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# WALKS WITH SMALL STEPS IN THE QUARTER PLANE (PRELIMINARY VERSION)

MIREILLE BOUSQUET-MÉLOU AND MARNI MISHNA

ABSTRACT. Let  $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ . We address the enumeration of plane lattice walks with steps in  $\mathcal{S}$ , that start from  $(0, 0)$  and always remain in the first quadrant  $i \geq 0, j \geq 0$ . *A priori*, there are  $2^8$  problems of this type, but some are trivial. Some others are equivalent to a model of walks confined to a half-plane: such models can be solved systematically using the kernel method, and are known to have algebraic generating functions. We focus on the remaining cases, and show that there are 79 inherently different problems to study.

To each of them, we associate a group  $G$  of birational transformations. We show that this group is finite (of order at most 8) in 23 cases. We give evidence that it is infinite in the 56 other cases.

We present a unified way of solving 22 of the 23 models associated with a finite group. For all of them, the generating function is found to be D-finite. The 23rd model, known as Gessel's walks, has recently been proved by Bostan *et al.* to have an algebraic (and hence D-finite) solution.

Our approach allows us to recover and refine some known results, and also obtain new results. For instance, we prove that walks with N, E, W, S, SW and NE steps have an algebraic generating function.

## 1. Introduction

The enumeration of lattice walks is a classical topic in combinatorics. It has numerous applications, as many combinatorial objects (trees, maps, permutations, lattice polygons, tableaux, queues...) can be encoded by lattice walks. Given a lattice, for instance the hypercubic lattice  $\mathbb{Z}^d$ , and a finite set of steps  $\mathcal{S} \subset \mathbb{Z}^d$ , a typical problem is to determine how many  $n$ -step walks with steps taken from  $\mathcal{S}$ , starting from  $(0, 0)$ , are confined to a certain region  $\mathcal{A}$  of the space (containing  $(0, 0)$ ). If  $\mathcal{A}$  is the whole space, then the length generating function of these walks is a simple rational series. If  $\mathcal{A}$  is a half-space, bounded by a rational hyperplane, the associated generating function is an algebraic series. Instances of the latter problem have been studied in many articles since at least the end of the 19th century [1, 6]. It is now understood that the *kernel method* provides a systematic solution to all such problems, which are, in essence, one-dimensional [13, 2]. Another generic approach to these problems is provided in [21].

A natural next class of problems is the enumeration of walks constrained to lie in the intersection of two rational half-spaces. Thus far, a number of instances have been solved, but no unified approach has emerged yet, and the problem is far from being completely understood. The generating functions that have been found demonstrate a more complicated structure than those of half-space problems. Some examples are algebraic, but for reasons that are poorly understood combinatorially [22, 12, 35, 8]. Some examples are, more generally, D-finite, meaning that the generating function satisfies a linear differential equation with polynomial coefficients [26, 14, 10, 41]. Some examples are not D-finite, having infinitely many singularities in the complex plane [14, 36].

The four examples of Figure 1, each walks in the plane confined to the first quadrant, illustrate this complexity nicely:

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- **Kreweras' walks (steps W, S, and NE)**: they were first counted in 1965 by Kreweras [32], who obtained a complicated expression for the number of  $n$ -step walks ending at  $(i, j)$ . It was then proved by Gessel that the associated 3-variable generating function is algebraic [22]. Simpler derivations of this series have been obtained since then [37, 12], including an automated proof [31], and a purely bijective one for walks returning to the origin [4].
- **Gessel's walks (steps E, W, NE and SW)**: Gessel conjectured a few years ago a simple hypergeometric formula for the number of  $n$ -step walks returning to the origin. This conjecture was only proved very recently by Kauers, Koutschan and Zeilberger [30]. Even more recently, Bostan and Kauers proved that the associated 3-variable generating function (which counts all quarter plane walks by the length and the coordinates of the endpoint) is in fact algebraic [8]. Strangely enough, the simple numbers conjectured by Gessel had not been recognized as the coefficients of an algebraic series before. Both approaches involve, among other tools, heavy computer algebra calculations.
- **Gouyou-Beauchamps walks (steps E, W, NW and SE)**: Gouyou-Beauchamps discovered in 1986 a simple hypergeometric formula for walks ending anywhere on the  $x$ -axis [24]. We derive in this paper similar expressions for the total number of walks, and for those ending at a prescribed position. The associated series are D-finite, but transcendental. These walks are related to Young tableaux of height at most 4 [25]. An affine deformation transforms them into square lattice walks (with N, S, E and W steps) confined to the wedge  $0 \leq j \leq i$ . An enumeration of these walks involving the number of visits to the diagonal appears in [27, 39, 38].
- **A non-D-finite case** is provided by walks with NE, NW and SE steps. Mishna and Reznitzky established a complicated expression for the generating function of these walks, from which they were able to prove that this series has infinitely many singularities, and thus cannot be D-finite [36].

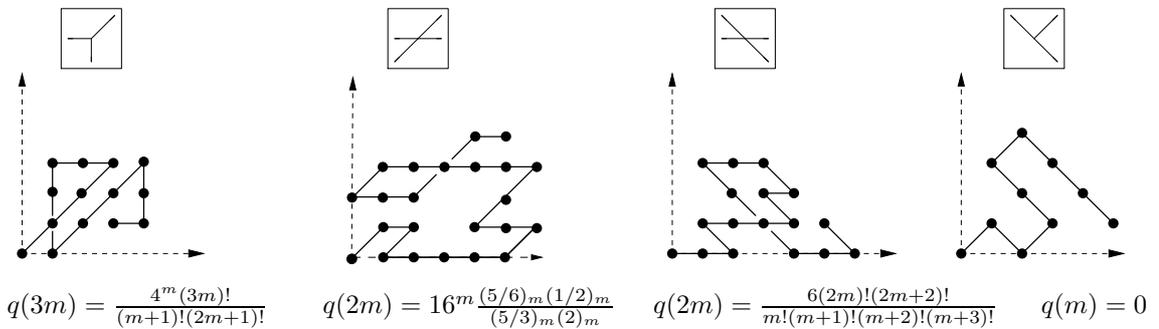


FIGURE 1. Four models of walks in the quarter plane: Kreweras' walks, Gessel's walks, Gouyou-Beauchamps's walks, and the non-D-finite example of Mishna and Reznitzky. The numbers  $q(n)$  count walks of length  $n$  confined to the quarter plane that start *and end* at the origin. We have used the notation  $(a)_n = a(a+1) \cdots (a+n-1)$ .

Observe that we have defined  $\mathcal{S}$ , the set of steps, as a subset of  $\mathbb{Z}^2$  but that we often use a more intuitive terminology, referring to  $(1, 1)$  as a NE step, for instance. We occasionally abuse the coordinate notation and directly write our steps using  $x$ 's and  $y$ 's, writing for example  $x\bar{y}$  for a SE step.

Ideally, we seek generic results or combinatorial conditions which assure D-finite (or even algebraic) generating functions. One criterion of this type states that if the walks are confined to the quarter plane, the set  $\mathcal{S}$  has a vertical symmetry, and consists of steps  $(i, j)$  such that  $|j| \leq 1$ ,

then the generating function is D-finite [14, 10]. This is also true when the number of confined walks can be derived from the number of unrestricted walks via the reflection principle [23]. We are not aware of any other such criteria.

In this paper, we restrict our attention to the quarter plane and *small* steps (*i.e.*, the step set  $\mathcal{S}$  is a subset of  $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ ). This includes the four above examples. We first narrow down the  $2^8$  possible cases to 79 distinct, non-trivial problems (Section 2). Their step sets  $\mathcal{S}$  are listed in Tables 1 to 4 in Section 8. These problems fall naturally into two categories, depending on whether a certain group associated with  $\mathcal{S}$  is finite or (conjecturally) infinite (Section 3). We are able to solve all but one model (namely, Gessel's walks) in the first category (finite group) in a uniform way. The solutions are all D-finite, and even algebraic in three cases (Sections 5 and 6). We conjecture that all the solutions to the second category are non-D-finite. Remarkably, the 23 models for which we found a finite group are

- those with a vertical symmetry, to which the criterion of [14, 10] can be applied,
- those in which the sum of all allowed steps is 0.

### Outline of the paper

The starting point of our approach is a functional equation defining the generating function  $Q(x, y; t)$  that counts quarter plane walks by the length (variable  $t$ ) and the coordinates of the endpoint (variables  $x$  and  $y$ ). This equation merely reflects a step by step construction of quarter plane walks. For instance, the equation obtained for Kreweras' walks (first example in Figure 1) reads:

$$(1 - t(1/x + 1/y + xy)) Q(x, y; t) = 1 - t/x Q(0, y; t) - t/y Q(x, 0; t).$$

Following Zeilberger's terminology [48], we say that the variables  $x$  and  $y$  are *catalytic*. This means that one cannot exploit the step by step construction without introducing these variables. Also, it is not clear how to derive from the above identity an equation for, say,  $Q(0, 0; t)$  or  $Q(1, 1; t)$ . One of our objectives is to provide some general principles that may be applied to the study of any such linear equation with two catalytic variables. The case of linear equations with *one* catalytic variable is well-understood, and the solutions are always algebraic [13].

One key tool in our approach is a certain group  $G(\mathcal{S})$  of birational transformations that leaves the *kernel* of the functional equation (that is, the coefficient of  $Q(x, y; t)$ ) unchanged (Section 3). We have borrowed this group from *Le petit livre jaune* by Fayolle, Iasnogorodski and Malyshev [18], in which the authors study the stationary distributions of Markov chains in the quarter plane. Ever since it was imported from probability theory to combinatorics, this idea has proved useful (in several disguises, like the *obstinate*, *algebraic*, or *iterated* kernel method) to solve various enumeration problems, dealing with walks [10, 12, 14, 28, 35, 36], but also with other objects, like permutations [11] or set partitions [16, 47]. All these problems boil down to solving a linear equation with two catalytic variables. A striking observation, which applies to all solutions obtained so far, is that the solution is D-finite if and only if the group is finite. We find that 23 out of our 79 quarter plane models give rise to a group of order at most 8. The remaining 56 models have a group of order at least 200, and we conjecture (and prove in some cases) that the group is actually infinite.

In Section 4 we develop certain general tools that apply to all models—in particular, we explain to write for each of them the functional equation that defines the generating function  $Q(x, y; t)$ . When the group  $G(\mathcal{S})$  is finite, we derive an identity between various specializations of  $Q(x, y; t)$  which we call the *orbit sum* (or *half-orbit sum*, when there is an  $x/y$  symmetry).

In Section 5, we show how to derive  $Q(x, y; t)$  from this orbit sum in 19 out of the 20 models that have a finite group and no  $x/y$  symmetry. The number of  $n$ -step walks ending at  $(i, j)$  is obtained by extracting the coefficient of  $x^i y^j t^n$  in a certain rational series  $R(x, y; t)$ , which is easily obtained from the set steps and the group. This implies that the generating function  $Q(x, y; t)$  is D-finite. The form of this solution is reminiscent of a formula obtained by Gessel and Zeilberger for the enumeration of walks confined to a Weyl chamber, when the set of steps is invariant under the associated Weyl group [23]. Indeed, when the quarter plane problem

happens to be a Weyl chamber problem, our method can be seen as an algebraic version of the reflection principle. However, its range of applications seems to be more general. We work out in details three cases: walks with N, W and SE steps (equivalent to Young tableaux with at most 3 rows), walks with N, S, E, W, SE and NW steps (which do not seem to have been solved before, but behave very much like the former case), and finally walks with E, W, NW and SE steps (studied in [24]), for which we obtain new explicit results.

