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# QUADRATIC FUNCTORS ON POINTED CATEGORIES

MANFRED HARTL; CHRISTINE VESPA

**ABSTRACT.** We study polynomial functors of degree 2, called quadratic, with values in the category of abelian groups  $Ab$ , and whose source category is an arbitrary category  $\mathcal{C}$  with null object such that all objects are finite coproducts of a generating object  $E$ . A functorial equivalence is established between quadratic functors from  $\mathcal{C}$  to  $Ab$  and certain minimal algebraic data. Applying this general result to the case where  $E$  is a cogroup these data take a particularly simple form. This application extends results of Baues and Pirashvili obtained for  $\mathcal{C}$  being the category of free groups or modules of finite rank.

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In their fundamental work on homology of spaces thereafter linked to their name [6] Eilenberg and MacLane introduced cross-effects and polynomial functors (see section 1 for definitions). Since then, these functors proved to play a crucial role in unstable homotopy theory; and during the last decade, homological algebra of polynomial functors turned out to be a powerful tool at the crossroad of various fields, such as algebraic K-theory, generic representation theory or cohomology of general linear groups.

In this paper we study polynomial functors of degree 2, called quadratic,  $F : \mathcal{T} \rightarrow Ab$  where  $Ab$  is the category of abelian groups and  $\mathcal{T}$  is an arbitrary pointed algebraic theory, i.e. a category  $\mathcal{T}$  with null object such that all objects are finite coproducts of a generating object  $E$ . For example, the category of finitely generated free objects in any pointed algebraic category is of this type, in particular the category of finitely generated free algebras over any reduced operad. Moreover, the homotopy category of finite one-point unions of a given space is a pointed algebraic theory, and metastable homotopy groups on such a category are examples of quadratic functors. This was one of the motivations for Baues and Pirashvili [1], [4] to study quadratic functors on several particular types of algebraic theories.

In the cited papers, and also in work of the same authors with Dreckmann and Franjou on polynomial functors of higher degree [2], a functorial equivalence is established between polynomial functors on  $\mathcal{T}$  and certain minimal algebraic data: these consist of the values of the cross-effects of the corresponding functor on the generating object  $E$ , and of certain maps relating them. For example, quadratic functors from the category of finitely generated free groups to  $Gr$  correspond to diagrams

$$(M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

called *square groups*, where  $M_e$  is a group,  $M_{ee}$  an abelian group,  $H$  a quadratic map and  $P$  a homomorphism (satisfying certain relations), see [4]. Those functors taking values in  $Ab$  hereby correspond to square groups for which  $H$  is linear, i.e. a homomorphism. Similarly, given a ring  $R$ , quadratic functors from the category of finitely generated free  $R$ -modules to  $Ab$  correspond to diagrams of the same type called *quadratic  $R$ -modules*, but where  $M_e$  is an abelian group endowed with a quadratic action of  $R$ ,  $M_{ee}$  is an  $R \otimes R$ -module, and  $H$  and  $P$  are homomorphisms compatible with the actions of  $R$ , see

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[1]. As a last example, quadratic functors from the category of finite pointed sets to  $Ab$  correspond to diagrams

$$(M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

containing no operator  $H$  but an involution  $T$  of  $M_{ee}$  instead, see [13].

In this paper we show that quadratic functors from an *arbitrary* pointed theory  $\mathcal{T}$  to  $Ab$  are functorially equivalent with diagrams

$$(T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} M_e \xrightarrow{\hat{H}} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

which we call *quadratic  $\mathcal{T}$ -modules*, where  $\Lambda$  is the reduced monoid ring of  $\mathcal{T}(E, E)$ ,  $M_e$  and  $M_{ee}$  are modules over  $\Lambda$  and  $\Lambda \otimes \Lambda$ , resp.,  $T_{11}(cr_2(U))(E, E)$  is a module over  $\Lambda \otimes \Lambda$  encoding maps from  $E$  to  $E \vee E$  in  $\mathcal{T}$ , the map  $T$  is an involution of  $M_{ee}$  and  $\hat{H}$  and  $P$  are homomorphisms compatible with these structures, see Definition 4.6 for details.

A particularly interesting case arises when  $E$  has a cogroup structure in  $\mathcal{T}$ ; note that this holds when  $\mathcal{T}$  is the category of finitely generated free algebras over an operad, or the homotopy category of finite one-point unions of a suspension. So if  $E$  is a cogroup our above data simplify considerably; notably, the map  $\hat{H}$  splits into two maps

$$M_e \xrightarrow{H_1} M_{ee} \xleftarrow{H_2} T_{11}cr_2\mathcal{T}(E, -)(E, E) \otimes coker(P)$$

the first of which generalizes the map  $H$  in the cited examples above, while the second one was not visible in these special cases where it is either trivial or determined by the remaining structure.

Our result shows that in order to model polynomial functors  $F : \mathcal{T} \rightarrow Ab$ , it is *not* sufficient to just add structure maps of the type  $H$  and  $P$  between the various cross-effects of  $F$  as is suggested by the special cases treated in the literature: the more complicated domain of the map  $\hat{H}$  destroys this idealistic picture. On the other hand, the map  $\hat{H}$  has the interesting structure of a morphism of symmetric  $\Lambda \otimes \Lambda$ -modules which had not become apparent so far.

The paper is organized as follows. In the first section we define cross-effects and polynomial functors with values in  $Gr$  and state a number of elementary properties. An apparently new feature is that we introduce bipolynomialization of bifunctors which is a crucial tool in our approach; this is more extensively studied in [7]. In section 2 we provide explicit formulas for the (bi)linearization and quadratization of functors with values in  $Ab$ . In section 3 we review the functorial equivalence between polynomial functors and certain modules essentially due to Johnson and McCarthy, in a more general framework. We here reprove and exploit only the (easy) case of (bi)linear functors. In section 4 we introduce quadratic  $\mathcal{C}$ -modules (for any pointed category  $\mathcal{C}$  with finite sums), and show how they are obtained from quadratic functors, while the inverse construction is provided in section 5: we construct a quadratic tensor product  $X \otimes M$  of an object  $X$  of  $\mathcal{C}$  with a quadratic  $\mathcal{C}$ -module  $M$ , generalizing the corresponding constructions in [1] and [4]. This allows to associate a quadratic functor to a quadratic  $\mathcal{C}$ -module  $M$ , defined by sending  $X$  to  $X \otimes M$ . The computation of  $E \otimes M$  and of the cross-effect of the functor  $- \otimes M$  allows to establish in section 6 the desired (adjoint) equivalence between quadratic functors from  $\mathcal{C}$  to  $Ab$  and quadratic  $\mathcal{C}$ -modules. In the final section we investigate the special case where  $E$  is equipped with a cogroup structure. In the cogroup case quadratic  $\mathcal{C}$ -modules turn out to take a particularly simple form:  $\hat{H}$  splits into two maps  $H_1$  and  $H_2$  as above, where the domain of  $H_2$  is the tensor product with  $M_e$  of a subquotient of the group of maps from  $E$  to  $E \vee E$ , and the involution  $T$  is determined by the remaining structure as  $T = H_1P - 1$ .

In subsequent work we plan to extend our results to quadratic functors with values in the category of *all* groups, starting from a pointed category  $\mathcal{C}$  having a generating

object  $E$  in the more general categorical sense: that is, such that any object of  $\mathcal{C}$  is a colimit of copies of  $E$ . In particular this holds when  $\mathcal{C}$  is the category of models of any algebraic theory.

We expect that our approach generalizes to polynomial functors of higher degree.

CONTENTS

1. Polynomial functors	3
1.1. Generalities on polynomial functors and bifunctors	3
1.2. Algebraic theories and polynomial functors	8
2. Explicit description of the (bi-)linearization and quadratization functor	9
2.1. Linearization and bilinearization functors	10
2.2. Quadratization functor	11
3. Equivalence between polynomial functors and suitable modules	12
3.1. Adjunction between reduced functors and $\Lambda$ -modules	13
3.2. Classification of linear functors	14
3.3. Classification of polynomial functors of arbitrary degree	15
3.4. Classification of bilinear bifunctors	16
4. Quadratic $\mathcal{C}$ -modules	16
4.1. Definition	17
4.2. Quadratic $\mathcal{C}$ -modules obtained from quadratic functors	19
5. Quadratic tensor product	20
5.1. Definition	20
5.2. The quadratic tensor product defines a quadratic functor	22
5.3. Computation of the quadratic tensor product $E \otimes M$	24
5.4. Computation of the cross-effect of $- \otimes M$	26
6. Equivalence between quadratic functors and quadratic $\mathcal{C}$ -modules	29
6.1. The co-unit of the adjunction	29
6.2. The unit of the adjunction	33
7. Application to theories of cogroups	33
7.1. Augmentation ideals: recollections and action of the linearization functor	34
7.2. Computation of $cr_2U(X, Y)$	35
7.3. Computation of $T_{11}cr_2U(X, Y)$	36
7.4. The action of the involution on $T_{11}cr_2U(X, Y)$	38
7.5. The homomorphism $\hat{H}$	38
7.6. The conditions $(QM1)$ and $(QM2)$	39
7.7. Proof of Theorem 7.1	40
7.8. Application: quadratic functors from free groups of finite rank to $Ab$	41
References	44

1. POLYNOMIAL FUNCTORS

**1.1. Generalities on polynomial functors and bifunctors.** Throughout this paper,  $\mathcal{C}$  denotes a pointed category (i.e. having a null object denoted by 0) with finite coproducts denoted by  $\vee$ . Let  $Gr$  and  $Ab$  denote the categories of groups and abelian groups, resp. We begin by giving a definition and basic properties of the cross-effect and of polynomial functors from  $\mathcal{C}$  to  $Gr$ , generalizing those given in [4] for linear and quadratic functors and those given by Eilenberg and Mac Lane in the case of functors from an abelian category to  $Ab$  [6].

In the sequel,  $\mathcal{D}$  denotes one of the categories  $Gr$  or  $Ab$ . Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. We often note  $f_* = F(f)$  for a morphism  $f$  in  $\mathcal{C}$ . For objects  $X_1, \dots, X_n$  of  $\mathcal{C}$  and  $1 \leq k \leq n$  let

$$X_k \xrightarrow{i_k^n} X_1 \vee \dots \vee X_n \xrightarrow{r_k^n} X_k$$

be the canonical injection and retraction, resp., the latter being defined by  $r_k^n i_k^n = 1_{X_k}$  and  $r_k^n i_p^n = 0$  if  $p \neq k$ .

**Definition 1.1.** *The  $n$ -th cross-effect of  $F$  is a functor  $cr_n F : \mathcal{C}^{\times n} \rightarrow \mathcal{D}$  (or a multi-functor) defined inductively by*

$$cr_1 F(X) = \ker(F(0) : F(X) \rightarrow F(0))$$

$$cr_2 F(X) = \ker((F(r_1^2), F(r_2^2))^t : F(X \vee Y) \rightarrow F(X) \times F(Y))$$

and, for  $n \geq 3$ , by

$$cr_n F(X_1, \dots, X_n) = cr_2(cr_{n-1}(-, X_3, \dots, X_n))(X_1, X_2).$$

In other words, to define the  $n$ -th cross-effect of  $F$  we consider the  $(n-1)$ -st cross-effect, we fix the  $n-2$  last variables and we consider the second cross-effect of this functor. One often writes  $F(X_1 | \dots | X_n) = cr_n F(X_1, \dots, X_n)$ .

Note that  $F(X) \simeq cr_1 F(X) \rtimes F(0)$  as  $F(0) : F(0) \rightarrow F(X)$  is a natural section of  $F(0) : F(X) \rightarrow F(0)$ . Moreover, one easily checks by induction that  $cr_n F(X_1, \dots, X_n)$  is a subgroup of  $F(X_1 \vee \dots \vee X_n)$ .

In this paper we are mainly interested in *reduced* functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ , that is satisfying  $F(0) = 0$ . We denote by  $Func_*(\mathcal{C}, \mathcal{D})$  the category of reduced functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

There is an alternative description of cross-effects for reduced functors. To state this let  $r_{12\dots(k-1)(k+1)\dots n}^n : X_1 \vee \dots \vee X_n \rightarrow X_1 \vee \dots \vee \hat{X}_k \vee \dots \vee X_n$  be the map whose restriction to  $X_i$  is its canonical injection for  $i \neq k$  and is the zero map if  $i = k$ .

**Proposition 1.2.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a reduced functor. Then the  $n$ -th cross-effect  $cr_n F(X_1, \dots, X_n)$  is equal to the kernel of the natural homomorphism*

$$F(X_1 \vee \dots \vee X_n) \xrightarrow{\prod_{k=1}^n F(r_{12\dots(k-1)(k+1)\dots n}^n)} \prod_{k=1}^n F(X_1 \vee \dots \vee \hat{X}_k \vee \dots \vee X_n).$$

As a consequence, we see that  $cr_n F(X_1, \dots, X_n)$  actually is a *normal* subgroup of  $F(X_1 \vee \dots \vee X_n)$ . Moreover, it follows that  $cr_n F(X_1, \dots, X_n)$  is symmetric in  $X_1, \dots, X_n$ . Finally, we see that the functor  $cr_n$  is *multi-reduced*, i.e.,  $cr_n F(X_1, \dots, X_n)$  vanishes if one of the  $X_k$  is the zero object since then  $F(r_{12\dots(k-1)(k+1)\dots n}^n)$  is an isomorphism.

The importance of cross-effects comes from the following property of functors with values in  $Ab$ .

**Proposition 1.3.** *Let  $F : \mathcal{C} \rightarrow Ab$  be a reduced functor. Then there is a natural decomposition*

$$F(X_1 \vee \dots \vee X_n) \simeq \bigoplus_{k=1}^n \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} cr_k F(X_{i_1}, \dots, X_{i_k}).$$

The cross-effects have the following crucial property.

**Proposition 1.4.** *The functor  $cr_n : Func(\mathcal{C}, \mathcal{D}) \rightarrow Func(\mathcal{C}^{\times n}, \mathcal{D})$  is exact for all  $n \geq 1$ .*

*Proof.* For  $n = 1$  it is a consequence of the natural decomposition  $F(X) \simeq cr_1 F(X) \rtimes F(0)$ . For  $n = 2$  this follows from the snake-lemma. For higher  $n$  use induction.  $\square$

**Definition 1.5.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is said to be polynomial of degree lower or equal to  $n$  if  $cr_{n+1}F = 0$ . Such a functor is called linear if  $n = 1$  and is called quadratic if  $n = 2$ . We denote by  $\text{Func}(\mathcal{C}, \mathcal{D})_{\leq n}$  the full subcategory of  $\text{Func}(\mathcal{C}, \mathcal{D})$  consisting of polynomial functors of degree lower or equal to  $n$ .

The category  $\text{Func}(\mathcal{C}, \mathcal{D})_{\leq n}$  has the following fundamental property which is an immediate consequence of Proposition 1.4.

**Proposition 1.6.** The category  $\text{Func}(\mathcal{C}, \mathcal{D})_{\leq n}$  is thick i.e. closed under quotients, subobjects and extensions.

Throughtout this paper we denote by  $\Delta_{\mathcal{C}}^n : \mathcal{C} \rightarrow \mathcal{C}^{\times n}$  the diagonal functor. For  $n = 2$  we write  $\Delta_{\mathcal{C}}$  instead of  $\Delta_{\mathcal{C}}^2$ .

**Definition 1.7.** For  $F \in \text{Func}(\mathcal{C}, \mathcal{D})$  and  $X \in \mathcal{C}$ , we denote by  $S_n^F$  the natural map  $S_n^F : (cr_n F)\Delta_{\mathcal{C}}^n \rightarrow F$  given by the composition

$$cr_n F(X, \dots, X) \xrightarrow{\text{inc}} F(\bigvee_{i=1}^n X) \xrightarrow{F(\nabla^n)} F(X)$$

where  $\nabla^n : \bigvee_{i=1}^n X \rightarrow X$  is the folding map.

Note that the image of  $S_n^F$  is normal in  $F(X)$ ; in fact,  $cr_n F(X, \dots, X)$  is normal in  $F(\bigvee_{i=1}^n X)$ , and  $F(\nabla^n)$  is surjective admitting  $F(i_1^n)$  as a section. This fact is used in the following:

**Definition 1.8.** For  $F \in \text{Func}(\mathcal{C}, \mathcal{D})$  the  $n$ -Taylorisation functor  $T_n : \text{Func}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Func}(\mathcal{C}, \mathcal{D})_{\leq n}$  is defined by:  $T_n F = \text{coker}((cr_{n+1} F)\Delta_{\mathcal{C}}^{n+1} \xrightarrow{S_{n+1}^F} F)$ . We call  $T_1$  the linearization functor and  $T_2$  the quadratization functor.

Let  $U_n : \text{Func}(\mathcal{C}, \mathcal{D})_{\leq n} \rightarrow \text{Func}(\mathcal{C}, \mathcal{D})$  denote the forgetful (i.e. inclusion) functor.

**Proposition 1.9.** The  $n$ -Taylorisation functor  $T_n : \text{Func}(\mathcal{C}, \mathcal{D}) \rightarrow (\text{Func}(\mathcal{C}, \mathcal{D}))_{\leq n}$  is a left adjoint of  $U_n$ . The unit of the adjunction is the natural epimorphism  $t_n : F \rightarrow T_n F$  which is an isomorphism if  $F$  is polynomial of degree  $\leq n$ .

Thus, we obtain the diagram:

$$\begin{array}{ccccccc} & & F & & & & \\ & t_{n+1} \swarrow & \downarrow t_n & \searrow t_{n-1} & & & \\ \cdots & \longrightarrow & T_{n+1}F & \xrightarrow{q_{n+1}} & T_n F & \xrightarrow{q_n} & T_{n-1}F & \xrightarrow{q_{n-1}} & \cdots & \longrightarrow & T_1 F & \longrightarrow & T_0 F = 0. \end{array}$$

The following calculations of  $T_1 F$  for special  $F$  are needed in section 7. Recall the following fact:

**Lemma 1.10** ([4] Lemma 1.6). For  $F : \mathcal{C} \rightarrow \text{Gr}$  a reduced linear functor,  $F(X)$  is an abelian group for  $X \in \mathcal{C}$ .

For a group  $G$  and  $a, b \in G$  let  $[a, b] = aba^{-1}b^{-1}$  and let  $G^{ab} = G/[G, G]$  denote the abelianization of  $G$ .

**Proposition 1.11.** Let  $\text{Id}_{\text{Gr}} : \text{Gr} \rightarrow \text{Gr}$  be the identity functor. There is a natural isomorphism of functors  $\text{Gr} \rightarrow \text{Gr}$

$$\Gamma_1 : T_1(\text{Id}_{\text{Gr}})(G) \xrightarrow{\cong} G^{ab}$$

such that for  $g \in G$ , one has  $\Gamma_1(\bar{g}) = t_1(g)$ .

*Proof.* The map  $\Gamma_1$  is welldefined by Lemma 1.10. The natural homomorphism  $ab : G \rightarrow G^{ab}$  factors through  $t_1$  followed by a map  $\overline{ab} : T_1(\text{Id}_{\text{Gr}})(G) \rightarrow G^{ab}$  since the abelianization functor  $G \mapsto G^{ab}$  is linear. It is straightforward to check that  $\overline{ab}$  is the inverse of  $\Gamma_1$ .  $\square$

Recall that a reduced functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called *diagonalizable* if there exists a bireduced bifunctor  $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  such that  $F = B\Delta_{\mathcal{C}}$ .

**Proposition 1.12.** *The linearization of a diagonalizable functor  $F$  is trivial.*

*Proof.* The section  $B(i_1^2, i_2^2)$  of the map  $B\Delta_{\mathcal{C}}(\nabla^2) : B(X \vee X, X \vee X) \rightarrow B(X, X)$  takes values in  $cr_2(B\Delta_{\mathcal{C}})(X, X)$  since  $B(r_k^2, r_k^2)B(i_1^2, i_2^2) = 0$  for  $k = 1, 2$ . Hence  $S_2^F$  is surjective and  $T_1F = 0$ . □

Since the cross-effect  $cr_2F$  of a functor is a bifunctor we need some general definitions and facts about bifunctors.

**Definition 1.13.** *A bifunctor  $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  is said to be bireduced if for all  $X \in \mathcal{C}$ ,  $B(X, 0) = B(0, X) = 0$ . We denote by  $BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, \mathcal{D})$  the category of bireduced bifunctors from  $\mathcal{C} \times \mathcal{C}$  to  $\mathcal{D}$ .*

*A bireduced bifunctor  $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  is said to be bipolynomial of bidegree  $\leq (n, m)$  if for all  $X \in \mathcal{C}$  the functors  $B(-, X), B(X, -) : \mathcal{C} \rightarrow \mathcal{D}$  are polynomial of degree  $\leq n$  and  $\leq m$  respectively. We denote by  $BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, \mathcal{D})_{\leq(n,m)}$  the category of bipolynomial bifunctors of bidegree  $\leq (n, m)$ .*

**Proposition 1.14.** *The category  $BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, \mathcal{D})_{\leq(n,m)}$  is thick.*

**Definition 1.15.** *Let  $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  be a bireduced bifunctor and  $n, m \geq 1$ . Then the bifunctor  $T_{n,m}B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$  is defined by  $T_{n,m}B(X, Y) = B(X, Y)/N_1N_2$  where*

$$N_1 = \text{im}(S_{n+1}^{B(-, Y)} : cr_{n+1}(-, Y)(X, \dots, X) \rightarrow B(X, Y)),$$

$$N_2 = \text{im}(S_{m+1}^{B(X, -)} : cr_{m+1}B(X, -)(Y, \dots, Y) \rightarrow B(X, Y)).$$

For  $(n, m) = (1, 1)$  we call  $T_{1,1}B$  the *bilinearization* of  $B$ .

Let  $U_{n,m} : (BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, \mathcal{D}))_{\leq(n,m)} \rightarrow BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, \mathcal{D})$  be the forgetful functor.

**Proposition 1.16.** *The  $(n, m)$ -Taylorisation functor  $T_{n,m} : BiFunc_{*,*}(\mathcal{C} \times \mathcal{C}, Ab) \rightarrow (BiFunc_{*,*}(\mathcal{C}, Ab))_{\leq(n,m)}$  is a left adjoint of  $U_{n,m}$ . The unit of this adjunction is given by the natural epimorphism  $t_{n,m} : B \rightarrow T_{n,m}B$ .*

**Notation.** For brevity we will often write  $T_{11}$  instead of  $T_{1,1}$  and  $\bar{x}$  instead of  $t_1(x)$  or  $t_{11}(x)$  for  $x \in F(X)$  or  $x \in B(X, Y)$ , resp.

**Example 1.17.** *For reduced functors  $F, G : \mathcal{C} \rightarrow Ab$  define the bifunctor  $F \boxtimes G : \mathcal{C} \times \mathcal{C} \rightarrow Ab$  by  $(F \boxtimes G)(X, Y) = F(X) \otimes F(Y)$ . Then there is a natural isomorphism*

$$T_{n,m}(F \boxtimes G) \simeq T_nF \boxtimes T_mG.$$

This is immediate from right-exactness of the tensor product. The following less trivial example is needed in section 7.

**Proposition 1.18.** *Let  $Id_{Gr} : Gr \rightarrow Gr$  be the identity functor. There is a natural isomorphism of bifunctors  $Gr \times Gr \rightarrow Ab$*

$$\Gamma_{11} : T_{11}cr_2(Id_{Gr})(G, H) \xleftarrow{\simeq} G^{ab} \otimes H^{ab}$$

such that for  $g \in G, h \in H$  one has  $\Gamma_{11}(\bar{g} \otimes \bar{h}) = t_{11}([i_1^2g, i_2^2h])$ .

*Proof.* Write  $G^* = G \setminus \{1\}$ , and for a set  $E$  let  $L(E)$  denote the free group with basis  $E$ . It is wellknown that there is an isomorphism

$$\sigma : L(G^* \times H^*) \xrightarrow{\simeq} cr_2(Id_{Gr})(G, H)$$

such that  $\sigma(g, h) = [i_1^2 g, i_2^2 h]$ , see [11]. Let  $B : Gr \times Gr \rightarrow Gr$  denote the bifunctor given by  $B(G, H) = L(G \times H)/N \simeq L(G^* \times H^*)$  where  $N$  is the normal subgroup generated by  $G \times \{1\} \cup \{1\} \times H$ . Let  $\pi : L(G \times H) \rightarrow B(G \times H)$  be the canonical projection. The natural homomorphism  $\Gamma'_{11} : B(G \times H) \rightarrow G^{ab} \otimes H^{ab}$  sending  $(g, h)$  to  $\bar{g} \otimes \bar{h}$  factors through  $t_{11}$  followed by a map  $\bar{\Gamma}'_{11} : T_{11}B(G, H) \rightarrow G^{ab} \otimes H^{ab}$  since the bifunctor sending  $(G, H)$  to  $G^{ab} \otimes H^{ab}$  is bilinear. So it remains to show that  $\bar{\Gamma}'_{11}$  is an isomorphism. For this it suffices to check that the map  $b : G \times H \rightarrow T_{11}B(G, H)$  sending  $(g, h)$  to  $t_{11}\pi(g, h)$  is bilinear, thus providing an inverse of  $\bar{\Gamma}'_{11}$ . To show that  $b$  is linear in  $h$  consider the map  $B(Id_G, \nabla^2) : B(G, H \vee H) \rightarrow B(G, H)$ . One has

$$x = \pi((g, h_1 h_2)(g, h_2)^{-1}(g, h_1)^{-1}) = B(Id_G, \nabla^2)(y)$$

with  $y = \pi((g, i_1^2(h_1)i_2^2(h_2))(g, i_2^2(h_2))^{-1}(g, i_1^2(h_1))^{-1})$ . But

$$B(Id_G, r_1^2)(y) = \pi((g, h_1)(g, 1)^{-1}(g, h_1)^{-1}) = 1$$

and

$$B(Id_G, r_2^2)(y) = \pi((g, h_2)(g, h_2)^{-1}(g, 1)^{-1}) = 1,$$

whence  $y \in B(G, -)(H | H)$ . Thus  $x \in im(S_2^{B(G, -)})$ , whence  $t_{11}(x) = 1$ . Thus  $b$  is linear in  $h$ . Similarly one shows that  $b$  is linear in  $g$ , as desired.  $\square$

In the following proposition we give another characterization of the quadratization functor which is useful in the sequel.

