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# DECAY OF MASS FOR NONLINEAR EQUATION WITH FRACTIONAL LAPLACIAN

AHMAD FINO AND GRZEGORZ KARCH

ABSTRACT. The large time behavior of nonnegative solutions to the reaction-diffusion equation  $\partial_t u = -(-\Delta)^{\alpha/2} u - u^p$ , ( $\alpha \in (0, 2]$ ,  $p > 1$ ) posed on  $\mathbb{R}^N$  and supplemented with an integrable initial condition is studied. We show that the anomalous diffusion term determines the large time asymptotics for  $p > 1 + \alpha/N$ , while nonlinear effects win if  $p \leq 1 + \alpha/N$ .

## 1. INTRODUCTION

We study the behavior, as  $t \rightarrow \infty$ , of solutions to the following initial value problem for the reaction-diffusion equation with the anomalous diffusion

$$(1) \quad \partial_t u = -\Lambda^\alpha u + \lambda u^p, \quad x \in \mathbb{R}^N, t > 0,$$

$$(2) \quad u(x, 0) = u_0(x),$$

where the pseudo-differential operator  $\Lambda^\alpha = (-\Delta)^{\alpha/2}$  with  $0 < \alpha \leq 2$  is defined by the Fourier transformation:  $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$ . Moreover, we assume that  $\lambda \in \{-1, 1\}$  and  $p > 1$ .

Nonlinear evolution problems involving fractional Laplacian describing *the anomalous diffusion* (or  $\alpha$ -stable Lévy diffusion) have been extensively studied in the mathematical and physical literature (see [2, 11, 5] for references). One of possible ways to understand the interaction between the anomalous diffusion operator (given by  $\Lambda^\alpha$  or, more generally, by the Lévy diffusion operator) and the nonlinearity in the equation under consideration is the study of the large time asymptotics of solutions to such equations. Our goal is to contribute to this theory and our results can be summarized as follows. For  $\lambda = -1$  in equation (1), nonnegative solutions to the Cauchy

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problem exist globally in time. Hence, we study the decay properties of the mass  $M(t) = \int_{\mathbb{R}^N} u(x, t) dx$  of the solutions  $u = u(x, t)$  to problem (1)-(2). We prove that  $\lim_{t \rightarrow \infty} M(t) = M_\infty > 0$  for  $p > 1 + \alpha/N$  (cf. Theorem 1, below), while  $M(t)$  tends to zero as  $t \rightarrow \infty$  if  $1 < p \leq 1 + \alpha/N$  (cf. Theorem 2). As a by-product of our analysis, we show the blow-up of all nonnegative solutions to (1)-(2) with  $\lambda = 1$  in the case of the critical nonlinearity exponent  $p = 1 + \alpha/N$  (see Theorem 3, below).

The idea which allows to express the competition between diffusive and nonlinear terms in an evolution equation by studying the large time behavior of the space integral of a solution was already introduced by Ben-Artzi & Koch [1] who considered the viscous Hamilton-Jacobi equation  $u_t = \Delta u - |\nabla u|^p$  (see also Pinsky [16]). An analogous result for the equation  $u_t = \Delta u + |\nabla u|^p$  (with the growing-in-time mass of solutions) was proved by Laurençot & Souplet [13]. Such questions concerning the asymptotic behavior of solutions to the Hamilton-Jacobi equation with the Lévy diffusion operator were answered in [11].

In the case of the classical reaction-diffusion equation (*i.e.* equation (1) with  $\alpha = 2$ ), for  $p < 1 + 2/N$ , Fujita [6] proved the nonexistence of nonnegative global-in-time solution for any nontrivial initial condition. On other hand, if  $p > 1 + 2/N$ , global solutions do exist for any sufficiently small nonnegative initial data. The proof of a blow-up of all nonnegative solutions in the critical case  $p = 1 + 2/N$  was completed in [9, 17, 12]. Analogous blow-up results for problem (1)-(2) with the fractional Laplacian (and with the critical exponent  $p = 1 + \alpha/N$  for the existence/nonexistence of solutions) are contained *e.g.* in [17, 7, 8, 3].