In Section 6, we show how to derive  $Q(x, y; t)$  from the half-orbit sum for the 3 models that have a finite group and an  $x/y$  symmetry. For each of them, the series  $Q(x, y; t)$  is found to be algebraic. We work out in details all cases: walks with S, W and NE steps (Kreweras' walks), walks with E, N and SW steps (the reverse steps of Kreweras' steps) and walks with N, S, E, W, NE and SW steps, which, to our knowledge, have never been studied before. We find that the series  $Q(1, 1; t)$  that counts all walks of this type, regardless of their endpoint, satisfies a remarkable quartic equation.

In Section 7 we present some comments and a list of open questions.

### Preliminaries and notation

We conclude this introduction with a few important definitions and some power series notation. Let  $A$  be a commutative ring and  $x$  an indeterminate. We denote by  $A[x]$  (resp.  $A[[x]]$ ) the ring of polynomials (resp. formal power series) in  $x$  with coefficients in  $A$ . If  $A$  is a field, then  $A(x)$  denotes the field of rational functions in  $x$ , and  $A((x))$  the field of Laurent series in  $x$ . These notations are generalized to polynomials, fractions and series in several indeterminates. We denote  $\bar{x} = 1/x$ , so that  $A[x, \bar{x}]$  is the ring of Laurent polynomials in  $x$  (with coefficients in  $A$ ). The coefficient of  $x^n$  in a Laurent series  $F(x)$  is denoted  $[x^n]F(x)$ . Recall that the *valuation* of a Laurent series  $F(x)$  is the smallest  $d$  such that  $x^d$  occurs in  $F(x)$  with a non-zero coefficient.

The main family of series that we use is that of power series in  $t$  with coefficients in  $A[x, \bar{x}]$ , that is, series of the form

$$F(x; t) = \sum_{n \geq 0, i \in \mathbb{Z}} f(i; n) x^i t^n,$$

where for all  $n$  almost all coefficients  $f(i; n)$  are zero. The *positive part* of  $F(x; t)$  in  $x$  is the following series, which has coefficients in  $\mathbb{Q}[x]$ :

$$[x^>]F(x; t) := \sum_{n \geq 0, i > 0} f(i; n) x^i t^n.$$

We define similarly the negative, non-negative and non-positive parts of  $F(x; t)$  in  $x$ , which we denote respectively by  $[x^<]F(x; t)$ ,  $[x^{\geq}]F(x; t)$  and  $[x^{\leq}]F(x; t)$ .

In our generating series, the indeterminate  $t$  keeps track of the length of the walks. We record the coordinates of the endpoints with the variables  $x$  and  $y$ . In order to simplify the notation, we often omit the dependence of our series in  $t$ , writing for instance  $Q(x, y)$  instead of  $Q(x, y; t)$  for the generating function of quarter plane walks.

Recall that a power series  $F(x_1, \dots, x_k) \in \mathbb{K}[[x_1, \dots, x_k]]$ , where  $\mathbb{K}$  is a field, is *algebraic* (over  $\mathbb{K}(x_1, \dots, x_k)$ ) if it satisfies a non-trivial polynomial equation  $P(x_1, \dots, x_k, F(x_1, \dots, x_k)) = 0$ . It is *transcendental* if it is not algebraic. It is *D-finite* (or *holonomic*) if the vector space over  $\mathbb{K}(x_1, \dots, x_k)$  spanned by all partial derivatives of  $F(x_1, \dots, x_k)$  has finite dimension. All algebraic series are D-finite. We refer to [33, 34] for a study of these series. We use in Section 5 the following result:

if  $F(x, y; t)$  is a rational power series in  $t$ , with coefficients in  $\mathbb{C}(x)[y, \bar{y}]$ , then  $F^+(x, y; t) := [y^>]F(x, y; t)$  is algebraic over  $\mathbb{C}(x, y, t)$ . If the latter series has coefficients in  $\mathbb{C}[x, \bar{x}, y]$ , its positive part in  $x$ , that is, the series  $[x^>][y^>]F(x, y; t)$ , is a 3-variable D-finite series (in  $x, y$  and  $t$ ).

The proof of the first statement is actually a simple adaptation of [21, Thm. 6.1]. The key tool is to expand  $F(x, y; t)$  in partial fractions of  $y$ . The second statement is more difficult, and relies on the fact that the diagonal of a D-finite series is D-finite. One first observes that there is some

$k$  such that  $F_+(x, y; tx^k)$  has polynomial coefficients in  $x$  and  $y$ , and then applies Remark (4), page 377 of [33].

We also use the fact that a series  $F(t)$  with complex coefficients such that  $[t^n]F(t) \sim \kappa\mu^n n^{-k}$  with  $k \in \{1, 2, 3, \dots\}$  cannot be algebraic [19].

## 2. The number of non-equivalent non-simple models

Since we restrict ourselves to walks with “small” steps (sometimes called steps with small variations), there is *a priori* a finite number of cases to study, namely  $2^8$ , the number of sets  $\mathcal{S}$  formed of small steps. However, some of these models are trivial (for instance  $\mathcal{S} = \emptyset$ , or  $\mathcal{S} = \{\bar{x}\}$ ). More generally, it often happens that one of the two constraints imposed by the quarter plane holds automatically, at least when the other constraint is satisfied. Such models are equivalent to problems of walks confined to a half-space: their generating function is always algebraic and can be derived automatically using the *kernel method* [13, 2]. We show in Section 2.1 that, out of the  $2^8 = 256$  models, only 138 are truly 2-constraint problems and are thus worth considering in greater detail. Then, some of the remaining problems coincide up to an  $x/y$  symmetry and are thus equivalent. As shown in Section 2.2, one finally obtains 79 inherently different, truly 2-constraint problems.

### 2.1. Easy algebraic cases

Let us say that a step  $(i, j)$  is  $x$ -positive if  $i > 0$ . We define similarly  $x$ -negative,  $y$ -positive and  $y$ -negative steps. There are a number of reasons that may make the enumeration of quarter plane walks with steps in  $\mathcal{S}$  a simple problem:

- (1) If  $\mathcal{S}$  contains no  $x$ -positive step, we can ignore its  $x$ -negative steps, which will never be used in a quarter plane walk: we are thus back to counting walks on a (vertical) half-line. This problem has a rational ( $\mathcal{S} = \emptyset$  or  $\mathcal{S} = \{y\}$ ) or algebraic ( $\mathcal{S} = \{y, \bar{y}\}$ ) solution;
- (2) Symmetrically, if  $\mathcal{S}$  contains no  $y$ -positive step, the problem is simple with an algebraic solution;
- (3) If  $\mathcal{S}$  contains no  $x$ -negative step, all walks with steps in  $\mathcal{S}$  that start from  $(0, 0)$  lie in the half-plane  $i \geq 0$ . Thus any walk lying above the  $x$ -axis is automatically a quarter plane walk, and the problem boils down to counting walks confined to the upper half-plane: the corresponding generating function is always algebraic;
- (4) Symmetrically, if  $\mathcal{S}$  contains no  $y$ -negative step, the problem is simple with an algebraic solution.

An inclusion-exclusion argument then shows that the number of sets  $\mathcal{S}$  containing an  $x$ -positive step, an  $x$ -negative step, a  $y$ -positive step and a  $y$ -negative step is 161. More precisely, the polynomial that counts these sets by cardinality is

$$\begin{aligned} P_1(z) &= (1+z)^8 - 4(1+z)^5 + 2(1+z)^2 + 4(1+z)^3 - 4(1+z) + 1 \\ &= 2z^2 + 20z^3 + 50z^4 + 52z^5 + 28z^6 + 8z^7 + z^8. \end{aligned}$$

All the other sets  $\mathcal{S}$  correspond to problems that are either trivial or can be solved automatically using the kernel method.

Among the remaining 161 sets  $\mathcal{S}$ , some do not contain any step with both coordinates non-negative: in this case the only quarter plane walk is the empty walk. These sets are subsets of  $\{\bar{x}, \bar{y}, x\bar{y}, \bar{x}\bar{y}, \bar{x}y\}$ . But, as we have assumed at this stage that  $\mathcal{S}$  contains  $x$ -positive and  $y$ -positive steps, both  $x\bar{y}$  and  $\bar{x}y$  must belong to  $\mathcal{S}$ . Hence we exclude  $2^3$  of our 161 step sets, which leaves us with 153 sets, the generating polynomial of which is

$$P_2(z) = P_1(z) - z^2(1+z)^3 = z^2 + 17z^3 + 47z^4 + 51z^5 + 28z^6 + 8z^7 + z^8.$$

Another, slightly less obvious, source of simplicity of the model is when one of the quarter plane constraint implies the other. Assume that all walks with steps in  $\mathcal{S}$  that end at a non-negative abscissa automatically end at a non-negative ordinate (we say, for short, that the  $x$ -condition

implies the  $y$ -condition). This implies in particular that the steps  $\bar{y}$  and  $x\bar{y}$  do not belong to  $\mathcal{S}$ . As we have assumed that  $\mathcal{S}$  contains a  $y$ -negative step,  $\bar{x}\bar{y}$  must be in  $\mathcal{S}$ . But then  $x$  cannot belong to  $\mathcal{S}$ , otherwise some walks with a non-negative final abscissa would have a negative final ordinate, like  $x$  followed by  $\bar{x}\bar{y}$ . We are left with sets  $\mathcal{S} \subset \{\bar{x}, y, xy, \bar{x}y, \bar{x}\bar{y}\}$  containing  $\bar{x}\bar{y}$ , and also  $xy$  (because we need at least one  $x$ -positive step). Conversely, it is easy to realize that for any such set, the  $x$ -condition implies the  $y$ -condition. The generating polynomial of such *super-diagonal* sets is  $z^2(1+z)^3$ . Similarly, we need not consider *sub-diagonal* cases. An inclusion-exclusion argument reduces the generating polynomial of non-simple cases to

$$P_3(z) = P_2(z) - 2z^2(1+z)^3 + z^2 = 11z^3 + 41z^4 + 49z^5 + 28z^6 + 8z^7 + z^8,$$

that is to say, to 138 sets  $\mathcal{S}$ .

## 2.2. Symmetries

The eight symmetries of the square act on the step sets. However, only the  $x/y$  symmetry (reflection across the first diagonal) leaves the quarter plane fixed. Thus two step sets obtained from one another by applying this symmetry lead to equivalent counting problems. As we want to count non-equivalent problems, we need to determine how many among the 138 sets  $\mathcal{S}$  that are left have the  $x/y$  symmetry. We repeat the argument of the previous subsection, keeping only symmetric models. We successively obtain

$$\begin{aligned} P_1^{\text{sym}} &= (1+z)^2(1+z^2)^3 - 2(1+z)(1+z^2) + 1, \\ P_2^{\text{sym}} &= P_1^{\text{sym}} - z^2(1+z)(1+z^2), \\ P_3^{\text{sym}} &= P_2^{\text{sym}} - z^2 = 3z^3 + 5z^4 + 5z^5 + 4z^6 + 2z^7 + z^8. \end{aligned}$$

The generating polynomial of (inherently different) models that are neither trivial, nor equivalent to a 1-constraint problem is thus

$$\frac{1}{2}(P_3 + P_3^{\text{sym}}) = 7z^3 + 23z^4 + 27z^5 + 16z^6 + 5z^7 + z^8.$$

This gives a total of 79 models, shown in Tables 1 to 4. From now on, we assume that  $\mathcal{S}$  is one of these sets.