**Proposition 1.19.** *For  $F \in Func_*(\mathcal{C}, \mathcal{D})$  and  $X \in \mathcal{C}$  we have:*

$$T_2(F) = coker\left(ker\left(cr_2F(X, X) \xrightarrow{t_{11}} (T_{11}cr_2F)(X, X)\right) \xrightarrow{S_2^F} F(X)\right).$$

*Proof.* We have the following commutative diagram

$$\begin{array}{ccccc} cr_3F(X, X, X) = cr_2(cr_2F(-, X))(X, X) & \xrightarrow{i_{(12)}^2} & cr_2F(-, X)(X \vee X) & \xrightarrow{i_{(12)}^3} & F(X \vee X \vee X) \\ & \searrow S_2^{F(-|X)} & \downarrow F(\nabla^2|1) & & \downarrow F(\nabla^3) \\ & & cr_2F(-, X)(X) & \xrightarrow{S_2^F} & F(X) \\ & & & \searrow t_1 & \\ & & & & T_1(cr_2F(-, X))(X) \end{array}$$

where the right hand square commutes by the following commutative diagram

$$\begin{array}{ccc} cr_2F(X \vee X, X) & \xrightarrow{\quad} & F(X \vee X \vee X) \\ \downarrow F(\nabla^2|1) & \nearrow F(\nabla^2 \vee 1) & \downarrow F(\nabla^3) \\ & F(X \vee X) & \\ \uparrow i_{(12)}^2 & \searrow F(\nabla^2) & \\ cr_2F(X, X) & \xrightarrow{S_2^F} & F(X). \end{array}$$

By Proposition 1.8

$$T_2F(X) = coker(cr_3F(X, X, X) \xrightarrow{S_3^F} F(X)) = coker(cr_3F(X, X, X) \xrightarrow{F(\nabla^3)i_{(12)}^3i_{(12)}^2} F(X)).$$



We deduce from the first diagram that

$$(1.19.1) \quad T_2F(X) = \operatorname{coker}(cr_3F(X, X, X) \xrightarrow{S_2^F S_2^{F(-|X)}} F(X))$$

$$(1.19.2) \quad = \operatorname{coker}\left(\ker\left(cr_2F(-, X)(X) \xrightarrow{t_1} T_1cr_2F(-, X)(X)\right) \xrightarrow{S_2^F} F(X)\right).$$

Considering  $cr_2F(X, -)$  instead of  $cr_2F(-, X)$ , we can write down similar commutative diagrams which imply

$$(1.19.3) \quad T_2F(X) = \operatorname{coker}\left(\ker\left(cr_2F(X, -)(X) \xrightarrow{t_1} T_1cr_2F(X, -)(X)\right) \xrightarrow{S_2^F} F(X)\right).$$

Combining 1.19.1 and 1.19.3 we deduce the result.  $\square$

Finally, the following easy lemma is useful in the sequel.

**Lemma 1.20.** *If  $B : \mathcal{C} \times \mathcal{C} \rightarrow Ab$  is a bilinear bireduced bifunctor then  $B\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow Ab$  is a quadratic functor.*

*Proof.* For  $X, Y \in \mathcal{C}$  we have:

$$\begin{aligned} B\Delta_{\mathcal{C}}(X \vee Y) &= B(X \vee Y, X \vee Y) = B(X, X) \oplus B(Y, Y) \oplus B(X, Y) \oplus B(Y, X) \\ &= B\Delta_{\mathcal{C}}(X) \oplus B\Delta_{\mathcal{C}}(Y) \oplus B(X, Y) \oplus B(Y, X) \end{aligned}$$

where the second equality follows from the bilinearity of  $B$ . We deduce that  $cr_2(B\Delta_{\mathcal{C}})(X, Y) = B(X, Y) \oplus B(Y, X)$  which is a bilinear functor. So  $B\Delta_{\mathcal{C}}$  is quadratic.  $\square$

From now on, we only consider functors on  $\mathcal{C}$  with values in  $Ab$ .

**1.2. Algebraic theories and polynomial functors.** When  $\mathcal{C}$  is an algebraic theory, the polynomial functors from  $\mathcal{C}$  to  $Ab$  of degree  $n$  have the crucial property that they are determined by their values on  $n$  objects of  $\mathcal{C}$ .

**Recollections on algebraic theories.** We here recall and discuss the definition of a pointed algebraic theory used in this paper and many others by Baues, Jibladze and Pirashvili.

**Definition 1.21.** *A pointed (algebraic) theory  $\mathcal{T}$  is a pointed category  $\mathcal{T}$  with an object  $E$  such that any object of  $\mathcal{T}$  is isomorphic to a finite sum of copies of  $E$ . In particular, for any object  $E$  of  $\mathcal{C}$  we denote by  $\langle E \rangle_{\mathcal{C}}$  the theory generated by  $E$ , i.e. the full subcategory of  $\mathcal{C}$  consisting of the objects  $E^{\vee n} = E \vee \dots \vee E$  ( $n$  times),  $n \geq 0$ , with  $E^{\vee 0} = 0$ .*

Note that this definition of an algebraic theory is dual to the classical one as being a category encoding algebraic operations, see [5]. Thus a model of a theory  $\mathcal{T}$  in our sense is a *contravariant* functor from  $\mathcal{T}$  to the category of sets transforming coproducts into products. The advantage of this definition is that here  $\mathcal{T}$  identifies with a full subcategory of its category of models, namely the category of free models of  $\mathcal{T}$  of finite rank [5]. This allows the quadratic functors we construct in section 5, from data depending only on  $\mathcal{T}$ , to be naturally defined on the whole category of models of  $\mathcal{T}$ ; indeed, on all of  $\mathcal{C}$  in the more general case where  $\mathcal{T} = \langle E \rangle_{\mathcal{C}}$ . We will show in subsequent work that actually all quadratic functors on  $\mathcal{C}$  with good properties are of this form if  $\mathcal{C}$  is generated by  $E$  in a suitable generalized sense.

**Polynomial functors on algebraic theories.** The following property of polynomial functors is crucial in the sequel.

**Proposition 1.22.** *Let  $F, G : \mathcal{C} \rightarrow Ab$  be two polynomial functors of degree lower or equal to  $n$  and  $\phi : F \rightarrow G$  be a natural transformation of functors. If  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$ , the following statements are equivalent:*

- (1)  $\phi$  is a natural isomorphism;
- (2)  $\forall k \leq n$ ,  $\phi_{E^{\vee k}}$  is an isomorphism;
- (3)  $\phi_{E^{\vee n}}$  is an isomorphism;
- (4)  $\forall k$  such that  $1 \leq k \leq n$ ,  $cr_k(\phi)_{E, \dots, E}$  is an isomorphism.

*Proof.* Clearly (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). The implication (3)  $\Rightarrow$  (4) is a consequence of the natural decomposition given in Proposition 1.3. To prove (4)  $\Rightarrow$  (1), let  $p \in \mathbb{N}$  and  $m = \min(p, n)$ . Proposition 1.3 provides a natural decomposition:

$$F(X_1 \vee \dots \vee X_p) \simeq \bigoplus_{k=1}^p \bigoplus_{1 \leq i_1 < \dots < i_k \leq p} cr_k F(X_{i_1}, \dots, X_{i_k}) \simeq \bigoplus_{k=1}^m \bigoplus_{1 \leq i_1 < \dots < i_k \leq p} cr_k F(X_{i_1}, \dots, X_{i_k})$$

since  $F$  is supposed to be polynomial of degree  $n$ . Using the analogous decomposition for  $G(X_1 \vee \dots \vee X_p)$  we have:

$$\phi_{X_1 \vee \dots \vee X_p} \simeq \bigoplus_{k=1}^m \bigoplus_{1 \leq i_1 < \dots < i_k \leq p} cr_k(\phi)_{X_{i_1}, \dots, X_{i_k}}.$$

For  $X_1 = \dots = X_p = E$  we deduce that  $\phi_{E^{\vee p}}$  is an isomorphism.  $\square$

Proposition 1.22 implies the following analogue for bipolynomial bifunctors.

**Corollary 1.23.** *If  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$ , for  $B, D : \mathcal{C} \times \mathcal{C} \rightarrow Ab$  two bipolynomial bifunctors of bidegree lower or equal to  $(n, m)$  and  $\phi : B \rightarrow D$  a natural transformation of functors. Then  $\phi$  is an natural equivalence if and only if  $\phi_{(E^{\vee k}, E^{\vee l})}$  is an isomorphism for all  $k \leq n$  and  $l \leq m$ .*

Let  $F : \mathcal{C} \rightarrow Ab$  be a reduced functor. Consider the natural map of bifunctors:

$$cr_2(t_2) : cr_2(F) \rightarrow cr_2(T_2 F).$$

Since  $T_{11}$  is the left adjoint of the forgetful functor  $U : (BiFunc_*(\mathcal{C}, Ab))_{\leq (1,1)} \rightarrow BiFunc_*(\mathcal{C}, Ab)$  we obtain that  $cr_2(t_2)$  factors through the unit map  $t_{11} : cr_2(F) \rightarrow T_{11}(cr_2(F))$ , thus providing a canonical morphism of bifunctors:

$$\overline{cr_2(t_2)} : T_{11}(cr_2(F)) \rightarrow cr_2(T_2 F).$$

The main result of this paper heavily relies on the following special case of a more general result in [7].

**Theorem 1.24.** *The morphism  $\overline{cr_2(t_2)} : T_{11}(cr_2(F)) \rightarrow cr_2(T_2 F)$  is an isomorphism of bifunctors.*

## 2. EXPLICIT DESCRIPTION OF THE (BI-)LINEARIZATION AND QUADRATIZATION FUNCTOR

The general principle of this section is to express the values  $T_n F(X)$  of the functor  $T_n F$  as a cokernel of a map  $F(X^{\vee(n+1)}) \rightarrow F(X)$  instead of a map  $cr_{n+1} F(X, \dots, X) \rightarrow F(X)$  since the elements of  $cr_{n+1} F(X, \dots, X)$  are more difficult to describe than the elements of  $F(X^{\vee(n+1)})$ . This requires some notations: in the diagram

$$\begin{array}{ccc}
& \xleftarrow{((i_{12\dots(k-1)(k+1)\dots n}^n)_*, (i_k^n)_*)} & \\
F(X_1 \vee \dots \vee X_n) & \xrightarrow{((r_{12\dots(k-1)(k+1)\dots n}^n)_*, (r_k^n)_*)^t} & F(X_1 \vee \dots \vee \hat{X}_k \vee \dots \vee X_n) \oplus F(X_k)
\end{array}$$

the map  $i_{12\dots(k-1)(k+1)\dots n}^n$  is the obvious injection.

Considering the kernel of  $((r_{12\dots(k-1)(k+1)\dots n}^n)_*, (r_k^n)_*)^t$  we obtain the maps:

$$\begin{array}{ccc}
& \xleftarrow{\rho_{(12\dots(k-1)(k+1)\dots n, k)}^n} & \\
F(X_1 \vee \dots \vee \hat{X}_k \vee \dots \vee X_n | X_k) & \xrightarrow{\iota_{(12\dots(k-1)(k+1)\dots n, k)}^n} & F(X_1 \vee \dots \vee X_n).
\end{array}$$

where  $\rho_{(12\dots(k-1)(k+1)\dots n, k)}^n$  is the retraction induced by the section  $((i_{12\dots(k-1)(k+1)\dots n}^n)_*, (i_k^n)_*)$  of  $((r_{12\dots(k-1)(k+1)\dots n}^n)_*, (r_k^n)_*)^t$ .

In particular we have the following split short exact sequence:

$$(2.0.1) \quad 0 \longrightarrow F(X|Y) \xrightarrow{\iota_{(1,2)}^2} F(X \vee Y) \xrightarrow{(i_{1*}^2, i_{2*}^2)} F(X) \oplus F(Y) \longrightarrow 0$$

which implies that:

$$(2.0.2) \quad Id_{F(X \vee Y)} = \iota_{(1,2)}^2 \circ \rho_{(1,2)}^2 + i_{1*}^2 \circ r_{1*}^2 + i_{2*}^2 \circ r_{2*}^2$$

so

$$Im(\iota_{(1,2)}^2) = Im(Id_{F(X \vee Y)} - i_{1*}^2 \circ r_{1*}^2 - i_{2*}^2 \circ r_{2*}^2).$$

Furthermore, we obtain a natural isomorphism of bifunctors

$$(2.0.3) \quad F(X|Y) \simeq F(X \vee Y) / i_{1*}^2 F(X) + i_{2*}^2 F(Y).$$

**2.1. Linearization and bilinearization functors.** In the following Proposition we give an explicit description of the linearization functor  $T_1$ .

**Proposition 2.1.** *For  $F \in Func_*(\mathcal{C}, Ab)$  and  $X \in \mathcal{C}$  we have:*

$$T_1(F)(X) = coker(F(X \vee X) \xrightarrow{S_2^F \circ \rho_{(1,2)}^2} F(X))$$

so

$$\begin{aligned}
T_1 F(X) &= F(X) / Im(S_2^F \circ \rho_{(1,2)}^2) = F(X) / \{\nabla_*^2(x) - r_{1*}^2(x) - r_{2*}^2(x) \mid x \in F(X \vee X)\} \\
&= F(X) / \{((1, 1)_* - (1, 0)_* - (0, 1)_*)(x) \mid x \in F(X \vee X)\}
\end{aligned}$$

*Proof.* The map  $\rho_{(1,2)}^2$  is surjective by the short exact sequence 2.0.1, hence

$$T_1(F)(X) = coker(F(X | X) \xrightarrow{S_2^F} F(X)) = coker(F(X \vee X) \xrightarrow{S_2^F \circ \rho_{(1,2)}^2} F(X))$$

and for  $x \in F(X \vee X)$  we have by 2.0.2:

$$\begin{aligned}
S_2^F \circ \rho_{(1,2)}^2 &= F(\nabla^2) \iota_{(1,2)}^2 \rho_{(1,2)}^2 = F(\nabla^2)(Id - i_{1*}^2 \circ r_{1*}^2 - i_{2*}^2 \circ r_{2*}^2) \\
&= \nabla_*^2 - r_{1*}^2 - r_{2*}^2.
\end{aligned}$$

□

Similarly we obtain:

**Proposition 2.2.** For  $B \in \text{BiFunc}_{*,*}(\mathcal{C} \times \mathcal{C}, \text{Ab})$  and  $X, Y \in \mathcal{C}$  we have:

$$T_{11}B(X, Y) = \text{coker} \left( B(X \vee X, Y) \oplus B(X, Y \vee Y) \xrightarrow{(S_2^{cr_2 B(-, Y)} \circ (\rho_{(1,2)}^2)^X, S_2^{cr_2 B(X, -)} \circ (\rho_{(1,2)}^2)^Y)} B(X, Y) \right)$$

where  $(\rho_{(1,2)}^2)^X : B(-, Y)(X \vee X) \rightarrow cr_2 B(-, Y)(X, X)$  and  $(\rho_{(1,2)}^2)^Y : B(X, -)(Y \vee Y) \rightarrow cr_2 B(X, -)(Y, Y)$ , so

$$\begin{aligned} T_{11}B(X, Y) &= B(X, Y) / \text{Im}((S_2^F \circ (\rho_{(1,2)}^2)^X, S_2^F \circ (\rho_{(1,2)}^2)^Y)) \\ &= B(X, Y) / \{B(\nabla^2, Id)(x) - B(r_1^2, Id)(x) - B(r_2^2, Id)(x) + B(Id, \nabla^2)(y) - B(Id, r_1^2)(y) - B(Id, r_2^2)(y) \\ &\quad | x \in B(X \vee X, Y), y \in B(X, Y \vee Y)\} \end{aligned}$$

Recall that for  $F \in \text{Func}_*(\mathcal{C}, \text{Ab})$  we have  $cr_2 F \in \text{BiFunc}_{*,*}(\mathcal{C} \times \mathcal{C}, \text{Ab})$ , so we can consider the bilinearization of the bifunctor  $cr_2 F$ .

Applying the previous proposition to  $cr_2 F$  give us a description of  $T_{11}cr_2 F(X, Y)$  as a quotient of  $cr_2 F(X, Y)$  where the relations are obtained from elements in  $cr_2 F(X \vee X, Y)$  and  $cr_2 F(X, Y \vee Y)$ . A more manageable description of  $T_{11}cr_2 F(X, Y)$ , as a quotient of  $F(X \vee Y)$ , is given as follows:

**Proposition 2.3.** For  $F \in \text{Func}_*(\mathcal{C}, \text{Ab})$  and  $X, Y \in \mathcal{C}$  we have:

$$\begin{aligned} T_{11}cr_2 F(X|Y) &= F(X \vee Y) / \{A(z_1), B(z_2), i_{1*}^2(x), i_{2*}^2(y) | x \in F(X), y \in F(Y), \\ &\quad z_1 \in F(X \vee X \vee Y), z_2 \in F(X \vee Y \vee Y)\} \end{aligned}$$

where

$$A = F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)$$

and

$$B = F(Id \vee \nabla^2) - F(Id \vee r_1^2) - F(Id \vee r_2^2).$$

*Proof.* By Proposition 2.1 we have

$$T_1(cr_2 F(-, Y))(X) = \text{coker}(F(\nabla^2 | Id) - F(r_1^2 | Id) - F(r_2^2 | Id)).$$

We obtain the term  $A$  from the following commutative diagram where the vertical arrows are isomorphisms of bifunctors by (2.0.3):

$$\begin{array}{ccc} cr_2 F(-, Y)(X \vee X) & \xrightarrow{F(\nabla^2 | Id) - F(r_1^2 | Id) - F(r_2^2 | Id)} & cr_2 F(-, Y)(X) \\ \simeq \downarrow & & \downarrow \simeq \\ F((X \vee X) \vee Y) / (i_{12*}^3 F(X \vee X) + i_{3*}^3 F(Y)) & \xrightarrow{F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)} & F(X \vee Y) / (i_{1*}^2 F(X) + i_{2*}^2 F(Y)). \end{array}$$

Similarly, considering  $T_1(cr_2 F(X, -))(Y)$  we obtain the term  $B$ .  $\square$

## 2.2. Quadraticization functor.

**Proposition 2.4.** For  $F \in \text{Func}_*(\mathcal{C}, \mathcal{A})$  and  $X \in \mathcal{C}$  we have:

$$\begin{aligned} T_2 F(X) &= F(X) / \{(\nabla_*^3 - (\nabla^2 r_{12}^3)_* - (\nabla^2 r_{13}^3)_* - (\nabla^2 r_{23}^3)_* + r_{1*}^3 + r_{2*}^3 + r_{3*}^3)(x) | x \in F(X \vee X \vee X)\} \\ &= F(X) / \{((1, 1, 1)_* - (1, 1, 0)_* - (1, 0, 1)_* - (0, 1, 1)_* + (1, 0, 0)_* + (0, 1, 0)_* + (0, 0, 1)_*)(x) | x \in F(X \vee X \vee X)\}. \end{aligned}$$

*Proof.* By the proof of Proposition 1.19 we have

$$T_2 F(X) = \text{coker}(cr_3 F(X, X, X) \xrightarrow{S_2^F S_2^{cr_2 F(-, X)}} F(X)).$$

By the following commutative diagram:

$$\begin{array}{ccccc}
& & \xleftarrow{\iota_{(1,2)}^2} & & \\
& & cr_2F(-, X)(X \vee X) & \xrightarrow{cr_2F(\nabla^2, Id)} & cr_2(cr_2F(-, X))(X) \\
& \uparrow Id - cr_2F(i_1^2 r_1^2, Id) - cr_2F(i_2^2 r_2^2, Id) & & & \downarrow S_2^{cr_2F(-, X)} \\
& cr_2F(-, X)(X \vee X) & \xrightarrow{F(\nabla^2 | Id) - F(r_1^2 | Id) - F(r_2^2 | Id)} & & cr_2F(-, X)(X) \\
& \uparrow \rho_{(12,3)}^3 & & & \downarrow \iota_{(1,2)}^2 \\
& F((X \vee X) \vee X) & \xrightarrow{F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)} & & F(X \vee X) \xrightarrow{F(\nabla^2)} F(X) \\
& \downarrow \iota_{(12,3)}^3 & & & \uparrow S_2^F
\end{array}$$

we have

$$\begin{aligned}
Im(S_2^F S_2^{cr_2F(-, X)}) &= Im(S_2^F cr_2F(\nabla^2, Id) \iota_{(1,2)}^2) \\
&= Im(S_2^F cr_2F(\nabla^2, Id) \iota_{(1,2)}^2 \rho_{(1,2)}^2) \text{ since } \rho_{(1,2)}^2 \text{ is surjective} \\
&= Im(S_2^F cr_2F(\nabla^2, Id) (Id - cr_2F(i_1^2 r_1^2, Id) - cr_2F(i_2^2 r_2^2, Id))) \text{ by 2.0.2} \\
&= Im(S_2^F (F(\nabla^2 | Id) - F(r_1^2 | Id) - F(r_2^2 | Id))) \\
&= Im(S_2^F (F(\nabla^2 | Id) - F(r_1^2 | Id) - F(r_2^2 | Id)) \rho_{(12,3)}^3) \text{ since } \rho_{(12,3)}^3 \text{ is surjective} \\
&= Im(F(\nabla^2)(F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)) \iota_{(1,2)}^2 \rho_{(12,3)}^3).
\end{aligned}$$

By the following short exact sequence

$$0 \longrightarrow F(X \vee X | Y) \xrightarrow[\iota_{(12,3)}^3]{\rho_{(12,3)}^3} F(X \vee X \vee Y) \xrightarrow[(r_{12*}^3, r_{3*}^3)]{(i_{12*}^3, i_{3*}^3)} F(X \vee X) \oplus F(Y) \longrightarrow 0$$

we obtain:

$$\begin{aligned}
&(F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)) \iota_{(12,3)}^3 \rho_{(12,3)}^3 \\
&= (F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id)) (Id - i_{12*}^3 r_{12*}^3 - i_{3*}^3 r_{3*}^3) \\
&= F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id) - F(\iota_1^2 \nabla^2 r_{12}^3) + F(i_1^2 r_1^3) + F(i_1^2 r_2^3) \\
&\quad - F(i_2^2 r_3^3) + F(i_2^2 r_3^3) + F(i_2^2 r_3^3) \\
&= F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id) - F(\iota_1^2 \nabla^2 r_{12}^3) + F(i_1^2 r_1^3) + F(i_1^2 r_2^3) + F(i_2^2 r_3^3).
\end{aligned}$$

Hence

$$\begin{aligned}
&Im(S_2^F S_2^{cr_2F(-, X)}) \\
&= Im(F(\nabla^2)(F(\nabla^2 \vee Id) - F(r_1^2 \vee Id) - F(r_2^2 \vee Id) - F(\iota_1^2 \nabla^2 r_{12}^3) + F(i_1^2 r_1^3) + F(i_1^2 r_2^3) + F(i_2^2 r_3^3))) \\
&= Im(F(\nabla^3) - F(\nabla^2 r_{13}^3) - F(\nabla^2 r_{23}^3) - F(\nabla^2 r_{12}^3) + F(r_1^3) + F(r_2^3) + F(r_3^3))
\end{aligned}$$

□

### 3. EQUIVALENCE BETWEEN POLYNOMIAL FUNCTORS AND SUITABLE MODULES

In this section we give a classification of polynomial functors by modules over suitable rings essentially due to Johnson and McCarthy in [9, 10]. Although this provides a classification of polynomial functors of all degrees, it is not satisfactory since the rings that appear are very complicated. So this complete classification does not seem to be manageable for functors of degree higher than 1. Therefore our aim is to describe polynomial functors by minimal data, which is achieved for quadratic functors in this paper.

We start with some definitions.

