldsymbol{W}_1^{0,p'}(\Omega)} + < \pi, h >_{W_{-1}^{-1,p}(\Omega), W_1^{1,p'}(\Omega)} = \\ &- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_1^{\frac{1}{p},p'}(\mathbb{R}^{n-1})}. \end{aligned}$$

Thus, noticing that $\boldsymbol{v} \in \boldsymbol{M}_0(\Omega)$, we deduce that $(\boldsymbol{u}, \pi)$ satisfies $(\mathcal{FV})$. $\square$

The next corollary relaxes the constraint (5.5) on the data. In order to establish this corollary, we give the following lemma.

Lemma 5.3. *For any $p > 1$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a function $s \in W_{-1}^{1,p}(\Omega)$ solution of*

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = g_0 \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \mathbb{R}^{n-1}.$$

Moreover, s satisfies

$$\|s\|_{W_{-1}^{1,p}(\Omega)} \leq C\ (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. Using results in the half space (see Theorem 3.7 in [4]), we know there exists $z \in W_{-1}^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta z = 0 \quad \text{in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \mathbb{R}^{n-1},$$

satisfying

$$\|z\|_{W^{1,p}_{-1}(\mathbb{R}^n_+)} \leq \ C \ \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}.$$

We have $g = g_0 - \dfrac{\partial z}{\partial \boldsymbol{n}} \in W^{-\frac{1}{p},p}(\Gamma_0)$ and it remains to solve the following problem : find $v \in W^{1,p}_{-1}(\Omega)$ solution of

$$\Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on } \mathbb{R}^{n-1}. \tag{5.7}$$

To solve this problem, we solve first the following one : find $y \in W^{1,p}_{-1}(\widetilde{\Omega})$ solution of

$$\Delta y = 0 \quad \text{in } \widetilde{\Omega}, \quad \frac{\partial y}{\partial \boldsymbol{n}} = \widetilde{g} \quad \text{on } \widetilde{\Gamma}_0 \tag{5.8}$$

such that

$$\|y\|_{W^{1,p}_{-1}(\widetilde{\Omega})} \leq \ C \ (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})});$$

here we remind that $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ with Ω' the symmetric region of Ω with respect to $\mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \partial\widetilde{\Omega}$ and that $\widetilde{g}$ is an extension of g in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ symmetric with respect to $\mathbb{R}^{n-1}$ (we refer to Theorem 3.3 in [5] to find such an extension). To find a solution y of (5.8), we split the proof into two cases. First, if $\dfrac{n}{p'} > 1$, we apply Theorem 3.11 in [8] (there is no condition of compatibility), so there exists $y \in W^{1,p}_0(\widetilde{\Omega}) \subset W^{1,p}_{-1}(\widetilde{\Omega})$ solution of (5.8) and satisfying the estimate. Next, if $\dfrac{n}{p'} \leq 1$, we set for any $\boldsymbol{x}$ in $\widetilde{\Omega}$

$$w(\boldsymbol{x}) = -\frac{1}{2\pi} \int_{\widetilde{\Gamma}_0} E(\boldsymbol{x} - \boldsymbol{y}) \ d\boldsymbol{x},$$

where E is the fundamental solution of the Laplacian and we easily show that $w \in W^{1,p}_{-1}(\widetilde{\Omega})$ (but $w \notin W^{1,p}_0(\widetilde{\Omega})$), that $\Delta w = 0$ in $\widetilde{\Omega}$ and $< \dfrac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0} \neq 0$. We define λ by

$$\lambda = \frac{< \widetilde{g}, 1 >_{\widetilde{\Gamma}_0}}{< \dfrac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0}},$$

so that the compatibility condition

$$< \widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0} = 0$$

is satisfied. Thanks to Theorem 3.11 in [8], there exists $u \in W^{1,p}_0(\widetilde{\Omega}) \subset W^{1,p}_{-1}(\widetilde{\Omega})$ solution of

$$\Delta u = 0 \quad \text{in } \widetilde{\Omega}, \quad \frac{\partial u}{\partial \boldsymbol{n}} = \widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}} \quad \text{on } \widetilde{\Gamma}_0$$

satisfying

$$\|u\|_{W^{1,p}_0(\widetilde{\Omega})} \leq \ C \ \|\widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_0)}.$$

Thus, $y = \lambda w + u$ is solution of (5.8) and satisfies the estimate. Now, let $y_0 \in W^{1,p}_{-1}(\widetilde{\Omega})$ a solution of (5.8) and let $s_0 \in W^{1,p}_{-1}(\widetilde{\Omega})$ be defined, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, by

$$s_0(\boldsymbol{x'}, x_n) = y_0(\boldsymbol{x'}, -x_n).$$

Thanks to the symmetry of $\widetilde{\Omega}$ and $\widetilde{g}$ with respect to $\mathbb{R}^{n-1}$, we prove that s_0 is also a solution of (5.8) (here again, for more details, we refer to the proof of Theorem 3.3 in [5]). Then, setting $v = \dfrac{1}{2}(y_0 + s_0)_{|\Omega} \in W^{1,p}_{-1}(\Omega)$, we show that v satisfies (5.7) and we have

$$\|v\|_{W^{1,p}_{-1}(\Omega)} \leq \; C \; (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}).$$

Finally, the function $s = z + v$ solves the problem and the estimate follows immediately. $\square$

Now, we have the following result.