## 2. STATEMENT OF RESULTS

In all theorems below, we always assume that  $u = u(x, t)$  is the nonnegative (possibly weak) solution of problem (1)-(2) corresponding to the nonnegative initial datum  $u_0 \in L^1(\mathbb{R}^N)$ . Let  $u_0 \not\equiv 0$ , for simplicity of the exposition. We refer the reader to [5] for several results on the existence, the uniqueness and the regularity of solutions to (1)-(2) as well as for the proof of the maximum principle (which assures that the solution is nonnegative if the corresponding initial datum is so).

First, we deal with the equation (1) containing the absorbing nonlinearity ( $\lambda = -1$ ) and we study the decay of the “mass”

$$(3) \quad M(t) \equiv \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^t \int_{\mathbb{R}^N} u^p(x, s) dx ds.$$

**Remark.** In order to obtain equality (3), it suffices to integrate equation (1) with respect to  $x$  and  $t$ . Another method which leads to (3) and which

requires weaker regularity assumptions on a solution consists in integrating with respect to  $x$  the integral formulation of problem (1)-(2) (see (12), below) and using the Fubini theorem.  $\square$

Since we limit ourselves to nonnegative solutions, the function  $M(t)$  defined in (3) is nonnegative and non-increasing. Hence, the limit  $M_\infty = \lim_{t \rightarrow \infty} M(t)$  exists and we answer the question whether it is equal to zero or not.

In our first theorem, the diffusion phenomena determine the large time asymptotics of solutions to (1)-(2).

**Theorem 1.** *Assume that  $u = u(x, t)$  is a nonnegative nontrivial solution of (1)-(2) with  $\lambda = -1$  and  $p > 1 + \alpha/N$ . Then  $\lim_{t \rightarrow \infty} M(t) = M_\infty > 0$ .*

*Moreover, for all  $q \in [1, \infty)$*

$$(4) \quad t^{\frac{N}{\alpha}(1-\frac{1}{q})} \|u(t) - M_\infty P_\alpha(t)\|_q \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

*where the function  $P_\alpha(x, t)$  denotes the fundamental solution of the linear equation  $u_t + \Lambda^\alpha u = 0$  (cf. equation (7) below).*

In the remaining range of  $p$ , the mass  $M(t)$  converges to zero and this phenomena can be interpreted as the domination of nonlinear effects in the large time asymptotic of solutions to (1)-(2). Note here that the mass  $M(t) = \int_{\mathbb{R}^N} u(x, t) dx$  of every solution to linear equation  $u_t + \Lambda^\alpha u = 0$  is constant in time.

**Theorem 2.** *Assume that  $u = u(x, t)$  is a nonnegative solution of problem (1)-(2) with  $\lambda = -1$  and  $1 < p \leq 1 + \alpha/N$ . Then  $\lim_{t \rightarrow \infty} M(t) = 0$ .*

Let us emphasize that the proof of Theorem 2 is based on the so-called the rescaled test function method which was used by Mitidieri & Pokhozhaev (cf. e.g. [14, 15] and the references therein) to prove the nonexistence of solutions to nonlinear elliptic and parabolic equations.

As the by-product of our analysis, we can also contribute to the theory on the blow-up of solutions to (1)-(2) with  $\lambda = +1$ . Recall that the method of the rescaled test function (which we also apply here) was used in [7, 8] to show the blow-up of all positive solutions to (1)-(2) with  $\lambda = 1$  and  $p < 1 + \alpha/N$ . Here, we complete that result by the simple proof of the blow-up in the critical case  $p = 1 + \alpha/N$ .