Let  $E$  be a fixed object of  $\mathcal{C}$ . For a set  $S$ , let  $\mathbb{Z}[S]$  denote the free abelian group with basis  $S$ . Since for all  $X \in \mathcal{C}$ ,  $\mathcal{C}(E, X)$  is pointed with basepoint the zero map, we can define a subfunctor  $\mathbb{Z}[0]$  of  $\mathbb{Z}[\mathcal{C}(E, -)]$  by  $\mathbb{Z}[0](X) = \mathbb{Z}[\{E \xrightarrow{0} X\}]$  for  $X \in \mathcal{C}$ . This allows us to give the following definition.

**Definition 3.1.** *The universal functor  $U_E : \mathcal{C} \rightarrow \text{Ab}$  relative to  $E$  is the quotient of  $\mathbb{Z}[\mathcal{C}(E, -)]$  by the subfunctor  $\mathbb{Z}[0]$ .*

To keep notation simple we write  $f$  also for the equivalence class in  $U_E(X)$  of an element  $f$  of  $\mathcal{C}(E, X)$  and we often omit the subscript  $E$  in  $U_E$ . We need the following obvious fact.

**Proposition 3.2.** *Composition in  $\mathcal{C}$  induces a ring structure on  $\Lambda := U(E)$  and a structure of right  $\Lambda$ -module on  $U(X)$  for any  $X \in \mathcal{C}$ .*

**3.1. Adjunction between reduced functors and  $\Lambda$ -modules.** In this section, we give an adjunction between reduced functors and  $\Lambda$ -modules which is the starting point of the equivalence between polynomial functors and module categories given in the sequel. We begin by the following straightforward lemma.

**Lemma 3.3.** *For  $F : \mathcal{C} \rightarrow \text{Ab}$  a reduced functor,  $F(E)$  is a left  $\Lambda$ -module via*

$$\alpha.x := F(\alpha)(x)$$

for  $\alpha \in \mathcal{C}(E, E)$  and  $x \in F(E)$ .

So, we can give the following definition.

**Definition 3.4.** *The functor*

$$\mathbb{S} : \text{Func}_*(\mathcal{C}, \text{Ab}) \rightarrow \Lambda\text{-Mod}$$

is defined by  $\mathbb{S}(F) = F(E)$  for  $F \in \text{Func}_*(\mathcal{C}, \text{Ab})$ .

A left adjoint of  $\mathbb{S}$  is provided in the following definition.

**Definition 3.5.** *The functor*

$$\mathbb{T} : \Lambda\text{-Mod} \rightarrow \text{Func}_*(\mathcal{C}, \text{Ab})$$

is defined by  $\mathbb{T}(M)(X) = U(X) \otimes_{\Lambda} M$  for  $M \in \Lambda\text{-Mod}$ .

**Proposition 3.6.** *The functor  $\mathbb{T}$  is a left adjoint of  $\mathbb{S}$ .*

*The **unit** of this adjunction is the canonical isomorphism*

$$u_M : M \xrightarrow{\cong} \Lambda \otimes_{\Lambda} M = \mathbb{S}\mathbb{T}(M) \quad \text{for } M \in \Lambda\text{-Mod}.$$

*The **co-unit** is*

$$(u'_F)_X : \mathbb{T}\mathbb{S}(F)(X) = U(X) \otimes_{\Lambda} F(E) \rightarrow F(X),$$

where  $(u'_F)_X(f \otimes x) = F(f)(x)$  for  $f \in \mathcal{C}(E, X)$ , and  $x \in F(E)$ .

We consider  $u'_F$  as a first order approximation of  $F$ ; if  $F$  is polynomial of degree  $n$  then  $u'_F$  may be reduced to a morphism

$$\overline{u'_F} : T_n(\mathbb{T}\mathbb{S}(F)) \rightarrow F.$$

This turns out to be an isomorphism for  $n = 1$  but is not for  $n > 1$ . So our approach to polynomial functors consists of inductively improving the approximation  $\overline{u'_F}$  in order to get an isomorphism again, by taking into account higher and higher cross-effects.

**3.2. Classification of linear functors.** Let  $Lin(\mathcal{C}, Ab)$  denote the category of linear reduced functors from  $\mathcal{C}$  to  $Ab$ . In this section we show that if  $\mathcal{C}$  is a pointed theory then the category  $Lin(\mathcal{C}, Ab)$  is equivalent to the category of modules over a suitable ring.

We begin by providing a number of equivalent characterizations of linear functors.

**Lemma 3.7.** *Let  $F : \mathcal{C} \rightarrow Ab$  be a reduced functor. Then the following conditions are equivalent*

- (1)  $F$  is linear;
- (2)  $S_2^F = 0$  where  $S_2^F : cr_2(F)\Delta_{\mathcal{C}} \rightarrow F$  is defined in Definition 1.7;
- (3) For  $X, Y \in \mathcal{C}$  one has

$$1_{F(X \vee Y)} = i_{1*}^2 r_{1*}^2 + i_{2*}^2 r_{2*}^2;$$

- (4) For  $X, Y \in \mathcal{C}$  and  $\xi \in \mathcal{C}(X, Y \vee Y)$  one has

$$F(\nabla^2 \xi) = F(r_1^2 \xi) + F(r_2^2 \xi).$$

*Proof.* By definition,  $F$  is linear if  $cr_2(F) = 0$ . Since  $S_2^F : cr_2(F)\Delta_{\mathcal{C}} \rightarrow F$  we have  $S_2^F = 0$ . Conversely, if  $S_2^F = 0$ , by Proposition 1.8,  $F \simeq T_1 F$  so  $F$  is linear and we proved that (1)  $\Leftrightarrow$  (2).

By the short exact sequence (2.0.1) we have (1)  $\Leftrightarrow$  (3).

By Proposition 1.8,  $F$  is linear if and only if  $F \simeq T_1 F$ , so by Proposition 2.1 this is equivalent to,  $\forall y \in F(Y \vee Y): F(\nabla^2)(y) = F(r_1^2)(y) + F(r_2^2)(y)$ . Applying the last equality to  $y = F(\xi)(x) \in F(Y \vee Y)$  where  $\xi \in \mathcal{C}(X, Y \vee Y)$  and  $x \in F(X)$ , we obtain (1)  $\Leftrightarrow$  (4).  $\square$

**Proposition 3.8.** *The abelian group  $\bar{\Lambda} := (T_1 U)(E)$  is a ring and  $T_1 U(X)$  has a right  $\bar{\Lambda}$ -module structure such that  $t_1$  is  $\Lambda$ -equivariant (i.e. for  $\lambda \in \Lambda$  and  $x \in U(X)$ ,  $t_1(x\lambda) = t_1(x)t_1(\lambda)$ .)*

*Proof.* For  $f \in \mathcal{C}(E, E)$  the relation  $(S_2^U)_E cr_2 U(f, f) = U(f)(S_2^U)_E$  shows that  $Im((S_2^U)_E)$  is a left ideal of  $\Lambda$ , and we deduce from the following commutative diagram that  $Ker(t_1) = Im((S_2^U)_E)$  is a right ideal, too:

$$\begin{array}{ccc} U(E) & \xrightarrow{\alpha^*} & U(E) \\ t_1 \downarrow & & \downarrow t_1 \\ T_1 U(E) & \xrightarrow{T_1(\alpha^*)} & T_1 U(E) \end{array}$$

where  $\alpha : E \rightarrow E$ . Consequently  $\bar{\Lambda}$  is a ring.

For  $X \in \mathcal{C}$ ,  $T_1 U(X)$  is a right  $\bar{\Lambda}$ -module via

$$t_1(f).t_1(\alpha) = t_1(f\alpha)$$

for  $f \in \mathcal{C}(E, X)$  and  $\alpha \in \mathcal{C}(E, E)$ ; this is welldefined again by naturality of  $S_2^U$ .  $\square$

**Lemma 3.9.** *For  $F : \mathcal{C} \rightarrow Ab$  a linear reduced functor,  $F(E)$  is a left  $\bar{\Lambda}$ -module via*

$$\bar{\alpha}.x := F(\alpha)(x)$$

for  $\alpha \in \mathcal{C}(E, E)$  and  $x \in F(E)$ .

*Proof.* By Proposition 2.1 we have

$$\bar{\Lambda} = \Lambda / \langle \nabla^2 \xi - r_1^2 \xi - r_2^2 \xi \mid \xi \in \mathcal{C}(E, E \vee E) \rangle.$$

But

$$(\nabla^2 \xi - r_1^2 \xi - r_2^2 \xi).x = \nabla^2 \xi.x - r_1^2 \xi.x - r_2^2 \xi.x = F(\nabla^2 \xi)(x) - F(r_1^2 \xi)(x) - F(r_2^2 \xi)(x)$$

$$= (F(\nabla^2\xi) - F(r_1^2\xi) - F(r_2^2\xi))(x) = 0$$

by Lemma 3.7 (4). □

This leads to the following definition.

**Definition 3.10.** *The functor*

$$\mathbb{S}_1 : \text{Lin}(\mathcal{C}, \text{Ab}) \rightarrow \overline{\Lambda} - \text{Mod}$$

is defined by  $\mathbb{S}_1(F) = F(E)$  for  $F \in \text{Lin}(\mathcal{C}, \text{Ab})$ .

**Definition 3.11.** *The functor*

$$\mathbb{T}_1 : \overline{\Lambda} - \text{Mod} \rightarrow \text{Lin}(\mathcal{C}, \text{Ab})$$

is defined by  $\mathbb{T}_1(M)(X) = T_1U(X) \otimes_{\overline{\Lambda}} M$  for  $M \in \overline{\Lambda} - \text{Mod}$ .

The following proposition connects the functors  $\mathbb{S}_1$  and  $\mathbb{T}_1$ .

**Proposition 3.12.** *The functor  $\mathbb{T}_1$  is the left adjoint of  $\mathbb{S}_1$ .*

*The **unit** of this adjunction is the canonical isomorphism*

$$u_M : M \xrightarrow{\cong} \overline{\Lambda} \otimes_{\overline{\Lambda}} M = \mathbb{S}_1\mathbb{T}_1(M) \quad \text{for } M \in \overline{\Lambda} - \text{Mod}.$$

*The **co-unit** is*

$$(u'_F)_X : \mathbb{T}_1\mathbb{S}_1(F)(X) = T_1U(X) \otimes_{\overline{\Lambda}} F(E) \rightarrow F(X),$$

where  $(u'_F)_X(t_1(f) \otimes x) = F(f)(x)$  for  $f \in \mathcal{C}(E, X)$ , and  $x \in F(E)$ .

A classification of linear functors is now given as follows:

**Theorem 3.13.** *If  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$  then the functors  $\mathbb{S}_1$  and  $\mathbb{T}_1$  form a pair of adjoint equivalences.*

*Proof.* It remains to show that the counit is an isomorphism. For  $F \in \text{Lin}(\mathcal{C}, \text{Ab})$  the source and target functor of  $u'_F$  are linear, so it suffices by Proposition 1.22 to prove that  $(u'_F)_E : \overline{\Lambda} \otimes_{\overline{\Lambda}} F(E) \rightarrow F(E)$  is an isomorphism. We have  $(u'_F)_E(t_1(f) \otimes x) = F(f)(x) = t_1(f).x$  so  $(u'_F)_E$  coincides with the canonical isomorphism. □

**3.3. Classification of polynomial functors of arbitrary degree.** Theorem 3.13 can be generalized to polynomial functors of arbitrary degree; this can be deduced from a more general result of Johnson and McCarthy on polynomial functors with values in categories of chain complexes [9, 10], by identifying the category  $\text{Ab}$  with the category of chain complexes concentrated in a given degree.

**Theorem 3.14.** *Suppose that  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$ . Then the group  $T_nU_{E^{\vee n}}(E^{\vee n})$  has a ring structure induced by composition in  $\mathcal{C}$ , and there is an equivalence of categories*

$$\mathbb{S}_n : \text{Func}(\mathcal{C}, \text{Ab})_{\leq n} \rightarrow T_nU_{E^{\vee n}}(E^{\vee n}) - \text{Mod}$$

given by  $\mathbb{S}_n(F) = F(E)$ .

This result generalizes a similar one for additive categories  $\mathcal{C}$  due to Pirashvili [12]. A simple direct proof of Theorem 3.14 is given in [7]. Further study of polynomial functors could thus be based on this theorem in analyzing the - very complicated - structure of the rings  $T_nU(E)$  and their representations; instead, we follow the basic idea of the work of Baues and Pirashvili: according to Proposition 1.22 a polynomial functor  $F$  of degree  $n$  is determined by the values of its cross-effects  $cr_k(F)$  on  $(E, \dots, E)$ ,  $1 \leq k \leq n$ ; so we seek for minimal extra structure relating them so as to make the correspondance between polynomial functors and these enriched cross-effects into a functorial equivalence.



**3.4. Classification of bilinear bifunctors.** Since cross-effects of quadratic functors are bilinear we need an analogue of Theorem 3.13 for bifunctors which goes as follows. Let  $\text{Bilin}(\mathcal{C} \times \mathcal{C}, Ab)$  denote the category of bilinear bireduced bifunctors from  $\mathcal{C} \times \mathcal{C}$  to  $Ab$ .

We begin by the following lemma.

**Lemma 3.15.** *For  $B : \mathcal{C} \times \mathcal{C} \rightarrow Ab$  a bilinear bireduced bifunctor,  $B(E, E)$  is a left  $\overline{\Lambda} \otimes \overline{\Lambda}$ -module via*

$$(\overline{\alpha} \otimes \overline{\beta}).x := B(\alpha, \beta)(x)$$

for  $\alpha, \beta \in \mathcal{C}(E, E)$  and  $x \in B(E, E)$ .

**Remark 3.16.** *Note that  $T_1U(X) \otimes T_1U(Y)$  is a right  $\Lambda \otimes \Lambda$ -module via  $(t_1(x) \otimes t_1(y))(\alpha \otimes \beta) = t_1(x\alpha) \otimes t_1(y\beta)$ ; this induces a structure of right  $\overline{\Lambda} \otimes \overline{\Lambda}$ -module.*

**Definition 3.17.** *The functor*

$$\mathbb{S}_{11} : \text{Bilin}(\mathcal{C} \times \mathcal{C}, Ab) \rightarrow \overline{\Lambda} \otimes \overline{\Lambda} - \text{Mod}$$

is defined by  $\mathbb{S}_{11}(B) = B(E, E)$  for  $B \in \text{Bilin}(\mathcal{C} \times \mathcal{C}, Ab)$ .

**Definition 3.18.** *The functor*

$$\mathbb{T}_{11} : \overline{\Lambda} \otimes \overline{\Lambda} - \text{Mod} \rightarrow \text{Bilin}(\mathcal{C} \times \mathcal{C}, Ab)$$

is defined by  $\mathbb{T}_{11}(M)(X, Y) = (T_1U(X) \otimes T_1U(Y)) \otimes_{\Lambda \otimes \Lambda} M$  for  $M \in \overline{\Lambda} \otimes \overline{\Lambda} - \text{Mod}$ .

The following proposition connects the functors  $\mathbb{T}_{11}$  and  $\mathbb{S}_{11}$ .

**Proposition 3.19.** *The functor  $\mathbb{T}_{11}$  is the left adjoint of  $\mathbb{S}_{11}$ .*

*The **unit** of this adjunction is the canonical isomorphism*

$$u_M : M \rightarrow (T_1U(E) \otimes T_1U(E)) \otimes_{\overline{\Lambda} \otimes \overline{\Lambda}} M$$

defined by  $u_M(m) = (t_1(1) \otimes t_1(1)) \otimes m$  where  $M \in \overline{\Lambda} \otimes \overline{\Lambda} - \text{Mod}$ .

*The **co-unit** is*

$$(u'_B)_{X,Y} : \mathbb{T}_{11}\mathbb{S}_{11}(B)(X, Y) = (T_1U(X) \otimes T_1U(Y)) \otimes_{\overline{\Lambda} \otimes \overline{\Lambda}} B(E, E) \rightarrow B(X, Y),$$

where  $(u'_B)_{X,Y}(t_1(f) \otimes t_1(g) \otimes x) = B(f, g)(x)$  for  $f \in \mathcal{C}(E, X), g \in \mathcal{C}(E, Y)$  and  $x \in B(E, E)$ .

The arguments in the proof of Theorem 3.13 are easily adapted to obtain:

**Theorem 3.20.** *If  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$  then the functors  $\mathbb{S}_{11}$  and  $\mathbb{T}_{11}$  form a pair of adjoint equivalences.*

#### 4. QUADRATIC $\mathcal{C}$ -MODULES

The aim of this section is to define quadratic  $\mathcal{C}$ -modules which generalize for any pointed category  $\mathcal{C}$  the quadratic  $\mathbb{Z}$ -modules considered by Baues in [1] for  $\mathcal{C} = Ab$ . Then we define a functor  $\mathbb{S}_2$  from quadratic functors to quadratic  $\mathcal{C}$ -modules which extends the functor  $\mathbb{S}_1 : \text{Lin}(\mathcal{C}, Ab) \rightarrow \overline{\Lambda} - \text{Mod}$  considered in Proposition 3.10. The functor  $\mathbb{S}_2$  is a part of the adjoint equivalence established in Theorem 6.1. Its left adjoint  $\mathbb{T}_2$  is constructed in section 5.

4.1. **Definition.** We begin by recalling the definition of a symmetric  $R \otimes R$ -module.

**Definition 4.1.** For a ring  $R$ , a symmetric  $R \otimes R$ -module is a left  $R \otimes R$ -module  $M$  equipped with a  $\mathbb{Z}$ -linear involution  $T$  (i.e.  $T^2 = \text{Id}$ ) such that for  $r, s \in R$  and  $m \in M$

$$T((r \otimes s)m) = (s \otimes r)T(m).$$

A morphism of symmetric  $R \otimes R$ -module is a morphism of  $R \otimes R$ -module compatible with the respective involutions.

In the following two propositions, we give examples of symmetric  $R \otimes R$ -modules for suitable  $R$  which are important in the sequel.

**Proposition 4.2.** For a reduced functor  $F : \mathcal{C} \rightarrow \text{Ab}$ , the group  $F(E|E)$  admits the structure of a symmetric  $\Lambda \otimes \Lambda$ -module such that for  $f, g \in \mathcal{C}(E, E)$  and  $\alpha \in F(E|E)$  we have

$$(f \otimes g).\alpha := (\iota_{(1,2)}^2)^{-1} F(f \vee g) \iota_{(1,2)}^2(\alpha) \in F(E|E)$$

and with involution:

$$T^F(\alpha) = (\iota_{(1,2)}^2)^{-1} F(\tau) \iota_{(1,2)}^2(\alpha) \in F(E|E)$$

where  $\tau : E \vee E \rightarrow E \vee E$  is the canonical switch.

*Proof.* By functoriality of  $F$  the first relation defines a module structure. The map  $T^F$  is clearly an involution and we have:

$$\begin{aligned} T^F((f \otimes g).\alpha) &= (\iota_{(1,2)}^2)^{-1} F(\tau) \iota_{(1,2)}^2((\iota_{(1,2)}^2)^{-1} F(f \vee g) \iota_{(1,2)}^2(\alpha)) \\ &= (\iota_{(1,2)}^2)^{-1} F(\tau(f \vee g)) \iota_{(1,2)}^2(\alpha) \\ &= (\iota_{(1,2)}^2)^{-1} F((g \vee f)\tau) \iota_{(1,2)}^2(\alpha) \\ &= (\iota_{(1,2)}^2)^{-1} F(g \vee f) \iota_{(1,2)}^2 (\iota_{(1,2)}^2)^{-1} F(\tau) \iota_{(1,2)}^2(\alpha) \\ &= (g \otimes f)T^F(\alpha) \end{aligned}$$

□

**Proposition 4.3.** For  $F : \mathcal{C} \rightarrow \text{Ab}$  a reduced functor, the group  $T_{11}(cr_2 F)(E, E)$  admits the structure of a symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -module, with involution denoted by  $\bar{T}^F$ , such that the projection  $t_{11} : F(E|E) \rightarrow T_{11}(cr_2 F)(E, E)$  is a morphism of symmetric  $\Lambda \otimes \Lambda$ -modules.

**Definition 4.4.** A pre-quadratic  $\mathcal{C}$ -module relative to  $E$  is a diagram of group homomorphisms:

$$M = (T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} M_e \xrightarrow{\hat{H}} M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

where

- $M_e$  is a left  $\Lambda$ -module;
- $M_{ee}$  is a symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -module with involution  $T$ ;
- $P : M_{ee} \rightarrow M_e$  is a homomorphism of  $\Lambda$ -modules with respect to the diagonal action of  $\Lambda$  on  $M_{ee}$ , i.e. for  $\alpha \in \mathcal{C}(E, E)$  and  $m \in M_{ee}$ :

$$P((\bar{\alpha} \otimes \bar{\alpha})m) = \alpha P(m),$$

and satisfies  $PT = P$ ;

- $\hat{H}$  is a homomorphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules such that for  $\xi \in \mathcal{C}(E, E \vee E)$  and  $a \in M_e$  the following relation holds:

$$(QM1) \quad (\nabla^2 \xi)a = (r_1^2 \xi)a + (r_2^2 \xi)a + P(\hat{H}(\overline{\rho_{12}^2(\xi)} \otimes a)).$$

**Remark 4.5.** By Proposition 2.1 the condition (QM1) implies that  $\text{coker}(P)$  is a  $\bar{\Lambda}$ -module.

**Definition 4.6.** A quadratic  $\mathcal{C}$ -module (relative to  $E$ ) is a pre-quadratic  $\mathcal{C}$ -module (relative to  $E$ ) as above satisfying the additional property that for  $\xi \in \mathcal{C}(E, E \vee E)$  and  $m \in M_{ee}$

$$(QM2) \quad \hat{H}(\overline{\rho_{12}^U(\xi)} \otimes Pm) = \overline{(r_1^2(\xi) \otimes r_2^2(\xi))}(m + Tm).$$

**Remark 4.7.** In view of the isomorphism in Proposition 2.3 the map  $\hat{H}$  in the definition of a (pre-)quadratic  $\mathcal{C}$ -module can be replaced by a group homomorphism:

$$\tilde{H} : U(E \vee E) \otimes_{\Lambda} M_e \rightarrow M_{ee}$$

satisfying the following relations for  $\alpha, \beta \in \mathcal{C}(E, E)$ ,  $\xi \in \mathcal{C}(E, E \vee E)$ ,  $\gamma \in \mathcal{C}(E, E \vee E \vee E)$ ,  $a \in M_e$ :

$$\begin{aligned} (H_1) \quad & \tilde{H}((\alpha \vee \beta)\xi \otimes a) = (\bar{\alpha} \otimes \bar{\beta})\tilde{H}(\xi \otimes a) \\ (H_2) \quad & \tilde{H}(\tau\xi \otimes a) = T\tilde{H}(\xi \otimes a) \\ (H_3) \quad & \tilde{H}(i_1^2\alpha \otimes a) = \tilde{H}(i_2^2\alpha \otimes a) = 0 \\ (H_4) \quad & \tilde{H}((\nabla^2 \vee Id - r_1^2 \vee Id - r_2^2 \vee Id)\gamma \otimes a) = 0 \\ (H_5) \quad & \tilde{H}((Id \vee \nabla^2 - Id \vee r_1^2 - Id \vee r_2^2)\gamma \otimes a) = 0. \end{aligned}$$

In fact,  $(H_1)$  translates the fact that  $\hat{H}$  is a morphism of  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules,  $(H_2)$  corresponds to the fact that  $\hat{H}$  is a morphism of symmetric modules, and  $(H_3)$ ,  $(H_4)$  and  $(H_5)$  correspond to the fact that the source of  $\hat{H}$  is  $(T_{11}(cr_2(U)))(E, E) \otimes_{\Lambda} M_e$ , see Proposition 2.3.

**Remark 4.8.** Suppose that  $\mathcal{C}$  and  $E$  are such that  $\mathcal{C}(E, E \vee E) = i_{1*}^2 \mathcal{C}(E, E) \cup i_{2*}^2 \mathcal{C}(E, E)$ . In particular, this holds when  $\mathcal{C}$  is the category  $\Gamma$  of finite pointed sets and  $E = [1] = \{0, 1\}$  is its canonical generator. Then  $U(E|E) = 0$  by (2.0.3), whence  $\Lambda = \bar{\Lambda}$  and the domain of  $\hat{H}$  is trivial. Thus a quadratic  $\mathcal{C}$ -module relative to  $E$  is a diagram

$$M = (M_{ee} \xrightarrow{T} M_{ee} \xrightarrow{P} M_e)$$

satisfying the properties in Definition 4.4 which do not involve  $\hat{H}$ ; in fact, the relations (QM1) and (QM2) are trivially satisfied. Together with Theorem 6.1 this reproduces a description of quadratic functors from  $\Gamma$  to  $Ab$  given in [13].

In the following proposition we give a useful equivalent formulation of condition (QM1).