## 3. The group of the walk

Let  $S(x, y)$  denote the generating polynomial of the steps of  $\mathcal{S}$ :

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j. \quad (1)$$

It is a Laurent polynomial in  $x$  and  $y$ . Recall that  $\bar{x}$  stands for  $1/x$ , and  $\bar{y}$  for  $1/y$ . Let us write

$$S(x, y) = A_{-1}(x)\bar{y} + A_0(x) + A_1(x)y = B_{-1}(y)\bar{x} + B_0(y) + B_1(y)x. \quad (2)$$

Clearly,  $S(x, y)$  is left unchanged by the following rational transformations:

$$\Phi : (x, y) \mapsto \left( \bar{x} \frac{B_{-1}(y)}{B_1(y)}, y \right) \quad \text{and} \quad \Psi : (x, y) \mapsto \left( x, \bar{y} \frac{A_{-1}(x)}{A_1(x)} \right).$$

Note that both  $\Phi$  and  $\Psi$  are involutions, and thus *birational* transformations. By composition, they generate a group of transformations that we denote  $G(\mathcal{S})$ , or  $G$  if there is no risk of confusion. This group is isomorphic to a dihedral group  $D_n$  of order  $2n$ , with  $n \in \mathbb{N} \cup \{\infty\}$ . For each  $g \in G$ , one has  $S(g(x, y)) = S(x, y)$ . The *sign* of  $g$  is 1 (resp.  $-1$ ) if  $g$  is the product of an even (resp. odd) number of generators  $\Phi$  and  $\Psi$ .

### Examples

**1.** Assume  $\mathcal{S}$  is left unchanged by a reflection across a vertical line. This is equivalent to saying that  $S(x, y) = S(\bar{x}, y)$ , or that  $B_1(y) = B_{-1}(y)$ , or that  $A_i(x) = A_i(\bar{x})$  for  $i = -1, 0, 1$ . Then the orbit of  $(x, y)$  under the action of  $G$  reads

$$(x, y) \xleftarrow{\Phi} (\bar{x}, y) \xleftarrow{\Psi} (\bar{x}, C(x)\bar{y}) \xleftarrow{\Phi} (x, C(x)\bar{y}) \xleftarrow{\Psi} (x, y),$$

with  $C(x) = \frac{A_{-1}(x)}{A_1(x)}$ , so that  $G$  is finite of order 4.

**2.** Consider the case  $\mathcal{S} = \{\bar{x}, y, x\bar{y}\}$ . We have  $A_{-1}(x) = x$ ,  $A_1(x) = 1$ ,  $B_{-1}(y) = 1$ ,  $B_1(y) = \bar{y}$ . The two transformations are

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}),$$

and they generate a group of order 6:

$$(x, y) \xleftarrow{\Phi} (\bar{x}y, y) \xleftarrow{\Psi} (\bar{x}y, \bar{x}) \xleftarrow{\Phi} (\bar{y}, \bar{x}) \xleftarrow{\Psi} (\bar{y}, x\bar{y}) \xleftarrow{\Phi} (x, x\bar{y}) \xleftarrow{\Psi} (x, y). \quad (3)$$

By composing the transformations  $\Phi$  and  $\Psi$ , we discovered among our 79 models 23 cases with a finite group, of order 4, 6 or 8. They are listed in Tables 1 to 3. These tables also give the orbit of  $(x, y)$  under the action of  $G$ , the elements being listed in the following order:  $(x, y)$ ,  $\Phi(x, y)$ ,  $\Psi \circ \Phi(x, y)$  and so on. We believe that the 56 remaining models, listed in Table 4, are associated with an infinite group. At least, we have computed, for each of them, the values  $(\Psi \circ \Phi)^n(1, 2)$ , for  $1 \leq n \leq 100$ , and found that they are always distinct from  $(1, 2)$ . Thus these groups have order at least 200.

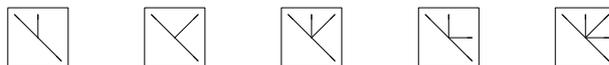


FIGURE 2. Five step sets with an infinite group.

The group of birational transformations of  $\mathbb{C}^2$  (or of the projective plane  $\mathbb{P}^2(\mathbb{C})$ ), called the Cremona group, is actually the object of many studies in algebraic geometry since the end of the 19th century [29, 46]. It seems likely that, given the attention already paid to the classification of finite subgroups of this group (see e.g. [7, 3]), there is a generic or automatic test for finiteness that can be applied to our examples. We are currently investigating the literature with this objective. Meanwhile, let us give an elementary proof for the five step sets of Figure 2. Note that these are the five sets of our collection for which all elements  $(i, j)$  satisfy  $i + j \geq 0$ .

**Proposition 1.** *If  $\mathcal{S}$  is one of the five sets of Figure 2, then the group associated with  $\mathcal{S}$  is infinite.*

*Proof.* Let  $z$  be an indeterminate, and assume that both  $x$  and  $y$  are Laurent series in  $z$  with coefficients in  $\mathbb{Q}$ , such that the coefficient of the smallest power of  $z$  that occurs (the *trailing coefficient*) is positive. Let  $a$  and  $b$  be the respective valuations of  $x$  and  $y$  in  $z$ . Then  $\Phi(x, y) = (x', y)$  is well-defined, the trailing coefficient of  $x'$  (and  $y$ ) is positive, and the valuation of  $x'$  (and  $y$ ) only depends on  $a$  and  $b$ :

$$\phi(a, b) := (\text{val}(x'), \text{val}(y)) = \begin{cases} \left( -a + b(v_{-1}^{(y)} - v_1^{(y)}), b \right) & \text{if } b \geq 0, \\ \left( -a + b(d_{-1}^{(y)} - d_1^{(y)}), b \right) & \text{if } b \leq 0, \end{cases}$$

where  $v_i^{(y)}$  (resp.  $d_i^{(y)}$ ) denotes the valuation (resp. degree) in  $y$  of  $B_i(y)$ , for  $i = \pm 1$ . Similarly,  $\Psi(x, y) = (x, y')$  is well-defined, and the valuations of  $x$  and  $y'$  only depend on  $a$  and  $b$ :

$$\psi(a, b) := (\text{val}(x), \text{val}(y')) = \begin{cases} \left( a, -b + a(v_{-1}^{(x)} - v_1^{(x)}) \right) & \text{if } a \geq 0, \\ \left( a, -b + a(d_{-1}^{(x)} - d_1^{(x)}) \right) & \text{if } a \leq 0, \end{cases}$$

where  $v_i^{(x)}$  (resp.  $d_i^{(x)}$ ) denotes the valuation (resp. degree) in  $x$  of  $A_i(x)$ , for  $i = \pm 1$ .

In order to prove that  $G$  is infinite, it suffices to prove that the group  $G'$  generated by  $\phi$  and  $\psi$  is infinite. To prove the latter statement, it suffices to exhibit  $(a, b) \in \mathbb{Z}^2$  such that the orbit of  $(a, b)$  under the action of  $G'$  is infinite.

Let  $\mathcal{S}$  be one of the five sets under consideration. Then  $A_{-1}(x) = x$  and  $B_{-1}(y) = y$ , so that  $v_{-1}^{(x)} = d_{-1}^{(x)} = v_{-1}^{(y)} = d_{-1}^{(y)} = 1$ . Also,  $v_1^{(x)} = v_1^{(y)} = -1$  as  $\mathcal{S}$  contains the steps  $NW \equiv (-1, 1)$  and  $SE \equiv (-1, -1)$ . Hence the transformations  $\phi$  and  $\psi$  read:

$$\phi(a, b) = \begin{cases} (-a + 2b, b) & \text{if } b \geq 0, \\ (-a + b(1 - d_1^{(y)}), b) & \text{if } b \leq 0, \end{cases} \quad \psi(a, b) = \begin{cases} (a, 2a - b) & \text{if } a \geq 0, \\ (a, -b + a(1 - d_1^{(x)})) & \text{if } a \leq 0. \end{cases}$$

It is easy to check, by induction on  $n \geq 0$ , that

$$(\psi \circ \phi)^n(1, 2) = (2n + 1, 2n + 2) \quad \text{and} \quad \phi(\psi \circ \phi)^n(1, 2) = (2n + 3, 2n + 2).$$

Hence the orbit of  $(1, 2)$  under the action of  $\phi$  and  $\psi$  is infinite, and so are the groups  $G'$  and  $G$ .  $\blacksquare$

**Remark.** It seems that for all other models of Table 4, the group  $G'$  is finite, and hence cannot be used to prove that  $G$  is infinite.

#### 4. General tools

Let  $\mathcal{S}$  be one of the 79 step sets of Tables 1 to 4. Let  $\mathcal{Q}$  be the set of walks that start from  $(0, 0)$ , take their steps from  $\mathcal{S}$  and always remain in the first quadrant. Let  $q(i, j; n)$  be the number of such walks that have length  $n$  and end at position  $(i, j)$ . Denote by  $Q(x, y; t) \equiv Q(x, y)$  the associated generating function:

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n.$$

It is a formal power series in  $t$  with coefficients in  $\mathbb{Q}[x, y]$ .

##### 4.1. A functional equation

**Lemma 2.** *As a power series in  $t$ , the generating function  $Q(x, y) \equiv Q(x, y; t)$  of walks with steps taken from  $\mathcal{S}$  starting from  $(0, 0)$  and staying in the first quadrant is characterized by the following functional equation:*

$$K(x, y)xyQ(x, y) = xy - txA_{-1}(x)Q(x, 0) - t\bar{y}B_{-1}(y)Q(0, y) + t\epsilon Q(0, 0)$$

where

$$K(x, y) = 1 - tS(x, y) = 1 - t \sum_{(i, j) \in \mathcal{S}} x^i y^j$$

is called the kernel of the equation, the polynomials  $A_{-1}(x)$  and  $B_{-1}(y)$  are the coefficients of  $\bar{y}$  and  $\bar{x}$  in  $S(x, y)$ , as described by (2), and  $\epsilon$  is 1 if  $(-1, -1)$  is one of the allowed steps, and 0 otherwise.

*Proof.* We construct walks step by step, starting from the empty walk and concatenating a new step at the end of the walk at each stage. The empty walk has weight 1. The generating function of walks obtained by adding a step of  $\mathcal{S}$  at the end of a walk of  $\mathcal{Q}$  is  $tS(x, y)Q(x, y)$ . However, some of these walks exit the quadrant: those obtained by concatenating a step with negative  $y$ -coordinate to a walk ending at ordinate 0, and those obtained by concatenating a step with negative  $x$ -coordinate to a walk ending at abscissa 0. Walks ending at ordinate (resp. abscissa) 0 are counted by the series  $Q(x, 0)$  (resp.  $Q(0, y)$ ). Hence we must subtract the series  $t\bar{y}A_{-1}(x)Q(x, 0)$  and  $t\bar{x}B_{-1}(y)Q(0, y)$ . However, if  $(-1, -1) \in \mathcal{S}$ , we have subtracted twice the series counting walks obtained by concatenating this step to a walk ending at  $(0, 0)$ : we must thus add the series  $t\bar{x}\bar{y}Q(0, 0)$ . This inclusion-exclusion argument gives

$$Q(x, y) = 1 + tS(x, y)Q(x, y) - t\bar{y}A_{-1}(x)Q(x, 0) - t\bar{x}B_{-1}(y)Q(0, y) + t\bar{x}\bar{y}Q(0, 0),$$

which, multiplied by  $xy$ , gives the equation of the lemma.