**Theorem 3.** *If  $\lambda = 1$ ,  $\alpha \in (0, 2]$  and  $p = 1 + \alpha/N$ , then any nonnegative nonzero solution of (1)-(2) blows up in a finite time.*

## 3. PROOFS OF THEOREMS 1, 2, AND 3

Note first that any (sufficiently regular) nonnegative solution to (1)-(2) satisfies

$$(5) \quad 0 \leq \int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx + \lambda \int_0^t \int_{\mathbb{R}^N} u^p(x, s) dx ds.$$

Hence, for  $\lambda = -1$  and  $u_0 \in L^1(\mathbb{R}^N)$ , we immediately obtain

$$(6) \quad u \in L^\infty([0, \infty), L^1(\mathbb{R}^N)) \cap L^p(\mathbb{R}^N \times (0, \infty)).$$

*Proof of Theorem 1.* First, we recall that the fundamental solution  $P_\alpha = P_\alpha(x, t)$  of the linear equation  $\partial_t u + \Lambda^\alpha u = 0$  can be written via the Fourier transform as follows

$$(7) \quad P_\alpha(x, t) = t^{-N/\alpha} P_\alpha(xt^{-1/\alpha}, 1) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi - t|\xi|^\alpha} d\xi.$$

It is well-known that for each  $\alpha \in (0, 2]$ , this function satisfies

$$(8) \quad P_\alpha(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad P_\alpha(x, t) \geq 0, \quad \int_{\mathbb{R}^N} P_\alpha(x, t) dx = 1,$$

for all  $x \in \mathbb{R}^N$  and  $t > 0$ . Hence, using the Young inequality for the convolution and the self-similar form of  $P_\alpha$ , we have

$$(9) \quad \|P_\alpha(t) * u_0\|_p \leq Ct^{-N(1-1/p)/\alpha} \|u_0\|_1,$$

$$(10) \quad \|\nabla P_\alpha(t)\|_p = Ct^{-N(1-1/p)/\alpha-1/\alpha},$$

$$(11) \quad \|P_\alpha(t) * u_0\|_p \leq \|u_0\|_p,$$

for all  $p \in [1, \infty]$  and  $t > 0$ .

In the next step, using the following well-known integral representation of solutions to (1)-(2)

$$(12) \quad u(t) = P_\alpha(t) * u_0 - \int_0^t P_\alpha(t-s) * u^p(s) ds,$$

we immediately obtain the estimate  $0 \leq u(x, t) \leq P_\alpha(x, t) * u_0(x)$ . Hence, by (9) and (11) we get

$$(13) \quad \begin{aligned} \|u(t)\|_p^p &\leq \|P_\alpha(t) * u_0\|_p^p \\ &\leq \min \{ Ct^{-N(p-1)/\alpha} \|u_0\|_1^p; \|u_0\|_p^p \} \equiv H(t, p, \alpha, u_0). \end{aligned}$$

Now, for fixed  $\varepsilon \in (0, 1]$ , we consider the solution  $u^\varepsilon = u^\varepsilon(x, t)$  of (1)-(2) with the initial condition  $\varepsilon u_0(x)$ . The comparison principle implies that  $0 \leq u^\varepsilon(x, t) \leq u(x, t)$  for every  $x \in \mathbb{R}^N$  and  $t > 0$ . Hence, it suffices to show that for small  $\varepsilon > 0$ , which will be determined later, we have

$$M_\infty^\varepsilon \equiv \lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} u^\varepsilon(x, t) dx > 0.$$

Note first the using equality (5) in the case of the solution  $u^\varepsilon$ , we obtain

$$(14) \quad M_\infty^\varepsilon = \varepsilon \left\{ \int_{\mathbb{R}^N} u_0(x) dx - \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt \right\}.$$

Now, we apply (13) with  $u$  replaced by  $u^\varepsilon$ . Observe that the function  $H$  defined in (13) satisfies  $H(t, p, \alpha, \varepsilon u_0) = \varepsilon^p H(t, p, \alpha, u_0)$ . Hence

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt &\leq \frac{1}{\varepsilon} \int_0^\infty H(t, p, \alpha, \varepsilon u_0) dt \\ &= \varepsilon^{1-p} \int_0^\infty H(t, p, \alpha, u_0) dt. \end{aligned}$$

It follows immediately from the definition of the function  $H$  that the integral on the right-hand side is convergent for  $p > 1 + \alpha/N$ . Consequently,

$$\frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x, t))^p dx dt \rightarrow 0 \quad \text{as } \varepsilon \searrow 0,$$

and the constant  $M_\infty^\varepsilon$  given by (14) is positive for sufficiently small  $\varepsilon > 0$ .