**Proposition 4.9.** Relation (QM1) means that the following diagram commutes

$$\begin{array}{ccc} M_{ee} & \xrightarrow{P} & M_e \\ \hat{H}(t_{11} \otimes 1) \uparrow & & \mu_e \uparrow \simeq \\ U(E|E) \otimes_{\Lambda} M_e & \xrightarrow{S_2^U \otimes 1} & U(E) \otimes_{\Lambda} M_e \end{array}$$

where  $\mu_e$  is the canonical isomorphism and  $S_2^U$  the map given in Definition 1.7.

*Proof.* We have, for  $\xi \in \mathcal{C}(E, E \vee E)$

$$\begin{aligned} \mu_e(S_2^U \otimes 1)(\rho_{(1,2)}^2(\xi) \otimes a) &= \mu_e(U(\nabla^2)t_{(1,2)}^2\rho_{(1,2)}^2(\xi) \otimes a) \text{ by definition of } S_2^U \text{ in 1.7} \\ &= \mu_e(U(\nabla^2)(Id - i_{1*}^2 \circ r_{1*}^2 - i_{2*}^2 \circ r_{2*}^2)(\xi) \otimes a) \text{ by 2.0.2} \\ &= \mu_e((U(\nabla^2)(\xi) - r_{1*}^2(\xi) - r_{2*}^2(\xi)) \otimes a) \\ &= \mu_e((\nabla^2\xi - r_1^2\xi - r_2^2\xi) \otimes a) \\ &= (\nabla^2\xi)a - (r_1^2\xi)a - (r_2^2\xi)a. \end{aligned}$$

So the diagram commutes if and only if for all  $\xi$  and  $a$

$$P(\hat{H}(\overline{\rho_{12}^2(\xi)} \otimes a)) = (\nabla^2\xi)a - (r_1^2\xi)a - (r_2^2\xi)a.$$

□

**Definition 4.10 (Morphisms of (pre)-quadratic  $\mathcal{C}$ -modules).** A morphism  $\phi : M \rightarrow M'$  of (pre)-quadratic  $\mathcal{C}$ -modules relative to  $E$  is a pair  $\phi = (\phi_e, \phi_{ee})$  where  $\phi_e : M_e \rightarrow M'_e$  is a morphism of  $\Lambda$ -modules and  $\phi_{ee} : M_{ee} \rightarrow M'_{ee}$  is a morphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules which commute with the structure maps  $\hat{H}$ ,  $T$  and  $P$ .

Composition of morphisms of (pre)-quadratic  $\mathcal{C}$ -modules is defined in the obvious way. This allows to define the category of (pre)-quadratic  $\mathcal{C}$ -modules.

**Definition 4.11.** The category  $QMod_{\mathcal{C}}^E$  is the category having as objects the quadratic  $\mathcal{C}$ -modules and as maps the morphisms of quadratic  $\mathcal{C}$ -modules.

**Remark 4.12.** There is a fully-faithful functor  $I_1 : \bar{\Lambda}\text{-Mod} \rightarrow QMod_{\mathcal{C}}^E$  given by

$$I_1(M) = (T_{11}(cr(U))(E, E) \otimes_{\Lambda} M \xrightarrow{\hat{H}} 0 \xrightarrow{T} 0 \xrightarrow{P} M).$$

In fact,  $I_1(M)$  satisfies (QM1) as  $M$  is a  $\bar{\Lambda}$ -module, and (QM2) is trivial.

**4.2. Quadratic  $\mathcal{C}$ -modules obtained from quadratic functors.** Let  $Quad(\mathcal{C}, Ab)$  denote the category of quadratic reduced functors from  $\mathcal{C}$  to  $Ab$ . The aim of this section is to define a functor  $\mathbb{S}_2 : Quad(\mathcal{C}, Ab) \rightarrow QMod_{\mathcal{C}}^E$ . In order to define  $\mathbb{S}_2$  we need the following lemma.

**Lemma 4.13.** For  $F \in Quad(\mathcal{C}, Ab)$  there exists a natural transformation of functors  $H^F : (T_{11}(cr_2(U)) \otimes_{\Lambda} F(E))\Delta_{\mathcal{C}} \rightarrow (cr_2 F)\Delta_{\mathcal{C}}$  such that the following natural diagram is commutative for  $X \in \mathcal{C}$ :

$$\begin{array}{ccc} T_{11}(cr_2(U))(X, X) \otimes_{\Lambda} F(E) & \xrightarrow{(H^F)_X} & F(X|X) \\ & \swarrow (t_{11} \otimes 1)_X & \nearrow (cr_2(u'_F))_X \\ & U(X|X) \otimes_{\Lambda} F(E) & \end{array}$$

*Proof.* Recall that the cross-effect of a quadratic functor is a bilinear bifunctor. Hence the existence of  $H^F$  follows from the universal property of  $t_{11}$ . □

We now associate a quadratic  $\mathcal{C}$ -module to any quadratic functor, as follows.

**Proposition 4.14.** For  $F \in Quad(\mathcal{C}, Ab)$  the object

$$M^F = (T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} F(E) \xrightarrow{(H^F)_E} F(E|E) \xrightarrow{T^F} F(E|E) \xrightarrow{(S_2^F)_E} F(E))$$

is a quadratic  $\mathcal{C}$ -module relative to  $E$ .

*Proof.* The abelian group  $F(E)$  is a  $\Lambda$ -module by Lemma 3.3, and since  $cr_2 F$  is bireduced and bilinear we deduce from Proposition 4.3 that  $F(E|E)$  is a symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -module. It is straightforward to see that  $(S_2^F)_E : F(E|E) \rightarrow F(E)$  is a morphism of  $\Lambda$ -modules.

Also the fact that  $(cr_2(u'_F))_E$  and hence  $(H^F)_E$  are morphisms of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules is easy to check.

Now the following commutative diagram shows that the equivalent characterization of property (QM1) given in Proposition 4.9 is satisfied by  $M^F$ :

$$\begin{array}{ccccc} T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} F(E) & \xrightarrow{(H^F)_E} & F(E|E) & \xrightarrow{(S_2^F)_E} & F(E) \\ \uparrow t_{11} \otimes 1 & & \uparrow (cr_2(u'_F))_E & & \uparrow (u'_F)_E = \mu_e \\ U(E|E) \otimes_{\Lambda} F(E) & \xrightarrow{\simeq} & (U(-) \otimes_{\Lambda} F(E))(E|E) & \xrightarrow{(S_2^{TS(F)})_E} & U(E) \otimes_{\Lambda} F(E) \end{array}$$

The property (QM2) is more intricate to check, so we postpone its proof to section 6 (see Corollary 6.7). □

The previous proposition leads to the following definition.

**Definition 4.15.** *The functor*

$$\mathbb{S}_2 : \text{Quad}(\mathcal{C}, \text{Ab}) \rightarrow \text{QMod}_{\mathcal{C}}^E$$

is defined by  $\mathbb{S}_2(F) = M^F$  for  $F \in \text{Quad}(\mathcal{C}, \text{Ab})$  and by  $\mathbb{S}_2(\varphi) = (\varphi_E, \text{cr}_2(\varphi)_{E,E})$  for  $\varphi : F \rightarrow G$  a morphism of quadratic functors.

The following proposition formalizes the fact that  $\mathbb{S}_2$  extends the functor  $\mathbb{S}_1$ .

**Proposition 4.16.** *The following diagram is commutative*

$$\begin{array}{ccc} \text{Quad}(\mathcal{C}, \text{Ab}) & \xrightarrow{\mathbb{S}_2} & \text{QMod}_{\mathcal{C}}^E \\ \uparrow & & \uparrow I_1 \\ \text{Lin}(\mathcal{C}, \text{Ab}) & \xrightarrow{\mathbb{S}_1} & \overline{\Lambda} - \text{Mod} \end{array}$$

where  $I_1$  is the functor defined in Remark 4.12.

## 5. QUADRATIC TENSOR PRODUCT

The left adjoint of the functor  $\mathbb{S}_2 : \text{Quad}(\mathcal{C}, \text{Ab}) \rightarrow \text{QMod}_{\mathcal{C}}^E$  is given by a construction which we call the quadratic tensor product. In fact, a special case of a quadratic tensor product first appeared in [1], providing a left adjoint of a functor  $\text{Quad}(\text{Ab}, \text{Ab}) \rightarrow \text{QMod}(\mathbb{Z})$  is defined explicitly by generators and relations; here  $\text{QMod}(\mathbb{Z})$  is the category of quadratic  $\mathbb{Z}$ -modules. Similarly, in [4] a left adjoint of  $\text{Quad}(\text{Gr}, \text{Gr}) \rightarrow \text{Square}$  is constructed by generators and relations; here  $\text{Square}$  is the category of square groups (see also section 7.8). In this paper, however, we give a more conceptual construction of the quadratic tensor product, by means of a push-out diagram, in our general setting. We expect to generalize this construction to polynomial functors of higher degree. A description of our quadratic tensor product in terms of generators and relations is nevertheless provided generalizing the constructions in [1] and [4]. We then compute the quadratic tensor product  $E \otimes M$  for  $M \in \text{QMod}_{\mathcal{C}}^E$  and the cross-effect of  $- \otimes M$  which are two essential tools in the proof of our main theorem in section 6.

**5.1. Definition.** We start with the following straightforward lemma.

**Lemma 5.1.** *If  $N$  is a symmetric  $\Lambda \otimes \Lambda$ -module with involution  $T$  and  $X \in \mathcal{C}$ , the group  $M = ((T_1U)(X) \otimes (T_1U)(X)) \otimes_{\Lambda \otimes \Lambda} N$  is equipped with an involution  $t$  defined by*

$$t(x \otimes y \otimes n) = y \otimes x \otimes T(n)$$

for  $x, y \in (T_1U)(X)$  and  $n \in N$ .

To define the quadratic tensor product, we need to consider the coinvariants by the action of the symmetric group  $\mathfrak{S}_2$ . So we introduce the following notation:

**Notation 5.2.** *If  $M$  is an abelian group equipped with an involution  $t$ , we denote by  $M_{\mathfrak{S}_2}$  the coinvariants of the action of the symmetric group  $\mathfrak{S}_2$  on  $M$  given by  $t$ , i. e.  $M_{\mathfrak{S}_2} = M/(1-t)M$ . Furthermore, we denote by  $\pi : M \rightarrow M_{\mathfrak{S}_2}$  the canonical projection.*

**Definition 5.3.** Let  $M$  be a quadratic  $\mathcal{C}$ -module relative to  $E$  and  $X \in \mathcal{C}$ . The quadratic tensor product  $X \otimes M \in Ab$  is the push-out of the following diagram of abelian groups:

$$\begin{array}{ccc} ((U(X) \otimes U(X)) \otimes_{\Lambda \otimes \Lambda} U(E|E) \otimes_{\Lambda} M_e) \oplus (U(X) \otimes M_{ee}) & \xrightarrow{\phi=(\phi_1, t_2 \otimes P)} & (T_2U)(X) \otimes_{\Lambda} M_e \\ \psi=(\psi_1, \pi(\delta \otimes 1)) \downarrow & & \downarrow \hat{\psi} \\ (((T_1U)(X) \otimes (T_1U)(X)) \otimes_{\Lambda \otimes \Lambda} M_{ee})_{\mathfrak{S}_2} & \xrightarrow{\hat{\phi}} & X \otimes M \end{array}$$

where  $\psi_1 = \pi(t_1 \otimes t_1 \otimes \hat{H}(t_{11} \otimes 1))$ ,  $\delta(f) = (t_1 f) \otimes (t_1 f)$  for  $f \in \mathcal{C}(E, X)$  and  $\phi_1(f \otimes g \otimes x \otimes a) = t_2 S_2^U U(f|g)(x) \otimes a$  for  $f, g \in \mathcal{C}(E, X)$ ,  $x \in U(E|E)$  and  $a \in M_e$ .

In the following proposition we give a description of the quadratic tensor product  $X \otimes M$  by generators and relations.

**Proposition 5.4.** Let  $M$  be a quadratic  $\mathcal{C}$ -module relative to  $E$  and  $X \in \mathcal{C}$ . The quadratic tensor product  $X \otimes M \in Ab$  is the abelian group generated by the symbols

$$\begin{aligned} f \otimes a, \quad f \in \mathcal{C}(E, X), \quad a \in M_e \\ [f, g] \otimes m, \quad f, g \in \mathcal{C}(E, X), \quad m \in M_{ee} \end{aligned}$$

subject to the following relations:

- (1)  $(f\beta) \otimes a = f \otimes (\beta a)$  for  $\beta \in \mathcal{C}(E, E)$
- (2)  $f \otimes (a + b) = f \otimes a + f \otimes b$
- (3)  $(1, 1, 1)\xi \otimes a - (1, 1, 0)\xi \otimes a - (1, 0, 1)\xi \otimes a - (0, 1, 1)\xi \otimes a + (1, 0, 0)\xi \otimes a + (0, 1, 0)\xi \otimes a + (0, 0, 1)\xi \otimes a = 0$  for  $\xi \in \mathcal{C}(E, X \vee X \vee X)$
- (4)  $[f\alpha, g\beta] \otimes m = [f, g] \otimes (\bar{\alpha} \otimes \bar{\beta})m$  for  $\alpha, \beta \in \mathcal{C}(E, E)$
- (5)  $[f, g] \otimes (m + n) = [f, g] \otimes m + [f, g] \otimes n$
- (6)  $[\nabla_*^2(\xi), g] \otimes m = [r_{1*}^2(\xi), g] \otimes m + [r_{2*}^2(\xi), g] \otimes m$  for  $\xi \in \mathcal{C}(E, X \vee X)$
- (7)  $[f, g] \otimes m = [g, f] \otimes T(m)$
- (8)  $[f, f] \otimes m = f \otimes P(m)$
- (9)  $(f, g)\gamma \otimes a = fr_1^2\gamma \otimes a + gr_2^2\gamma \otimes a + [f, g] \otimes \hat{H}(t_{11}\rho_{(1,2)}^2(\gamma) \otimes a)$  for  $\gamma \in \mathcal{C}(E, E \vee E)$ .

*Proof.* The symbol  $f \otimes a$  corresponds to a generator of  $U(X) \otimes_{\Lambda} M_e$  and  $[f, g] \otimes m$  corresponds to a generator of  $(U(X) \otimes_{\Lambda} U(X)) \otimes_{\Lambda \otimes \Lambda} M_{ee}$ .

For the elements  $f \otimes a$ , relation (1) corresponds to the fact that the tensor product is taken over  $\Lambda$ , (2) translates the linearity in  $M_e$ , (3) corresponds to the fact that the element  $f \otimes a$  is in  $T_2U(X) \otimes_{\Lambda} M_e$  where we use Proposition 2.4 which describes  $T_2F$  as a quotient of  $F$ .

For the elements  $[f, g] \otimes m$ , (4) corresponds to the fact that the tensor product is taken over  $\Lambda \otimes \Lambda$ , (5) translates the linearity in  $M_{ee}$ , (6) (also using (7)) corresponds to the fact that the element  $[f, g] \otimes m$  lies in  $((T_1U)(X) \otimes (T_1U)(X)) \otimes_{\Lambda \otimes \Lambda} M_{ee}$ , and (7) translates the fact that we take the coinvariants by the action of  $\mathfrak{S}_2$ .

Finally, (8) and (9) correspond to the fact that the diagram in Definition 5.3 is a pushout.  $\square$

We need the following technical lemma.

**Lemma 5.5.** The quadratic tensor product  $X \otimes M$  is equal to the push-out  $P$  of the following diagram of abelian groups

$$\begin{array}{ccc} ((T_1UX \otimes T_1UX) \otimes_{\Lambda \otimes \Lambda} T_{11}cr_2U(E, E) \otimes_{\Lambda} M_e)_{\mathfrak{S}_2} \oplus (UX \otimes M_{ee}) & \xrightarrow{\bar{\phi}=(\bar{\phi}'_1 \otimes 1, t_2 \otimes P)} & T_2UX \otimes_{\Lambda} M_e \\ \bar{\psi}=(\bar{\psi}_1, \pi(\delta \otimes 1)) \downarrow & & \downarrow \hat{\psi} \\ (((T_1U)(X) \otimes (T_1U)(X)) \otimes_{\Lambda \otimes \Lambda} M_{ee})_{\mathfrak{S}_2} & \xrightarrow{\hat{\phi}} & P \end{array}$$

where  $UX := U(X)$ ,  $T_i UX := T_i U(X)$  for  $i \in \{1, 2\}$ ,  $\overline{\psi_1} = \overline{1 \otimes 1 \otimes \hat{H}}$  and  $\phi'_1$  is the following composite map:

$$\begin{array}{ccc} (T_1 U(X) \otimes T_1 U(X)) \otimes_{\Lambda \otimes \Lambda} T_{11}(cr_2 U)(E, E) & \xrightarrow{\overline{\phi'_1}} & T_2 U(X) \\ \downarrow 1 \otimes 1 \otimes \overline{cr_2(t_2)} & & \uparrow S_2^{T_2 U} \\ (T_1 U(X) \otimes T_1 U(X)) \otimes_{\Lambda \otimes \Lambda} cr_2(T_2 U)(E, E) & \xrightarrow{u'_{cr(T_2 U)}} & cr_2(T_2 U)(X, X). \end{array}$$

*Proof.* We have the following surjection

$$(UX \otimes UX) \otimes_{\Lambda \otimes \Lambda} U(E|E) \otimes_{\Lambda} M_e \xrightarrow{\pi(t_1 \otimes t_1 \otimes t_{11} \otimes 1)} (((T_1 U)X \otimes (T_1 U)X) \otimes_{\Lambda \otimes \Lambda} T_{11}(cr_2 U)(E, E) \otimes_{\Lambda} M_e)_{\mathfrak{S}_2}$$

which verifies

$$\psi_1 = \pi(t_1 \otimes t_1 \otimes \hat{H}(t_{11} \otimes 1)) = \pi(1 \otimes 1 \otimes \hat{H})(t_1 \otimes t_1 \otimes t_{11} \otimes 1).$$

We first check that  $\phi_1 = (\phi'_1 \otimes 1)(t_1 \otimes t_1 \otimes t_{11} \otimes 1)$ . For  $f, g \in \mathcal{C}(E, X)$  and  $x \in U(E|E)$  we have:

$$\begin{aligned} \phi'_1(t_1 \otimes t_1 \otimes t_{11})(f \otimes g \otimes x) &= S_2^{T_2 U} u'_{cr_2(T_2 U)}(1 \otimes 1 \otimes \overline{cr_2(t_2)})(t_1(f) \otimes t_1(g) \otimes t_{11}(x)) \\ &= S_2^{T_2 U} cr_2(T_2 U)(f, g) cr_2(t_2)(x) \quad \text{by definition of } u'_{cr_2(T_2 U)} \\ &= (T_2 U)(\nabla^2) \iota_{(1,2)}^2 cr_2(T_2 U)(f, g) cr_2(t_2)(x) \quad \text{by definition of } S_2^{T_2 U} \\ &= (T_2 U)(\nabla^2)(T_2 U)(f \vee g) t_2 \iota_{(1,2)}^2(x) \\ &= t_2 U(\nabla^2) U(f \vee g) \iota_{(1,2)}^2(x) \\ &= t_2 U(\nabla^2) \iota_{(1,2)}^2 U(f|g)(x) \\ &= t_2 S_2^U U(f|g)(x). \end{aligned}$$

It remains to check that  $\phi'_1$  factors through the coinvariants, that is  $\phi'_1(t_1(f) \otimes t_1(g) \otimes t_{11}x) = \phi'_1(t(t_1(f) \otimes t_1(g) \otimes t_{11}x))$ . We have:

$$\begin{aligned} \phi'_1(t(t_1(f) \otimes t_1(g) \otimes t_{11}x)) &= \phi'_1(t_1(g) \otimes t_1(f) \otimes t_{11}(\iota_{(1,2)}^2)^{-1} U(\tau) \iota_{(1,2)}^2 x) \\ &= t_2 S_2^U U(g|f)(\iota_{(1,2)}^2)^{-1} U(\tau)(\iota_{(1,2)}^2 x) \quad \text{by the previous calculation} \\ &= t_2 U(\nabla^2) \iota_{(1,2)}^2 U(g|f)(\iota_{(1,2)}^2)^{-1} U(\tau)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2) U(g \vee f) \iota_{(1,2)}^2 (\iota_{(1,2)}^2)^{-1} U(\tau)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2) U(g \vee f) U(\tau)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2) U(\tau) U(f \vee g)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2 \tau) U(f \vee g)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2) U(f \vee g)(\iota_{(1,2)}^2 x) \\ &= t_2 U(\nabla^2) \iota_{(1,2)}^2 U(f|g)(x) \\ &= t_2 S_2^U U(f|g)(x) \\ &= \phi'_1(t_1(f) \otimes t_1(g) \otimes t_{11}x) \end{aligned}$$

□

**5.2. The quadratic tensor product defines a quadratic functor.** The aim of this section is to prove the following result.

**Proposition 5.6.** *For  $M \in QMod_{\mathcal{C}}^E$  the functor given by the quadratic tensor product:  $- \otimes M : \mathcal{C} \rightarrow Ab$  is a quadratic functor.*

The proof of this proposition relies on the following lemma.

**Lemma 5.7.** *For  $M$  a pre-quadratic  $\mathcal{C}$ -module, the following natural transformation of functors from  $\mathcal{C}$  to  $Ab$  is pointwise surjective:*

$$cr_2(\hat{\phi}) : cr_2(\left(\left(\left(T_1U(-) \otimes T_1U(-)\right) \otimes_{\Lambda \otimes \Lambda} M_{ee}\right) \Delta_{\mathcal{C}}\right)_{\mathfrak{S}_2}) \rightarrow cr_2(- \otimes M);$$

here  $\hat{\phi}$  is the map in the push-out diagram in Definition 5.3.

**Remark 5.8.** *In Theorem 5.18 below we give an improved version of this lemma.*

*Proof of Lemma 5.7.* To simplify notation we write  $\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}$  instead of  $\left(\left(\left(T_1U(-) \otimes T_1U(-)\right) \otimes_{\Lambda \otimes \Lambda} M_{ee}\right) \Delta_{\mathcal{C}}\right)$ . Recall that the functor  $\mathbb{T}_{11}$  is the functor defined in 3.18.

By the universal property of a push-out, we obtain the existence of a map  $f : X \otimes M \rightarrow coker(\phi)$  making the following diagram of abelian groups commutative:

$$\begin{array}{ccc} (UX \otimes UX) \otimes_{\Lambda \otimes \Lambda} U(E|E) \otimes_{\Lambda} M_e \oplus UX \otimes M_{ee} & \xrightarrow{\phi} & (T_2U)(X) \otimes_{\Lambda} M_e \\ \psi \downarrow & & \downarrow \hat{\psi} \\ \left(\left(\left(T_1U(X) \otimes T_1U(X)\right) \otimes_{\Lambda \otimes \Lambda} M_{ee}\right)_{\mathfrak{S}_2}\right) & \xrightarrow{\hat{\phi}} & X \otimes M \\ & \searrow 0 & \swarrow f \\ & & coker(\phi). \end{array}$$

We deduce the existence of a natural exact sequence

$$(5.8.1) \quad \left(\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}\right)_{\mathfrak{S}_2}(X) \xrightarrow{\hat{\phi}} X \otimes M \xrightarrow{f} coker(\phi) \rightarrow 0.$$

In the sequel we show that the functor  $coker(\phi)$  is linear. Recall that  $\phi_1 = \phi'_1(t_1 \otimes t_{11}) \otimes 1$  where  $(\phi'_1(t_1 \otimes t_1 \otimes t_{11}))_X$  is given by the following compositon:

$$(U(X) \otimes U(X)) \otimes_{\Lambda \otimes \Lambda} U(E|E) \xrightarrow{\alpha} cr_2(T_2U)(X, X) \xrightarrow{(S_2^{T_2U})_X} (T_2U)(X)$$

where  $\alpha = (u'_{cr_2(T_2U)})_{X,X}(1 \otimes 1 \otimes cr_2(t_2))(t_1 \otimes t_1 \otimes t_{11})$ .

The functor  $T_2U$  is quadratic by definition of  $T_2$ , so  $cr_2(T_2U)$  is a bilinear functor, and so is  $(T_1U(-) \otimes T_1U(-)) \otimes_{\Lambda \otimes \Lambda} cr_2(T_2U)(E, E)$ . Since  $(u'_{cr_2(T_2U)})_{E,E}$  is an isomorphism according to Theorem 3.20, we deduce by Proposition 1.23 that  $(u'_{cr_2(T_2U)})$  is a natural equivalence and  $(u'_{cr_2(T_2U)})_{X,X}$  is an isomorphism. Furthermore  $1 \otimes 1 \otimes cr_2(t_2)$  and  $t_1 \otimes t_{11}$  are surjective by construction. It follows that

$$coker((\phi_1)_X) = coker((S_2^{T_2U})_X) = T_1(T_2U)(X)$$

by Proposition 1.8.