The fact that it characterizes  $Q(x, y; t)$  completely (as a power series in  $t$ ) comes from the fact that the coefficient of  $t^n$  in  $Q(x, y; t)$  can be computed inductively using this equation. This

is of course closely related to the fact that we have used a recursive description of walks in  $\mathcal{Q}$  to obtain the equation.  $\blacksquare$

#### 4.2. Orbit sums

We have seen in Section 3 that all transformations  $g$  of the group  $G$  associated with the step set  $\mathcal{S}$  leave the polynomial  $S(x, y)$  unchanged. Hence they also leave the kernel  $K(x, y) = 1 - tS(x, y)$  unchanged. Write the equation of Lemma 2 as

$$K(x, y)xyQ(x, y) = xy - F(x) - G(y) + t\epsilon Q(0, 0),$$

with  $F(x) = txA_{-1}(x)Q(x, 0)$  and  $G(y) = tyB_{-1}(y)Q(0, y)$ . Replacing  $(x, y)$  by  $\Phi(x, y) = (x', y)$  gives

$$K(x, y)x'yQ(x', y) = x'y - F(x') - G(y) + t\epsilon Q(0, 0).$$

The difference between the former and latter identities reads:

$$K(x, y)(xyQ(x, y) - x'yQ(x', y)) = xy - x'y - F(x) + F(x').$$

The term  $G(y)$  has disappeared. We can repeat this process, and add to this identity the functional equation of Lemma 2, evaluated at  $(x', y') = \Psi(x', y)$ . This gives:

$$K(x, y)(xyQ(x, y) - x'yQ(x', y) + x'y'Q(x', y')) = xy - x'y + x'y' - F(x) - G(y') + t\epsilon Q(0, 0).$$

Now the term  $F(x')$  has disappeared. If  $G$  is finite of order  $2n$ , we can repeat the procedure until we come back to  $(\Psi \circ \Phi)^n(x, y) = (x, y)$ . That is to say, we form the alternating sum of the equations over the orbit of  $(x, y)$ . All unknown functions on the right-hand side finally vanish, giving the following identity. Observe that the right-hand side is a rational function in  $x, y$  and  $t$ .

**Proposition 3 (Orbit sums).** *Assume the group  $G(\mathcal{S})$  is finite. Then*

$$\sum_{g \in G} \text{sign}(g)g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G} \text{sign}(g)g(xy). \quad (4)$$

We show in Section 5 that this identity implies immediately that 19 of the 23 models having a finite group have a D-finite solution.

The 4 remaining models are Gessel's model  $\{x, \bar{x}, xy, \bar{x}\bar{y}\}$  (which we do not solve in this paper) and the three models with steps  $\mathcal{S}_1 = \{\bar{x}, \bar{y}, xy\}$ ,  $\mathcal{S}_2 = \{x, y, \bar{x}\bar{y}\}$ , and  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ . For each of these three models, the orbit of  $(x, y)$  is

$$(x, y) \xleftarrow{\Phi} (\bar{x}\bar{y}, y) \xleftarrow{\Psi} (\bar{x}\bar{y}, x) \xleftarrow{\Phi} (y, x) \xleftarrow{\Psi} (y, \bar{x}\bar{y}) \xleftarrow{\Phi} (x, \bar{x}\bar{y}) \xleftarrow{\Psi} (x, y),$$

and exhibits an  $x/y$  symmetry. That is,  $(y, x)$  belongs to the orbit of  $(x, y)$ . Moreover, if  $g((x, y)) = (y, x)$ , then  $\text{sign}(g) = -1$ . Thus the right-hand side of (4) vanishes, leaving

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = xyQ(y, x) - \bar{x}Q(y, \bar{x}\bar{y}) + \bar{y}Q(x, \bar{x}\bar{y}).$$

But this identity directly follows from the obvious relation  $Q(x, y) = Q(y, x)$  and does not bring much information. In Section 6, we solve these three models by summing the functional equation over one half of the orbit only. The identity resulting from this half-orbit summation is given by the following proposition.

**Proposition 4 (Half-orbit sums).** *Let  $\mathcal{S}_1 = \{\bar{x}, \bar{y}, xy\}$  and  $\mathcal{S}_2 = \{x, y, \bar{x}\bar{y}\}$ . If the set of steps is  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  or  $\mathcal{S}_1 \cup \mathcal{S}_2$ , then*

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = \frac{xy - \bar{x} + \bar{y} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0)}{K(x, y)}.$$

**Remark.** For Gessel's walks, the orbit of  $(x, y)$  is shown in Table 3. Proposition 3 reads

$$\sum_{g \in G} \text{sign}(g)g(xyQ(x, y)) = 0,$$

although no obvious symmetry explains this identity.

### 4.3. The roots of the kernel

Recall that the kernel of the main functional equation (Lemma 2) is

$$K(x, y) = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j.$$

**Lemma 5.** *Let*

$$\Delta(x) = (1 - tA_0(x))^2 - 4t^2 A_{-1}(x)A_1(x).$$

*Let  $Y_0(x)$  and  $Y_1(x)$  denote the two roots of the kernel  $K(x, y)$ , seen as a polynomial in  $y$ . These roots are Laurent series in  $t$  with coefficients in  $\mathbb{Q}(x)$ :*

$$Y_{0,1}(x) = \frac{1 - tA_0(x) \mp \sqrt{\Delta(x)}}{2tA_1(x)}. \quad (5)$$

*Their valuations in  $t$  are respectively 1 and  $-1$ . Moreover,  $1/K(x, y)$  is a power series in  $t$  with coefficients in  $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ , and the coefficient of  $y^j$  in this series can be easily extracted using*

$$\frac{1}{K(x, y)} = \frac{1}{\sqrt{\Delta(x)}} \left( \frac{1}{1 - \bar{y}Y_0(x)} + \frac{1}{1 - y/Y_1(x)} - 1 \right). \quad (6)$$

*Proof.* The equation  $K(x, Y) = 0$  also reads

$$Y = t(A_{-1}(x) + YA_0(x) + Y^2A_1(x)). \quad (7)$$

Solving this quadratic provides the expressions of  $Y_0(x)$  and  $Y_1(x)$  given above. As  $\Delta(x) = 1 + O(t)$ , the series  $Y_1$  has valuation  $-1$  in  $t$ , and first term  $1/(tA_1(x))$ . The equation

$$Y_0(x)Y_1(x) = \frac{A_{-1}(x)}{A_1(x)}$$

then shows that  $Y_0(x)$  has valuation 1. This is also easily seen on (7), which also implies that  $Y_0$  has coefficients in  $\mathbb{Q}[x, \bar{x}]$ . The equation

$$Y_0(x) + Y_1(x) = \frac{1}{tA_1(x)} - \frac{A_0(x)}{A_1(x)}$$

shows that for  $n \geq 1$ , the coefficient of  $t^n$  in  $Y_1(x)$  is also a Laurent polynomial in  $x$ . This is not true, in general, of the coefficients of  $t^{-1}$  and  $t^0$ .

The identity (6) results from a partial fraction expansion in  $y$ . Note that, as both  $Y_0$  and  $1/Y_1$  have valuation 1 in  $t$ , the expansion in  $y$  of  $1/K(x, y)$  gives

$$[y^j] \frac{1}{K(x, y)} = \begin{cases} \frac{Y_0(x)^{-j}}{\sqrt{\Delta(x)}} & \text{if } j \leq 0 \\ \frac{Y_1(x)^{-j}}{\sqrt{\Delta(x)}} & \text{if } j \geq 0. \end{cases}$$

■

### 4.4. Canonical factorization of series with coefficients in $\mathbb{Q}[x, \bar{x}]$

Consider the discriminant of the the kernel, seen as a polynomial in  $y$ :

$$\Delta(x) = (1 - tA_0(x))^2 - 4t^2 A_{-1}(x)A_1(x).$$

Say it has valuation  $-\delta$  in  $x$ , and degree  $d$  in  $x$ . As a polynomial in  $x$ , it admits  $\delta + d$  roots  $X_i \equiv X_i(t)$ , for  $1 \leq i \leq \delta + d$ , which are Puiseux series in  $t$  with complex coefficients. Exactly  $\delta$  of them, say  $X_1, \dots, X_\delta$ , are finite (and actually vanish) at  $t = 0$ . The remaining  $d$  roots,  $X_{\delta+1}, \dots, X_{\delta+d}$ , have a negative valuation in  $t$  and thus diverge at  $t = 0$ . (We refer to [45, Chapter 6] for generalities on solutions of algebraic equations with coefficients in  $\mathbb{C}(t)$ .) We write

$$\Delta(x) = \Delta_0 \Delta_-(\bar{x}) \Delta_+(x),$$

with

$$\begin{aligned}\Delta_-(\bar{x}) \equiv \Delta_-(\bar{x}; t) &= \prod_{i=1}^{\delta} (1 - \bar{x}X_i), \\ \Delta_+(x) \equiv \Delta_+(x; t) &= \prod_{i=\delta+1}^{\delta+d} (1 - x/X_i),\end{aligned}$$

and

$$\Delta_0 \equiv \Delta_0(t) = (-1)^{\delta} \frac{[\bar{x}^{\delta}] \Delta(x)}{\prod_{i=1}^{\delta} X_i} = (-1)^d [x^d] \Delta(x) \prod_{i=\delta+1}^{\delta+d} X_i.$$

It can be seen that  $\Delta_0$  (resp.  $\Delta_-(\bar{x})$ ,  $\Delta_+(x)$ ) is a power series in  $t$  with constant term 1 and coefficients in  $\mathbb{C}$  (resp.  $\mathbb{C}[\bar{x}]$ ,  $\mathbb{C}[x]$ ). The above factorization is an instance of the *canonical factorization* of series in  $\mathbb{Q}[x, \bar{x}][[t]]$ , which was introduced by Gessel [21], and has proved useful in several walk problems since then [9, 15, 12].

## 5. Solutions via orbit sums

### 5.1. A general result

**Proposition 6.** *For the 23 models associated with a group of order at most 8, except from the 4 cases  $\mathcal{S} = \{\bar{x}, \bar{y}, xy\}$ ,  $\mathcal{S} = \{x, y, \bar{x}\bar{y}\}$ ,  $\mathcal{S} = \{x, y, \bar{x}, \bar{y}, xy, \bar{x}\bar{y}\}$  and  $\mathcal{S} = \{x, \bar{x}, xy, \bar{x}\bar{y}\}$ , the following holds. The rational function*

$$R(x, y; t) = \frac{1}{K(x, y; t)} \sum_{g \in G} \text{sign}(g) g(xy)$$

is a power series in  $t$  with coefficients in  $\mathbb{Q}(x)[y, \bar{y}]$  (Laurent polynomials in  $y$ , having coefficients in  $\mathbb{Q}(x)$ ). However, the positive part in  $y$  of  $R(x, y; t)$ , denoted  $R^+(x, y; t)$ , is a power series in  $t$  with coefficients in  $\mathbb{Q}[x, \bar{x}, y]$ . Extracting the positive part in  $x$  of  $R^+(x, y; t)$  gives  $xyQ(x, y; t)$ . In brief,

$$xyQ(x, y; t) = [x^>][y^>]R(x, y; t).$$

In particular,  $Q(x, y; t)$  is  $D$ -finite. The number of  $n$ -step walks ending at  $(i, j)$  is

$$q(i, j; n) = [x^{i+1}y^{j+1}] \left( \sum_{g \in G} \text{sign}(g) g(xy) \right) S(x, y)^n \quad (8)$$

where

$$S(x, y) = \sum_{(p, q) \in \mathcal{S}} x^p y^q.$$

*Proof.* We begin with the 16 models associated with a group of order 4 (Table 1). We will then address the three remaining cases,  $\mathcal{S} = \{\bar{x}, y, x\bar{y}\}$ ,  $\mathcal{S} = \{x, \bar{x}, x\bar{y}, \bar{x}y\}$  and  $\mathcal{S} = \{x, \bar{x}, y, \bar{y}, x\bar{y}, \bar{x}y\}$ .