From now on, the proof of the asymptotic relation (4) is standard, hence, we shall be brief in details. First we recall that for every  $u_0 \in L^1(\mathbb{R}^N)$  we have

$$(15) \quad \lim_{t \rightarrow \infty} \|P_\alpha(t) * u_0 - MP_\alpha(t)\|_1 = 0,$$

where  $M = \int_{\mathbb{R}^N} u_0(x) dx$ . This is the immediate consequence of the Taylor argument combined with an approximation argument. Details of this reasoning can be found in [2, Lemma 3.3].

Now, to complete the proof of Theorem 1, we adopt the reasoning from [13]. It follows from the integral equation (12) and inequality (3.7) with  $p = 1$  that

$$\|u(t) - P_\alpha(t - t_0) * u(t_0)\|_1 \leq \int_{t_0}^t \|u(s)\|_p^p ds \quad \text{for all } t \geq t_0 \geq 0.$$

Hence, using the triangle inequality we infer

$$(16) \quad \begin{aligned} \|u(t) - M_\infty P_\alpha(t)\|_1 &\leq \int_{t_0}^t \|u(s)\|_p^p ds \\ &+ \|P_\alpha(t - t_0) * u(t_0) - M(t_0)P_\alpha(t - t_0)\|_1 \\ &+ \|M(t_0)(P_\alpha(t - t_0) - P_\alpha(t))\|_1 \\ &+ \|P_\alpha(t)\|_1 |M(t_0) - M_\infty|. \end{aligned}$$

Applying first (8) and (15) with  $u_0 = u(t_0)$ , and next passing to the limit as  $t \rightarrow \infty$  on the right-hand side of (16), we obtain

$$\limsup_{t \rightarrow \infty} \|u(t) - M_\infty P_\alpha(t)\|_1 \leq \int_{t_0}^\infty \|u(s)\|_p^p ds + |M(t_0) - M_\infty|.$$

By letting  $t_0$  go to  $+\infty$  and using (6) we conclude that

$$(17) \quad \|u(t) - M_\infty P_\alpha(t)\|_1 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In order to obtain the asymptotic term for  $p > 1$ , observe that by the integral equation (12) and estimate (9), for each  $m \in [1, \infty]$ , we have

$$(18) \quad \|u(t)\|_m \leq \|P_\alpha(t) * u_0\|_m \leq Ct^{-N(1-1/m)/\alpha} \|u_0\|_1.$$

Hence, for every  $q \in [1, m)$ , using the Hölder inequality, we obtain

$$\begin{aligned} \|u(t) - M_\infty P_\alpha(t)\|_q &\leq \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta} (\|u(t)\|_m^\delta + \|M_\infty P_\alpha(t)\|_m^\delta) \\ &\leq Ct^{-N(1-1/q)/\alpha} \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta}, \end{aligned}$$

with  $\delta = (1 - 1/q)/(1 - 1/m)$ . Finally, applying (17) we complete the proof of Theorem 1.  $\square$

*Proof of Theorem 2.* Let us define the function  $\varphi(x, t) = (\varphi_1(x))^\ell (\varphi_2(t))^\ell$  where

$$\ell = \frac{2p-1}{p-1}, \quad \varphi_1(x) = \psi\left(\frac{|x|}{BR}\right), \quad \varphi_2(t) = \psi\left(\frac{t}{R^\alpha}\right), \quad R > 0,$$

and  $\psi$  is a smooth non-increasing function on  $[0, \infty)$  such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