Since  $Im(\phi_1) \subset Im(\phi)$  we see that  $coker(\phi)$  is a quotient of  $coker(\phi_1) = T_1(T_2U) \otimes_{\Lambda} M_e$  which is a linear functor. Thus  $coker(\phi)$  is a linear functor by Proposition 1.6. But the cross-effect functor is exact by Proposition 1.4, so applying it to sequence (5.8.1) shows that  $cr_2(\hat{\phi})$  is pointwise surjective.  $\square$

*Proof of Proposition 5.6.* The functor  $\mathbb{T}_{11}(M_{ee})$  is bilinear and bireduced, so by Lemma 1.20,  $\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}$  is quadratic. By Proposition 1.14, we deduce that  $\left(\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}\right)_{\mathfrak{S}_2}$  is quadratic. Consequently  $cr_2\left(\left(\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}\right)_{\mathfrak{S}_2}\right)$  is bilinear. Since  $cr_2(- \otimes M)$  is a quotient of  $cr_2\left(\left(\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}}\right)_{\mathfrak{S}_2}\right)$  by Lemma 5.7, the functor  $cr_2(- \otimes M)$  is also bilinear, so  $- \otimes M$  is quadratic.  $\square$

Proposition 5.6 leads to the following definition.



**Definition 5.9.** *The functor*

$$\mathbb{T}_2 : QMod_{\mathcal{C}}^E \rightarrow Quad(\mathcal{C}, Ab)$$

is defined as follows: for  $M \in QMod_{\mathcal{C}}^E$  let  $\mathbb{T}_2(M) = - \otimes M$ , and for a morphism of quadratic  $\mathcal{C}$ -modules  $(\phi_e, \phi_{ee}) : M \rightarrow N$  let  $\mathbb{T}_2(\phi_e, \phi_{ee}) = \varphi$  where  $\varphi_X : X \otimes M \rightarrow X \otimes N$  is given by the universal property of a push-out.

**Remark 5.10.** *The functor  $\mathbb{T}_2$  extends the tensor product functor  $\mathbb{T}_1$  in the following sense: it improves the approximation of a quadratic functor  $F$  by  $\overline{u'} : T_2U(-) \otimes_{\Lambda} F(E) \rightarrow F$  by taking into account the cross-effect of  $F$  in "amalgamating"  $T_2U(X) \otimes_{\Lambda} F(E)$  with the image of  $F(X|X)$  under  $(S_2^F)_X$ . We expect that this idea can be extended to polynomial functors of higher degree.*

**Remark 5.11.** *Since the functor  $- \otimes M$  is quadratic by Proposition 5.6, the computation of  $(- \otimes M)(E^{\vee n})$  for  $n \geq 1$  reduces to computing  $(- \otimes M)(E)$  and  $cr_2(- \otimes M)(E, E)$ , see Proposition 1.3.*

**5.3. Computation of the quadratic tensor product  $E \otimes M$ .** In section 6, in order to obtain the desired equivalence between quadratic functors and quadratic  $\mathcal{C}$ -modules, we need to compute  $E \otimes M$  for  $M \in QMod_{\mathcal{C}}^E$ , as follows.

**Proposition 5.12.** *For  $M \in QMod_{\mathcal{C}}^E$ , there exists an isomorphism of abelian groups*

$$E \otimes M \xrightarrow{\cong} M_e.$$

In order to define this isomorphism we need the following lemmas.

**Lemma 5.13.** *For  $M \in QMod_{\mathcal{C}}^E$ , there exists an isomorphism  $\overline{\mu}_e : (T_2U)(E) \otimes_{\Lambda} M_e \rightarrow M_e$  making the following diagram commutative*

$$\begin{array}{ccc} U(E) \otimes_{\Lambda} M_e & \xrightarrow{\mu_e} & M_e \\ & \searrow^{t_2 \otimes 1} & \nearrow^{\overline{\mu}_e} \\ & (T_2U)(E) \otimes_{\Lambda} M_e & \end{array}$$

where  $\mu_e$  is the canonical isomorphism. In particular, the map  $t_2 \otimes 1$  is an isomorphism.

*Proof.* To prove that  $\overline{\mu}_e$  exists we have to prove that  $\mu_e(Ker(t_2 \otimes 1)) = 0$ . By Proposition 1.19 we have

$$T_2(F) = coker(ker(cr_2F(X|X)) \xrightarrow{t_{11}} (T_{11}cr_2F)(X|X)) \xrightarrow{S_2^U} F(X).$$

So we have  $ker(t_2) = S_2^U(ker(t_{11}))$  and

$$\mu_e(ker(t_2 \otimes 1)) = \mu_e(S_2^U \otimes 1)(ker(t_{11}) \otimes M_e) = P\hat{H}(t_{11} \otimes Id)(ker(t_{11}) \otimes M_e) = 0$$

where the second equality is given by Proposition 4.9.

Since  $t_2 \otimes 1$  is surjective and  $\mu_e$  is an isomorphism we deduce that  $\overline{\mu}_e$  is an isomorphism.

As a consequence we obtain that  $t_2 \otimes 1$  is also an isomorphism.  $\square$

**Lemma 5.14.** *For  $M \in QMod_{\mathcal{C}}^E$ , there exists a morphism  $\overline{P} : (M_{ee})_{\mathfrak{S}_2} \rightarrow M_e$  making the following diagram commutative*

$$\begin{array}{ccc} M_{ee} & \xrightarrow{P} & M_e \\ & \searrow^{\pi} & \nearrow^{\overline{P}} \\ & (M_{ee})_{\mathfrak{S}_2} & \end{array}$$

*Proof.* This is due to the relation  $PT = P$  in Definition 4.6.  $\square$

**Lemma 5.15.** *There exists an isomorphism*

$$\overline{\mu_{ee}} : ((T_1U)(E) \otimes (T_1U)(E)) \otimes_{\Lambda \otimes \Lambda} M_{ee} \otimes_{\mathfrak{S}_2} \rightarrow (M_{ee}) \otimes_{\mathfrak{S}_2}$$

making the following diagram commutative

$$\begin{array}{ccc} ((T_1U)(E) \otimes (T_1U)(E)) \otimes_{\Lambda \otimes \Lambda} M_{ee} & \xrightarrow{\mu_{ee}} & M_{ee} \\ \pi \downarrow & & \pi \downarrow \\ (((T_1U)(E) \otimes (T_1U)(E)) \otimes_{\Lambda \otimes \Lambda} M_{ee}) \otimes_{\mathfrak{S}_2} & \xrightarrow{\overline{\mu_{ee}}} & (M_{ee}) \otimes_{\mathfrak{S}_2} \end{array}$$

where  $((T_1U)(E) \otimes (T_1U)(E)) \otimes_{\Lambda \otimes \Lambda} M_{ee} \xrightarrow{\mu_{ee}} M_{ee}$  is the canonical isomorphism.

*Proof.* This is immediate from the fact that the canonical isomorphism

$$((T_1U)(E) \otimes (T_1U)(E)) \otimes_{\Lambda \otimes \Lambda} M_{ee} \xrightarrow{\mu_{ee}} M_{ee}$$

is compatible with the involutions.  $\square$

Proposition 5.12 is a direct consequence of the following lemma.

**Lemma 5.16.** *For  $M \in QMod_{\mathcal{C}}^E$ , there exists an isomorphism*

$$(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e}) : E \otimes M \rightarrow M_e$$

such that  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e}) \hat{\phi} = \overline{P} \overline{\mu_{ee}}$  and  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e}) \hat{\psi} = \overline{\mu_e}$  where the maps  $\overline{P}$ ,  $\overline{\mu_{ee}}$  and  $\overline{\mu_e}$  are defined in Lemma 5.14, 5.15 and 5.13 respectively and the maps  $\hat{\phi}$  and  $\hat{\psi}$  appear in the pushout diagram of Proposition 5.3.

*Proof.* To prove that the morphism  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e})$  exists, it is sufficient to prove that  $\overline{P} \overline{\mu_{ee}} \psi = \overline{\mu_e} \phi$  by the universal property of the push-out.

For  $f \in \mathcal{C}(E, E)$  and  $m \in M_{ee}$  we have:

$$\begin{aligned} \overline{P} \overline{\mu_{ee}} \pi(\delta \otimes 1)(f \otimes m) &= P \mu_{ee}(\bar{f} \otimes \bar{f} \otimes m) && \text{by Lemmas 5.14 and 5.15} \\ &= P((\bar{f} \otimes \bar{f}).m) && \text{by definition of } \mu_{ee} \\ &= f P(m) && \text{by Definition 4.4} \\ &= \mu_e(f \otimes P(m)) && \text{by definition of } \mu_e \\ &= \overline{\mu_e}(t_2 \otimes Id)(f \otimes P(m)) && \text{by Lemma 5.13} \\ &= \overline{\mu_e}(t_2 \otimes P)(f \otimes m). \end{aligned}$$

For  $f, g \in \mathcal{C}(E, E)$ ,  $x \in U(E|E)$ ,  $a \in M_e$  we have:

$$\begin{aligned} &\overline{P} \overline{\mu_{ee}} \pi(t_1 \otimes t_1 \otimes \hat{H}(t_{11} \otimes 1))(f \otimes g \otimes x \otimes a) \\ &= P \mu_{ee}(\bar{f} \otimes \bar{g} \otimes \hat{H}(t_{11}(x) \otimes a)) && \text{by Lemmas 5.14 and 5.15} \\ &= P((\bar{f} \otimes \bar{g}) \hat{H}(t_{11}(x) \otimes a)) && \text{by definition of } \mu_{ee} \\ &= P \hat{H}((\bar{f} \otimes \bar{g})(t_{11}(x) \otimes a)) && \text{since } \hat{H} \text{ is a morphism of } \bar{\Lambda} \otimes \bar{\Lambda}\text{-modules by Definition 4.4} \\ &= P \hat{H}(t_{11}(U(f|g)(x)) \otimes a) && \text{by the structure of } \bar{\Lambda} \otimes \bar{\Lambda}\text{-module of } T_{11}cr_2U(E, E) \\ &= P \hat{H}(t_{11} \otimes Id)(U(f|g)(x) \otimes a) \\ &= \mu_e(S_2^U \otimes Id)(U(f|g)(x) \otimes a) && \text{by Proposition 4.9} \\ &= \overline{\mu_e}(t_2 \otimes Id)(S_2^U \otimes Id)(U(f|g)(x) \otimes a) && \text{by Lemma 5.13} \\ &= \overline{\mu_e}(t_2 S_2^U U(f|g)(x) \otimes a). \end{aligned}$$

Hence the morphism  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e}) : E \otimes M \rightarrow M_e$  exists.

For  $m \in M_{ee}$ , we have:

$$\overline{\mu_{ee}} \pi(\delta \otimes 1)(1 \otimes m) = \bar{m}.$$

Since  $\overline{\mu_{ee}}$  is an isomorphism by Lemma 5.15 we deduce that  $\pi(\delta \otimes 1)$  is surjective. Consequently  $\psi$  is surjective and by general properties of push-out diagrams we obtain

that  $\hat{\psi} : (T_2U)(E) \otimes M_e \rightarrow E \otimes M$  is surjective. Now, since  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e})\hat{\psi} = \overline{\mu_e}$  is an isomorphism by Lemma 5.13, it follows that  $(\overline{P} \overline{\mu_{ee}}, \overline{\mu_e})$  is an isomorphism.  $\square$

In particular, by Proposition 4.14, for  $F$  a quadratic functor,  $M^F$  is a quadratic  $\mathcal{C}$ -module, so we can apply the previous lemma to  $M^F$  to obtain:

**Lemma 5.17.** *For  $F \in \text{Quad}(\mathcal{C}, \text{Ab})$  the morphism*

$$((\overline{S_2^F})_E \overline{\mu_{ee}}, \overline{\mu_e}) : E \otimes M^F \rightarrow F(E).$$

*is an isomorphism.*

**5.4. Computation of the cross-effect of  $- \otimes M$ .** The aim of this section is to prove the following theorem which allows to compute the cross-effect of  $- \otimes M$  in Corollary 5.21 and to complete the proof of Proposition 4.14 in Corollary 6.7.

**Theorem 5.18.** *For  $\mathcal{C}$  a pointed theory generated by  $E$  and  $M$  a pre-quadratic  $\mathcal{C}$ -module relative to  $E$ , the following natural transformation of functors is an equivalence if and only if  $M$  is a quadratic  $\mathcal{C}$ -module.*

$$cr_2(\hat{\phi}) : cr_2(((T_1U(-) \otimes T_1U(-)) \otimes_{\Lambda \otimes \Lambda} M_{ee})\Delta_{\mathcal{C}})_{\mathfrak{S}_2} \rightarrow cr_2(- \otimes M)$$

Here  $\hat{\phi}$  is the map in the push-out diagram in Definition 5.3.

The proof of this theorem relies on the following lemmas.

**Lemma 5.19.** *If  $B : \mathcal{C} \times \mathcal{C} \rightarrow \text{Ab}$  is a bilinear bireduced bifunctor, the natural map given by*

$$B(X, Y) \xrightarrow{(\iota_{(1,2)}^2)^{-1} \pi B(i_1^2, i_2^2)} cr_2(B\Delta_{\mathcal{C}})_{\mathfrak{S}_2}(X, Y)$$

*for  $X, Y \in \mathcal{C}$ , is a natural equivalence.*

*Proof.* For  $X, Y \in \mathcal{C}$ , we have  $B\Delta_{\mathcal{C}}(X \vee Y) = B(X, X) \oplus B(X, Y) \oplus B(Y, X) \oplus B(Y, Y)$  since  $B$  is a bilinear functor. Let  $T : B(X \vee Y, X \vee Y) \rightarrow B(X \vee Y, X \vee Y)$  be the involution we have  $T(B(X, X)) = B(X, X)$ ,  $T(B(Y, Y)) = B(Y, Y)$  and  $T(B(X, Y)) = T(B(Y, X))$ . So  $(B\Delta_{\mathcal{C}})_{\mathfrak{S}_2}(X \vee Y) = B(X, X)_{\mathfrak{S}_2} \oplus B(Y, Y)_{\mathfrak{S}_2} \oplus (B(X, Y) \oplus B(Y, X))_{\mathfrak{S}_2}$  and  $cr_2(B\Delta_{\mathcal{C}})_{\mathfrak{S}_2}(X \vee Y) = (B(X, Y) \oplus B(Y, X))_{\mathfrak{S}_2}$ . We have  $(\iota_{(1,2)}^2)^{-1} \pi B(i_1^2, i_2^2)(x) = (x, 0)$ .

An inverse of this map is defined as follows: let  $B(X, Y) \oplus B(Y, X) \xrightarrow{f} B(X, Y)$  such that  $f(x, y) = x + Ty$ . We have  $f((x, y) - T(x, y)) = f((x, y) - (T(y), T(x))) = f(x - T(y), y - T(x)) = 0$ . So,  $f$  defines a map  $(B(X, Y) \oplus B(Y, X))_{\mathfrak{S}_2} \xrightarrow{\overline{f}} B(X, Y)$ , and one easily checks that  $\overline{f}$  is the inverse of  $(\iota_{(1,2)}^2)^{-1} \pi B(i_1^2, i_2^2)$ .  $\square$

**Lemma 5.20.** *For a  $\overline{\Lambda} \otimes \overline{\Lambda}$ -module  $A$ , let  $\mu_A : \mathbb{T}_{11}(A)(E, E) \rightarrow A$  denote the canonical isomorphism. Then the following diagram is commutative:*

$$\begin{array}{ccc} N \oplus U(E|E) \otimes M_{ee} & \xrightarrow[\simeq]{k_1} & cr_2((\mathbb{T}_{11}(N)\Delta_{\mathcal{C}})_{\mathfrak{S}_2})(E, E) \oplus U(E|E) \otimes M_{ee} \\ \downarrow k_2 & & \downarrow cr_2(\overline{\psi}) \\ M_{ee} & \xrightarrow[\simeq]{k_3} & cr_2((\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}})_{\mathfrak{S}_2})(E, E) \end{array}$$

where:  $N = T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} M_e$ ,  $k_1 = (\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\mu_N)^{-1} \oplus 1$ ,  $k_2 = (\hat{H}, k'_2)$  with  $k'_2(\rho_{(1,2)}^2(\xi) \otimes m) = (r_1^2 \xi \otimes r_2^2 \xi) \cdot (m + Tm)$  for  $\xi \in \mathcal{C}(E, E \vee E)$  and  $m \in M_{ee}$ , and  $k_3 = (\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(M_{ee})(i_1^2, i_2^2)(\mu_{M_{ee}})^{-1}$ .

*Proof.* The morphisms  $k_1$  and  $k_3$  are isomorphisms since  $\mu_{M_{ee}}$  and  $\mu_N$  are the canonical isomorphisms and  $(\iota_{(1,2)}^2)^{-1}\pi\mathbb{T}_{11}(N)(i_1^2, i_2^2)$  and  $(\iota_{(1,2)}^2)^{-1}\pi\mathbb{T}_{11}(M_{ee})(i_1^2, i_2^2)$  are isomorphisms by Lemma 5.19. The fact that  $k_2$  is welldefined follows from the fact that  $k_3$  is an isomorphism and by commutativity of the diagram which we prove now.

By Lemma 5.5,  $cr_2(\bar{\psi}) = (cr_2(1 \otimes 1 \otimes \hat{H}), cr_2(\pi(\delta \otimes 1)))$ .

The commutativity of the diagram on  $N$  follows from the naturality of  $\mu$  and by the fact that  $\mathbb{T}_{11}(-)(-, -)$  is a trifunctor.

To prove the commutativity of the diagram on  $U(E|E) \otimes M_{ee}$  we consider the injection  $\iota_{(1,2)}^2 : cr_2((\mathbb{T}_{11}(M_{ee})\Delta_C)\mathfrak{S}_2)(E, E) \rightarrow (\mathbb{T}_{11}(M_{ee})\Delta_C)\mathfrak{S}_2(E \vee E)$  and we prove that  $\iota_{(1,2)}^2 cr_2(\bar{\psi}) k_1 = \iota_{(1,2)}^2 k_3 k_2$  on  $U(E|E) \otimes M_{ee}$ .

$$\begin{aligned}
 & \iota_{(1,2)}^2 cr_2(\bar{\psi}) k_1(\rho_{(1,2)}^2(\xi) \otimes m) \\
 &= \iota_{(1,2)}^2 cr_2(\pi(\delta \otimes 1))(\rho_{(1,2)}^2(\xi) \otimes m) \\
 &= \pi(\delta \otimes 1)\iota_{(1,2)}^2(\rho_{(1,2)}^2(\xi) \otimes m) \\
 &= \pi(\delta \otimes 1)((\xi - i_1^2 r_1^2 \xi - i_2^2 r_2^2 \xi) \otimes m) \text{ by 2.0.2} \\
 &= \pi((\bar{\xi} \otimes \bar{\xi} - \overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} - \overline{i_2^2 r_2^2 \xi} \otimes \overline{i_1^2 r_1^2 \xi}) \otimes m) \\
 &= \pi((\overline{i_1^2 r_1^2 \xi} + \overline{i_2^2 r_2^2 \xi}) \otimes (\overline{i_1^2 r_1^2 \xi} + \overline{i_2^2 r_2^2 \xi}) - \overline{i_1^2 r_1^2 \xi} \otimes \overline{i_1^2 r_1^2 \xi} - \overline{i_2^2 r_2^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi}) \otimes m) \\
 & \text{by Lemma 3.7 (3) since } T_1 U \text{ is linear} \\
 &= \pi(\overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes m + \overline{i_2^2 r_2^2 \xi} \otimes \overline{i_1^2 r_1^2 \xi} \otimes m) \\
 &= \pi(\overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes m + t(\overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes Tm)) \text{ where } t \text{ is defined in Lemma 5.1} \\
 &= \pi(\overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes m + \overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes Tm) \text{ since } \pi(tx) = \pi(x) \\
 &= \pi(\overline{i_1^2 r_1^2 \xi} \otimes \overline{i_2^2 r_2^2 \xi} \otimes (m + Tm)) \\
 &= \pi \mathbb{T}_{11}(M_{ee})(i_1^2, i_2^2)(r_1^2 \xi \otimes r_2^2 \xi \otimes (m + Tm)) \\
 &= \pi \mathbb{T}_{11}(M_{ee})(i_1^2, i_2^2)(\bar{1} \otimes \bar{1} \otimes (r_1^2 \xi \otimes r_2^2 \xi).(m + Tm)) \\
 &= \pi \mathbb{T}_{11}(M_{ee})(i_1^2, i_2^2)\mu_{M_{ee}}^{-1}((r_1^2 \xi \otimes r_2^2 \xi).(m + Tm)) \\
 &= \iota_{(1,2)}^2 k_3((r_1^2 \xi \otimes r_2^2 \xi).(m + Tm)) \\
 &= \iota_{(1,2)}^2 k_3 k_2(\rho_{(1,2)}^2(\xi) \otimes m)
 \end{aligned}$$

□

*Proof of Theorem 5.18 .* By Lemma 1.20 and Proposition 1.14 the source functor is bilinear and we deduce from Proposition 5.6 that the target functor is bilinear. So according to Proposition 1.23 it suffices to check that  $cr_2(\hat{\phi})_{(E,E)}$  is an isomorphism if and only if  $M$  is a quadratic  $\mathcal{C}$ -module. By Lemma 5.7 we know that, for  $M$  a pre-quadratic  $\mathcal{C}$  module,  $cr_2(\hat{\phi})_{(E,E)}$  is surjective. So it is sufficient to prove that  $cr_2(\hat{\phi})_{(E,E)}$  is injective if and only if the condition (QM2) is satisfied.

As a pushout of abelian groups can be written as a right exact sequence in an obvious way and as the cross-effect functor is exact, we deduce from Lemma 5.5 that the following diagram is also a pushout.

$$\begin{array}{ccc}
 cr_2((\mathbb{T}_{11}(N)\Delta_C)\mathfrak{S}_2)(E, E) \oplus cr_2(U)(E, E) \otimes M_{ee} & \xrightarrow{cr_2(\bar{\phi})_{E,E}} & cr_2(T_2 U)(E, E) \otimes_{\Lambda} M_e \\
 \downarrow cr_2(\bar{\psi})_{E,E} & & \downarrow cr_2(\hat{\psi})_{E,E} \\
 cr_2((\mathbb{T}_{11}(M_{ee})\Delta_C)\mathfrak{S}_2)(E, E) & \xrightarrow{cr_2(\hat{\phi})_{E,E}} & cr_2(- \otimes M)(E, E).
 \end{array}$$

By the previous Lemma we obtain the following commutative diagram (where, for simplicity, we omit the subscript  $E, E$ ):

$$\begin{array}{ccccc}
N \oplus U(E|E) \otimes M_{ee} & \xrightarrow[\simeq]{k_1} & cr_2((\mathbb{T}_{11}(N)\Delta_{\mathcal{C}})_{\mathfrak{S}_2})(E, E) \oplus cr_2(U)(E, E) \otimes M_{ee} & \xrightarrow{cr_2(\bar{\phi})} & cr_2(T_2U)(E, E) \otimes_{\Lambda} M_e \\
\downarrow k_2 & & \downarrow cr_2(\bar{\psi}) & & \downarrow cr_2(\hat{\psi}) \\
M_{ee} & \xrightarrow[k_3]{\simeq} & cr_2((\mathbb{T}_{11}(M_{ee})\Delta_{\mathcal{C}})_{\mathfrak{S}_2})(E, E) & \xrightarrow{cr_2(\hat{\phi})} & cr_2(- \otimes M)(E, E).
\end{array}$$

As  $k_1$  and  $k_3$  are isomorphisms, we deduce that the exterior diagram is a pushout too.