All models with a group of order 4 exhibit a vertical symmetry. That is,  $K(x, y) = K(\bar{x}, y)$ . As discussed in Section 3, the orbit of  $(x, y)$  reads

$$(x, y) \xleftrightarrow{\Phi} (\bar{x}, y) \xleftrightarrow{\Psi} (\bar{x}, C(x)\bar{y}) \xleftrightarrow{\Phi} (x, C(x)\bar{y}) \xleftrightarrow{\Psi} (x, y),$$

with  $C(x) = \frac{A_{-1}(x)}{A_1(x)}$ . The orbit sum of Proposition 3 reads

$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}C(x)Q(\bar{x}, C(x)\bar{y}) + x\bar{y}C(x)Q(x, C(x)\bar{y}) = R(x, y).$$

Clearly, both sides of this identity are series in  $t$  with coefficients in  $\mathbb{Q}(x)[y, \bar{y}]$ . Let us extract the positive part in  $y$ : only the first two terms of the left-hand side contribute, and we obtain

$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) = R^+(x, y).$$

Extracting now the positive part in  $x$  gives the result, since the second term of the left-hand side does not contribute.

Let us now examine the cases  $\mathcal{S} = \{\bar{x}, y, x\bar{y}\}$ ,  $\mathcal{S} = \{x, \bar{x}, x\bar{y}, \bar{x}y\}$  and  $\mathcal{S} = \{x, \bar{x}, y, \bar{y}, x\bar{y}, \bar{x}y\}$ . For each of them, the orbit of  $(x, y)$  (shown in Tables 2 and 3) consists of pairs of the form  $(x^a y^b, x^c y^d)$  for integers  $a, b, c$  and  $d$ . This shows that  $R(x, y)$  is a series in  $t$  with coefficients in  $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ . When extracting the positive part in  $x$  and  $y$  from the orbit sum of Proposition 3, it is easily checked, in each of the three cases, that only the term  $xyQ(x, y)$  remains in the left-hand side. The result follows.  $\blacksquare$

## 5.2. Two models with algebraic specializations: $\{\bar{x}, y, x\bar{y}\}$ and $\{x, \bar{x}, y, \bar{y}, x\bar{y}, \bar{x}y\}$

Consider the case  $\mathcal{S} = \{\bar{x}, y, x\bar{y}\} \equiv \{W, N, SE\}$ . A walk  $w$  with steps taken from  $\mathcal{S}$  remains in the first quadrant if each of its prefixes contains more N steps than SE steps, and more SE steps than W steps. These walks are thus in bijection with the *Yamanouchi words* that encode Young tableaux of height at most 3 (Figure 3), or, via the Schensted correspondence [43], involutions with no decreasing subsequence of length 4. The enumeration of Young tableaux is a well-understood topic. In particular, the number of tableaux of a given shape—and hence the number of  $n$ -step walks ending at a prescribed position—is given by the hook-length formula [43]. It is also known that the total number of tableaux of size  $n$  is the  $n$ th Motzkin number [42]. We recover and refine those results here, using orbit and half-orbit sums. Then, we show that the case  $\mathcal{S} = \{x, \bar{x}, y, \bar{y}, x\bar{y}, \bar{x}y\}$ , which, to our knowledge, has never been solved, behaves very similarly. In particular, the total number of  $n$ -step walks confined to the quadrant is also related to Motzkin numbers (Proposition 8).

6	7	10	
3	5	9	
1	2	4	8

N–N–SE–N–SE–W–W–N–SE–V

FIGURE 3. A Young tableau of height 3 and the corresponding quarter plane walk with steps taken from  $\{W, N, SE\}$ .

**Proposition 7.** *The generating function of walks with steps  $W, N, SE$  confined to the quarter plane is the non-negative part (in  $x$  and  $y$ ) of a rational series in  $t$  having coefficients in  $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ :*

$$Q(x, y; t) = [x^{\geq} y^{\geq}] \tilde{R}(x, y; t), \quad \text{with} \quad \tilde{R}(x, y; t) = \frac{(1 - \bar{x}\bar{y})(1 - \bar{x}^2 y)(1 - x\bar{y}^2)}{1 - t(\bar{x} + y + x\bar{y})}.$$

In particular,  $Q(x, y)$  is  $D$ -finite. The number of walks of length  $n = 3m + 2i + j$  ending at  $(i, j)$  is

$$q(i, j; n) = \frac{(i+1)(j+1)(i+j+2)(3m+2i+j)!}{m!(m+i+1)!(m+i+j+2)!}.$$

In particular,

$$q(0, 0; 3m) = \frac{2(3m)!}{m!(m+1)!(m+2)!} \sim \sqrt{3} \frac{3^{3m}}{\pi m^4},$$

so that  $Q(0, 0; t)$ —and hence  $Q(x, y; t)$ —is transcendental.

However, the specialization  $Q(x, 1/x; t)$  is algebraic of degree 2:

$$Q(x, 1/x; t) = \frac{1 - t\bar{x} - \sqrt{1 - 2\bar{x}t + t^2\bar{x}^2 - 4t^2x}}{2xt^2}.$$

In particular, the total number of  $n$ -step walks confined to the quadrant is the  $n^{\text{th}}$  Motzkin number:

$$Q(1, 1; t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2} = \sum_{n \geq 0} t^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{n}{2k} \binom{2k}{k}.$$

*Proof.* The orbit of  $(x, y)$  under the action of  $G$  is shown in (3). The first result of the proposition is a direct application of Proposition 6, with  $\tilde{R}(x, y) = R(x, y)/(xy)$ . It is then an easy task to extract the coefficient of  $x^i y^j t^n$  in  $\tilde{R}(x, y; t)$ , using

$$[x^i y^j](\bar{x} + y + x\bar{y})^n = \frac{(3m + 2i + j)!}{m!(m+i)!(m+i+j)!}$$

if  $n = 3m + 2i + j$ .

The algebraicity of  $Q(x, \bar{x})$  can be proved as follows: let us form the alternating sum of the three equations obtained from Lemma 2 by replacing  $(x, y)$  by the first 3 elements of the orbit. These elements are those in which  $y$  occurs with a non-negative exponent. This gives

$$K(x, y)(xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x})) = xy - \bar{x}y^2 + \bar{x}^2y - tx^2Q(x, 0) - t\bar{x}Q(0, \bar{x}).$$

We now specialize this equation to two values of  $y$ . First, replace  $y$  by  $\bar{x}$ : the second and third occurrence of  $Q$  in the left-hand side cancel out, leaving

$$K(x, \bar{x})Q(x, \bar{x}) = 1 - tx^2Q(x, 0) - t\bar{x}Q(0, \bar{x}).$$

For the second specialization, replace  $y$  by the root  $Y_0(x)$  of the kernel (see (5)). This is a well-defined substitution, as  $Y_0(x)$  has valuation 1 in  $t$ . The left-hand side vanishes, leaving

$$0 = xY_0(x) - \bar{x}Y_0(x)^2 + \bar{x}^2Y_0(x) - tx^2Q(x, 0) - t\bar{x}Q(0, \bar{x}).$$

By combining the last two equations, we obtain

$$K(x, \bar{x})Q(x, \bar{x}) = 1 - xY_0(x) + \bar{x}Y_0(x)^2 - \bar{x}^2Y_0(x).$$

The expression of  $Q(x, \bar{x})$  follows. ■

The case  $\mathcal{S} = \{N, S, W, E, SE, NW\}$  is very similar to the previous one. In particular, the orbit of  $(x, y)$  is the same in both cases. The proof of the previous proposition translates almost verbatim. Remarkably, Motzkin numbers occur again.

**Proposition 8.** *The generating function of walks with steps  $N, S, W, E, SE, NW$  confined to the quarter plane is the non-negative part (in  $x$  and  $y$ ) of a rational function:*

$$Q(x, y; t) = [x^{\geq} y^{\geq}] \tilde{R}(x, y; t), \quad \text{with} \quad \tilde{R}(x, y; t) = \frac{(1 - \bar{x}\bar{y})(1 - \bar{x}^2y)(1 - x\bar{y}^2)}{1 - t(x + y + \bar{x} + \bar{y} + x\bar{y} + \bar{x}y)}.$$

In particular,  $Q(x, y; t)$  is  $D$ -finite. The specialization  $Q(x, 1/x; t)$  is algebraic of degree 2:

$$Q(x, 1/x; t) = \frac{1 - tx - t\bar{x} + \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2(1 + x)(1 + \bar{x})}}{2t^2(1 + x)(1 + \bar{x})}.$$

In particular, the total number of  $n$ -step walks confined to the quadrant is  $2^n$  times the  $n^{\text{th}}$  Motzkin number:

$$Q(1, 1; t) = \frac{1 - 2t - \sqrt{(1 + 2t)(1 - 6t)}}{8t^2} = \sum_{n \geq 0} t^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^n}{k+1} \binom{n}{2k} \binom{2k}{k}.$$

### 5.3. The case $\mathcal{S} = \{x, \bar{x}, x\bar{y}, \bar{x}y\}$

Consider quadrant walks made of E, W, NW and SE steps. The linear transformation  $(i, j) \mapsto (i+j, j)$  maps them bijectively to walks with E, W, N and S steps confined to  $\{(i, j) : 0 \leq j \leq i\}$ . These walks were studied by Gouyou-Beauchamps, and shown to be in bijection with Young tableaux of height at most 4 [24, 25]. In particular, he proved that the number of  $n$ -step walks ending on the  $x$ -axis is a product of Catalan numbers.

Here, we first specialize Proposition 6 to obtain the number of  $n$ -step walks ending at  $(i, j)$ . Then, we perform an indefinite summation on  $i, j$ , or both  $i$  and  $j$  to obtain additional closed form expressions, included Gouyou-Beauchamps's.

**Proposition 9.** *The generating function of walks with steps  $E, W, NW, SE$  confined to the quarter plane is the non-negative part (in  $x$  and  $y$ ) of a rational function:*

$$Q(x, y; t) = [x^{\geq} y^{\geq}] \tilde{R}(x, y; t),$$

with

$$\tilde{R}(x, y; t) = \frac{(1 - \bar{x})(1 + \bar{x})(1 - \bar{y})(1 - \bar{x}^2 \bar{y})(1 - x\bar{y})(1 + x\bar{y})}{1 - t(x + \bar{x} + x\bar{y} + \bar{x}y)}.$$

In particular,  $Q(x, y; t)$  is  $D$ -finite. The number of walks of length  $n = 2m + i$  ending at  $(i, j)$  is

$$q(i, j; 2m + i) = \frac{(i + 1)(j + 1)(i + j + 2)(i + 2j + 3)}{(2m + i + 1)(2m + i + 2)(2m + i + 3)^2} \binom{2m + i + 3}{m - j} \binom{2m + i + 3}{m + 1}.$$

In particular,

$$q(0, 0; 2m) = \frac{6(2m)!(2m + 2)!}{m!(m + 1)!(m + 2)!(m + 3)!} \sim \frac{24 \cdot 4^{2m}}{\pi m^5}$$

so that  $Q(0, 0; t)$  —and hence  $Q(x, y; t)$ — is transcendental.

Before we prove this proposition, let us state a corollary. Recall that a hypergeometric sequence  $(f(k))_k$  is *Gosper summable* (in  $k$ ) if there exists another hypergeometric sequence  $(g(k))_k$  such that  $f(k) = g(k + 1) - g(k)$ . In this case, indefinite summation can be performed in closed form [40, Chapter 5]:

$$\sum_{k=k_0}^{k_1} f(k) = g(k_1 + 1) - g(k_0).$$

The numbers  $q(i, j; n)$  of Proposition 9 have remarkable Gosper properties, from which we now derive Gouyou-Beauchamps's result for walks ending on the  $x$ -axis, and more.