The constant  $B > 0$  in the definition of  $\varphi_1$  is fixed and will be chosen later. In fact, it plays some role in the critical case  $p = 1 + \alpha/N$  only while in the subcritical case  $p < 1 + \alpha/N$  we simply put  $B = 1$ . In the following, we denote by  $\Omega_1$  and  $\Omega_2$  the supports of  $\varphi_1$  and  $\varphi_2$ , respectively:

$$\Omega_1 = \{x \in \mathbb{R}^N : |x| \leq 2BR\}, \quad \Omega_2 = \{t \in [0, \infty) : t \leq 2R^\alpha\}.$$

Now, we multiply equation (1) by  $\varphi(x, t)$  and integrate with respect to  $x$  and  $t$  to obtain

$$\begin{aligned} &\int_{\Omega_1} u_0(x) \varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \\ &= \int_{\Omega_2} \int_{\mathbb{R}^N} u(x, t) \varphi_2(t)^\ell \Lambda^\alpha (\varphi_1(x))^\ell dx dt \\ (19) \quad &\quad - \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_1(x)^\ell \partial_t \varphi_2(t)^\ell dx dt \\ &\leq \ell \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_2(t)^\ell \varphi_1(x)^{\ell-1} \Lambda^\alpha \varphi_1(x) dx dt \\ &\quad - \ell \int_{\Omega_2} \int_{\Omega_1} u(x, t) \varphi_1(x)^\ell \varphi_2(t)^{\ell-1} \partial_t \varphi_2(t) dx dt. \end{aligned}$$

In (19), we have used the inequality  $\Lambda^\alpha \varphi_1^\ell \leq \ell \varphi_1^{\ell-1} \Lambda^\alpha \varphi_1$ , (see [4, Prop. 2.3] and [10, Prop. 3.3] for its proof) which is valid for all  $\alpha \in (0, 2]$ ,  $\ell \geq 1$ , and any sufficiently regular, nonnegative, decaying at infinity function  $\varphi_1$ .

Hence, by the  $\varepsilon$ -Young inequality  $ab \leq \varepsilon a^p + C(\varepsilon)b^{\ell-1}$  (note that  $1/p + 1/(\ell - 1) = 1$ ) with  $\varepsilon > 0$ , we deduce from (19)

$$(20) \quad \int_{\Omega_1} u_0(x)\varphi(x, 0) dx - (1 + 2\ell\varepsilon) \int_{\Omega_2} \int_{\Omega_1} u^p(x, t)\varphi(x, t) dx dt \leq C(\varepsilon)\ell \left\{ \int_{\Omega_2} \int_{\Omega_1} \varphi_1 \varphi_2^\ell |\Lambda^\alpha \varphi_1|^{\ell-1} dx dt + \int_{\Omega_2} \int_{\Omega_1} \varphi_1^\ell \varphi_2 |\partial_t \varphi_2|^{\ell-1} dx dt \right\}.$$

Recall now that the functions  $\varphi_1$  and  $\varphi_2$  depend on  $R > 0$ . Hence changing the variables  $\xi = R^{-1}x$  and  $\tau = R^{-\alpha}t$ , we easily obtain from (20) the following estimate

$$(21) \quad \int_{\Omega_1} u_0(x)\varphi(x, 0) dx - (1 + 2\ell\varepsilon) \int_{\Omega_2} \int_{\Omega_1} u^p(x, t)\varphi(x, t) dx dt \leq CR^{N+\alpha-\alpha(\ell-1)},$$

where the constant  $C$  on the right hand side of (21) is independent of  $R$ . Note that  $N + \alpha - \alpha(\ell - 1) \leq 0$  if and only if  $p \leq 1 + \alpha/N$ . Now, we consider two cases.

For  $p < 1 + \alpha/N$ , we have  $N + \alpha - \alpha(\ell - 1) < 0$ . Hence, computing the limit  $R \rightarrow \infty$  in (21) and using the Lebesgue dominated convergence theorem, we obtain

$$M_\infty = \int_{\mathbb{R}^N} u_0(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \leq 2\ell\varepsilon \int_0^\infty \int_{\mathbb{R}^N} u^p dx dt.$$

Since  $u \in L^p(\mathbb{R}^N \times (0, \infty))$  (cf. (6)) and since  $\varepsilon > 0$  can be chosen arbitrary small, we immediately obtain that  $M_\infty = 0$ .