By a general property of pushouts in  $Ab$  we deduce that:

$$(5.20.1) \quad \ker(cr_2(\hat{\phi})) = k_3 \ker(cr_2(\hat{\phi})k_3) = k_3 k_2 \ker(cr_2(\bar{\phi})k_1).$$

Recall that  $cr_2(\bar{\phi}) = (cr_2(\overline{\phi'_1 \otimes 1}), cr_2(t_2) \otimes P)$ . So, we have

$$cr_2(\bar{\phi})k_1 = (cr_2(\overline{\phi'_1 \otimes 1})\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\mu_N)^{-1}, cr_2(t_2) \otimes P).$$

In the sequel, we compute  $cr_2(\overline{\phi'_1 \otimes 1})\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\mu_N)^{-1}$ . For  $x \in cr_2(U)(E, E)$  and  $a \in M_e$  we have:

$$\begin{aligned}
& \iota_{(1,2)}^2 cr_2(\overline{\phi'_1 \otimes 1})\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\mu_N)^{-1}(t_{11}(x) \otimes a) \\
&= (\overline{\phi'_1 \otimes 1})\iota_{(1,2)}^2 (\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\mu_N)^{-1}(t_{11}(x) \otimes a) \\
&= (\phi'_1 \otimes 1) \pi \mathbb{T}_{11}(N)(i_1^2, i_2^2)(\bar{1} \otimes \bar{1} \otimes (t_{11}(x) \otimes a)) \\
&= (\phi'_1 \otimes 1)(i_1^2 \otimes i_2^2 \otimes (t_{11}(x) \otimes a)) \\
&= \phi'_1(t_1 \otimes t_1 \otimes t_{11})(i_1^2 \otimes i_2^2 \otimes x) \otimes a \\
&= t_2 S_2^U U(i_1^2 | i_2^2)(x) \otimes a \quad \text{by the proof of Lemma 5.5} \\
&= t_2 U(\nabla^2) U(i_1^2 \vee i_2^2) \iota_{(1,2)}^2(x) \otimes a \\
&= t_2 \iota_{(1,2)}^2(x) \otimes a \\
&= \iota_{(1,2)}^2 cr_2(t_2)(x) \otimes a \\
&= \iota_{(1,2)}^2 (\overline{cr_2(t_2) \otimes 1})(t_{11}(x) \otimes a).
\end{aligned}$$

We deduce that:

$$cr_2(\bar{\phi})k_1 = (\overline{cr_2(t_2) \otimes 1}, cr_2(t_2) \otimes P).$$

By Theorem 1.24,  $\overline{cr_2(t_2) \otimes 1}$  is an isomorphism, whence we have the following exact sequence:

$$U(E | E) \otimes_{\Lambda} M_{ee} \xrightarrow{k_4} N \oplus (U(E | E) \otimes_{\Lambda} M_{ee}) \xrightarrow{cr_2(\bar{\phi})k_1} T_2U(E | E) \otimes_{\Lambda} M_e$$

where:  $k_4 = ((\overline{cr_2(t_2) \otimes 1})^{-1}(cr_2(t_2) \otimes P), -1)^t = ((\overline{cr_2(t_2) \otimes 1})^{-1}cr_2(t_2) \otimes P, -1)^t = (t_{11} \otimes P, -1)^t$ . Since  $k_3$  is an isomorphism, we deduce from 5.20.1 and the last exact sequence that:

$$\ker(cr_2(\hat{\phi})k_3) = k_2 \ker(cr_2(\bar{\phi})k_1) = k_2 \text{Im}(k_4).$$

For  $\xi \in \mathcal{C}(E, E \vee E)$  and  $m \in M_{ee}$  we have:

$$\begin{aligned}
k_2 k_4(\rho_{(1,2)}^2(\xi) \otimes m) &= k_2(t_{11}\rho_{(1,2)}^2(\xi) \otimes Pm, -\rho_{(1,2)}^2(\xi) \otimes m) \\
&= \hat{H}(t_{11}\rho_{(1,2)}^2(\xi) \otimes Pm) - (r_1^2\xi \otimes r_2^2\xi).(m + Tm).
\end{aligned}$$

It follows that  $\ker(cr_2(\hat{\phi})) = 0$  if and only if the condition (QM2) holds.  $\square$

We now are ready to compute the cross-effect of  $- \otimes M$ .

**Corollary 5.21.** *Let  $M \in QMod_{\mathcal{C}}^E$ , there is a natural isomorphism of bifunctors:*

$$\gamma : (T_1U(-) \otimes T_1U(-)) \otimes_{\Lambda \otimes \Lambda} M_{ee} \xrightarrow{\simeq} cr_2(- \otimes M)(-, -)$$

such that, for all  $X, Y \in \langle E \rangle_{\mathcal{C}}$  :

$$(\gamma)_{X,Y}(t_1(f) \otimes t_1(g) \otimes m) = (\iota_{(1,2)}^2)^{-1} \hat{\phi} \pi(t_1(i_1 f) \otimes t_1(i_2 g) \otimes m)$$

where  $f \in \mathcal{C}(E, X)$ ,  $g \in \mathcal{C}(E, Y)$ ,  $m \in M_{ee}$ ,  $i_1 : X \rightarrow X \vee Y$  and  $i_2 : Y \rightarrow X \vee Y$ .

*Proof.* Recall that  $\mathbb{T}_{11}(M_{ee})(X, Y) = (T_1 U(X) \otimes T_1 U(Y)) \otimes_{\Lambda \otimes \Lambda} M_{ee}$  by Definition 3.18. We have the following natural factorization of  $(\gamma)_{X,Y}$ :

$$\mathbb{T}_{11}(M_{ee})(X, Y) \xrightarrow{(\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(M_{ee})(i_1, i_2)} cr_2((\mathbb{T}_{11}(M_{ee}) \Delta_{\mathcal{C}})_{\mathfrak{S}_2})(X, Y) \xrightarrow{cr_2(\hat{\phi})} cr_2(- \otimes M)(X, Y).$$

Since  $\mathbb{T}_{11}(M_{ee})$  is bilinear we deduce from Proposition 5.19 that  $(\iota_{(1,2)}^2)^{-1} \pi \mathbb{T}_{11}(M_{ee})(i_1, i_2)$  is a natural equivalence, and we deduce from Theorem 5.18 that  $cr_2(\hat{\phi})$  is an isomorphism. Hence  $\gamma$  is a natural equivalence, too.  $\square$

## 6. EQUIVALENCE BETWEEN QUADRATIC FUNCTORS AND QUADRATIC $\mathcal{C}$ -MODULES

The aim of this section is to prove the following theorem which is the main result of this paper.

**Theorem 6.1.** *The functors*

$$Quad(\mathcal{C}, Ab) \begin{array}{c} \xleftarrow{\mathbb{T}_2} \\ \xrightarrow{\mathfrak{S}_2} \end{array} QMod_{\mathcal{C}}^E$$

where  $\mathfrak{S}_2$  is defined in Definition 4.15 and  $\mathbb{T}_2(M) = - \otimes M$ , form a pair of adjoint functors extending  $\mathbb{T}_1$  and  $\mathfrak{S}_1$  given in Proposition 3.12 (see also Proposition 4.16).

For  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$ , the functors  $\mathfrak{S}_2$  and  $\mathbb{T}_2$  form a pair of adjoint equivalences.

To prove this theorem we begin by constructing the co-unit and prove that it is an isomorphism thanks to the computations of  $E \otimes M$  given in Proposition 5.12 and of the cross-effect of  $- \otimes M$  given in Corollary 5.21. Then we construct the unit of this adjunction and prove that it is an isomorphism.

**6.1. The co-unit of the adjunction.** The co-unit of the adjunction of Theorem 6.1 is a natural map of quadratic functors  $\epsilon : - \otimes M^F \rightarrow F$ . To define this map we need a sequence of lemmas.

**Lemma 6.2.** *For  $F \in Quad(\mathcal{C}, Ab)$ , there exists a natural transformation  $\overline{u'_F} : (T_2 U)(-) \otimes_{\Lambda} F(E) \rightarrow F(-)$  such that we have the following natural commutative diagram*

$$\begin{array}{ccc} U(X) \otimes_{\Lambda} F(E) & \xrightarrow{(u'_F)_X} & F(X) \\ & \searrow (t_2 \otimes 1)_X & \nearrow (\overline{u'_F})_X \\ & (T_2 U)(X) \otimes_{\Lambda} F(E) & \end{array}$$

for  $X \in \mathcal{C}$  and  $u'_F$  the co-unit of the adjunction between  $\mathfrak{S}$  and  $\mathbb{T}$  given in Proposition 3.6.

*Proof.* This is immediate from the universal property of  $t_2$ .  $\square$

**Lemma 6.3.** *For  $F \in Quad(\mathcal{C}, Ab)$ , there exists a morphism  $(\overline{S_2^F})_X : F(X|X)_{\mathfrak{S}_2} \rightarrow F(X)$  making the following diagram commutative*

$$\begin{array}{ccc} F(X|X) & \xrightarrow{(S_2^F)_X} & F(X) \\ & \searrow \pi & \nearrow (\overline{S_2^F})_X \\ & F(X|X)_{\mathfrak{S}_2} & \end{array}$$

for  $X \in \mathcal{C}$ .

*Proof.* To prove that  $\overline{S_2^F}$  exists we have to prove that  $S_2^F T^F = S_2^F$  where  $T^F$  is the involution of  $F(-|-)$ . Recall that  $T^F$  is defined by the following commutative diagram

$$\begin{array}{ccc} F(X|X) & \xrightarrow{\iota_{(1,2)}^2} & F(X \vee X) \\ T^F \downarrow & & \downarrow F(\tau) \\ F(X|X) & \xrightarrow{\iota_{(1,2)}^2} & F(X \vee X) \end{array}$$

where  $\tau$  is the canonical switch. Thus:

$$(S_2^F)_X T^F = F(\nabla^2) \iota_{(1,2)}^2 T^F = F(\nabla^2 \tau) \iota_{(1,2)}^2 = F(\nabla^2) \iota_{(1,2)}^2 = (S_2^F)_X.$$

□

**Lemma 6.4.** *For  $F \in \text{Quad}(\mathcal{C}, \text{Ab})$ , there exists a natural transformation of functors:*

$$\overline{u'_{cr(F)}} : (((T_1 U)(-) \otimes (T_1 U)(-)) \otimes_{\Lambda \otimes \Lambda} F(E|E)) \Delta_{\mathcal{C}} \mathfrak{S}_2 \rightarrow (cr_2 F(-, -) \Delta_{\mathcal{C}}) \mathfrak{S}_2$$

where  $\Delta_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is the diagonal functor, making the following diagram naturally commutative.

$$\begin{array}{ccc} ((T_1 U)(X) \otimes (T_1 U)(X)) \otimes_{\Lambda \otimes \Lambda} F(E|E) & \xrightarrow{(u'_{cr_2(F)})_{X,X}} & F(X|X) \\ \pi \downarrow & & \pi \downarrow \\ (((T_1 U)(X) \otimes (T_1 U)(X)) \otimes_{\Lambda \otimes \Lambda} F(E|E)) \mathfrak{S}_2 & \xrightarrow{(\overline{u'_{cr_2(F)}})_{X,X}} & (F(X|X)) \mathfrak{S}_2 \end{array}$$

Here  $X \in \mathcal{C}$ , and  $((T_1 U)(X) \otimes (T_1 U)(X)) \otimes_{\Lambda \otimes \Lambda} F(E|E) \xrightarrow{(u'_{cr_2(F)})_{X,X}} F(X|X)$  is the co-unit of the adjunction in Proposition 3.19.

*Proof.* For  $x, y \in U(X)$  and  $m \in F(E|E)$  we have:

$$\begin{aligned} T^F u'_{cr_2(F)}(t_1(f) \otimes t_1(g) \otimes m) &= T^F (cr_2 F(f, g))(m) \\ &= (\iota_{(1,2)}^2)^{-1} F(\tau) \iota_{(1,2)}^2 cr_2 F(f, g)(m) \\ &= (\iota_{(1,2)}^2)^{-1} F(\tau) F(f \vee g) \iota_{(1,2)}^2(m) \\ &= (\iota_{(1,2)}^2)^{-1} F(g \vee f) F(\tau) \iota_{(1,2)}^2(m) \\ &= cr_2 F(g, f) (\iota_{(1,2)}^2)^{-1} F(\tau) \iota_{(1,2)}^2(m) \\ &= cr_2 F(g, f) (T^F(m)) \\ &= u'_{cr_2(F)}(t_1(g) \otimes t_1(f) \otimes T^F(m)) \\ &= u'_{cr_2(F)}(t_1(f) \otimes t_1(g) \otimes m) \end{aligned}$$

Thus  $u'_{cr_2(F)}$  is  $\mathfrak{S}_2$ -equivariant and hence passes to coinvariants. □

We deduce the following proposition.

**Proposition 6.5.** *For  $F \in \text{Quad}(\mathcal{C}, \text{Ab})$ , there exists a natural map  $\epsilon : - \otimes M^F \rightarrow F$  given by:*

$$\epsilon_X = ((\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X, (\overline{u'_F})_X) : X \otimes M^F \rightarrow F(X)$$

for  $X \in \mathcal{C}$  such that

$$((\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X, (\overline{u'_F})_X) \hat{\phi} = (\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X$$

and

$$((\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X, (\overline{u'_F})_X) \hat{\psi} = (\overline{u'_F})_X$$

where the natural maps  $\overline{u'_F}$ ,  $\overline{S_2^F}$  and  $\overline{u'_{cr_2(F)}}$  are defined in Lemmas 6.2, 6.3 and 6.4 respectively and the maps  $\hat{\phi}$  and  $\hat{\psi}$  appear in the pushout diagram of Proposition 5.3.

*Proof.* To prove the existence of  $\epsilon_X$  it is sufficient to show that  $(\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X \psi = (\overline{u'_F})_X \phi$  where  $\psi$  and  $\phi$  are the maps in the pushout diagram of Proposition 5.3.

For  $f \in \mathcal{C}(E, X)$  and  $m \in F(E|E)$  we have:

$$\begin{aligned} (\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X \pi(\delta \otimes 1)(f \otimes m) &= (\overline{S_2^F})_X \pi(u'_{cr_2(F)})_X (\delta \otimes 1)(f \otimes m) \text{ by Lemma 6.4} \\ &= (S_2^F)_X (u'_{cr_2(F)})_X (\delta \otimes 1)(f \otimes m) \text{ by Lemma 6.3} \\ &= (S_2^F)_X (u'_{cr_2(F)})_X (t_1(f) \otimes t_1(f) \otimes m) \text{ by definition of } \delta \\ &= (S_2^F)_X cr_2 F(f, f)(m) \text{ by definition of } u'_{cr_2(F)} \\ &= F(\nabla^2) \iota_{(1,2)}^2 cr_2 F(f, f)(m) \text{ by definition of } S_2^F \\ &= F(\nabla^2) F(f \vee f) \iota_{(1,2)}^2(m) \text{ by definition of } cr_2 F(f, f) \\ &= F(\nabla^2(f \vee f)) \iota_{(1,2)}^2(m) \text{ by functoriality} \\ &= F(f \nabla^2) \iota_{(1,2)}^2(m) \text{ since } \nabla^2(f \vee f) = f \nabla^2 \\ &= F(f) F(\nabla^2) \iota_{(1,2)}^2(m) \text{ by functoriality} \\ &= (u'_F)_X (f \otimes F(\nabla^2) \iota_{(1,2)}^2(m)) \text{ by definition of } u'_F \\ &= (\overline{u'_F})_X (t_2 \otimes Id)(f \otimes F(\nabla^2) \iota_{(1,2)}^2(m)) \text{ by Lemma 6.2} \\ &= (\overline{u'_F})_X (t_2 \otimes Id)(f \otimes S_2^F(m)) \text{ by definition of } S_2^F \\ &= (\overline{u'_F})_X (t_2 \otimes S_2^F)(f \otimes m) \end{aligned}$$

For  $f, g \in \mathcal{C}(E, X)$ ,  $\xi \in \mathcal{C}(E, E \vee E)$ ,  $x = (\iota_{(1,2)}^2)^{-1}(\xi - i_{1*}^2 r_{1*}^2(\xi) - i_{2*}^2 r_{2*}^2(\xi))$  ( $x \in U(E|E)$  by 2.0.2) and  $a \in F(E)$  we have:

$$\begin{aligned} &(\overline{S_2^F})_X (\overline{u'_{cr_2(F)}})_X \psi_1(f \otimes g \otimes x \otimes a) \\ &= (S_2^F)_X (u'_{cr_2(F)})_X \pi(t_1 \otimes t_1 \otimes (H^F)_E(t_{11} \otimes 1))(f \otimes g \otimes x \otimes a) \\ &= (S_2^F)_X (u'_{cr_2(F)})_X (\bar{f} \otimes \bar{g} \otimes (H^F)_E(t_{11} \otimes 1)(x \otimes a)) \text{ by Lemmas 6.3 and 6.4} \\ &= (S_2^F)_X (u'_{cr_2(F)})_X (\bar{f} \otimes \bar{g} \otimes (cr_2(u'_F))_E(x \otimes a)) \text{ by Lemma 4.13} \\ &= (S_2^F)_X cr_2 F(f, g)(cr_2(u'_F))_E(x \otimes a) \text{ by definition of } u'_{cr_2(F)} \\ &= F(\nabla^2) \iota_{(1,2)}^2 cr_2 F(f, g)(cr_2(u'_F))_E(x \otimes a) \\ &= F(\nabla^2) F(f \vee g)(u'_F)_E(\iota_{(1,2)}^2 x \otimes a) \\ &= F(\nabla^2) F(f \vee g) F(\iota_{(1,2)}^2 x)(a) \text{ by definition of } u'_F \\ &= F(\nabla^2(f \vee g)) \iota_{(1,2)}^2 x(a) \text{ by functoriality} \\ &= (u'_F)_X (\nabla^2(f \vee g) \iota_{(1,2)}^2 x \otimes a) \text{ by definition of } u'_F \\ &= (u'_F)_X (U(\nabla^2) U(f \vee g) \iota_{(1,2)}^2 x \otimes a) \\ &= (u'_F)_X (U(\nabla^2) \iota_{(1,2)}^2 cr_2 U(f, g) x \otimes a) \\ &= (\overline{u'_F})_X (t_2 \otimes 1)(U(\nabla^2) \iota_{(1,2)}^2 cr_2 U(f, g) x \otimes a) \text{ by Lemma 6.2} \\ &= (\overline{u'_F})_X (t_2 S_2^U cr_2 U(f, g)(x) \otimes a) \\ &= (\overline{u'_F})_X \phi_1(f \otimes g \otimes x \otimes a). \end{aligned}$$

□

In the following proposition we compute the cross-effect of  $\epsilon$ .

**Proposition 6.6.** *Let  $F : \mathcal{C} \rightarrow Ab$  be a quadratic functor. Then the natural map:*

$$cr_2(\epsilon)_{X,Y} : cr_2(- \otimes M^F)(X, Y) \rightarrow cr_2 F(X, Y)$$

*is an isomorphism for all  $X, Y \in \langle E \rangle_{\mathcal{C}}$ .*



*Proof.* For  $X, Y$  arbitrary objects of  $\mathcal{C}$  we have the following diagram whose right hand square commutes by definition of  $cr_2(\epsilon)_{X,Y}$ :

$$\begin{array}{ccccc}
cr_2(\mathbb{T}_{11}(F(E|E))\Delta_{\mathcal{C}})_{\mathfrak{S}_2}(X, Y) & \xrightarrow{cr_2(\hat{\phi})} & cr_2(- \otimes M^F)(X, Y) & \xrightarrow{\iota_{(1,2)}^2} & (- \otimes M^F)(X \vee Y) \\
\uparrow \simeq & & \downarrow cr_2(\epsilon)_{X,Y} & & \downarrow (\epsilon)_{X \vee Y} \\
(\iota_{(1,2)}^2)^{-1}\pi\mathbb{T}_{11}(F(E|E))(i_1, i_2) & & & & \\
\mathbb{T}_{11}(F(E|E))(X, Y) & \xrightarrow{u'_{cr_2 F}} & cr_2 F(X, Y) & \xrightarrow{\iota_{(1,2)}^2} & F(X \vee Y).
\end{array}$$

In the following, we prove that the left hand square is commutative. Since  $\iota_{(1,2)}^2$  is injective, it is sufficient to prove that

$$\iota_{(1,2)}^2 cr_2(\epsilon)_{X,Y} cr_2(\hat{\phi})(\iota_{(1,2)}^2)^{-1}\pi\mathbb{T}_{11}(F(E|E))(i_1, i_2) = \iota_{(1,2)}^2 u'_{cr_2 F}.$$

For  $f \in \mathcal{C}(E, X)$ ,  $g \in \mathcal{C}(E, Y)$  and  $m \in F(E|E)$  we have:

$$\begin{aligned}
& \iota_{(1,2)}^2 cr_2(\epsilon)_{X,Y} cr_2(\hat{\phi})(\iota_{(1,2)}^2)^{-1}\pi\mathbb{T}_{11}(F(E|E))(i_1, i_2)(\bar{f} \otimes \bar{g} \otimes m) \\
&= (\epsilon)_{X \vee Y} \iota_{(1,2)}^2 cr_2(\hat{\phi})(\iota_{(1,2)}^2)^{-1}\pi(\bar{i}_1 \bar{f} \otimes \bar{i}_2 \bar{g} \otimes m) \\
&= (\epsilon)_{X \vee Y} \hat{\phi} \pi(\bar{i}_1 \bar{f} \otimes \bar{i}_2 \bar{g} \otimes m) \\
&= (\overline{S_2^F})_{X \vee Y} (\overline{u'_{cr(F)}})_{X \vee Y} \pi(\bar{i}_1 \bar{f} \otimes \bar{i}_2 \bar{g} \otimes m) \quad \text{by definition of } \epsilon \text{ given in Proposition 6.5} \\
&= (\overline{S_2^F})_{X \vee Y} (\overline{u'_{cr(F)}})_{X \vee Y} (\bar{i}_1 \bar{f} \otimes \bar{i}_2 \bar{g} \otimes m) \\
&= (\overline{S_2^F})_{X \vee Y} cr_2 F(i_1 f, i_2 g)(m) \quad \text{by definition of } u'_{cr(F)} \text{ given in Proposition 3.19} \\
&= F(\nabla^2) \iota_{(1,2)}^2 cr_2 F(i_1 f, i_2 g)(m) \quad \text{by definition of } \overline{S_2^F} \\
&= F(\nabla^2) F(i_1 f \vee i_2 g) \iota_{(1,2)}^2(m) \\
&= F(f \vee g) \iota_{(1,2)}^2(m) \quad \text{since } \nabla^2(i_1 f \vee i_2 g) = f \vee g \\
&= \iota_{(1,2)}^2 cr_2 F(f, g)(m) \\
&= \iota_{(1,2)}^2 u'_{cr_2(F)}(\bar{f} \otimes \bar{g} \otimes m) \quad \text{by definition of } u'_{cr_2(F)}.
\end{aligned}$$

Since  $F$  is quadratic,  $cr_2(F)$  is bilinear and by Theorem 3.20,  $(u'_{cr_2(F)})_{X,Y}$  is an isomorphism. By commutativity of the left hand square this implies that  $cr_2(\hat{\phi})$  is injective and by Lemma 5.7 we know that  $cr_2(\hat{\phi})$  is surjective. Thus  $cr_2(\hat{\phi})$  is an isomorphism, hence so is  $cr(\epsilon)_{X,Y}$ .  $\square$

In particular, we obtain the following corollary which achieves the proof of Proposition 4.14.

**Corollary 6.7.** *The property (QM2) is satisfied for  $M^F$ .*

*Proof.* By Theorem 5.18 it is sufficient to prove that

$$cr_2(\hat{\phi}) : cr_2((((T_1 U(-) \otimes T_1 U(-)) \otimes_{\Lambda \otimes \Lambda} F(E|E))\Delta_{\mathcal{C}})_{\mathfrak{S}_2}) \rightarrow cr_2(- \otimes M^F)$$

is an equivalence. This is part of the proof of Proposition 6.6.  $\square$

**Proposition 6.8.** *If  $\mathcal{C} = \langle E \rangle_{\mathcal{C}}$ , for  $F \in \text{Quad}(\mathcal{C}, Ab)$  the natural map  $\epsilon : - \otimes M^F \rightarrow F$  is a natural equivalence.*

*Proof.* Since  $F$  is quadratic by hypothesis and  $- \otimes M^F$  is quadratic by Proposition 5.6, it is sufficient to prove that  $(\epsilon)_E$  and  $cr_2(\epsilon)_{(E,E)}$  are isomorphisms according to Proposition 1.22.

By Lemma 5.17 we know that  $(\epsilon)_E$  is an isomorphism since

$$((\overline{S_2^F})_E (\overline{u'_{cr(F)}})_{E, (\overline{u'_F})_E}) = ((\overline{S_2^F})_E \overline{\mu_{ee}}, \overline{\mu_e}).$$

By Proposition 6.6 we know that  $cr_2(\epsilon)_{(E,E)}$  is an isomorphism.  $\square$

**Proposition 6.9.** *The natural map  $\epsilon : - \otimes M^F \rightarrow F$  is the counit of the adjunction given in Theorem 6.1.*

*Proof.* For  $M \in QMod_{\mathcal{C}}^E$  and  $\alpha : - \otimes M \rightarrow F$  we want to prove that there exists a unique  $\beta : M \rightarrow M^F$  such that  $\epsilon(- \otimes M)(\beta) = \alpha$ . Since  $\epsilon$  is a natural equivalence the latter relation implies that:  $(- \otimes M)(\beta) = \epsilon^{-1}\alpha$ . One can check that  $\beta$  such that  $\beta_e = \alpha_e(\overline{P\mu_{ee}}, \overline{\mu_e})^{-1}$  and  $\beta_{ee} = cr_2(\alpha)_{E,E}\gamma_{E,E}(\mu_{ee})^{-1}$  is the unique solution.  $\square$

## 6.2. The unit of the adjunction.

**Definition 6.10.** *Let  $\eta : Id_{QMod_{\mathcal{C}}^E} \rightarrow \mathbb{S}_2\mathbb{T}_2$  be the natural map such that, for  $M \in QMod_{\mathcal{C}}^E$ ,  $\eta_M : M \rightarrow \mathbb{S}_2\mathbb{T}_2(M)$  is given by  $(\eta_M)_e = (\overline{P\mu_{ee}}, \overline{\mu_e})^{-1}$  and  $(\eta_M)_{ee} = (\gamma)_{E,E}\mu_{ee}^{-1}$  where  $\mu_{ee}$  is the canonical isomorphism and  $\gamma$  is the natural equivalence of Corollary 5.21.*

**Proposition 6.11.** *The natural map  $\eta : Id_{QMod_{\mathcal{C}}^E} \rightarrow \mathbb{S}_2\mathbb{T}_2$  is a natural equivalence.*

*Proof.* By Lemma 5.16 and Corollary 5.21,  $(\eta_M)_e$  and  $(\eta_M)_{ee}$  are isomorphisms.  $\square$

**Proposition 6.12.** *The natural map  $\eta : Id_{QMod_{\mathcal{C}}^E} \rightarrow \mathbb{S}_2\mathbb{T}_2$  is the unit of the adjunction in Theorem 6.1.*

*Proof.* For  $F \in Quad(\mathcal{C}, Ab)$  and  $\alpha : M \rightarrow M^F$  a morphism in  $QMod_{\mathcal{C}}^E$ , we want to prove that there exists a unique natural map  $\beta : - \otimes M \rightarrow F$  such that  $\mathbb{S}_2(\beta)\eta_M = \alpha$ . Let  $\beta$  be the composition  $\epsilon(- \otimes \alpha)$ . We check that  $\beta$  is the unique solution of  $\mathbb{S}_2(\beta)\eta_M = \alpha$ .  $\square$

Together with Proposition 6.11, 6.9 and 6.8 this achieves the proof of Theorem 6.1.