**Corollary 10.** *The numbers  $q(i, j; n)$  are Gosper-summable in  $i$  and in  $j$ . Hence sums of the form  $\sum_{i=i_0}^{i_1} q(i, j; n)$  and  $\sum_{j=j_0}^{j_1} q(i, j; n)$  have closed form expressions. In particular, the number of walks of length  $n$  ending at ordinate  $j$  is*

$$q(-, j; n) := \sum_{i \geq 0} q(i, j; n) = \begin{cases} \frac{(j + 1)(2m)!(2m + 2)!}{(m - j)!(m + 1)!^2(m + j + 2)!} & \text{if } n = 2m, \\ \frac{2(j + 1)(2m + 1)!(2m + 2)!}{(m - j)!(m + 1)!(m + 2)!(m + j + 2)!} & \text{if } n = 2m + 1. \end{cases}$$

Similarly, the number of walks of length  $n = 2m + i$  ending at abscissa  $i$  is

$$q(i, -; 2m + i) := \sum_{j \geq 0} q(i, j; 2m + i) = \frac{(i + 1)(2m + i)!(2m + i + 2)!}{m!(m + 1)!(m + i + 1)!(m + i + 2)!}.$$

In particular, when  $n$  is even, as many  $n$ -step walks end on the  $x$ - and  $y$ -axes.

The numbers  $q(i, -; n)$  and  $q(-, j; n)$  defined above are again Gosper-summable in  $i$  and  $j$  respectively. Hence sums of the form  $\sum_{i=i_0}^{i_1} q(i, -; n) = \sum_{i=i_0}^{i_1} \sum_j q(i, j; n)$  and  $\sum_{j=j_0}^{j_1} q(-, j; n) = \sum_{j=j_0}^{j_1} \sum_i q(i, j; n)$ , which count walks ending between certain vertical or horizontal lines, have simple closed form expressions. In particular, the total number of quarter plane walks of length  $n$  is

$$q(-, -; n) := \sum_{i, j \geq 0} q(i, j; n) = \begin{cases} \frac{(2m)!(2m + 1)!}{m!^2(m + 1)!^2} & \text{if } n = 2m, \\ \frac{(2m + 1)!(2m + 2)!}{m!(m + 1)!^2(m + 2)!} & \text{if } n = 2m + 1. \end{cases}$$

The asymptotic behaviour of these numbers is found to be  $c \cdot 4^n / n^2$  for  $q(-, -; n)$  and  $c \cdot 4^n / n^3$  for  $q(0, -; n)$  and  $q(-, 0; n)$ , which shows that the series  $Q(1, 1; t)$ ,  $Q(0, 1; t)$  and  $Q(1, 0; t)$  are transcendental.

*Proof.* The orbit of  $(x, y)$  under the action of  $G$  is shown in Table 3. In particular,  $G$  has order 8. The expression of  $Q(x, y; t)$  given in Proposition 9 is a direct application of Proposition 6, with  $\tilde{R}(x, y) = R(x, y)/(xy)$ . It is then an easy task to extract the coefficient of  $x^i y^j t^n$  in  $\tilde{R}(x, y; t)$ , using

$$x + \bar{x} + \bar{x}y + x\bar{y} = (1 + \bar{y})(x + \bar{x}y) \quad \text{and} \quad [x^i y^j](x + \bar{x} + \bar{x}y + x\bar{y})^n = \binom{n}{m+i} \binom{n}{m-j}$$

for  $n = 2m + i$ .

For the first part of corollary, that is, the expressions of  $q(-, j; n)$  and  $q(i, -; n)$ , it suffices to check the following identities, which we have obtained using the implementation of Gosper's algorithm found in the `sumtools` package of Maple:

$$\begin{aligned} q(2i, j; 2m) &= g_1(i, j; m) - g_1(i+1, j; m), \\ q(2i+1, j; 2m+1) &= g_2(i, j; m) - g_2(i+1, j; m), \\ q(i, j; 2m+i) &= g_3(i, j; m) - g_3(i, j+1; m), \end{aligned}$$

with

$$\begin{aligned} g_1(i, j; m) &= \frac{2(1+j)(m+1+2i(i+1+j))(2m)!(2m+1)!}{(m-i-j)!(m-i+1)!(m+i+1)!(m+i+j+2)!}, \\ g_2(i, j; m) &= \frac{2(1+j)(m+1+j(1+2i)+2(1+i)^2)(2m+1)!(2m+2)!}{(m-i-j)!(m-i+1)!(m+i+2)!(m+i+j+3)!}, \\ g_3(i, j; m) &= \frac{(1+i)(m+1+(i+j+1)(1+j))(2m+i)!(2m+i+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+2)!}. \end{aligned}$$

These three identities respectively lead to

$$q(-, j; 2m) = g_1(0, j; m), \quad q(-, j; 2m+1) = g_2(0, j; m) \quad \text{and} \quad q(i, -; 2m+i) = g_3(i, 0; m)$$

as stated in the corollary.

For the second part, we have used the following identities, also discovered (and proved) using Maple:

$$\begin{aligned} q(2i, -; 2m) &= g_4(i; m) - g_4(i+1; m), \\ q(2i+1, -; 2m+1) &= g_5(i; m) - g_5(i+1; m), \\ q(-, j; 2m) &= g_6(j; m) - g_6(j+1; m), \\ q(-, j; 2m+1) &= g_7(j; m) - g_7(j+1; m), \end{aligned}$$

with

$$\begin{aligned} g_4(i; m) &= \frac{(2m)!(2m+1)!}{(m-i)!(m-i+1)!(m+i)!(m+i+1)!}, \\ g_5(i; m) &= \frac{(2m+1)!(2m+2)!}{(m-i)!(m-i+1)!(m+i+1)!(m+i+2)!}, \\ g_6(j; m) &= \frac{(2m)!(2m+1)!}{(m-j)!m!(m+1)!(m+j+1)!}, \\ g_7(j; m) &= \frac{(2m+1)!(2m+2)!}{(m-j)!(m+1)!(m+2)!(m+j+1)!}. \end{aligned}$$

These identities give two ways to determine the total number of  $n$ -step walks in the quadrant, as

$$q(-, -; 2m) = g_4(0; m) = g_6(0; m) \quad \text{and} \quad q(-, -; 2m+1) = g_5(0; m) = g_7(0; m). \quad \blacksquare$$

## 6. Solutions via half-orbit sums

In this section we solve in a unified manner the 3 models whose orbit has an  $x/y$  symmetry:  $\mathcal{S}_1 = \{\bar{x}, \bar{y}, xy\}$ ,  $\mathcal{S}_2 = \{x, y, \bar{x}\bar{y}\}$  and  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ . Remarkably, in all three cases the generating function  $Q(x, y; t)$  is found to be algebraic. The first problem was solved in 1965 by Kreweras, who obtained expressions of the numbers  $q(i, j; n)$ . It was then proved by Gessel that the series  $Q(x, y; t)$  is algebraic [22]. Simpler solutions were derived later [12, 31, 37]. See also [20, 18], where the stationary distribution of a related random walk in the quarter plane is obtained, and found to be algebraic. The case  $\mathcal{S} = \mathcal{S}_2$  was solved by the second author in [35], and the case  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$  is, to our knowledge, new.

Recall from Proposition 4 that for each of these three models,

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = \frac{xy - \bar{x} + \bar{y} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0)}{K(x, y)}.$$

Extract from this equation the coefficient of  $y^0$ : in the left-hand side, only the second term contributes, and its contribution is  $\bar{x}Q_d(\bar{x})$ , where  $Q_d(x) \equiv Q_d(x; t)$  is the generating function of walks ending on the diagonal:

$$Q_d(x; t) = \sum_{n, i \geq 0} t^n x^i q(i, i; n).$$

The coefficient of  $y^0$  in the right-hand side can be easily extracted using the expansion of  $1/K(x, y)$  given in Lemma 5. This gives

$$-\bar{x}Q_d(\bar{x}) = \frac{1}{\sqrt{\Delta(x)}} \left( xY_0(x) - \bar{x} + \frac{1}{Y_1(x)} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0) \right),$$

or, given the expression (5) of  $Y_0$  and the fact that  $Y_0Y_1 = \bar{x}$ ,

$$\frac{x}{tA_1(x)} - \bar{x}Q_d(\bar{x}) = \frac{1}{\sqrt{\Delta(x)}} \left( \frac{x(1 - tA_0(x))}{tA_1(x)} - \bar{x} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0) \right).$$

Let us write the canonical factorization of  $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$  (see Section 4.4). Multiplying the equation by  $A_1(x)\sqrt{\Delta_-(\bar{x})}$  gives

$$\begin{aligned} \sqrt{\Delta_-(\bar{x})} \left( \frac{x}{t} - \bar{x}A_1(x)Q_d(\bar{x}) \right) = \\ \frac{1}{\sqrt{\Delta_0\Delta_+(x)}} \left( \frac{x(1 - tA_0(x))}{t} - \bar{x}A_1(x) - 2txA_{-1}(x)A_1(x)Q(x, 0) + t\epsilon A_1(x)Q(0, 0) \right). \end{aligned} \quad (9)$$

Each term in this equation is a power series in  $t$  with coefficients in  $\mathbb{Q}[x, \bar{x}]$ . Moreover, very few positive powers of  $x$  occur in the left-hand side, while very few negative powers in  $x$  occur in the right-hand side. We will extract from this equation the positive and negative parts in  $x$ , and this will give algebraic expressions for the unknown series  $Q_d(x)$  and  $Q(x, 0)$ . At this stage, we have to consider each model separately.

### 6.1. The case $\mathcal{S} = \{\bar{x}, \bar{y}, xy\}$

We have  $A_{-1}(x) = 1$ ,  $A_0(x) = \bar{x}$ ,  $A_1(x) = x$  and  $\epsilon = 0$ . The discriminant  $\Delta(x)$  reads  $(1 - t\bar{x})^2 - 4t^2x$ . The curve  $\Delta(x; t) = 0$  has a rational parametrization in terms of the series  $W \equiv W(t)$ , defined as the only power series in  $t$  satisfying

$$W = t(2 + W^3). \quad (10)$$

Replacing  $t$  by  $W/(2 + W^3)$  in  $\Delta(x)$  gives the canonical factorization as  $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\bar{x})$  with

$$\Delta_0 = \frac{4t^2}{W^2}, \quad \Delta_+(x) = 1 - xW^2, \quad \Delta_-(\bar{x}) = 1 - \frac{W(W^3 + 4)}{4x} + \frac{W^2}{4x^2}.$$

Extracting the positive part in  $x$  from (9) gives

$$\frac{x}{t} = -\frac{(2t^2x^2Q(x,0) - x + 2t)W}{2t^2\sqrt{1-xW^2}} + \frac{W}{t},$$

from which we obtain an expression of  $Q(x,0)$  in terms of  $W$ . Extracting the non-positive part in  $x$  from (9) gives

$$\sqrt{1 - \frac{W(W^3 + 4)}{4x} + \frac{W^2}{4x^2}} \left( \frac{x}{t} - Q_d(\bar{x}) \right) - \frac{x}{t} = -\frac{W}{t},$$

from which we obtain an expression of  $Q_d(\bar{x})$ . We recover the results of [12].