Corollary 5.4. *For any $p > 1$ satisfying $\dfrac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$ there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of (5.1)-(5.4). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq \; C \; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. The uniqueness comes from Theorem 5.2. For the existence, thanks to the previous lemma, there exists $s \in W^{1,p}_{-1}(\Omega)$ solution of

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_0} \cdot \boldsymbol{n} \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_1} \cdot \boldsymbol{n} \quad \text{on } \mathbb{R}^{n-1}.$$

Now, we define $\boldsymbol{w}$ by $\boldsymbol{w} = \nabla s \in \boldsymbol{W}^{0,p}_{-1}(\Omega)$ and we easily show that traces of $\boldsymbol{w}$ on Γ_0 and $\mathbb{R}^{n-1}$ have a sense respectively in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$. We set $\boldsymbol{g_0}^* = \boldsymbol{g_0} - \boldsymbol{w}_{|\Gamma_0}$ and $\boldsymbol{g_1}^* = \boldsymbol{g_0} - \boldsymbol{w}_{|\mathbb{R}^{n-1}}$ and we notice that the functions $\boldsymbol{g_0}^*$ and $\boldsymbol{g_1}^*$ satisfy (5.5). So we can apply the previous theorem and there exists $(\boldsymbol{v}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{g_0}^* & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{g_1}^* \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

and satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq \; C \; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}),$$

Finally, $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi)$ is solution of (5.1)-(5.4) and the estimates follows immediately. $\square$

Now, we describe a result for the case $\ell = 1$.

Theorem 5.5. *For any $p > 1$ such that $\dfrac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying (5.5) and the following compatibility condition if $p \leq \dfrac{n}{n-1}$: for each $(\boldsymbol{z}, p) \in \mathcal{S}_0^{p'}(\Omega)$*

$$< \boldsymbol{g_0}, \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} = 0,$$

there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ solution of (5.1)-(5.4). Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W_0^{-1,p}(\Omega)} \leq \ C \ (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof. In fact, here again, we solve $(\mathcal{FV})$. For this, we apply a duality argument. Like $\dfrac{n}{p} \neq 1$, thanks to Theorem 4.4, we can say that for any $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ and $h \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega)$, there exists a unique $(\boldsymbol{v}, \eta) \in (\boldsymbol{W}_0^{2,p'}(\Omega) \times W_0^{1,p'}(\Omega))/\mathcal{S}_0^{p'}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = h \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

satisfying

$$\inf_{(\boldsymbol{z},p) \in \mathcal{S}_0^{p'}(\Omega)} (\|\boldsymbol{v} + \boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p'}(\Omega)} + \|\eta + p\|_{W_0^{1,p'}(\Omega)}) \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|h\|_{W_0^{1,p'}(\Omega)}).$$

Then, for any $(\boldsymbol{z}, p) \in \mathcal{S}_0^{p'}(\Omega)$

$$\begin{aligned} & | < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ & = | < \boldsymbol{g_0}, \frac{\partial}{\partial \boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{z}) >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} \\ & \qquad + < \boldsymbol{g_1}, \frac{\partial}{\partial \boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{z}) >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ & \leq \ C \ (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})})(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|h\|_{W_0^{1,p'}(\Omega)}). \end{aligned}$$

We deduce from this that the linear mapping T defined on $\boldsymbol{L}^{p'}(\Omega) \times \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega)$ by (5.6) is continuous. So, according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ such that

$$\begin{aligned} & < \boldsymbol{u}, \boldsymbol{f} >_{\boldsymbol{L}^p(\Omega), \boldsymbol{L}^{p'}(\Omega)} + < \pi, h >_{W_0^{-1,p}(\Omega), W_0^{1,p'}(\Omega)} = \\ & - < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} . \end{aligned}$$

Thus, noticing that $\boldsymbol{v} \in \boldsymbol{M}_1(\Omega)$, we deduce that $(\boldsymbol{u}, \pi)$ satisfies $(\mathcal{FV})$. $\square$

Here again, with a similar proof as in Corollary 5.4, we want to relax the constraint on the data :

Corollary 5.6. *For any* $p > \dfrac{n}{n-1}$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ *and* $\boldsymbol{g_1} \in \boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, *there exists a unique* $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ *solution of (5.1)-(5.4). Moreover,* $(\boldsymbol{u}, \pi)$ *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W_0^{-1,p}(\Omega)} \leq \ C \ (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C *is a real positive constant which depends only on* p *and* ω_0.

Proof. When $p > \dfrac{n}{n-1}$, we follow the same reasoning as in Corollary 5.4 using Theorem 3.3 in [5] to find $s \in W_0^{1,p}(\Omega)$ such that

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_0} \cdot \boldsymbol{n} \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_1} \cdot \boldsymbol{n} \quad \text{on } \mathbb{R}^{n-1}.$$

and using the previous theorem. $\square$

Remark : When $1 < p \leq \dfrac{n}{n-1}$, we notice that, because of the compatibility condition of Theorem 5.5, we can not prove a result similar to Corollary 5.4.

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