## 7. APPLICATION TO THEORIES OF COGROUPS

In this section, we apply Theorem 6.1 to the particular case where  $\mathcal{C}$  is a theory of cogroups  $\mathcal{T}$  with generator  $E$ . The interest in this case comes from the paper [4] where the authors define, in Proposition 3.6, a functor:

$$u : Quad(\mathcal{T}, Gr) \rightarrow Square$$

to the category *Square* of so-called square groups, sending  $F \in Quad(\mathcal{T}, Gr)$  to the square group  $u(F)(E) = (F(E) \xrightarrow{H} F(E \mid E) \xrightarrow{P} F(E))$ , see Definition 7.12 below. They prove that  $u$  is an equivalence of categories when  $\mathcal{T}$  is the category  $\langle \mathbb{Z} \rangle_{Gr}$  of free groups of finite rank. This is no longer true for more general theories of cogroups, even when restricting to quadratic functors with values in  $Ab$ . In fact, we here show how for general  $\mathcal{T}$  the structure of square group has to be enriched in order to obtain an equivalence with quadratic functors from  $\mathcal{T}$  to  $Ab$ . In particular, a second map relating  $F(E)$  to  $F(E \mid E)$  has to be added to the picture; in the case where  $\mathcal{T} = \langle \mathbb{Z} \rangle_{Gr}$  this turns out to provide a new interpretation of the map  $\Delta$  associated with a square group in [4]. Recall that a theory of cogroups is a pointed category  $\mathcal{T}$  with finite sums such that each object  $X$  has the structure of a cogroup given by the maps  $\mu : X \rightarrow X \vee X$  and  $\tau : X \rightarrow X$ . Then, for  $X, Y \in \mathcal{T}$ ,  $\mathcal{T}(X, Y)$  is a group with  $f \bullet g = (f, g)\mu$  and  $f^{-1} = f\tau$ . The identity element of  $\bullet$  is the nul map 0. Moreover, for  $Z \in \mathcal{T}$  and  $h \in \mathcal{T}(Y, Z)$  the map  $h_* : \mathcal{T}(X, Y) \rightarrow \mathcal{T}(X, Z)$  is a homomorphism of groups.

**Notation.** For brevity, we write in this section  $i_1, i_2, r_1$  and  $r_2$  instead of  $i_1^2, i_2^2, r_1^2$  and  $r_2^2$ , respectively.

For a theory of cogroups  $\mathcal{T}$  we have a simpler description of a quadratic  $\mathcal{T}$ -module relative to  $E$  given in the following theorem. In particular, note that the functor  $U$  disappears from the picture.

**Theorem 7.1.** *Let  $\mathcal{T}$  be a theory of cogroups. The category of quadratic  $\mathcal{T}$ -modules relative to  $E$  is equivalent to the category of diagrams of group homomorphisms:*

$$M = \left( \begin{array}{ccc} & M_e & \\ & \searrow^{H_1} & \\ & & M_{ee} \xrightarrow{P} M_e \\ & \nearrow_{H_2} & \\ T_{11}cr_2(\mathcal{T}(E, -))(E, E) \otimes_{\Lambda} \text{coker}(P) & & \end{array} \right)$$

where

- $M_e$  is a left  $\Lambda$ -module;
- $M_{ee}$  is a left  $\bar{\Lambda} \otimes \bar{\Lambda}$ -module;
- $P : M_{ee} \rightarrow M_e$  is a homomorphism of  $\Lambda$ -modules with respect to the diagonal action of  $\Lambda$  on  $M_{ee}$ ;
- $H_1$  is a homomorphism of abelian groups;
- $H_2$  is a homomorphism of  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules,

satisfying the following relations for  $a \in M_e$ ,  $m \in M_{ee}$ ,  $\gamma \in cr_2(\mathcal{T}(E, -))(E, E)$ ,  $\alpha, \beta \in \mathcal{T}(E, E)$ :

$$\begin{aligned} (T1) \quad & PH_1P = 2P \\ (T2) \quad & H_1P((\bar{\alpha} \otimes \bar{\beta})m) - (\bar{\beta} \otimes \bar{\alpha})H_1P(m) = (\bar{\alpha} \otimes \bar{\beta})m - (\bar{\beta} \otimes \bar{\alpha})m \\ (T3) \quad & H_1PH_1(a) = 2H_1(a) + H_2(t_{11}([i_2, i_1]) \otimes \bar{a}) \\ (T4) \quad & H_1PH_2(t_{11}\gamma \otimes \bar{a}) = H_2(t_{11}(\gamma \bullet \tau\gamma) \otimes \bar{a}) \\ (T5) \quad & (\nabla^2\gamma \bullet \alpha \bullet \beta)a = \alpha a + \beta a + P((\bar{\alpha} \otimes \bar{\beta})H_1(a)) + PH_2(t_{11}\gamma \otimes \bar{a}) \\ (T6) \quad & H_1(\alpha a) = H_2(t_{11}h(\alpha) \otimes \bar{a}) + (\bar{\alpha} \otimes \bar{\alpha})H_1(a) \end{aligned}$$

where  $[i_2, i_1] = i_2 \bullet i_1 \bullet (i_2)^{-1} \bullet (i_1)^{-1}$ ,  $\tau$  is the canonical switch of  $E \vee E$  and  $h : \mathcal{T}(E, E) \rightarrow \mathcal{T}(E, E | E)$  is given by

$$h(\alpha) = (i_{(1,2)}^2)^{-1}(((i_1 \bullet i_2)\alpha) \bullet (i_2\alpha)^{-1} \bullet (i_1\alpha)^{-1}).$$

Note that the map  $h$  describes the deviation of endomorphisms of  $E$  from being morphisms of cogroups.

The plan of the proof of Theorem 7.1 is as follows: we first compute, in several steps, the symmetric bifunctor  $T_{11}(cr_2U)$ . The result is then used to split up the map  $\hat{H}$  in a given quadratic  $\mathcal{T}$ -module into the two maps  $H_1$  and  $H_2$  occurring in Theorem 7.1, and to translate the properties of the map  $\hat{H}$  in terms of  $H_1$  and  $H_2$ , which leads to the relations (T1)-(T6).

As a main tool we need some elementary facts about augmentation ideals of group rings.

**7.1. Augmentation ideals: recollections and action of the linearization functor.** Recall that the augmentation ideal  $IG$  of a group  $G$  is the kernel of the augmentation map  $\epsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$  of the group ring  $\mathbb{Z}[G]$ .  $IG$  is a free  $\mathbb{Z}$ -module generated by the elements  $g - 1$  for  $g \in G \setminus \{1\}$ . For a subset  $H$  of  $G$  we denote by  $IH$  the subgroup of  $IG$  generated by the elements of the form  $h - 1$  for  $h \in H$ . The key ideas of this section are to use the natural isomorphism  $\Xi : I(\mathcal{T}(E, X)) \rightarrow U(X)$  such that  $\Xi(f - 1) = f$  and the trivial but useful formula  $a \cdot b - 1 = (a - 1) + (b - 1) + (a - 1) \cdot (b - 1)$  in  $IG$ . Moreover, we need the following elementary result which is wellknown.

**Proposition 7.2.** *For  $G \in Gr$  we have a natural isomorphism of abelian groups:*

$$\theta : IG/(IG)^2 \xrightarrow{\cong} G/[G, G]$$

such that for  $g \in G$ ,  $\theta(\overline{g-1}) = \bar{g}$ .

**7.2. Computation of  $cr_2U(X, Y)$ .** In a first step we compute the cross-effect of the functor  $U$ , which by the isomorphism  $\Xi$  above comes down to computing the cross-effect of the composite functor  $IT(E, -)$  where  $I : Gr \rightarrow Ab$  is given by taking the augmentation ideal. More generally, we have the following result.

**Proposition 7.3.** *Let  $F : \mathcal{C} \rightarrow Gr$  be a reduced functor. There is an isomorphism of bifunctors  $\mathcal{T} \times \mathcal{T} \rightarrow Ab$ :*

$$\Theta : I(F(X | Y)) \oplus (IF(X) \otimes IF(Y)) \oplus (I(F(X | Y)) \otimes I(F(X) \times F(Y))) \xrightarrow{\cong} cr_2(IF)(X, Y)$$

given by

$$\Theta(x - 1, 0, 0) = x - 1$$

$$\Theta(0, (y - 1) \otimes (z - 1), 0) = (i_{1*}y - 1) \cdot (i_{2*}z - 1)$$

$$\Theta(0, 0, (u - 1) \otimes (v - 1)) = (u - 1) \cdot (sv - 1)$$

where  $s : F(X) \times F(Y) \rightarrow F(X \vee Y)$  is the map defined by  $s(x, y) = i_{1*}x \cdot i_{2*}y$ , the point  $\cdot$  denoting the group structure on  $F(X \vee Y)$ .

The proof requires the following elementary fact.

**Lemma 7.4.** *Let  $G$  be a group and  $H, K$  be two subsets of  $G$  containing 1 such that each  $g \in G$  admits a unique decomposition  $g = h_g k_g$  with  $h_g \in H$  and  $k_g \in K$ . Then the map:*

$$\phi : IH \oplus IK \oplus (IH \otimes IK) \rightarrow IG$$

given by  $\phi(h - 1, 0, 0) = h - 1$ ,  $\phi(0, k - 1, 0) = k - 1$  and  $\phi(0, 0, (h' - 1) \otimes (k' - 1)) = (h' - 1) \cdot (k' - 1)$  is an isomorphism of  $\mathbb{Z}$ -modules.

*Proof.* We define

$$\psi : IG \rightarrow IH \oplus IK \oplus (IH \otimes IK)$$

by

$$\psi(g - 1) = (h_g - 1, k_g - 1, (h_g - 1) \otimes (k_g - 1)).$$

One readily checks that  $\psi$  is the inverse of  $\phi$ , using the fact that the unique decomposition of  $h \in H \subset G$  (resp.  $k \in K \subset G$ ) in  $HK$  is  $h \cdot 1$  (resp.  $1 \cdot k$ ).  $\square$

*Proof of Proposition 7.3.* For  $X, Y \in \mathcal{C}$  we have a short exact sequence:

$$1 \rightarrow F(X | Y) \rightarrow F(X \vee Y) \xrightarrow{(r_{1*}, r_{2*})^t} F(X) \times F(Y) \rightarrow 1.$$

Since  $i_{1*}$  and  $i_{2*}$  are group morphisms we have  $s(1, 1) = 1$  and since  $F$  is reduced,  $s$  is a set-theoretic section of  $(r_{1*}, r_{2*})^t$  natural in  $X$  and  $Y$ . Hence the subsets  $H = F(X | Y)$  and  $K = s(F(X) \times F(Y))$  of the group  $F(X \vee Y)$  satisfy the conditions of Lemma 7.4 and we have an isomorphism of bifunctors:

$$I(F(X \vee Y)) \simeq I(F(X | Y)) \oplus I(s(F(X) \times F(Y))) \oplus (I(F(X | Y)) \otimes I(s(F(X) \times F(Y)))).$$

Since the map  $s$  is a bijection of  $F(X) \times F(Y)$  onto the set  $K$ , applying Proposition 7.4 to  $F(X) \times F(Y)$  provides an isomorphism of bifunctors:

$$\begin{aligned} I(F(X \vee Y)) &\simeq I(F(X | Y)) \oplus (IF(X) \oplus IF(Y) \oplus IF(X) \otimes IF(Y)) \\ &\quad \oplus (I(F(X | Y)) \otimes I(F(X) \times F(Y))) \end{aligned}$$

Now the assertion follows from (2.0.3).  $\square$

The following application of Proposition 7.3 illustrates the power of the linearization functor  $T_1$ .

**Proposition 7.5.** *Let  $F : \mathcal{C} \rightarrow Gr$  be a reduced functor. Then we have a natural isomorphism:*

$$\theta^F : T_1(IF) \rightarrow T_1F$$

defined by  $\theta^F(t_1(x-1)) = t_1(x)$  for  $X \in \mathcal{C}$  and  $x \in F(X)$ .

*Proof.* By the isomorphism  $\Theta$  in Proposition 7.3 we have

$$\begin{aligned} T_1(IF)(X) &= \text{coker}(cr_2(IF)(X, X) \xrightarrow{S_2^{IF(-)}} IF(X)) \\ &= \text{coker}(I(F(X | X)) \oplus (IF(X) \otimes IF(X)) \oplus (I(F(X | X)) \otimes I(F(X) \times F(X))) \xrightarrow{S_2^{IF(-)} \Theta} IF(X)) \\ S_2^{IF(-)} \Theta(x-1, 0, 0) &= S_2^{IF(-)}(x-1) = I(S_2^F)(x-1) \\ S_2^{IF(-)} \Theta(0, (y-1) \otimes (z-1), 0) &= S_2^{IF(-)}((i_{1*}y-1) \cdot (i_{2*}z-1)) = IF(\nabla^2)((i_{1*}y-1) \cdot (i_{2*}z-1)) \\ &= (\nabla_*^2 i_{1*}y - 1) \cdot (\nabla_*^2 i_{2*}z - 1) = (y-1) \cdot (z-1) \in (IF(X))^2 \\ S_2^{IF(-)} \Theta(0, 0, (u-1) \otimes (v-1)) &= IF(\nabla^2)((u-1) \cdot (sv-1)) = (\nabla^2 u - 1) \cdot (\nabla^2 sv - 1) \in (IF(X))^2. \end{aligned}$$

By Proposition 7.2 we obtain:

$$\begin{aligned} T_1(IF)(X) &= IF(X) / (Im(I(S_2^F)) + (IF(X))^2) \simeq F(X)^{ab} / ab(Im(S_2^F)) \\ &\simeq (F(X) / Im(S_2^F))^{ab} \simeq (T_1F(X))^{ab} \simeq T_1F(X) \text{ by Proposition 1.10} \end{aligned}$$

where  $-^{ab} : Gr \rightarrow Ab$  is the abelianization functor and  $ab : F(X) \rightarrow F(X)^{ab}$ .  $\square$

**7.3. Computation of  $T_{11}cr_2U(X, Y)$ .** In order to compute  $T_{11}cr_2U(E, E)$  as a right  $\Lambda$ -module we apply the results of section 7.2 to the reduced functor  $\mathcal{T}(E, -) : \mathcal{C} \rightarrow Gr$ .

**Notation 7.6.** *In the sequel we use the following abbreviations:*

$$\begin{aligned} T_1IT(E, X) &:= T_1(IT(E, -))(X); \\ I(\mathcal{T}(E, X | Y)) &:= Icr_2(\mathcal{T}(E, -))(X, Y); \\ T_{11}IT(E, X | Y) &:= T_{11}(Icr_2(\mathcal{T}(E, -)))(X, Y). \end{aligned}$$

**Proposition 7.7.** *There is a binatural isomorphism:*

$$(7.7.1) \quad \Upsilon : T_{11}\mathcal{T}(E, X | Y) \oplus (T_1\mathcal{T}(E, X) \otimes T_1\mathcal{T}(E, Y)) \xrightarrow{\cong} T_{11}cr_2(IT(E, -))(X, Y)$$

such that

$$\Upsilon(t_{11}\xi, t_1f \otimes t_1g) = t_{11}(t_{(1,2)}^2)^{-1}((\xi - 1) + (i_1f - 1) \bullet (i_2g - 1)).$$

Moreover, the right action of  $\Lambda$  on  $T_{11}cr_2(IT(E, -))(X, Y)$  induced by precomposition in  $\mathcal{T}$  is given by

$$(7.7.2) \quad \Upsilon(t_{11}\xi, 0) \cdot \alpha = \Upsilon(t_{11}(\xi\alpha), 0)$$

$$(7.7.3) \quad \Upsilon(0, t_1f \otimes t_1g) \cdot \alpha = \Upsilon(t_{11}((f \vee g)h(\alpha)), t_1(f\alpha) \otimes t_1(g\alpha))$$

for  $\alpha \in \mathcal{T}(E, E)$ .

*Proof.* Taking  $F = \mathcal{T}(E, -)$  in Proposition 7.3 gives an isomorphism of bifunctors

$$\begin{aligned} I(\mathcal{T}(E, X | Y)) \oplus (I(\mathcal{T}(E, X)) \otimes I(\mathcal{T}(E, Y))) \oplus (I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X) \times \mathcal{T}(E, Y))) \\ \xrightarrow{\Theta} cr_2(IT(E, -))(X, Y). \end{aligned}$$

We have the following binatural isomorphisms

$$\begin{aligned} I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X) \times \mathcal{T}(E, Y)) &\simeq (I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X))) \\ &\quad \oplus (I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, Y))) \\ &\quad \oplus (I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X)) \otimes I(\mathcal{T}(E, Y))). \end{aligned}$$

Thus  $I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X) \times \mathcal{T}(E, Y))$  is a sum of bifunctors which are diagonalizable as functors in  $X$  or in  $Y$ . So Proposition 1.12 implies that  $T_{11}(I(\mathcal{T}(E, X | Y)) \otimes I(\mathcal{T}(E, X) \times \mathcal{T}(E, Y))) = 0$ . Using Proposition 7.5 and Example 1.17 the isomorphism  $T_{11}(\Theta)$  thus becomes the desired isomorphism  $\Upsilon$ .

The structure of right  $\Lambda$ -module on  $T_{11}IT(E, X | Y)$  is induced by the inclusion  $\mathcal{T}(E, X | Y) \rightarrow \mathcal{T}(E, X \vee Y)$ ; this implies relation (7.7.2). To prove relation (7.7.3) let  $f \in \mathcal{T}(E, X)$  and  $g \in \mathcal{T}(E, Y)$ . Then:

$$\begin{aligned} \iota_{(1,2)}^2 \Theta(0, (f-1) \otimes (g-1), 0). \alpha &= ((i_1 f - 1) \bullet (i_2 g - 1)). \alpha \\ &= ((i_1 f \bullet i_2 g - 1) - (i_1 f - 1) - (i_2 g - 1)). \alpha \\ &= ((i_1 f \bullet i_2 g) \circ \alpha - 1) - (i_1 f \circ \alpha - 1) - (i_2 g \circ \alpha - 1) \\ &= (\omega \bullet (i_1 f \alpha) \bullet (i_2 g \alpha) - 1) - (i_1 f \alpha - 1) - (i_2 g \alpha - 1) \end{aligned}$$

where  $\omega = ((i_1 f \bullet i_2 g) \circ \alpha) \bullet (i_2 g \circ \alpha)^{-1} \bullet (i_1 f \circ \alpha)^{-1}$ . Hence:

$$\begin{aligned} \iota_{(1,2)}^2 \Theta(0, (f-1) \otimes (g-1), 0). \alpha &= (\omega - 1) + ((i_1 f \alpha) \bullet (i_2 g \alpha) - 1) + (\omega - 1) \bullet ((i_1 f \alpha) \bullet (i_2 g \alpha) - 1) - (i_1 f \alpha - 1) \\ &\quad - (i_2 g \alpha - 1) \\ &= (\omega - 1) + (i_1 f \alpha - 1) \bullet (i_2 g \alpha - 1) + (\omega - 1) \bullet ((i_1 f \alpha) \bullet (i_2 g \alpha) - 1). \end{aligned}$$

Now observe that

$$(7.7.4) \quad (i_1 f) \bullet (i_2 g) = (f \vee g)_*(i_1 \bullet i_2).$$

Thus  $\omega = (f \vee g)_*((i_1 \bullet i_2) \alpha) \bullet (f \vee g)_*(i_2 \alpha)^{-1} \bullet (f \vee g)_*(i_1 \alpha)^{-1} = (f \vee g)_*(h(\alpha))$ . It follows that  $\omega = \mathcal{T}(E, f | g)(h(\alpha)) \in \mathcal{T}(E, X | Y)$ . Hence

$$(\omega - 1) \bullet ((i_1 f \alpha) \bullet (i_2 g \alpha) - 1) \in \mathcal{T}(E, X | Y) \otimes I(\mathcal{T}(E, X) \times \mathcal{T}(E, Y)),$$

and we deduce that

$$\begin{aligned} \Upsilon(0, t_1 f \otimes t_1 g). \alpha &= t_{11}(\Theta(0, (f-1) \otimes (g-1), 0). \alpha) \\ &= \Upsilon(t_{11}((f \vee g)h(\alpha)), t_1(f \alpha) \otimes t_1(g \alpha)), \end{aligned}$$

as desired.  $\square$

Using the natural isomorphism  $\Xi : I(\mathcal{T}(E, X)) \rightarrow U(X)$  we finally obtain:

**Corollary 7.8.** *We have a natural isomorphism of right  $\Lambda$ -modules:*

$$T_{11}cr_2(\Xi)\Upsilon : T_{11}\mathcal{T}(E, X | Y) \oplus (T_1\mathcal{T}(E, X) \otimes T_1\mathcal{T}(E, Y)) \rightarrow T_{11}cr_2U(X, Y)$$

such that:

$$T_{11}cr_2(\Xi)\Upsilon(t_{11}\xi, t_1 f \otimes t_1 g) = t_{11}(\xi + \rho_{(1,2)}^2(i_1 f \bullet i_2 g))$$

and where the action of  $\Lambda$  on the domain of  $T_{11}cr_2(\Xi)\Upsilon$  is given by:

$$(t_{11}\xi, t_1 f \otimes t_1 g). \alpha = (t_{11}((\xi \alpha) \bullet (f \vee g)h(\alpha)), t_1(f \alpha) \otimes t_1(g \alpha)).$$

*Proof.* It suffices to combine Proposition 7.7 with the isomorphism  $T_{11}cr_2(\Xi)$ . In fact,

$$\begin{aligned} T_{11}cr_2(\Xi)\Upsilon(t_{11}\xi, t_1 f \otimes t_1 g) &= T_{11}cr_2(\Xi)t_{11}(\iota_{(1,2)}^2)^{-1}((\xi - 1) + (i_1 f - 1) \bullet (i_2 g - 1)) \\ &= t_{11}(\iota_{(1,2)}^2)^{-1}\Xi((\xi - 1) + (i_1 f - 1) \bullet (i_2 g - 1)) \\ &= t_{11}(\iota_{(1,2)}^2)^{-1}\Xi((\xi - 1) + (i_1 f \bullet i_2 g - 1) - (i_1 f - 1) - (i_2 g - 1)) \\ &= t_{11}(\iota_{(1,2)}^2)^{-1}(\xi + i_1 f \bullet i_2 g - i_1 f - i_2 g) \\ &= t_{11}(\xi + \rho_{12}^2(i_1 f \bullet i_2 g)) \text{ by (2.0.2)} \end{aligned}$$

$\square$

**7.4. The action of the involution on  $T_{11}cr_2U(X, Y)$ .** Since  $\hat{H}$  is a morphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules we have to understand the action of the involution on the two components of  $T_{11}cr_2U(E, E)$  according to Corollary 7.8.

The canonical switch  $\tau$  induces a map  $\tau_* : \mathcal{T}(E, X | Y) \rightarrow \mathcal{T}(E, Y | X)$ ; for  $X = Y = E$  we have  $\tau_* = T^{\mathcal{T}(E, -)}$  and thus

$$\bar{T}^U t_{11}(\xi) = t_{11} T^{\mathcal{T}(E, -)}(\xi)$$

for  $\xi \in \mathcal{T}(E, E | E)$ , see Propositions 4.2 and 4.3.