**Proposition 11.** *Let  $W \equiv W(t)$  be the power series in  $t$  defined by (10). Then the generating function of quarter plane walks formed of  $W$ ,  $S$  and  $NE$  steps, and ending on the  $x$ -axis is*

$$Q(x,0;t) = \frac{1}{tx} \left( \frac{1}{2t} - \frac{1}{x} - \left( \frac{1}{W} - \frac{1}{x} \right) \sqrt{1-xW^2} \right).$$

Consequently, the length generating function of walks ending at  $(i,0)$  is

$$[x^i]Q(x,0;t) = \frac{W^{2i+1}}{2.4^i t} \left( C_i - \frac{C_{i+1}W^3}{4} \right),$$

where  $C_i = \binom{2i}{i}/(i+1)$  is the  $i$ -th Catalan number. The Lagrange inversion formula gives the number of such walks of length  $m = 3m + 2i$  as

$$q(i,0;3m+2i) = \frac{4^m(2i+1)}{(m+i+1)(2m+2i+1)} \binom{2i}{i} \binom{3m+2i}{m}.$$

The generating function of walks ending on the diagonal is

$$Q_d(x;t) = \frac{W - \bar{x}}{t\sqrt{1-xW(1+W^3/4) + x^2W^2/4}} + \bar{x}/t.$$

Note that  $Q(0,0)$  is algebraic of degree 3,  $Q(x,0)$  is algebraic of degree 6, and  $Q(x,y)$  (which can be expressed in terms of  $Q(x,0)$ , and  $Q(0,y) = Q(y,0)$  using the functional equation we started from) is algebraic of degree 12.

## 6.2. The case $\mathcal{S} = \{x, y, \bar{x}\bar{y}\}$

Observe that the steps of this model are obtained by reversing the steps of the former model. In particular, the series  $Q(0,0)$  counting walks that start and end at the origin is the same in both models. This observation was used in [35] to solve the latter case. We present here a self-contained solution.

We have  $A_{-1}(x) = \bar{x}$ ,  $A_0(x) = x$ ,  $A_1(x) = 1$  and  $\epsilon = 1$ . The discriminant  $\Delta(x)$  is now  $(1-tx)^2 - 4t^2\bar{x}$ , and is obtained by replacing  $x$  by  $\bar{x}$  in the discriminant of the previous model. In particular, the canonical factors of  $\Delta(x)$  are

$$\Delta_0 = \frac{4t^2}{W^2}, \quad \Delta_+(x) = 1 - \frac{W(W^3+4)}{4}x + \frac{W^2}{4}x^2, \quad \Delta_-(\bar{x}) = 1 - \bar{x}W^2,$$

where  $W \equiv W(t)$  is the power series in  $t$  defined by (10). Extracting the coefficient of  $x^0$  in (9) gives

$$-\frac{W^2}{2t} = -\frac{W(W^4 + 4W + 8tQ(0,0))}{16t}$$

from which we obtain an expression of  $Q(0,0)$ . Extracting the non-negative part in  $x$  from (9) gives

$$\frac{x}{t} - \frac{W^2}{2t} = -\frac{(2xt^2Q(x,0) - xt^2Q(0,0) + t - x^2 + x^3t)W}{2xt^2\sqrt{1-xW(W^3+4)/4 + x^2W^2/4}} + \frac{W}{2xt}$$

from which we obtain an expression of  $Q(x, 0)$ . Finally, extracting the negative part in  $x$  from (9) gives

$$\left(\frac{x}{t} - \frac{Q_d(\bar{x})}{x}\right) \sqrt{1 - \frac{W^2}{x} - \frac{x}{t} + \frac{W^2}{2t}} = -\frac{W}{2xt},$$

from which we obtain an expression of  $Q_d(\bar{x})$ . We have thus recovered, and completed, the results of [35]. Note in particular how simple the number of walks of length  $n$  ending at a diagonal point  $(i, i)$  is.

**Proposition 12.** *Let  $W \equiv W(t)$  be the power series in  $t$  defined by (10). Then the generating function of quarter plane walks formed of  $N$ ,  $E$  and  $SW$  steps and ending on the  $x$ -axis is*

$$Q(x, 0; t) = \frac{W(4 - W^3)}{16t} - \frac{t - x^2 + tx^3}{2xt^2} - \frac{(2x^2 - xW^2 - W) \sqrt{1 - xW(W^3 + 4)/4 + x^2W^2/4}}{2txW}.$$

The generating function of walks ending on the diagonal is

$$Q_d(x; t) = \frac{xW(x + W) - 2}{2tx^2 \sqrt{1 - xW^2}} + \frac{1}{tx^2}.$$

Consequently, the length generating function of walks ending at  $(i, i)$  is

$$[x^i]Q_d(x; t) = \frac{W^{2i+1}}{4^{i+1}t(i+2)} \binom{2i}{i} (2i + 4 - (2i + 1)W^3).$$

The Lagrange inversion formula gives the number of such walks of length  $n = 3m + 2i$  as

$$q(i, i; 3m + 2i) = \frac{4^m(i+1)^2}{(m+i+1)(2m+2i+1)} \binom{2i+1}{i} \binom{3m+2i}{m}.$$

Note that  $Q(0, 0)$  is algebraic of degree 3,  $Q(x, 0)$  is algebraic of degree 6, and  $Q(x, y)$  (which can be expressed in terms of  $Q(x, 0)$ ,  $Q(0, y) = Q(y, 0)$  and  $Q(0, 0)$  using the functional equation we started from) is algebraic of degree 12.

### 6.3. The case $\mathcal{S} = \{x, y, \bar{x}, \bar{y}, xy, \bar{x}\bar{y}\}$

We have  $A_{-1}(x) = 1 + \bar{x}$ ,  $A_0(x) = x + \bar{x}$ ,  $A_1(x) = 1 + x$  and  $\epsilon = 1$ . The discriminant  $\Delta(x)$  is now  $(1 - t(x + \bar{x}))^2 - 4t^2(1 + x)(1 + \bar{x})$ , and is symmetric in  $x$  and  $\bar{x}$ . Two of its roots, say  $X_1$  and  $X_2$ , have valuation 1 in  $t$ , and the other two roots are  $1/X_1$  and  $1/X_2$ . By studying the two elementary symmetric functions of  $X_1$  and  $X_2$  (which are the coefficients of  $\Delta_-(\bar{x})$ ), we are led to introduce the power series  $Z \equiv Z(t)$ , satisfying

$$Z = t \frac{1 - 2Z + 6Z^2 - 2Z^3 + Z^4}{(1 - Z)^2} \quad (11)$$

and having no constant term. Replacing  $t$  by its expression in terms of  $Z$  in  $\Delta(x)$  provides the canonical factors of  $\Delta(x)$  as

$$\Delta_0 = \frac{t^2}{Z^2}, \quad \Delta_+(x) = 1 - 2Z \frac{1 + Z^2}{(1 - Z)^2} x + Z^2 x^2, \quad \Delta_-(\bar{x}) = \Delta_+(\bar{x}).$$

As in the previous case, extracting from (9) the coefficient of  $x^0$  gives an expression of  $Q(0, 0)$ :

$$Q(0, 0) = \frac{Z(1 - 2Z - Z^2)}{t(1 - Z)^2}.$$

Extracting then the positive and negative parts of (9) in  $x$  gives expressions of  $Q(x, 0)$  and  $Q_d(\bar{x})$ .

**Proposition 13.** *Let  $Z \equiv Z(t)$  be the power series with no constant term satisfying (11), and denote*

$$\Delta_+(x) = 1 - 2Z \frac{1 + Z^2}{(1 - Z)^2} x + Z^2 x^2.$$

Then the generating function of quarter plane walks formed of  $N$ ,  $S$ ,  $E$ ,  $W$ ,  $SE$  and  $NW$  steps, and ending on the  $x$ -axis is

$$Q(x, 0; t) = \frac{(Z(1-Z) + 2xZ - (1-Z)x^2) \sqrt{\Delta_+(x)}}{2txZ(1-Z)(1+x)^2} - \frac{Z(1-Z)^2 + Z(Z^3 + 4Z^2 - 5Z + 2)x - (1-2Z + 7Z^2 - 4Z^3)x^2 + x^3Z(1-Z)^2}{2txZ(1-Z)^2(1+x)^2}.$$

The generating function of walks ending on the diagonal is

$$Q_d(x; t) = \frac{1 - Z - 2xZ + x^2Z(Z-1)}{tx(1+x)(Z-1)\sqrt{\Delta_+(x)}} + \frac{1}{tx(1+x)}$$

Note that  $Q(0, 0)$  is algebraic of degree 4,  $Q(x, 0)$  is algebraic of degree 8, and  $Q(x, y)$  (which can be expressed in terms of  $Q(x, 0)$ ,  $Q(0, y) = Q(y, 0)$  and  $Q(0, 0)$  using the functional equation we started from) is algebraic of degree 16. However,  $Q \equiv Q(1, 1)$  has degree 4 only, and the algebraic equation it satisfies has a remarkable form:

$$Q(1+tQ)(1+2tQ+2t^2Q^2) = \frac{1}{1-6t}.$$

Also, Motzkin numbers seem to be lurking around, as in Proposition 8.

**Corollary 14.** *Let  $N \equiv N(t)$  be the only power series in  $t$  satisfying*

$$N = t(1 + 2N + 4N^2).$$

*Up to a factor  $t$ , this series is the generating function of the numbers  $2^n M_n$ , where  $M_n$  is the  $n^{\text{th}}$  Motzkin number:*

$$N = \sum_{n \geq 0} t^{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^n}{k+1} \binom{n}{2k} \binom{2k}{k}.$$

Then the generating function of all walks in the quadrant with steps  $N$ ,  $S$ ,  $E$ ,  $W$ ,  $SE$  and  $NW$  is

$$Q(1, 1; t) = \frac{1}{2t} \left( \sqrt{\frac{1+2N}{1-2N}} - 1 \right),$$

and the generating function of walks in the quadrant ending at the origin is

$$Q(0, 0; t) = \frac{(1+4N)^{3/2}}{2Nt} - \frac{1}{2t^2} - \frac{2}{t}.$$

The proof is elementary once the algebraic equation satisfied by  $Q(1, 1)$  and  $Q(0, 0)$  are obtained.

## 7. Final comments and questions

The above results raise, in our opinion, numerous natural questions. Here are some of them.

- (1) **Combinatorial explanations of closed form expressions:** we have obtained remarkable hypergeometric expressions for the number of walks in many classes (Propositions 7 to 12). Are there direct explanations? For instance, the fact that walks with  $N$ ,  $E$ ,  $S$  and  $W$  steps are counted by simple numbers has been explained combinatorially in [17, 5] (for walks starting and ending at the origin) and in [26] (for more general walks, but using the reflection principle). The fact that Kreweras' walks ending at the origin (Proposition 11) are counted by simple numbers has also been explained combinatorially very recently, in connection with the enumeration of planar triangulations [4]. To our knowledge, it remains to explain why Kreweras' walks ending at  $(i, 0)$  are counted by simple numbers. Note the similarities between the latter numbers and those counting walks with  $N$ ,  $E$  and  $SW$  steps ending at  $(i, i)$  (Proposition 12). Another natural problem is to relate the occurrences of Motzkin numbers in Propositions 7 and 8.

- (2) **Combinatorial explanations of algebraic series:** a related problem is to explain combinatorially why the three models solved in Section 6 have algebraic generating functions. And what about Gessel's walks (with steps E, W, NE and SW), which we have not solved in this paper, but have very recently been proved to have an algebraic generating function as well [8]?
- (3) **Counting contacts:** some authors have studied the enumeration of walks in a wedge, keeping track not only of the number of steps, but also of the number of *contacts*, or *visits* to the boundary lines [27, 39, 38]. Can our approach be extended to solve such refined problems?
- (4) **Models with an infinite group:** We conjecture that the 56 models of Table 4 are associated with an infinite group. Two of them have been proved to have a non-D-finite generating function [36]. Would this hold for all models with an infinite group? For instance, what about the case  $\mathcal{S} = \{N, E, NE, SW\}$ ? And what is the importance, if any, that can be attributed to the fact that the step sets with finite groups either exhibit a symmetry across the  $y$ -axis or have a vector sum of zero?