In the critical case  $p = 1 + \alpha/N$ , we estimate first term on the right hand side of inequality (19) using again by the  $\varepsilon$ -Young inequality and the second term by the Hölder inequality (with  $\bar{p} = p/(p - 1) = \ell - 1$ ) as follows

$$(22) \quad \begin{aligned} & \int_{\Omega_1} u_0(x)\varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p \varphi(x, t) dx dt \\ & \leq \ell\varepsilon \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\ & \quad + C(\varepsilon) \int_{\Omega_2} \int_{\Omega_1} \varphi_2^{\ell\bar{p}}(t) \varphi_1^{(\ell-1)\bar{p}}(x) |\Lambda^\alpha \varphi_1(x)|^{\bar{p}} dx dt \\ & \quad + \ell \left( \int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} \\ & \quad \times \left( \int_{\Omega_2} \int_{\Omega_1} \varphi_1^{\ell\bar{p}}(x) \varphi_2^{(\ell-1)\bar{p}}(t) |\partial_t \varphi_2(t)|^{\bar{p}} dx dt \right)^{1/\bar{p}}. \end{aligned}$$

Here,  $\Omega_3 = \{t \in [0, \infty) : R^\alpha \leq t \leq 2R^\alpha\}$  is the support of  $\partial_t \varphi_2$ . Note that

$$\int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

because  $u \in L^p(\mathbb{R}^N \times [0, \infty))$  (cf. (6)).

Now, introducing the new variables  $\xi = (BR)^{-1}x$ ,  $\tau = R^{-\alpha}t$  and recalling that  $p = 1 + \alpha/N$ , we rewrite (22) as follows

$$(23) \quad \int_{\Omega_1} u_0(x) \varphi(x, 0) dx - \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt - \varepsilon \ell \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\ \leq C_1 B^{N/\bar{p}} \left( \int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} + C_2 C(\varepsilon) B^{-\alpha},$$

where the constants  $C_1, C_2$  are independent of  $R, B$ , and of  $\varepsilon$ . Passing in (23) to the limit as  $R \rightarrow +\infty$  and using the Lebesgue dominated convergence theorem we get

$$(24) \quad \int_{\mathbb{R}^N} u_0(x) dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt - \varepsilon \ell \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \\ \leq C_2 C(\varepsilon) B^{-\alpha}.$$

Finally, computing the limit  $B \rightarrow \infty$  in (24) we infer that  $M_\infty = 0$  because  $\varepsilon > 0$  can be arbitrarily small. This complete the proof of Theorem 2.  $\square$

*Proof of Theorem 3.* The proof proceeds by contradiction. Let  $u$  be a non-negative non-trivial solution of (1)-(2) with  $\lambda = 1$ . Take the test function  $\varphi$  the same as in the proof of Theorem 2. Repeating the estimations which lead to (23), we obtain

$$\int_{\Omega_1} u_0(x) \varphi(x, 0) dx + \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) \varphi(x, t) dx dt \\ - \varepsilon \ell \int_{\Omega_2} \int_{\Omega_1} u^p(x, t) dx dt \\ \leq C_1 B^{N/\bar{p}} \left( \int_{\Omega_3} \int_{\Omega_1} u^p(x, t) dx dt \right)^{1/p} + C_2 C(\varepsilon) B^{-\alpha}.$$

Now, we chose  $\varepsilon = 1/(2\ell)$  in (25) and we pass to the following limits: first  $R \rightarrow \infty$ , next  $B \rightarrow \infty$ . Using the Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N} u_0(x) dx + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} u^p(x, t) dx dt \leq 0.$$

Hence,  $u(x, t) = 0$  which contradicts our assumption imposed on  $u$ .  $\square$

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