For  $f, g \in \mathcal{T}(E, X)$  we have

$$\begin{aligned} I(\tau)\iota_{(1,2)}^2 \Theta(0, (f-1) \otimes (g-1), 0) &= I(\tau)((i_1 f - 1) \bullet (i_2 g - 1)) \\ &= (i_2 f - 1) \bullet (i_1 g - 1) \\ &= (i_2 f \bullet i_1 g - 1) - (i_2 f - 1) - (i_1 g - 1) \\ &= ([i_2 f, i_1 g] \bullet i_1 g \bullet i_2 f - 1) - (i_2 f - 1) - (i_1 g - 1) \\ &= ([i_2 f, i_1 g] - 1) + (i_1 g \bullet i_2 f - 1) + \\ &\quad ([i_2 f, i_1 g] - 1) \bullet (i_1 g \bullet i_2 f - 1) - (i_2 f - 1) - (i_1 g - 1) \\ &= ([i_2 f, i_1 g] - 1) + (i_1 g - 1) \bullet (i_2 f - 1) + \\ &\quad ([i_2 f, i_1 g] - 1) \bullet (i_1 g \bullet i_2 f - 1) \end{aligned}$$

But  $[i_2 f, i_1 g] \in \mathcal{T}(E, X | Y)$ , hence we obtain for  $f = \alpha$  and  $g = \beta$  in  $\mathcal{T}(E, E)$ :

$$\begin{aligned} \bar{T}^U \Upsilon(0, t_1 \alpha \otimes t_1 \beta) &= t_{11} (\iota_{(1,2)}^2)^{-1} I(\tau) \iota_{(1,2)}^2 \Theta(0, (\alpha-1) \otimes (\beta-1), 0) \\ &= \Upsilon(t_{11}[i_2 \alpha, i_1 \beta], t_1 \beta \otimes t_1 \alpha) \end{aligned}$$

Combining this relation with (7.4) and Corollary 7.8 we obtain

$$(7.8.1) \quad \bar{T}^U t_{11}(\xi + \rho_{(1,2)}^2((i_1 \alpha) \bullet (i_2 \beta))) = t_{11} (T^{\mathcal{T}(E, -)}(\xi) + [i_2 \alpha, i_1 \beta] + \rho_{(1,2)}^2((i_1 \beta) \bullet (i_2 \alpha)))$$

**7.5. The homomorphism  $\hat{H}$ .** By definition of a quadratic  $\mathcal{C}$ -module the map

$$\hat{H} : T_{11}(cr_2(U))(E, E) \otimes_{\Lambda} M_e \rightarrow M_{ee}$$

is a morphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules. In the case of cogroups we have the following equivalent description of  $\hat{H}$ :

**Proposition 7.9.** *For a theory of cogroups  $\mathcal{T}$ , the morphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules  $\hat{H}$  is equivalent to the following data:*

(1) *a morphism of abelian groups:*

$$H_1 : M_e \rightarrow M_{ee}$$

(2) *a morphism of symmetric  $\bar{\Lambda} \otimes \bar{\Lambda}$ -modules*

$$H'_2 : T_{11}\mathcal{T}(E, X | Y) \otimes_{\Lambda} M_e \rightarrow M_{ee}$$

*satisfying the following relations for  $\alpha \in \mathcal{T}(E, E)$  and  $a \in M_e$ :*

$$(7.9.1) \quad H_1(\alpha a) = H'_2(t_{11}h(\alpha) \otimes a) + (\bar{\alpha} \otimes \bar{\alpha})H_1(a)$$

$$(7.9.2) \quad TH_1(a) = H_1(a) + H'_2(t_{11}([i_2, i_1]) \otimes a).$$

*Proof.* Using Corollary 7.8 and relation (7.7.4) one easily verifies that when dropping the symmetry conditions, the map  $\hat{H}$  is equivalent with the maps  $H_1$  and  $H'_2$  as in (1) and (2) satisfying relation (7.9.1). In fact, given  $\hat{H}$  we may define

$$(7.9.3) \quad H_1(a) = \hat{H}(t_{11}\rho_{(1,2)}^2(i_1 \bullet i_2) \otimes a)$$

$$(7.9.4) \quad H'_2(t_{11}\xi \otimes a) = \hat{H}(t_{11}\xi \otimes a).$$

Conversely, given  $H_1$  and  $H'_2$  the associated map  $\hat{H}$  is determined by the relations (7.9.4) and

$$\hat{H}(t_{11}\rho_{(1,2)}((i_1\alpha) \bullet (i_2\beta)) \otimes a) = (\bar{\alpha} \otimes \bar{\beta})H_1(a)$$

for  $\alpha, \beta \in \mathcal{T}(E, E)$ . It remains to check that  $\hat{H}$  commutes with the respective involutions if and only if relation (7.9.2) holds. In fact,

$$\begin{aligned} & \hat{H}\bar{T}^U(t_{11}(\xi + \rho_{(1,2)}^2((i_1\alpha) \bullet (i_2\beta))) \otimes a) \\ &= \hat{H}\left(t_{11}\left(T^{\mathcal{T}(E,-)}(\xi) + [i_2\alpha, i_1\beta] + \rho_{(1,2)}^2((i_1\beta) \bullet (i_2\alpha))\right) \otimes a\right) \text{ by (7.7.4)} \\ &= \hat{H}\left(t_{11}\left(T^{\mathcal{T}(E,-)}(\xi) + [i_2\alpha, i_1\beta] + cr_2U(\beta, \alpha)\rho_{(1,2)}^2(i_1 \bullet i_2)\right) \otimes a\right) \\ &= H'_2\left(t_{11}\left(T^{\mathcal{T}(E,-)}(\xi) + [i_2\alpha, i_1\beta]\right) \otimes a\right) + (\bar{\beta} \otimes \bar{\alpha})H_1(a) \\ &= H'_2\left(t_{11}T^{\mathcal{T}(E,-)}(\xi) \otimes a\right) + (\bar{\beta} \otimes \bar{\alpha})H'_2([i_2, i_1] \otimes a) + (\bar{\beta} \otimes \bar{\alpha})H_1(a) \end{aligned}$$

since  $[i_2\alpha, i_1\beta] = (\beta \vee \alpha)_*[i_2, i_1] = \mathcal{T}(E, \beta \mid \alpha)([i_2, i_1])$  and  $H'_2$  is  $\bar{\Lambda} \otimes \bar{\Lambda}$ -linear. On the other hand,

$$\begin{aligned} T\hat{H}(t_{11}(\xi + \rho_{(1,2)}^2((i_1\alpha) \bullet (i_2\beta))) \otimes a) &= T\hat{H}(t_{11}(\xi + cr_2U(\alpha, \beta)\rho_{(1,2)}^2(i_1 \bullet i_2)) \otimes a) \\ &= TH'_2(t_{11}\xi \otimes a) + T((\bar{\alpha} \otimes \bar{\beta})H_1(a)) \\ &= H'_2(t_{11}T^{\mathcal{T}(E,-)}(\xi) \otimes a) + (\bar{\beta} \otimes \bar{\alpha})TH_1(a). \end{aligned}$$

Thus if relation (7.9.2) holds,  $\hat{H}$  commutes with the respective involutions; the converse is also true as we may take  $\alpha = \beta = 1_E$ .  $\square$

**7.6. The conditions (QM1) and (QM2).** In this section we translate the conditions (QM1) and (QM2) to the case of cogroups. We begin by the remark that for each  $\xi \in \mathcal{T}(E, E \vee E)$ , there exists  $\gamma \in \mathcal{T}(E, E \mid E)$  such that  $\xi = \gamma \bullet i_1r_1\xi \bullet i_2r_2\xi$ . In fact,  $(r_{1*}, r_{2*})^t(\xi) = (r_1\xi, r_2\xi) = (r_{1*}, r_{2*})^t(i_1r_1\xi \bullet i_2r_2\xi)$  hence  $\xi \bullet (i_1r_1\xi \bullet i_2r_2\xi)^{-1} \in \mathcal{T}(E, E \mid E)$ . Hence any element of  $\mathcal{T}(E, E \vee E)$  can be written in the form  $\gamma \bullet i_1\alpha \bullet i_2\beta$  for  $\alpha, \beta \in \mathcal{T}(E, E)$  and  $\gamma \in \mathcal{T}(E, E \mid E)$ .

**Proposition 7.10.** *For a theory of cogroups  $\mathcal{T}$ , the condition (QM1) is equivalent to the following equation:*

$$(\nabla^2\gamma \bullet \alpha \bullet \beta).a = \alpha a + \beta a + P((\alpha \otimes \beta)H_1(a)) + PH'_2(t_{11}\gamma \otimes a)$$

for  $\alpha, \beta \in \mathcal{T}(E, E)$ ,  $a \in M_e$  and  $\gamma \in \mathcal{T}(E, E \mid E)$ .

*Proof.* For  $\xi \in \mathcal{T}(E, E \vee E)$  such that  $\xi = \gamma \bullet i_1\alpha \bullet i_2\beta$  and  $a \in M_e$ , we have the following equations in  $U(E \vee E)$ :

$$\begin{aligned} \iota_{(1,2)}^2 \circ \rho_{(1,2)}^2(\xi) &= \iota_{(1,2)}^2 \circ \rho_{(1,2)}^2(\gamma \bullet i_1\alpha \bullet i_2\beta) \\ &= \gamma \bullet i_1\alpha \bullet i_2\beta - i_1r_1(\gamma \bullet i_1\alpha \bullet i_2\beta) - i_2r_2(\gamma \bullet i_1\alpha \bullet i_2\beta) \text{ by 2.0.2} \\ &= \gamma \bullet i_1\alpha \bullet i_2\beta - 0 \bullet i_1\alpha \bullet 0 - 0 \bullet 0 \bullet i_2\beta \\ &= \gamma \bullet i_1\alpha \bullet i_2\beta - i_1\alpha - i_2\beta. \end{aligned}$$

Hence:

$$\begin{aligned} & (\Xi)^{-1}(\iota_{(1,2)}^2 \circ \rho_{(1,2)}^2(\xi)) \\ &= (\gamma \bullet i_1\alpha \bullet i_2\beta - 1) - (i_1\alpha - 1) - (i_2\beta - 1) \\ &= (\gamma - 1) + (i_1\alpha \bullet i_2\beta - 1) + (\gamma - 1) \bullet (i_1\alpha \bullet i_2\beta - 1) - (i_1\alpha - 1) - (i_2\beta - 1) \\ &= (\gamma - 1) + (i_1\alpha - 1) \bullet (i_2\beta - 1) + (\gamma - 1) \bullet (i_1\alpha \bullet i_2\beta - 1) \end{aligned}$$

and:

$$(\Upsilon)^{-1}(T_{11}cr_2(\Xi))^{-1}(t_{11}\rho_{12}^2(\xi)) = (t_{11}\gamma, t_1\alpha \otimes t_1\beta).$$



We deduce that:

$$\begin{aligned}
\hat{H}(t_{11}\rho_{12}^2(\xi) \otimes a) &= \hat{H}(T_{11}cr_2(\Xi)\Upsilon(t_{11}\gamma, t_1\alpha \otimes t_1\beta) \otimes a) \\
&= \hat{H}(t_{11}(\gamma + \rho_{(1,2)}^2(i_1\alpha \bullet i_2\beta)) \otimes a) \text{ by Corollary 7.8.} \\
&= H_2'(t_{11}\gamma \otimes a) + (t_1\alpha \otimes t_1\beta)H_1(a).
\end{aligned}$$

Hence

$$(7.10.1) \quad P(\hat{H}(t_{11}\rho_{12}^2(\xi) \otimes a)) = PH_2'(t_{11}\gamma \otimes a) + P((t_1\alpha \otimes t_1\beta)H_1(a)).$$

On the other hand

$$\begin{aligned}
(\nabla^2\xi)a - (r_1^2\xi)a - (r_2^2\xi)a &= (\nabla^2(\gamma \bullet i_1\alpha \bullet i_2\beta))a - (r_1^2(\gamma \bullet i_1\alpha \bullet i_2\beta))a - (r_2^2(\gamma \bullet i_1\alpha \bullet i_2\beta))a \\
&= (\nabla^2\gamma \bullet \alpha \bullet \beta)a - (0 \bullet \alpha \bullet 0)a - (0 \bullet 0 \bullet \beta)a \\
&= (\nabla^2\gamma \bullet \alpha \bullet \beta)a - \alpha a - \beta a.
\end{aligned}$$

□

**Proposition 7.11.** *For a theory of cogroups  $\mathcal{T}$ , the condition (QM2) is equivalent to the following equations for  $m \in M_{ee}$  and  $\gamma \in \mathcal{T}(E, E \mid E)$ :*

$$(QM2 - 1) \quad H_2'(t_{11}\gamma \otimes Pm) = 0$$

$$(QM2 - 2) \quad H_1(Pm) = m + Tm.$$

.

*Proof.* For  $m \in M_{ee}$  and  $\xi \in \mathcal{T}(E, E \vee E)$  such that  $\xi = \gamma \bullet i_1\alpha \bullet i_2\beta$  as above, we have:

$$\hat{H}(t_{11}\rho_{(1,2)}^2(\xi) \otimes Pm) = H_2'(t_{11}\gamma \otimes Pm) + (t_1\alpha \otimes t_1\beta)H_1(Pm)$$

by (7.10.1). On the other hand:

$$(t_1r_1(\xi) \otimes t_1r_2(\xi))(m + Tm) = (t_1\alpha \otimes t_1\beta)(m + Tm).$$

Hence the condition (QM2) is equivalent to the condition:

$$H_2'(t_{11}\gamma \otimes Pm) + (t_1\alpha \otimes t_1\beta)H_1(Pm) = (t_1\alpha \otimes t_1\beta)(m + Tm).$$

Since  $\alpha, \beta$  and  $\gamma$  are independent, taking  $\alpha = 0$  we deduce (QM2 - 1) and taking  $\gamma = 0$  and  $\alpha = \beta = 1_E$  we deduce (QM2 - 2). The converse is clear. □

**7.7. Proof of Theorem 7.1.** This is now an easy combination of Corollary 7.8 and the Propositions 7.9 and 7.11: let  $M$  be a quadratic  $\mathcal{C}$ -module relative to  $E$ . First use Proposition 7.9 to replace the map  $\hat{H}$  by  $H_1$  and  $H_2'$ , then  $H_2'$  by  $H_2$  using relation (QM2-1). Now the main feature is that relation (QM2-2) implies that  $T$  is determined by  $H_1$  and  $P$ , as

$$T = H_1P - 1.$$

Using this, the relation  $PT = P$  becomes (T1), which, in the converse proof, implies that  $T$  is an involution. Next (T2) translates the relation  $T((\alpha \otimes \beta)m) = (\beta \otimes \alpha)T(m)$ . Relation (7.9.2) becomes (T3), while (T4) translates the fact that  $H_2$  is compatible with the respective involutions. Finally, (T5) is relation (QM1) and (T6) is (7.9.1).

**7.8. Application: quadratic functors from free groups of finite rank to  $Ab$ .** In this section we apply Theorem 7.1 to the category  $\mathcal{C} = \langle \mathbb{Z} \rangle_{Gr}$  of free groups of finite rank. Baues and Pirashvili described these quadratic functors in terms of simpler data in [4]. We start by recalling the simplified version of this description given in [3].

**Definition 7.12** ([4]). *A square group is a diagram*

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

where  $M_e$  is a group and  $M_{ee}$  is an abelian group. Both groups are written additively. Moreover  $P$  is a homomorphism and  $H$  is a quadratic map, that is, the cross effect

$$(a \mid b)_H = H(a + b) - H(a) - H(b)$$

is linear in  $a, b \in M_e$ . In addition the following identities are satisfied for all  $x, y \in M_e$  and  $a, b \in M_{ee}$ :

$$(Pa \mid y)_H = 0 = (x \mid Pb)_H;$$

$$P(x \mid y)_H = -x - y + x + y;$$

$$PHP(a) = 2P(a).$$

**Theorem 7.13** ([4]). *The category of quadratic functors from  $\langle \mathbb{Z} \rangle_{Gr}$  to  $Ab$  is equivalent to the category of square groups with  $H$  linear.*

*Proof.* Since  $\langle \mathbb{Z} \rangle_{Gr}$  is a theory of cogroups we can apply Theorem 7.1. Let  $M$  be a diagram as in this theorem. Condition (T1) shows that

$$Square(M) = (M_e \xrightarrow{H_1} M_{ee} \xrightarrow{P} M_e)$$

is a square group such that  $H_1$  is linear. So we have to prove that the remaining structure of  $M$  is determined and welldefined by  $Square(M)$  such that the conditions (T2) - (T6) hold.

First of all,

$$\begin{aligned} \bar{\Lambda} &= (T_1U)(\mathbb{Z}) \text{ by Proposition 3.8} \\ &\simeq (T_1I(Gr(\mathbb{Z}, -)))(\mathbb{Z}) \\ &\simeq T_1(Gr(\mathbb{Z}, -))(\mathbb{Z}) \text{ by Proposition 7.5} \\ &\simeq T_1(Id_{Gr})(\mathbb{Z}) \\ &\simeq \mathbb{Z} \text{ by Proposition 1.11.} \end{aligned}$$

We deduce that  $\bar{\Lambda} \otimes \bar{\Lambda} \simeq \mathbb{Z}$  hence condition (T2) is trivially satisfied.

The isomorphism of endofunctors of  $Gr$ :

$$\nu : Gr(\mathbb{Z}, -) \rightarrow Id_{Gr}, \quad \nu_G(f) = f(1)$$

induces an isomorphism of bifunctors

$$T_{11}(cr_2\nu) : T_{11}cr_2(Gr(\mathbb{Z}, -)) \rightarrow T_{11}cr_2(Id_{Gr})$$

such that

$$(7.13.1) \quad T_{11}(cr_2\nu)_{G,H}(t_{11}(\xi)) = t_{11}(\xi(1))$$

for  $\xi \in cr_2Gr(\mathbb{Z}, -)(G, H)$ . So by Proposition 1.18,

$$(7.13.2) \quad T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z};$$

and  $T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z})$  is generated by the commutator  $t_{11}[i_2, i_1]$ . We deduce that condition (T3) means that  $H_2$  is determined by  $Square(M)$  and satisfies

$$PH_2 = 0 \quad \text{and} \quad H_2(t_{11}\gamma \otimes \bar{a}) = 0$$

by (T1). However, starting out with the square group  $Square(M)$ , condition (T3) a priori only gives rise to a map

$$\tilde{H}_2 : T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} coker(P) \rightarrow M_{ee};$$

in order to check that it factors through the tensor product over  $\Lambda$  we must first consider the action of  $\Lambda$  on  $M_e$ .

Let  $n : \mathbb{Z} \rightarrow \mathbb{Z}$  be the homomorphism such that  $1 \mapsto n$ . Consider condition (T5). By (7.13.2) we have  $\gamma = k[i_2, i_1]$ ,  $k \in \mathbb{Z}$ , whence  $\nabla^2 \gamma = k[Id, Id] = 0$ . As we know that  $PH_2 = 0$ , condition (T5) is equivalent to the relation

$$(7.13.3) \quad ([n] \bullet [m])a = [n]a + [m]a + P((\overline{[n]} \otimes \overline{[m]})H_1(a)).$$

which by induction is equivalent to

$$(7.13.4) \quad [n]a = na + \binom{n}{2} PH_1(a).$$

So (T5) means that the action of  $\Lambda$  on  $M_e$  is determined by  $Square(M)$ , via (7.13.4); the property to be an action is a formal consequence of the identity

$$\binom{nm}{2} = m \binom{n}{2} + n \binom{m}{2} + 2 \binom{n}{2} \binom{m}{2}$$

for  $n, m \in \mathbb{Z}$ .

By (7.13.4),  $[n]\bar{a} = n\bar{a}$  in  $coker(P)$ . On the other hand, for  $\gamma \in cr_2(Gr(\mathbb{Z}, \mathbb{Z} | \mathbb{Z}))$  we have:

$$\gamma([n + m]) = \gamma([n] \bullet [m]) = \gamma([n]) \bullet \gamma([m]).$$

Since  $\gamma([1]) = \gamma$  we deduce that  $\gamma([n]) = (\gamma([1]))^{\bullet n} = \gamma^{\bullet n}$ . Hence the right action of  $\Lambda$  on  $T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z})$  is given by

$$t_{11}(\gamma) \cdot [n] = t_{11}(\gamma[n]) = t_{11}(\gamma^{\bullet n}) = nt_{11}(\gamma).$$

We deduce that:

$$T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z}) \otimes_{\Lambda} coker(P) = T_{11}cr_2(Gr(\mathbb{Z}, -))(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} coker(P) \simeq coker(P).$$

So finally we obtain that  $H_2 = \tilde{H}_2$  is welldefined.

The fact that  $P$  is  $\Lambda$ -equivariant is equivalent to condition (T1) by (7.13.4).

Condition (T4) is trivially satisfied since  $PH_2 = 0$  and  $\gamma \bullet \tau \gamma = (k[i_2, i_1]) \bullet (k[i_1, i_2]) = 0$ .

So it remains to show that condition (T6) is a consequence of the others. For this we need the following lemma:

**Lemma 7.14.** *Let  $G$  and  $H$  be two groups. Then for  $g \in G$ ,  $h \in H$  and  $n \in \mathbb{Z}$ , the following identity holds in  $T_{11}cr_2(Id_{Gr})(G, H)$ :*

$$t_{11}\left(\left((i_1g)(i_2h)\right)^n (i_2h)^{-n} (i_1g)^{-n}\right) = t_{11}([i_2h, i_1g]^{\binom{n}{2}}).$$

*Proof.* Consider the following diagram where  $\gamma_3(G) = [[G, G], G]$ , the isomorphism  $\Gamma'_{11}$  is defined in the proof of Proposition 1.18,  $p$  is the canonical projection, and the map  $c$  is given by  $c(\bar{g} \otimes \bar{h}) = [i_1g, i_2h]$ .

$$\begin{array}{ccc} cr_2(Id_{Gr})(G, H) & \xrightarrow{\quad inc \quad} & G \vee H \\ \downarrow t_{11} & & \downarrow p \\ T_{11}cr_2(Id_{Gr})(G, H) & \xrightarrow[\Gamma'_{11}]{} G^{ab} \otimes H^{ab} \xrightarrow{c} & (G \vee H) / \gamma_3(G \vee H) \end{array}$$

The diagram commutes as is easily checked on the canonical generators  $t_{11}[i_1g, i_2h]$  of  $T_{11}cr_2(Id_{Gr})(G, H)$ , see Proposition 1.18. Thus

$$\begin{aligned} (c\Gamma'_{11})t_{11}\left(\left((i_1g)(i_2h)\right)^n(i_2h)^{-n}(i_1g)^{-n}\right) &= p\left(\left((i_1g)(i_2h)\right)^n(i_2h)^{-n}(i_1g)^{-n}\right) \\ &= p([i_2h, i_1g]^{\binom{n}{2}}) \text{ by the Hall-Petrescu formula} \\ &= (c\Gamma'_{11})t_{11}([i_2h, i_1g]^{\binom{n}{2}}). \end{aligned}$$

But the map  $c$  is injective, see [8, Proposition 1.2].  $\square$

This implies that

$$t_{11}(h([n])) = \binom{n}{2} t_{11}[i_2, i_1];$$

in fact,

$$\begin{aligned} T_{11}(cr_2\nu)(t_{11}(h([n]))) &= t_{11}(h([n])(1)) \text{ by (7.13.1)} \\ &= t_{11}\left(\left((i_1 \bullet i_2)[n]\right) \bullet (i_2[n])^{-1} \bullet (i_1[n])^{-1}\right)(1) \\ &= t_{11}\left(\left((i_1 \bullet i_2)(n)\right) i_2(n)^{-1} i_1(n)^{-1}\right) \\ &= t_{11}\left(\left((i_1 \bullet i_2)(1)\right)^n i_2(1)^{-n} i_1(1)^{-n}\right) \\ &= t_{11}\left(\left((i_1(1) i_2(1))\right)^n i_2(1)^{-n} i_1(1)^{-n}\right) \\ &= t_{11}([i_2(1), i_1(1)]^{\binom{n}{2}}) \text{ by Lemma 7.14} \\ &= t_{11}([i_2, i_1]^{\bullet \binom{n}{2}}(1)) \\ &= T_{11}(cr_2\nu)(t_{11}([i_2, i_1]^{\bullet \binom{n}{2}})) \\ &= T_{11}(cr_2\nu)\left(\binom{n}{2} t_{11}([i_2, i_1])\right). \end{aligned}$$

Thus

$$\begin{aligned} H_2(t_{11}h([n]) \otimes \bar{a}) + n^2 H_1(a) &= \binom{n}{2} H_2(t_{11}[i_2, i_1] \otimes \bar{a}) + n^2 H_1(a) \\ &= \binom{n}{2} (H_1 P H_1(a) - 2H_1(a)) + n^2 H_1(a) \text{ by condition (T3)} \\ &= \binom{n}{2} H_1 P H_1(a) + n H_1(a) \\ &= H_1([n]a) \text{ by (7.13.4),} \end{aligned}$$

as desired.  $\square$

**Remark 7.15.** *In the definition of square group given in [4] the authors consider the map:  $\Delta(a) = HPH(a) + H(2a) - 4H(a)$ . When  $H$  is linear we have  $\Delta(a) = HPH(a) - H(2a)$ . So the map  $\Delta$  corresponds to our map  $H_2$  according to condition (T3).*

**Remark 7.16.** *The case of the theory of free groups of finite rank is very simple compared to a general theory of cogroups since a quadratic Gr-module relative to  $\mathbb{Z}$  must satisfy only the single condition (T1) instead of the six conditions (T1) - (T6) in the general situation.*

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