### 8. Tables

The tables below list the 79 step sets  $\mathcal{S}$  we consider, classified according to the cardinality of the group  $G(\mathcal{S})$ . The first three tables contain sets for which  $G(\mathcal{S})$  has cardinality 4, 6 and 8. The orbit of  $(x, y)$  under the action of this group is given in the second column. The third column lists the steps of  $\mathcal{S}$ . The fourth one displays the numbers  $q(1, 1; n)$  and  $q(0, 0; n)$  that respectively count all quarter plane walks and quarter plane walks ending at the origin. We have given the reference of these sequences when they appear in the Encyclopedia of Integer Sequences [44]. In the rightmost column, we give references on this model, both in this paper and in other papers.

#	$G(\mathcal{S})$	$\mathcal{S}$	$q(1, 1; n)$ $q(0, 0; n)$	References		
1	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y}), (x, \bar{y})$		$1, 2, 6, 18, 60, 200, 700, 2450, 8820, 31752$ (A005566) <hr/> $1, 0, 2, 0, 10, 0, 70, 0, 588, 0, 5544, 0, 56628$ (A005568)	Prop. 6, [10, 17, 5, 23, 26]		
2			$1, 1, 4, 9, 36, 100, 400, 1225, 4900, 15876$ (A018224) <hr/> $1, 0, 1, 0, 4, 0, 25, 0, 196, 0, 1764, 0, 17424$ (A001246)		Prop. 6, [10, 23, 41]	
3			$1, 2, 10, 39, 210, 960, 5340, 26250, 148610, 761796$ <hr/> $1, 0, 2, 0, 18, 0, 255, 0, 4522, 0, 91896, 0, 2047452$			Prop. 6, [10, 23]
4			$1, 3, 18, 105, 684, 4550, 31340, 219555, 1564080$ <hr/> $1, 0, 3, 6, 38, 160, 905, 4830, 28308, 166992$			
5	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{1}{x+\bar{x}}),$ $(x, \bar{y} \frac{1}{x+\bar{x}})$		$1, 1, 3, 7, 19, 49, 139, 379, 1079, 3011$ <hr/> $1, 0, 0, 0, 2, 0, 0, 0, 28, 0, 0, 0, 660, 0, 0, 0, 20020$	Prop. 6, [10, 35]		
6			$1, 2, 9, 34, 151, 659, 2999, 13714, 63799, 298397,$ $1408415, 6678827$ <hr/> $1, 0, 1, 3, 4, 20, 65, 175, 742, 2604, 9072, 36960, 139392$		Prop. 6, [10]	
7	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{1}{x+1+\bar{x}}),$ $(x, \bar{y} \frac{1}{x+1+\bar{x}})$		$1, 2, 7, 23, 84, 301, 1127, 4186, 15891, 60128, 230334$ <hr/> $1, 0, 1, 0, 4, 0, 20, 0, 126, 0, 882, 0, 6732$	Prop. 6, [10]		
8			$1, 3, 15, 74, 392, 2116, 11652, 64967, 365759, 2074574$ <hr/> $1, 0, 2, 3, 12, 40, 145, 560, 2240, 9156, 38724, 166320,$ $728508$		Prop. 6, [10]	
9	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{x+\bar{x}}{x+1+\bar{x}}),$ $(x, \bar{y} \frac{x+\bar{x}}{x+1+\bar{x}})$		$1, 2, 8, 29, 129, 535, 2467, 10844, 50982, 231404$ <hr/> $1, 0, 1, 0, 6, 0, 55, 0, 644, 0, 8694, 0, 128964$	Prop. 6, [10]		
10			$1, 3, 16, 86, 509, 3065, 19088, 120401, 771758$ <hr/> $1, 0, 2, 3, 20, 60, 345, 1400, 7770, 36876, 204876$		Prop. 6, [10]	
11	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} (x+1+\bar{x})),$ $(x, \bar{y} (x+1+\bar{x}))$		$1, 1, 3, 5, 17, 34, 121, 265, 969, 2246, 8351, 20118$ <hr/> $1, 0, 1, 0, 4, 0, 20, 0, 126, 0, 882, 0, 6732$	Prop. 6, [10]		
12			$1, 2, 7, 26, 105, 444, 1944, 8728, 39999, 186266$ <hr/> $1, 0, 2, 3, 12, 40, 145, 560, 2240, 9156, 38724, 166320,$ $728508$		Prop. 6, [10]	

13	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y} \frac{x+1+\bar{x}}{x+\bar{x}}),$ $(x, \bar{y} \frac{x+1+\bar{x}}{x+\bar{x}})$		$1, 1, 5, 13, 61, 199, 939, 3389, 16129, 61601, 295373$ <hr/> $1, 0, 1, 0, 6, 0, 55, 0, 644, 0, 8694, 0, 128964$	Prop. 6, [10]
14			$1, 2, 11, 49, 277, 1479, 8679, 49974, 301169, 1805861$ <hr/> $1, 0, 2, 3, 20, 60, 345, 1400, 7770, 36876, 204876$	Prop. 6, [10]
15	$(x, y), (\bar{x}, y),$ $(\bar{x}, \bar{y}(x + \bar{x})),$ $(x, \bar{y}(x + \bar{x}))$		$1, 1, 2, 3, 8, 15, 39, 77, 216, 459, 1265, 2739, 7842$ <hr/> $1, 0, 0, 0, 2, 0, 0, 0, 28, 0, 0, 0, 660$	Prop. 6, [10, 35]
16			$1, 2, 6, 21, 76, 290, 1148, 4627, 19038, 79554, 336112$ <hr/> $1, 0, 1, 3, 4, 20, 65, 175, 742, 2604, 9072, 36960$	Prop. 6, [10]

Table 1: The group is isomorphic to  $D_2$ . All these models have a D-finite generating function.

#	$G(\mathcal{S})$	$\mathcal{S}$	$q(1, 1; n)$ $q(0, 0; n)$	References
1	$(x, y), (\bar{x}y, y),$ $(\bar{x}y, \bar{x}), (\bar{y}, \bar{x}),$ $(\bar{y}, \bar{y}x), (x, \bar{y}x)$		$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798$ (A001006) <hr/> $1, 0, 0, 1, 0, 0, 5, 0, 0, 42, 0, 0, 462$ (A005789)	Prop. 7, [23]
2			$1, 2, 8, 32, 144, 672, 3264, 16256, 82688$ (A129400) <hr/> $1, 0, 2, 2, 12, 30, 130, 462, 1946, 7980, 34776, 153120$	Prop. 8, [23]
3	$(x, y), (\bar{x}\bar{y}, y),$ $(\bar{x}\bar{y}, x), (y, x),$ $(y, \bar{x}\bar{y}), (x, \bar{x}\bar{y})$		$1, 1, 3, 7, 17, 47, 125, 333, 939, 2597, 7183$ <hr/> $1, 0, 0, 2, 0, 0, 16, 0, 0, 192, 0, 0, 2816$ (A006335)	Prop. 11, [32, 12]
4			$1, 2, 4, 10, 26, 66, 178, 488, 1320, 3674, 10318$ <hr/> $1, 0, 0, 2, 0, 0, 16, 0, 0, 192, 0, 0, 2816$ (A006335)	Prop. 12, [35]
5			$1, 3, 14, 67, 342, 1790, 9580, 52035, 285990$ <hr/> $1, 0, 3, 4, 26, 80, 387, 1596, 7518, 34656, 167310$	Prop. 13

Table 2: The group is isomorphic to  $D_3$ . All these models have a D-finite generating function, and the last three even have an algebraic generating function.

#	$G(\mathcal{S})$	$\mathcal{S}$	$q(1, 1; n)$ $q(0, 0; n)$	References
1	$(x, y), (y\bar{x}, y),$ $(y\bar{x}, y\bar{x}^2), (\bar{x}, y\bar{x}^2),$ $(\bar{x}, \bar{y}), (x\bar{y}, \bar{y}),$ $(x\bar{y}, x^2\bar{y}), (x, \bar{y}x^2)$		$1, 1, 3, 6, 20, 50, 175, 490, 1764, 5292$ (A005558) $1, 0, 1, 0, 3, 0, 14, 0, 84, 0, 594, 0, 4719$ (A005700)	Prop. 9, [23, 24, 39]
2	$(x, y), (\bar{x}\bar{y}, y),$ $(\bar{x}\bar{y}, x^2y), (\bar{x}, x^2y),$ $(\bar{x}, \bar{y}), (xy, \bar{y}),$ $(xy, \bar{x}^2\bar{y}), (x, \bar{y}\bar{x}^2)$		$1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880$ (A060900) $1, 0, 2, 0, 11, 0, 85, 0, 782, 0, 8004, 0, 88044$ (A135404)	[30, 8]

Table 3: The group is isomorphic to  $D_4$ . Both models have a D-finite generating function, and the second one even has an algebraic generating function.

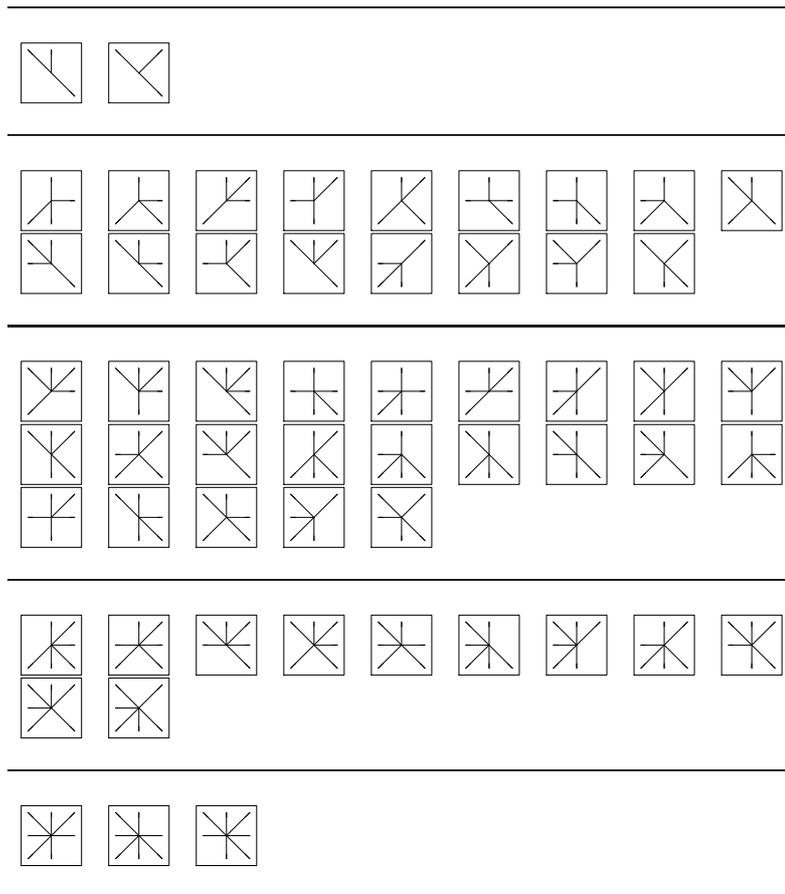


TABLE 4. Step sets with a group of order  $\geq 200$ , conjectured to be infinite.

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